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Spherical matrix functions and Banach representability for compactly generated locally compact motion groups

Dedicated to Professor Takeshi Hirai on his 60th birthday

By

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Introduction

Let G be a locally compact group, and K a compact subgroup of G. Let $\{\mathfrak{H}, T(x)\}$ be a topologically irreducible representation of G on a locally convex complete Hausdorff topological vector space $\mathfrak{H}$. For any $\delta \in \widehat{K}$, we shall denote by $\mathfrak{H}(\delta)$ the space of vectors which transform according to δ under $k \mapsto T(k)$. Then the operator

$$T^{\circ}(x) = \int_K T(kxk^{-1}) d_K(k)$$

leaves $\mathfrak{H}(\delta)$ invariant, $d_K(k)$ being the normalized Haar measure on K. If $pd = \dim \mathfrak{H}(\delta)$ is a positive integer, then there exists a $p \times p$ matrix-valued continuous function $U_{\delta}(x)$ on G such that

$$T^{\circ}(x)|_{\mathfrak{H}(\delta)} = U_{\delta}(x) \otimes I_d$$

where d is the degree of δ and I_d is the $d \times d$ unit matrix. This function $U_{\delta}(x)$ is called a *spherical matrix function of type* δ *of height* p. Put $\chi_{\delta} = d \cdot \operatorname{trace} \delta$, then the function $U = U_{\delta}(x)$ satisfies

(a) $\int_K U(xk^{-1}) \chi_{\delta}(k) d_K(k) = U(x)$,

(b) $\int_K U(xkyk^{-1}) d_K(k) = U(x) U(y)$,

(c) $\{U(x) \mid x \in G\}$ is an irreducible family of $p \times p$ matrices.

Conversely, any $p \times p$ matrix-valued continuous function $U(x)$ which satisfies these three conditions is a spherical matrix function of type δ of height p [11].

Let $U(x)$ be the one which satisfies the above conditions (a)-(c). If $\phi(x) = \operatorname{trace} U(x)$ is positive definite, then $U(x)$ is bounded and is given by an irreducible unitary representation of G. In some cases the boundedness of

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$U(x)$ means that it is given by an irreducible unitary representation [8]. If $U(x)$ is quasi-bounded [4], then it is defined by an irreducible Banach representation, and the converse is also true. Spherical functions were treated by many people, under the assumption of boundedness or quasi-boundedness (e.g., [1], [8], [9], [12], [15]), or, for groups with good structure, such as connected semisimple Lie groups, in which we can deduce the quasi-boundedness of them even if it is not assumed expicitely (e.g., [2], [5], [6]).

Let $U_\delta(x)$ be the spherical matrix function of type δ of height p defined by a topologically irreducible representation $\{\mathfrak{H}, T(x)\}$ of G. If it is equal to a spherical matrix function defined by some Banach representation, then we say that $\{\mathfrak{H}, T(x)\}$ is essentially a Banach representation. In this case $U_\delta(x)$ is quasi-bounded. Of course there exist essentially "non-Banach" representations in general [14].

Assume that G be decomposed into the product $G = SK$, where S is a closed abelian subgroup and K a compact subgroup. Then it is likely to be true that all topologically irreducible representations containing some $\delta \in \widehat{K}$ finitely many times are subquotients of $\operatorname{Ind}_{S\uparrow G} \Lambda$ for some 1-dimensional representation Λ of S, that is, all such representations are likely to be essentially Banach representations. The author tried to prove it, but he has succeeded only when G is a compactly generated locally compact motion group, which means, after R.Gangolli [3], that $G = S \rtimes K$ is a semidirect product of a compactly generated abelian group S and a compact group K. This paper is devoted to prove it.

In §1, we recall the definition of spherical matrix functions and some relationships between spherical matrix functions and representations. A representation $\{\mathfrak{H}, T(x)\}$ is called *$\widehat{K}$-finite* if $\dim \mathfrak{H}(\delta) < +\infty$ for all $\delta \in \widehat{K}$. Then one of the most important theorem in §1 is Theorem 4, which asserts that, if $G_0 = SK_0$ (where K_0 is an open subgroup of K) is a subgroup of $G = SK$, then every $\widehat{K}$-finite topologically irreducible representation of G is essentially a Banach representation if and only if every $\widehat{K}_0$-finite topologically irreducible representation of G_0 is essentially a Banach representation.

In §2, we reduce our problem to the one for the group $G = S \rtimes K$ where $S = \mathbf{Z}^n \mathbf{T} \mathbf{R}^m$ (direct product), here $\mathbf{Z}$ denotes the set of integers, $\mathbf{T}$ the 1-dimensional torus, and $\mathbf{R}$ the real number field.

In §3, we define some algebras on G. One of them, $\mathscr{L}_\mu(G)$, is constructed using functions on $\mathbf{Z}^n \times K$ and the universal enveloping algebra of the complexification of the Lie algebra of $\mathbf{TR}^m$. $\mathscr{L}_\mu(G)$ plays an important role in place of the group algebra on G.

In §4, we complete the proof of the result which is stated in Theorem 8.

§1. Spherical matrix functions and topologically irreducible representations

1.1. Some general theorems. Let G be a locally compact group. We shall denote by $C_c(G)$ the convolution algebra of compactly supported continuous functions on G. If K is a compact subgroup of G, then $\widehat{K}$ denotes the unitary dual of K and χ_δ does the normalized trace of $\delta \in \widehat{K}$, that is, $\chi_\delta = d \cdot \text{trace } \delta$ where $d = d(\delta)$ is the degree of δ. The normalized Haar measure on K will be always denoted by $d_K(k)$. For every function $f \in C_c(G)$ we define

$$\overline{\chi_\delta} * f * \overline{\chi_\delta}(x) = \int_{K \times K} f(k_1^{-1} x k_2^{-1}) \overline{\chi_\delta(k_1) \chi_\delta(k_2)} d_K(k_1) d_K(k_2).$$

The subalgebra of these functions $\overline{\chi_\delta} * f * \overline{\chi_\delta}$ is denoted by $C_{c,\delta}(G)$. For every function $f \in C_c(G)$ we put

$$f^\circ(x) = \int_K f(kxk^{-1}) d_K(k),$$

then the set of all functions f° for $f \in C_{c,\delta}(G)$ is a subalgebra of $C_{c,\delta}(G)$ and is denoted by $I_{c,\delta}(G)$.

The space $\mathfrak{H}$ of a representation $\{\mathfrak{H}, T(x)\}$ of G is, if nothing is stated, understood to be a locally convex Hausdorff topological vector space. We assume usual continuity condition for representations (see [10]), and an integrability condition, that is, for every closed subgroup H of G and $\varphi \in C_c(H)$, the integral

$$T(\varphi) = \int_H \varphi(h) T(h) d_H(h),$$

where $d_H(h)$ denotes a left Haar measure on H, converges strongly and defines a continuous linear operator $T(\varphi)$ on $\mathfrak{H}$. If some $\delta \in \widehat{K}$ is contained in $k \mapsto T(k)$ with finite multiplicity, then the representatinn $\{\mathfrak{H}, T(x)\}$ of G is called *δ-finite*. If the multiplicities of δ are finite or zero for all $\delta \in \widehat{K}$, then it is called *$\widehat{K}$-finite*. For any $\delta \in \widehat{K}$ we put

$$E(\mathfrak{H}, \delta) = \int_K \overline{\chi_\delta(k)} T(k) d_K(k),$$

then the subspace $\mathfrak{H}(\delta) = E(\mathfrak{H}, \delta)\mathfrak{H}$ consists of all vectors which transform according to δ under $k \mapsto T(k)$. If a representation $\{\mathfrak{H}, T(x)\}$ is $\widehat{K}$-finite, then $\dim \mathfrak{H}(\delta) < +\infty$ for all $\delta \in \widehat{K}$. If it is δ-finite for some $\delta \in \widehat{K}$, then $0 < \dim \mathfrak{H}(\delta) < +\infty$. Denoting by $d_G(x)$ a (left) Haar measure on G, we put

$$T(f) = \int_G f(x) T(x) d_G(x) \qquad (f \in C_c(G)),$$

then $f \mapsto T(f)$ is a representation of $C_c(G)$ on $\mathfrak{H}$, and $T(f)\mathfrak{H}(\delta) \subset \mathfrak{H}(\delta)$ for any $f \in C_{c,\delta}(G)$. If the representation $\{\mathfrak{H}, T(x)\}$ of G is topologically irreducible, so is the represeentation of $C_{c,\delta}(G)$ on $\mathfrak{H}(\delta)$.

Assume G is unimodular and $\{\mathfrak{H}, T(x)\}$ is a δ-finite topologically irreducible representation of G. Let p be the multiplicity of δ, then dim $\mathfrak{H}(\delta)$ $=pd$ where $d=d(\delta)$. The operators

$$T^{\circ}(x)=\int_{K} T(kxk^{-1})\,d_{K}(k)$$

leave $\mathfrak{H}(\delta)$ invariant, and $T^{\circ}(x)|_{\mathfrak{H}(\delta)}$ commute with all $T(k)$ for $k \in K$. Now we decompose the subspace $\mathfrak{H}(\delta)$ into a direct sum of K-irreducible subspaces

$$\mathfrak{H}(\delta)=V_1 \oplus \cdots \oplus V_p,$$

and we get a basis of $\mathfrak{H}(\delta)$ by gathering up those of V_i, then $T^{\circ}(x)|_{\mathfrak{H}(\delta)}$ are represented by matrices

$$T^{\circ}(x)|_{\mathfrak{H}(\delta)}=\begin{bmatrix} u_{11}(x)I_d & \cdots & u_{1p}(x)I_d \\ \vdots & \ddots & \vdots \\ u_{p1}(x)I_d & \cdots & u_{pp}(x)I_d \end{bmatrix}=U_{\delta}(x)\otimes I_d$$

with respect to this basis. Here, I_d denotes the $d \times d$ unit matrix and $U_{\delta}(x)$ does a $p \times p$ matrix with coefficients $u_{ij}(x) \in \mathbf{C}$ ($\mathbf{C}$ is the complex number field). Then the $M(p, \mathbf{C})$-valued continuous function $U_{\delta}(x)$, where $M(p, \mathbf{C})$ is the set of all $p \times p$ complex matrices, is called a *spherical matrix function* of type δ of height p defined by $\{\mathfrak{H}, T(x)\}$ [11]. Note that $U_{\delta}(x)$ satisfies

(a) $\chi_{\delta} * U_{\delta}=U_{\delta}$,

(b) $\int_{K} U_{\delta}(xkyk^{-1})\,d_{K}(k)=U_{\delta}(x)\,U_{\delta}(y)$,

(c) $\{U_{\delta}(x) \mid x \in G\}$ is an irreducible family of matrices.

We should also note that these properties conversely characterize a spherical matrix function [11]. Define

$$U_{\delta}(f)=\int_{G} f(x)\,U_{\delta}(x)\,d_{G}(x)$$

for $f \in I_{c,\delta}(G)$, then $f \mapsto U_{\delta}(f)$ is an irreducible p-dimensional representation of the algebra $I_{c,\delta}(G)$. The continuous function

$$\phi_{\delta}(x)=d \cdot \text{trace } U_{\delta}(x)$$

is called a *spherical function* of type δ of height p [4].

Let $\{\mathfrak{H}, T(x)\}$ be a δ-finite topologically irreducible representation of G. We take any $v \in \mathfrak{H}(\delta)$, $v \neq 0$, and define the *fundamental subspace*

$$\mathfrak{H}_0=\{T(f)v \mid f \in C_c(G)\}.$$

This is invariant, algebraically irreducible under $T(f)$ for $f \in C_c(G)$, and independent of v. Moreover, if the representation $\{\mathfrak{H}, T(x)\}$ is δ'-finite for some $\delta' \in \widehat{K}'$ for another compact subgroup K', then $\mathfrak{H}_0=\{T(f)v' \mid f \in C_c(G)\}$

for any $v' \in \mathfrak{H}(\delta') = E(\mathfrak{H}, \delta')\mathfrak{H}$, $v' \neq 0$ [10].

We have the following relationships between spherical matrix functions, spherical functions, and representations.

Theorem 1. *Let* $\{\mathfrak{H}^1, T^1(x)\}$, $\{\mathfrak{H}^2, T^2(x)\}$ *be two topologically irreducible representations of* G. *Assume there exists at least one pair* (K, δ) *of compact subgroup* K *of* G *and* $\delta \in \widehat{K}$ *such that both* $\{\mathfrak{H}^i, T^i(x)\}$ $(i=1, 2)$ *are* δ*-finite. Denote by* $\mathfrak{H}_0^i$, $U_\delta^i(x)$, *and* $\phi_\delta^i(x)$ *for* $i = 1, 2$ *the corresponding fundamental subspaces, spherical matrix functions, and spherical functions. Then the following four conditions are equivalent.*

(i) *There exists a pair* (K, δ) *such that* $\phi_\delta^1 = \phi_\delta^2$.

(ii) *For every pair* (K, δ) *for which* $\{\mathfrak{H}^i, T^i(x)\}$ $(i=1, 2)$ *are* δ*-finite, we have* $\phi_\delta^1 = \phi_\delta^2$.

(iii) *There exists a pair* (K, δ) *such that* U_δ^1 *and* U_δ^2 *are equivalent, i.e., they are of the same degree and* $U_\delta^1(x) = PU_\delta^2(x)P^{-1}$ *for some invertible matrix* P.

(iv) *For every pair* (K, δ) *for which* $\{\mathfrak{H}^i, T^i(x)\}$ $(i=1, 2)$ *are* δ*-finite,* U_δ^1 *and* U_δ^2 *are equivalent.*

(v) *There exists a bijective linear map* I: $\mathfrak{H}_0^1 \rightarrow \mathfrak{H}_0^2$ *such that*

$$IT^1(f) = T^2(f)I \qquad (\forall f \in C_c(G)).$$

Proof. It is proved in [10] that (i), (ii), and (v) are equivalent. By the definition of spherical matrix function (iii) and (iv) follow from (v). The statements (i), (ii) clearly follow from (iii), (iv), respectively.

Remark. Assume the condition (v) in the above Theorem 1 be satisfied. Then it is easy to see

$$IT^1(x) = T^2(x)I \qquad (\forall x \in G).$$

Moreover, for every closed subgroup H of G, we have

$$IT^1(\varphi) = T^2(\varphi)I \qquad (\forall \varphi \in C_c(H)),$$

because $IT^1(\varphi)T^1(f)v = IT^1(\varphi * f)v = T^2(\varphi * f)Iv = T^2(\varphi)T^2(f)Iv = T^2(\varphi)IT^1(f)v$ for every $T^1(f)v \in \mathfrak{H}_0^1$, where $\varphi * f(x) = \int_H \varphi(h) f(h^{-1}x)\, d_H(h)$. In particular, note that

$$IE(\mathfrak{H}^1, \delta) = E(\mathfrak{H}^2, \delta)I$$

holds for any (K, δ).

Definition. When the conditions (i) ~ (v) in Theorem 1 are satisfied, we call $\{\mathfrak{H}^1, T^1(x)\}$ and $\{\mathfrak{H}^2, T^2(x)\}$ are *SF-equivalent*.

A positive valued lower semi-continuous function $\rho(x)$ on G is called a semi-norm if it is bounded on every compact subset and satisfies $\rho(xy) \leqq \rho(x)\rho(y)$ for all $x, y \in G$. A function f is *quasi-bounded* if there exists a semi-norm $\rho(x)$ such that $|f(x)| \leqq \rho(x)$. If all matrix coefficients of a spherical matrix function $U_\delta(x)$ are quasi-bounded, then we say U_δ is quasi-bounded.

Now we know the following fact [10], [11].

Theorem 2. *Let $\{\mathfrak{H}, T(x)\}$ be a topologically irreducible representation of G which is δ-finite for some (K, δ). Then the following four conditions are equivalent.*

(i) *ϕ_δ is quasi-bounded.*

(ii) *U_δ is quasi-bounded.*

(iii) *ϕ_δ, U_δ are defined by some topologically irreducible Banach representation $\{\mathfrak{B}, T_{\mathfrak{B}}(x)\}$ of G.*

(iv) *$\{\mathfrak{H}, T(x)\}$ is SF-equivalent to some topologically irreducible Banach representation $\{\mathfrak{B}, T_{\mathfrak{B}}(x)\}$ of G.*

Definition. When the conditions (i) ~ (iv) in Theorem 2 are satisfied, we say $\{\mathfrak{H}, T(x)\}$ is *essentially a Banach representation.*

Theorem 3. *Let G be a locally compact unimodular group, S a closed subgroup, and K a compact subgroup. Assume that $G=SK$, $S \cap K=\{e\}$, where e is the unit element of G, and that the decomposition is continuous. Let $\{\mathfrak{H}, T(x)\}$ be a $\widehat{K}$-finite topologically irreducible representation of G. Then $\{\mathfrak{H}, T(x)\}$ is essentially a Banach representation if and only if there exists a topologically irreducible Banach representation $\{\mathscr{B}, \Lambda(s)\}$ of S and a non-zero linear map α: $\mathfrak{H}_0 \to \mathscr{B}$ such that*

$$\alpha T(s) = \Lambda(s)\alpha \quad (\forall s \in S),$$
$$\alpha T(\varphi) = \Lambda(\varphi)\alpha \quad (\forall \varphi \in C_c(S)).$$

Proof. Assume $\{\mathfrak{H}, T(x)\}$ be essentially a Banach representation, then, by the definition of essential Banach representability, there exists a topologically irreducible Banach representation $\{\mathfrak{B}, T_{\mathfrak{B}}(x)\}$ of G and a bijective linear map $I: \mathfrak{H}_0 \to \mathfrak{B}_0$ such that

$$IT(f) = T_{\mathfrak{B}}(f)I \qquad (\forall f \in C_c(G)).$$

Then, as is noted in Remark after Theorem 1, we also have

$$IT(s) = T_{\mathfrak{B}}(s)I \quad (\forall s \in S),$$
$$IT(\varphi) = T_{\mathfrak{B}}(\varphi)I \quad (\forall \varphi \in C_c(S)).$$

For the semi-norm $\rho(x) = \|T_{\mathfrak{B}}(x)\|$ we put

$$\|f\|_\rho = \int_G |f(x)| \rho(x) d_G(x)$$

and denote by $L_\rho(G)$ the Banach algebra of all functions satisfying $\|f\|_\rho < +\infty$. We choose $\delta \in \widehat{K}$ for which $\mathfrak{B}(\delta) \neq \{0\}$, take a vector $v \in \mathfrak{B}(\delta)$, $v \neq 0$, and put

$$\mathfrak{V}_\rho = \{T_{\mathfrak{V}}(f)v \mid f \in L_\rho(G)\}.$$

This subspace is independent of such δ and v. We proved in [12] that there exists a topologically irreducible Banach representation $\{\mathscr{B}, \Lambda(s)\}$ of S and a surjective linear map $\beta : \mathfrak{V}_\rho \to \mathscr{B}$ such that

$$\beta T_{\mathfrak{V}}(s) = \Lambda(s)\beta \quad (\forall s \in S),$$
$$\beta T_{\mathfrak{V}}(\varphi) = \Lambda(\varphi)\beta \quad (\forall \varphi \in C_c(S)).$$

Then the linear map $\alpha = \beta \circ I : \mathfrak{H}_0 \to \mathscr{B}$ is not identically zero and clearly satisfies

$$\alpha T(s) = \Lambda(s)\alpha \quad (\forall s \in S),$$
$$\alpha T(\varphi) = \Lambda(\varphi)\alpha \quad (\forall \varphi \in C_c(S)).$$

Conversely, let $\{\mathscr{B}, \Lambda(s)\}$ be a topologically irreducible Banach representation of S and $\alpha : \mathfrak{H}_0 \to \mathscr{B}$ a non-zero linear map such that

$$\alpha T(s) = \Lambda(s)\alpha \quad (\forall s \in S),$$
$$\alpha T(\varphi) = \Lambda(\varphi)\alpha \quad (\forall \varphi \in C_c(S)).$$

We now induce a representation $\{\mathfrak{H}^\Lambda, T^\Lambda(x)\}$ of G from $\{\mathscr{B}, \Lambda(s)\}$. That is, $\mathfrak{H}^\Lambda$ is the Banach space of all continuous $\mathscr{B}$-valued function ξ on K in which the norm is given by

$$\|\xi\| = \sup_{k \in K} \|\xi(k)\|_{\mathscr{B}}.$$

The operators $T^\Lambda(x)$ on $\mathfrak{H}^\Lambda$ are given by

$$[T^\Lambda(x)\xi](k) = \Lambda(\sigma(kx))\xi(\kappa(kx))$$

where $kx = \sigma(kx)\kappa(kx)$, $\sigma(kx) \in S$, $\kappa(kx) \in K$. For the induced representation $\{\mathfrak{H}^\Lambda, T^\Lambda(x)\}$, which is not topologically irreducible in general, we denote by $(\mathfrak{H}^\Lambda)_0$ the $C_c(G)$-invariant subspace of $\mathfrak{H}^\Lambda$ generated by $\mathfrak{H}^\Lambda(\delta) = E(\mathfrak{H}^\Lambda, \delta)\mathfrak{H}^\Lambda$. Then, by the Frobenius reciprocity theorem [13], there exists a non-zero linear map $I : \mathfrak{H}_0 \to (\mathfrak{H}^\Lambda)_0$ satisfying

$$IT(f) = T^\Lambda(f)I \qquad (\forall f \in C_c(G)).$$

Then we clearly have

$$IE(\mathfrak{H}, \delta) = E(\mathfrak{H}^\Lambda, \delta)I$$

for any $\delta \in \widehat{K}$. Since $\mathfrak{H}_0$ is algebraically irreducible under the action of $C_c(G)$, the map I is injective. Denote by $\mathfrak{V} = \overline{I(\mathfrak{H}_0)}$ the closure of $I(\mathfrak{H}_0)$ in $\mathfrak{H}^\Lambda$, then $\{\mathfrak{V}, T^\Lambda(x)|_{\mathfrak{V}}\}$ is a Banach representation of G.

We will show that the Banach representation $\{\mathfrak{V}, T^\Lambda(x)|_{\mathfrak{V}}\}$ is topologically irreducible. Let $\mathscr{V}(\subsetneqq \mathfrak{V})$ be a G-invariant closed subspace of $\mathfrak{V}$. Since $I(\mathfrak{H}_0)$ is algebraically irreducible under the action of $C_c(G)$, we have $\mathscr{V} \cap I(\mathfrak{H}_0) = \{0\}$.

For every $\delta\in\widehat{K}$ we have $E(\mathfrak{H}^A, \delta)\mathcal{V}\subset\mathcal{V}$, $E(\mathfrak{H}^A, \delta)I(\mathfrak{H}_0)\subset I(\mathfrak{H}_0)$, and hence

$$\mathfrak{B}(\delta)=E(\mathfrak{H}^A, \delta)\mathfrak{B}\supset E(\mathfrak{H}^A, \delta)\mathcal{V}\oplus E(\mathfrak{H}^A, \delta)I(\mathfrak{H}_0).$$

On the other hand, $\dim E(\mathfrak{H}^A, \delta)I(\mathfrak{H}_0)=\dim \mathfrak{H}(\delta)<+\infty$ implies

$$\mathfrak{B}(\delta)=E(\mathfrak{H}^A, \delta)\overline{I(\mathfrak{H}_0)}=E(\mathfrak{H}^A, \delta)I(\mathfrak{H}_0).$$

This means $E(\mathfrak{H}^A, \delta)\mathcal{V}=\{0\}$ for every $\delta\in\widehat{K}$, namely, $\mathcal{V}=\{0\}$. Thus Banach representation $\{\mathfrak{B}, T^A(x)|_{\mathfrak{B}}\}$ is topologically irreducible.

We choose $\delta\in\widehat{K}$ such that $\mathfrak{H}(\delta)\neq\{0\}$ and a vector $v\in\mathfrak{H}(\delta)$, $v\neq 0$, then the injective linear map $I: \mathfrak{H}_0\rightarrow\mathfrak{B}_0$ is bijective because

$$I(\mathfrak{H}_0)=\{IT(f)v\mid f\in C_c(G)\}=\{T^A(f)Iv\mid f\in C_c(G)\}=\mathfrak{B}_0.$$

Clearly we have $IT(f)=T^A(f)|_{\mathfrak{B}}I$ for all $f\in C_c(G)$, and, by Theorem 1, this implies that the representation $\{\mathfrak{H}, T(x)\}$ is essentially a Banach representation.

1.2. Proof of Theorem 4. As is stated in Introduction, we want to study on Banach representability for a group $G=S\rtimes K$ where S is a compactly generated abelian group. To do this we will often use Theorem 4 below. Our aim here is to prove it.

Theorem 4. *Let G be a locally compact unimodular group, S a closed subgroup, and K a compact subgroup. Assume that $G=SK$, $S\cap K=\{e\}$, and that the decomposition is continuous. Let K_0 be an open subgroup of K and assume $G_0=SK_0$ is a closed subgroup of G. Then every $\widehat{K}$-finite topologically irreducible representation of G is essentially a Banach representation if and only if so is every $\widehat{K}_0$-finite topologically irreducible representation of G_0.*

First we take a left Haar measure $d_S(s)$ on S and the normalized Haar measures $d_K(k)$, $d_{K_0}(u)$ on K, K_0, respectively. Then $d_G(x)=d_S(s)d_K(k)$ $(x=sk)$, $d_{G_0}(y)=d_S(s)d_{K_0}(u)$ $(y=su)$ are Haar measures on G, G_0, respectively.

Let us begin with the proof of 'if' part. Take an arbitrary $\widehat{K}$-finite topologically irreducible representation $\{\mathfrak{H}, T(x)\}$ of G. We have to show it is essentially a Banach representation.

We choose $\delta_0\in\widehat{K}$ such that $\mathfrak{H}(\delta_0)\neq\{0\}$. Let $\tau_1,\dots,\tau_l$ be all distinct elements in $\widehat{K}_0$ contained in δ_0. Put

$$P=\sum_{j=1}^{l}E(\mathfrak{H}, \tau_j),$$

then it is clear that $\mathfrak{H}(\delta_0)\subset P\mathfrak{H}_0$. Since K_0 has a finite index in K, the Frobenius reciprocity theorem $[\delta|_{K_0}:\tau]=[\operatorname{Ind}_{K_0\uparrow K}\tau:\delta]$ means that the number of elements $\delta\in\widehat{K}$ which contain a given $\tau\in\widehat{K}_0$ is finite. This implies that the representation $\{\mathfrak{H}, T(x)\}$ is $\widehat{K}_0$-finite and hence $\dim P\mathfrak{H}_0 < +\infty$. Choose a vector $v_0 \in \mathfrak{H}(\delta_0)$, $v_0 \neq 0$. We decompose K into K_0-cosets

$$K = \bigcup_{i=0}^{n} K_0 k_i \qquad (\text{where } k_0 = e),$$

and define vectors $v_i \in \mathfrak{H}(\delta_0)$ as

$$v_i = T(k_i) v_0 \qquad (0 \leqq i \leqq n).$$

Consider the family $\mathfrak{F}$ of (not necessarily closed) proper subspaces of the fundamental subspace $\mathfrak{H}_0$ which are invariant under $E(\mathfrak{H}, \tau)$ for all $\tau \in \widehat{K}_0$ and $T(h)$ for all $h \in C_c(G_0)$.

Lemma 1. *For every $\mathcal{K} \in \mathfrak{F}$, there exists at least one $v_i \notin P\mathcal{K}$.*

Proof. Assume $P\mathcal{K}$ contains all v_i $(0 \leqq i \leqq n)$. Then $v_i \in \mathcal{K}$ $(0 \leqq i \leqq n)$ because $\mathcal{K} \supset P\mathcal{K}$.

On the other hand, let $T(f)v_0$ $(f \in C_c(G))$ be an arbitrary vector in $\mathfrak{H}_0$, then,

$$\begin{aligned} T(f)v_0 &= \int_{S\times K} f(sk)\, T(s)\, T(k) v_0 \, d_S(s) d_K(k) \\ &= \int_S T(s) \Big(\sum_{i=0}^{n} \int_{K_0} f(skk_i)\, T(kk_i) v_0 \, d_K(k) \Big) d_S(s). \end{aligned}$$

Here we put $h_i(su) = f(suk_i)$ for $s \in S$, $u \in K_0$, then $h_i \in C_c(G_0)$ and

$$\begin{aligned} T(f)v_0 &= \frac{1}{n+1} \int_S \Big(\sum_{i=0}^{n} \int_{K_0} h_i(su)\, T(su) v_i \, d_{K_0}(u) \Big) d_S(s) \\ &= \frac{1}{n+1} \sum_{i=0}^{n} T(h_i) v_i \in \mathcal{K}. \end{aligned}$$

Thus we have $\mathfrak{H}_0 = \mathcal{K}$, a contradiction.

Lemma 2. *There exists at least one maximal element in $\mathfrak{F}$, and it is closed in $\mathfrak{H}_0$.*

Proof. Let $\{\mathcal{K}_\lambda\}$ be a totally ordered subset in $\mathfrak{F}$. By Lemma 1 there exists a vector v_i such that $v_i \notin P\mathcal{K}_\lambda$ for all λ. Since the subspace $\mathcal{K} = \bigcup_\lambda \mathcal{K}_\lambda$ satisfies $P\mathcal{K} = \bigcup_\lambda P\mathcal{K}_\lambda \not\ni v_i$, we know $\mathcal{K} \in \mathfrak{F}$. Therefore, by Zorn's lemma, there exists a maximal element $\mathcal{K}_\infty$ in $\mathfrak{F}$. Choose v_j such that $v_j \notin P\mathcal{K}_\infty$. We denote by $\overline{\mathcal{K}_\infty}$ the closure of $\mathcal{K}_\infty$ in $\mathfrak{H}$. It is easy to see that $\overline{\mathcal{K}_\infty} \cap \mathfrak{H}_0$ is invariant under $T(h)$ $(\forall h \in C_c(G_0))$ and $E(\mathfrak{H}, \tau)$ $(\forall \tau \in \widehat{K}_0)$. Moreover, $\overline{\mathcal{K}_\infty} \cap \mathfrak{H}_0$ is a proper subspace of $\mathfrak{H}_0$ because $P(\overline{\mathcal{K}_\infty} \cap \mathfrak{H}_0) = P\mathcal{K}_\infty \not\ni v_j$. Thus we obtain $\overline{\mathcal{K}_\infty} \cap \mathfrak{H}_0 \in \mathfrak{F}$, which implies $\overline{\mathscr{K}_\infty} \cap \mathfrak{H}_0 = \mathscr{K}_\infty$.

By the above Lemma 2 we can pick up a maximal elemnt $\mathscr{K}$ in $\mathfrak{F}$, then it is closed in $\mathfrak{H}_0$. Denote by $\overline{\mathscr{K}}$ the closure of $\mathscr{K}$ in $\mathfrak{H}$.

Lemma 3. (i) *$\overline{\mathscr{K}}$ is G_0- and $C_c(G_0)$-invariant.*

(ii) $\mathfrak{H}_0 \cap \overline{\mathscr{K}} = \mathscr{K}$.

(iii) *$\overline{\mathscr{K}}$ is maximal in the family of $C_c(G_0)$-invariant proper closed subspaces of $\mathfrak{H}$.*

Proof. (i) Easy.

(ii) Already proved in the proof of Lemma 2.

(iii) Suppose there exists a $C_c(G_0)$-invariant closed subspace $\mathscr{V}$ of $\mathfrak{H}$ such that $\overline{\mathscr{K}} \subset \mathscr{V} \subsetneqq \mathfrak{H}$. $\mathscr{V}$ is clearly invariant under $E(\mathfrak{H}, \tau)$ ($\forall \tau \in \widehat{K}_0$). Since $\mathscr{V}$ is closed and $\mathscr{V} \subsetneqq \mathfrak{H}$, we have $\mathfrak{H}_0 \not\subset \mathscr{V}$. Thus $\mathfrak{H}_0 \supsetneqq \mathfrak{H}_0 \cap \mathscr{V} \supset \mathfrak{H}_0 \cap \overline{\mathscr{K}} = \mathscr{K}$, but this means $\mathfrak{H}_0 \cap \mathscr{V} = \mathscr{K}$ because $\mathscr{K}$ is maximal in $\mathfrak{F}$. The $\widehat{K}_0$-finiteness of the representation $\{\mathfrak{H}, T(x)\}$ implies $E(\mathfrak{H}, \tau)\mathfrak{H} \subset \mathfrak{H}_0$. Therefore, for every $\tau \in \widehat{K}_0$, we obtain

$$E(\mathfrak{H}, \tau)\mathscr{V} \subset \mathscr{V} \cap \mathfrak{H}_0 = \mathscr{K},$$

and hence

$$\mathscr{V} = \overline{\oplus \sum_{\tau \in \widehat{K}_0} E(\mathfrak{H}, \tau)\mathscr{V}} \subset \overline{\mathscr{K}},$$

which means $\mathscr{V} = \overline{\mathscr{K}}$.

Now, by Lemma 3, we can naturally define a topologically irreducible representation of G_0 on $\mathfrak{H}/\overline{\mathscr{K}}$. We shall denote it by $\{\mathfrak{H}/\overline{\mathscr{K}}, \widetilde{T}(y)\}$. This representation is of course $\widehat{K}_0$-finite. Take $\tau_0 \in \widehat{K}_0$ such that $E(\mathfrak{H}, \tau_0)\mathfrak{H} \not\subset \overline{\mathscr{K}}$ and choose a vector $w_0 \in E(\mathfrak{H}, \tau_0)\mathfrak{H}$, $w_0 \notin \overline{\mathscr{K}}$. We shall denote by $\widetilde{w}_0$ the class of w_0, then the fundamental subspace for $\{\mathfrak{H}/\overline{\mathscr{K}}, \widetilde{T}(y)\}$ is given by

$$(\mathfrak{H}/\overline{\mathscr{K}})_0 = \{\widetilde{T}(h)\widetilde{w}_0 \mid h \in C_c(G_0)\}.$$

Now we need the following general

Lemma 4. *If a representation $\{\mathfrak{H}, T(x)\}$ of G is $\widehat{K}$-finite and topologically irreducible, then, for every $\tau \in \widehat{K}_0$ and $w \in \mathfrak{H}(\tau)$, $w \neq 0$, we have*

$$\{T(h)w \mid h \in C_c(G_0)\} \subset \mathfrak{H}_0.$$

Proof. There exist distinct $\delta_1, \ldots, \delta_m \in \widehat{K}$ such that $w \in \oplus \sum_{i=1}^m \mathfrak{H}(\delta_i)$. Let

$$w=w_1+\cdots+w_m,\quad w_i\in\mathfrak{H}(\delta_i)\ (0\leqq i\leqq m).$$

Each $\mathfrak{H}(\delta_i)$ is $C_{c,\delta_i}(G)$-irreducible, so there exists a function $f_i\in C_{c,\delta_i}(G)$ such that $T(f_i)w_i=w_i$ by Burnside's theorem. Then the function $f_0=f_1+\cdots+f_m\in C_c(G)$ satisfies

$$T(f_0)w=\sum_{i,j=1}^{m}T(f_i)w_j=\sum_{i=1}^{m}T(f_i)w_i=w.$$

Therefore, for every function $h\in C_c(G_0)$, we have

$$T(h)w=T(h)T(f_0)w=T(h*f_0)w\in\mathfrak{H}_0,$$

which completes the proof.

Let us return to our situation. Since Lemma 4 shows that $\{T(h)w_0|\ h\in C_c(G_0)\}\subset\mathfrak{H}_0$, we know

$$(\mathfrak{H}/\overline{\mathcal{K}})_0\subset\mathfrak{H}_0/(\mathfrak{H}_0\cap\overline{\mathcal{K}})=\mathfrak{H}_0/\mathcal{K}.$$

Let $\mathcal{V}$ be the subspace of $\mathfrak{H}_0$ such that $\mathcal{V}\supsetneqq\mathcal{K}$ and that $\mathcal{V}/\mathcal{K}=(\mathfrak{H}/\overline{\mathcal{K}})_0$. Then $\mathcal{V}$ is $C_c(G_0)$-invariant, and also invariant under $E(\mathfrak{H},\tau)$ for all $\tau\in\widehat{K}_0$, but this implies $\mathcal{V}=\mathfrak{H}_0$ because $\mathcal{K}$ is maximal in $\mathfrak{F}$. Thus we have

$$(\mathfrak{H}/\overline{\mathcal{K}})_0=\mathfrak{H}_0/\mathcal{K}.$$

We now assume that every $\widehat{K}_0$-finite topologically irreducible representation of G_0 is essentially a Banach representation. Then the representation $\{\mathfrak{H}/\overline{\mathcal{K}},\ \widetilde{T}(y)\}$ defined above is also essentially a Banach representation. So, by Theorem 3, there exists a topologically irreducible Banach representation $\{\mathcal{B},\Lambda(s)\}$ of S and a non-zero linear map $\beta:(\mathfrak{H}/\overline{\mathcal{K}})_0\to\mathcal{B}$ such that

$$\beta\widetilde{T}(s)=\Lambda(s)\beta\quad(\forall s\in S),$$
$$\beta\widetilde{T}(\varphi)=\Lambda(\varphi)\beta\quad(\forall\varphi\in C_c(S)).$$

As we saw above, we may consider that β is defined on $\mathfrak{H}_0/\mathcal{K}$. So, if we denote by γ the canonical projection of $\mathfrak{H}_0$ onto $\mathfrak{H}_0/\mathcal{K}$, then the non-zero linear map

$$\alpha=\beta\circ\gamma:\mathfrak{H}_0\to\mathcal{B}$$

satisfies

$$\alpha T(s)=(\beta\circ\gamma)T(s)=\beta\widetilde{T}(s)\gamma=\Lambda(s)(\beta\circ\gamma)=\Lambda(s)\alpha$$

for all $s\in S$, and

$$\alpha T(\varphi)=(\beta\circ\gamma)T(\varphi)=\beta\widetilde{T}(\varphi)\gamma=\Lambda(\varphi)(\beta\circ\gamma)=\Lambda(\varphi)\alpha$$

for all $\varphi \in C_c(S)$. Therefore, by Theorem 3 again, the representation $\{\mathfrak{H}, T(x)\}$ turns out to be essentially a Banach representation.

Before starting the proof of 'only if' part of Theorem 4, we shall prove two lemmas.

Lemma 5. *Let $\{\mathfrak{H}, T(x)\}$ be a $\hat{K}$-finite representation of G, and $\tau \in \hat{K}_0$ an equivalence class such that $\mathfrak{H}(\tau) \neq \{0\}$. Then we can find $\delta \in \hat{K}$ and a $C_c(G)$-invariant closed subspace $\mathscr{H}$ of $\mathfrak{H}$ which satisfy the following conditions.*

(i) *δ contains τ.*

(ii) *$\mathscr{H}(\delta) = E(\mathfrak{H}, \delta)\mathscr{H}$ is a $C_{c,\delta}(G)$-irreducible non-zero subspace of $\mathscr{H}$.*

(iii) *For any vector $v \in \mathscr{H}(\delta)$, $v \neq 0$, the subspace*

$$\mathscr{H}_0 = \{T(f)v \mid f \in C_c(G)\}$$

is $C_c(G)$-invariant, dense in $\mathscr{H}$, and independent of $v \in \mathscr{H}(\delta)$, $v \neq 0$.

(iv) *There exists the largest $C_c(G)$-invariant subspace $\mathscr{K}$ of $\mathscr{H}$ satisfying $E(\mathfrak{H}, \delta)\mathscr{K} = \{0\}$, and it is closed.*

(v) *$E(\mathfrak{H}, \tau)\mathscr{K} = \{0\}$.*

Proof. We shall denote by $\delta_1, \dots, \delta_m$ the all distinct elements in $\hat{K}$ which contain $\tau \in \hat{K}_0$ and appear in $\{\mathfrak{H}, T(x)\}$. Take a vector v_1 in some $C_{c,\delta_1}(G)$-irreducible subspace of $\mathfrak{H}(\delta_1)$, then the subspace

$$\mathscr{H}_{v_1} = \overline{\{T(f)v_1 \mid f \in C_c(G)\}}$$

of $\mathfrak{H}$ is $C_c(G)$-invariant and closed (here the bar indicates the closure in $\mathfrak{H}$), and the subspace $E(\mathfrak{H}, \delta_1)\mathscr{H}_{v_1}$ is clearly non-zero and $C_{c,\delta_1}(G)$-irreducible. Let $\mathscr{K}_{v_1}$ be the union of all $C_c(G)$-invariant subspaces $\mathscr{K}$ of $\mathscr{H}_{v_1}$ such that $E(\mathfrak{H}, \delta_1)\mathscr{K} = \{0\}$. Then it is clear that $\mathscr{K}_{v_1}$ is closed and the largest one among such subspaces $\mathscr{K}$. If $E(\mathfrak{H}, \tau)\mathscr{K}_{v_1} = \{0\}$, then δ_1 and $\mathscr{H}_{v_1}$ are just one of those pairs we wanted.

If $E(\mathfrak{H}, \tau)\mathscr{K}_{v_1} \neq \{0\}$, then we can find one of $\delta_2, \dots \delta_m$, say δ_2, such that $E(\mathfrak{H}, \delta_2)\mathscr{K}_{v_1} \neq \{0\}$. We take a vector $v_2 \neq 0$ in some $C_{c,\delta_2}(G)$-irreducible subspace of $E(\mathfrak{H}, \delta_2)\mathscr{K}_{v_1}$ and get

$$\mathscr{H}_{v_2} = \overline{\{T(f)v_2 \mid f \in C_c(G)\}}$$

which is a $C_c(G)$-invariant subspace of $\mathfrak{H}$, and $E(\mathfrak{H}, \delta_2)\mathscr{H}_{v_2}$ is a non-zero $C_{c,\delta_2}(G)$-irreducible subspace of $\mathscr{H}_{v_2}$. Let $\mathscr{K}_{v_2}$ be the largest $C_c(G)$-invariant subspace of $\mathscr{H}_{v_2}$ which lies in the kernel of $E(\mathfrak{H}, \delta_2)$. Then $\mathscr{K}_{v_2}$ is closed. If $E(\mathfrak{H}, \tau)\mathscr{K}_{v_2} = \{0\}$, then δ_2 and $\mathscr{H}_{v_2}$ satisfy the conditions (i) ~ (v) in Lemma 5. We have only to repeat this procedure at most m times to get what we need.

Lemma 6. *Let $\{\mathfrak{B}, T(x)\}$ be a $\hat{K}$-finite topologically irreducible Banach representation of G, and $\{\mathscr{E}, \Theta(y)\}$ a $\hat{K}_0$-finite topologically irreducible representation of G_0. If there exists a non-zero linear map $J: \mathfrak{B}_0 \to \mathscr{E}$ satisfying*

$$JT(h) = \Theta(h)J \qquad (\forall h \in C_c(G_0)),$$
$$JE(\mathfrak{B}, \tau) = E(\mathscr{E}, \tau)J \quad (\forall \tau \in \widehat{K}_0),$$

then the representation $\{\mathscr{E}, \Theta(y)\}$ *of* G_0 *is essentially a Banach representation.*

Proof. Let us first prove that $\ker(J)$, the kernel of J, is closed in $\mathfrak{B}_0$. To do this we denote by $\overline{\ker(J)}$ the closure of $\ker(J)$ in $\mathfrak{B}$. If $\ker(J)$ is not closed in $\mathfrak{B}_0$, then we can find a vector $v \in \mathfrak{B}_0 \cap \overline{\ker(J)}$ such that $v \notin \ker(J)$. Since $J(v) \neq 0$, $E(\mathscr{E}, \tau)J(v) \neq 0$ for some $\tau \in \widehat{K}_0$. This means

$$JE(\mathfrak{B}, \tau)(\mathfrak{B}_0 \cap \overline{\ker(J)}) \neq \{0\},$$

which contradicts the equality

$$JE(\mathfrak{B}, \tau)(\mathfrak{B}_0 \cap \overline{\ker(J)}) = JE(\mathfrak{B}, \tau)\ker(J) = E(\mathscr{E}, \tau)J(\ker(J)) = \{0\}.$$

Therefore, $\ker(J)$ is closed in $\mathfrak{B}_0$.

Since $\overline{\ker(J)}$ is G_0- and $C_c(G_0)$-invariant, we can naturally define a representation of G_0 on the Banach space $\widetilde{\mathfrak{B}} = \mathfrak{B}/\overline{\ker(J)}$. We shall denote it by $\{\widetilde{\mathfrak{B}}, \widetilde{T}(y)\}$. The $\widehat{K}_0$-finiteness of $\{\mathfrak{B}, T(x)\}$ implies that of $\{\widetilde{\mathfrak{B}}, \widetilde{T}(y)\}$.

To show that $\{\widetilde{\mathfrak{B}}, \widetilde{T}(y)\}$ is topologically irreducible, we assume the contrary. Then there exists a closed G_0-invariant subspace $\mathscr{V}$ of $\mathfrak{B}$ such that $\overline{\ker(J)} \subsetneqq \mathscr{V} \subsetneqq \mathfrak{B}$. Then, for any $\tau \in \widehat{K}_0$, we have $E(\mathfrak{B}, \tau)(\mathfrak{B}_0 \cap \mathscr{V}) = \mathfrak{B}(\tau) \cap E(\mathfrak{B}, \tau)\mathscr{V} = E(\mathfrak{B}, \tau)\mathscr{V}$. Now choose $\tau_1 \in \widehat{K}_0$ such that $E(\mathfrak{B}, \tau_1)\mathscr{V} \not\subset \overline{\ker(J)}$. Then, denoting by $\overline{J(\mathfrak{B}_0 \cap \mathscr{V})}$ the closure of $J(\mathfrak{B}_0 \cap \mathscr{V})$ in $\mathscr{E}$, it follows that

$$\begin{aligned} E(\mathscr{E}, \tau_1)\overline{J(\mathfrak{B}_0 \cap \mathscr{V})} &= E(\mathscr{E}, \tau_1)J(\mathfrak{B}_0 \cap \mathscr{V}) = JE(\mathfrak{B}, \tau_1)(\mathfrak{B}_0 \cap \mathscr{V}) \\ &= JE(\mathfrak{B}, \tau_1)\mathscr{V} \neq \{0\}. \end{aligned}$$

This means $\overline{J(\mathfrak{B}_0 \cap \mathscr{V})} \neq \{0\}$. On the other hand, if we choose $\tau_2 \in \widehat{K}_0$ such that $E(\mathfrak{B}, \tau_2)\mathscr{V} \subsetneqq \mathfrak{B}(\tau_2)$ and $\mathfrak{B}(\tau_2) \not\subset \overline{\ker(J)}$, we have

$$\begin{aligned} E(\mathscr{E}, \tau_2)\overline{J(\mathfrak{B}_0 \cap \mathscr{V})} &= E(\mathscr{E}, \tau_2)J(\mathfrak{B}_0 \cap \mathscr{V}) = JE(\mathfrak{B}, \tau_2)(\mathfrak{B}_0 \cap \mathscr{V}) \\ &= JE(\mathfrak{B}, \tau_2)\mathscr{V} \subsetneqq J\mathfrak{B}(\tau_2) \\ &= E(\mathscr{E}, \tau_2)J(\mathfrak{B}_0) \subset \mathscr{E}(\tau_2). \end{aligned}$$

From this we know $\overline{J(\mathfrak{B}_0 \cap \mathscr{V})} \subsetneqq \mathscr{E}$. Therefore, the $C_c(G_0)$-invariant closed subspace $\overline{J(\mathfrak{B}_0 \cap \mathscr{V})}$ of $\mathscr{E}$ is non-trivial. This contadicts the topological irreducibility of $\{\mathscr{E}, \Theta(y)\}$.

Choose $\tau \in \widehat{K}_0$ such that $\mathfrak{B}(\tau) \not\subset \overline{\ker(J)}$ and a vector $v \in \mathfrak{B}(\tau)$, $v \notin \overline{\ker(J)}$, and denote by $\widetilde{v}$ the class of v in $\widetilde{\mathfrak{B}}$, then the fundamental subspace of $\{\widetilde{\mathfrak{B}},$

$\widetilde{T}(y)\}$ is

$$\widetilde{\mathfrak{B}}_0=\{\widetilde{T}(h)\widetilde{v}\mid h\in C_c(G_0)\}.$$

Since we know $T(h)v\in\mathfrak{B}_0$ for all $h\in C_c(G_0)$ by Lemma 4, and since $\mathfrak{B}_0\cap\overline{\ker(J)}=\ker(J)$ as is shown above, we have

$$\widetilde{\mathfrak{B}}_0\subset\mathfrak{B}_0/(\mathfrak{B}_0\cap\overline{\ker(J)})=\mathfrak{B}_0/\ker(J).$$

The linear map $J:\mathfrak{B}_0\to\mathscr{E}$ naturally induces a non-zero linear map

$$\widetilde{J}:\mathfrak{B}_0/\ker(J)\to\mathscr{E},$$

and, if we denote by the same $\widetilde{J}$ the restriction of $\widetilde{J}$ on $\widetilde{\mathfrak{B}}_0$, the non-zero linear map $\widetilde{J}:\widetilde{\mathfrak{B}}_0\to\mathscr{E}$ is injective, and satisfies

$$\widetilde{J}\widetilde{T}(h)=\Theta(h)\widetilde{J}\quad(\forall h\in C_c(G_0)).$$

Since $\widetilde{J}(\widetilde{v})\in\mathscr{E}(\tau)$, $\widetilde{J}(\widetilde{v})\neq 0$, algebraic irreducibility of $\mathscr{E}_0$ under the $C_c(G_0)$-action implies $\widetilde{J}(\widetilde{\mathfrak{B}}_0)=\mathscr{E}_0$. Therefore, the linear map $\widetilde{J}:\widetilde{\mathfrak{B}}_0\to\mathscr{E}_0$ is bijective. This means, by Theorem 1, the representation $\{\mathscr{E},\Theta(y)\}$ of G_0 is SF-equivalent to the Banach representation $\{\widetilde{\mathfrak{B}},\widetilde{T}(y)\}$.

Now let us start the proof of 'only if' part of Theorem 4. Let $\{\mathscr{E},\Theta(y)\}$ be a $\widehat{K}_0$-finite topologically irreducible representation of G_0. We shall denote by $\mathfrak{H}^\Theta$ the vector space of all $\mathscr{E}$-valued continuous functions ξ on K satisfying

$$\xi(uk)=\Theta(u)\xi(k)\qquad(\forall u\in K_0).$$

For a family of semi-norms $\{|\cdot|_\iota\}_{\iota\in I}$ defining the locally convex topology of $\mathscr{E}$, we put

$$\|\xi\|_\iota=\sup_{k\in K}|\xi(k)|_\iota\qquad(\iota\in I),$$

and give a locally convex topology in $\mathfrak{H}^\Theta$ by these semi-norms $\|\cdot\|_\iota$. For every $x\in G$ and $\xi\in\mathfrak{H}^\Theta$, we define

$$[T^\Theta(x)\xi](k)=\Theta(\sigma(kx))\xi(\kappa(kx)),$$

where $kx=\sigma(kx)\kappa(kx)$, $\sigma(kx)\in S$, $\kappa(kx)\in K$ as before. In this way we got the induced reprepentation $\{\mathfrak{H}^\Theta, T^\Theta(x)\}$.

Lemma 7. *The induced representation $\{\mathfrak{H}^\Theta, T^\Theta(x)\}$ is $\widehat{K}$-finite.*

Proof. For every $\delta\in\widehat{K}$ and $\xi\in\mathfrak{H}^\Theta$, we have

$$\begin{aligned}[E(\mathfrak{H}^\Theta,\delta)\xi](k)&=\int_K\overline{\chi_\delta(k')}\,[T^\Theta(k')\xi](k)d_K(k')\\&=\int_K\overline{\chi_\delta(k')}\,\xi(kk')d_K(k')=\int_K\overline{\chi_\delta(k^{-1}k')}\,\xi(k')d_K(k').\end{aligned}$$

Let $k \mapsto D(k)$ be a unitary matrix representation of K belonging to δ whose matrix coefficients are $d_{ij}(k)$, $d = d(\delta)$ the degree of δ, and $k_0, \ldots, k_n$ the set of representative elements of K modulo K_0 as before. Then it follows that

$$[E(\mathfrak{H}^\theta, \delta)\xi](k) = d\sum_{i,j=1}^{d}\left(\int_K \overline{d_{ij}(k')}\,\xi(k')\,d_K(k')\right)d_{ij}(k)$$

$$= \frac{d}{n+1}\sum_{i,j=1}^{d}\left(\sum_{l=0}^{n}\int_{K_0}\overline{d_{ij}(uk_l)}\,\xi(uk_l)\,d_{K_0}(u)\right)d_{ij}(k)$$

$$= \frac{d}{n+1}\sum_{i,j=1}^{d}\left(\sum_{l=0}^{n}\int_{K_0}\overline{d_{ij}(uk_l)}\,\Theta(u)\,\xi(k_l)\,d_{K_0}(u)\right)d_{ij}(k).$$

Note that, for each l, the vector

$$\int_{K_0}\overline{d_{ij}(uk_l)}\,\Theta(u)\,\xi(k_l)\,d_{K_0}(u)$$

belongs to the direct sum of $\mathscr{E}(\tau)$'s, for all τ contained in δ, which is finite-dimensional because $\{\mathscr{E}, \Theta(y)\}$ is $\widehat{K}_0$-finite. From this, it is easy to see that the vectors $E(\mathfrak{H}^\theta, \delta)\,\xi$, $\xi \in \mathfrak{H}^\theta$, belong to some fixed finite-dimensional subspace of $\mathfrak{H}^\theta$.

Now we fix $\tau_0 \in \widehat{K}_0$ such that $\mathscr{E}(\tau_0) \neq \{0\}$, then clearly $\mathfrak{H}^\theta(\tau_0) \neq \{0\}$. So we can apply Lemma 5 to the induced representation $\{\mathfrak{H}^\theta, T^\theta(x)\}$ and this $\tau_0 \in \widehat{K}_0$. Let $\delta \in \widehat{K}$, $C_c(G)$-invariant closed subspace $\mathscr{H}$ of $\mathfrak{H}^\theta$, and the largest $C_c(G)$-invariant closed subspace $\mathscr{K}$ of $\mathscr{H}$ satisfying $E(\mathfrak{H}^\theta, \delta)\mathscr{K} = \{0\}$, be all given in the sense of Lemma 5 for $\{\mathfrak{H}^\theta, T^\theta(x)\}$ and $\tau_0 \in \widehat{K}_0$. Consider the linear map $\beta : \mathscr{H} \to \mathscr{E}$ given by

$$\beta(\xi) = \xi(e) \qquad (\xi \in \mathscr{H}).$$

This map β is not identically zero, because, for $\xi \neq 0$, we can find an element $k \in K$ such that $[T^\theta(k)\xi](e) = \xi(k) \neq 0$. The following equalities

$$\beta T^\theta(h) = \Theta(h)\beta \qquad (\forall h \in C_c(G_0)),$$

$$\beta E(\mathfrak{H}^\theta, \tau) = E(\mathscr{E}, \tau)\beta \quad (\forall \tau \in \widehat{K}_0)$$

are also easy to check. Hence $\beta(\mathscr{K})$ is a $C_c(G_0)$-invariant subspace of $\mathscr{E}$. It can be seen that $\beta(\mathscr{K}) = \{0\}$. In fact, assume $\beta(\mathscr{K}) \neq \{0\}$.Then $\overline{\beta(\mathscr{K})} = \mathscr{E}$ because of the topological irreducibility of $\{\mathscr{E}, \Theta(y)\}$. But the condition $E(\mathfrak{H}^\theta, \tau_0)\mathscr{K} = \{0\}$ means

$$\mathscr{E}(\tau_0) = E(\mathscr{E}, \tau_0)\mathscr{E} = E(\mathscr{E}, \tau_0)\overline{\beta(\mathscr{K})} = E(\mathscr{E}, \tau_0)\beta(\mathscr{K}) = \beta E(\mathfrak{H}^\theta, \tau_0)\mathscr{K} = \{0\},$$

which is a contradiction. Therefore, the linear map $\beta : \mathscr{H} \to \mathscr{E}$ naturally induces a non-zero linear map

$$\tilde{\beta}: \mathscr{H}/\mathscr{K} \to \mathscr{E}.$$

We denote by $\{\mathscr{H}/\mathscr{K}, \tilde{T}^{\Theta}(x)\}$ the topologically irreducible representation of G on $\mathscr{H}/\mathscr{K}$. This representation is $\hat{K}$-finite since $\{\mathfrak{H}^{\Theta}, T^{\Theta}(x)\}$ is, and clearly

$$\tilde{\beta}\tilde{T}^{\Theta}(h) = \Theta(h)\tilde{\beta} \qquad (\forall h \in C_c(G_0)),$$
$$\tilde{\beta}E(\mathscr{H}/\mathscr{K}, \tau) = E(\mathscr{E}, \tau)\tilde{\beta} \quad (\forall \tau \in \hat{K}_0).$$

Now we assume that every $\hat{K}$-finite topologically irreducible representation of G is essentially a Banach representation, then $\{\mathscr{H}/\mathscr{K}, \tilde{T}^{\Theta}(x)\}$ is, too. Thus there exists a $\hat{K}$-finite topologically irreducible representation $\{\mathfrak{B}, T(x)\}$ of G and a bijective linear map $I: \mathfrak{B}_0 \to (\mathscr{H}/\mathscr{K})_0$ such that

$$IT(f) = \tilde{T}^{\Theta}(f)I \qquad (\forall f \in C_c(G)).$$

Since I satisfies

$$IE(\mathfrak{B}, \tau) = E(\mathscr{H}/\mathscr{K}, \tau)I \qquad (\forall \tau \in \hat{K}_0)$$

as is pointed out in Remark following Theorem 1, non-zero linear map

$$J = \tilde{\beta} \circ I: \mathfrak{B}_0 \to \mathscr{E}$$

clearly satisfies

$$JT(h) = \Theta(h)J \qquad (\forall h \in C_c(G_0)),$$
$$JE(\mathfrak{B}, \tau) = E(\mathscr{E}, \tau)J \quad (\forall \tau \in \hat{K}_0).$$

Therefore, by Lemma 6, $\{\mathscr{E}, \Theta(y)\}$ is essentially a Banach representation of G_0. This completes the proof of Theorem 4.

§2. Two steps of reductions for the group $G = S \rtimes K$

2.1. First reduction. Let $G = S \rtimes K$ be a locally compact group which is a semi-direct product of a compactly generated abelian group S and a compact group K. It is well known that we may assume $S = \mathbf{Z}^n F \mathbf{R}^m$ (direct product) for some $m, n \in \mathbf{N}$, where F is a compact abelian group [7]. ($\mathbf{Z}$ is the ring of integers, $\mathbf{R}$ the real number field, and $\mathbf{N}$ the set of positive integers.) Our aim in this paper is to prove that every $\hat{K}$-finite topologically irreducible representation of G is essentially a Banach representation.

Let $\{\mathfrak{H}, T(x)\}$ be a δ-finite (accordingly $\hat{K}$-finite in this case) topologically irreducible representation of G. We note that $kFk^{-1} \subset F$ for any $k \in K$, because $\mathbf{Z}^n\mathbf{R}^m$ has no non-trivial compact subgroups. Therefore, FK is a compact subgroup of G. We shall denote by $\mathscr{H}(\mathfrak{H}(\delta))$ the FK-invariant subspace of $\mathfrak{H}$ generated by $\mathfrak{H}(\delta)$. Since the representation $\{\mathfrak{H}, T(x)\}$ is

$(FK)^\wedge$-finite, we have

$$\mathfrak{H}_0 \supset \mathfrak{H}^{FK} = \oplus \sum_{\sigma \in (FK)^\wedge} \mathfrak{H}(\sigma) \qquad \text{(algebraic direct sum)}.$$

Let $\sigma_1, \ldots, \sigma_l$ be all distinct elements in $(FK)^\wedge$ such that $\mathfrak{H}(\sigma_i) \neq \{0\}$ and $[\sigma_i|_K : \delta] \neq 0$. Then it is clear that

$$\mathfrak{H}(\delta) \subset \mathfrak{H}(\sigma_1) \oplus \cdots \oplus \mathfrak{H}(\sigma_l).$$

From this, it follows that

$$\dim \mathscr{H}(\mathfrak{H}(\delta)) \leq \sum_{i=1}^{l} \dim \mathfrak{H}(\sigma_i) < +\infty.$$

We now decompose the representation of F on $\mathscr{H}(\mathfrak{H}(\delta))$ into a direct sum of irreducible ones, and denote by $\lambda_1, \ldots, \lambda_q \in \hat{F}$ all distinct characters which appear in it.

Lemma 8. *Put $K_i = \{k \in K | \lambda_i(k^{-1}\gamma k) = \lambda_i(\gamma)$ for all $\gamma \in F\}$, then $K_0 = \bigcap_{i=1}^{q} K_i$ is an open normal subgroup of K.*

Proof. Let $w \in \mathscr{H}(\mathfrak{H}(\delta))$ be a vector which satisfies $T(\gamma)w = \lambda_i(\gamma)w$ for all $\gamma \in F$, then $T(\gamma)T(k)w = T(k)T(k^{-1}\gamma k)w = \lambda_i(k^{-1}\gamma k)T(k)w$ for any $k \in K$. Thus we know that the character $\gamma \mapsto \lambda_i(k^{-1}\gamma k)$ is also a member in $\{\lambda_1, \ldots, \lambda_q\}$. In this sense, K acts on the set $\{\lambda_1, \ldots, \lambda_q\}$. So we get the inequality $[K: K_i] \leq q$ and know that K_i is an open subgroup of K.

Let k be an arbitrary element in K. If j is a number such that $\lambda_i(k^{-1}\gamma k) = \lambda_j(\gamma)$ for all $\gamma \in F$, then, for any $u \in K_0$, we have

$$\lambda_i(k^{-1}u^{-1}k\gamma k^{-1}uk) = \lambda_j(u^{-1}k\gamma k^{-1}u) = \lambda_j(k\gamma k^{-1}) = \lambda_i(\gamma),$$

i.e., $k^{-1}uk \in K_0$, thus K_0 is normal.

Now we put $G_0 = S \rtimes K_0$. As we have seen in Lemma 3, there exists a maximal $C_c(G_0)$-invariant proper closed subspace $\mathscr{K}$ of $\mathfrak{H}$. Then $\mathfrak{H}(\delta) \not\subset \mathscr{K}$, because $\mathfrak{H}(\delta) \subset \mathscr{K}$ implies $\mathfrak{H}_0 \subset \mathscr{K}$ and hence $\mathscr{K} = \mathfrak{H}$. Denote by $\tilde{T}(y)$ $(y \in G_0)$ the naturally induced operator, from $T(y)$, on the factor space $\tilde{\mathfrak{H}} = \mathfrak{H}/\mathscr{K}$. Then the representation $\{\tilde{\mathfrak{H}}, \tilde{T}(y)\}$ of G_0 is $\hat{K}_0$-finite and topologically irreducible. If we observe the argument done in some paragraphs after the proof of Lemma 4, we know that $\{\mathfrak{H}, T(x)\}$ is essentially a Banach representation provided that the representation $\{\tilde{\mathfrak{H}}, \tilde{T}(y)\}$ of G_0 is.

Lemma 9. *Let $\{\tilde{\mathfrak{H}}, \tilde{T}(y)\}$ be the representation of G_0 defined above. Then there exists a character $\lambda \in \hat{F}$ such that $\tilde{T}(\gamma) = \lambda(\gamma) I_{\tilde{\mathfrak{H}}}$ $(\forall \gamma \in F)$ where $I_{\tilde{\mathfrak{H}}}$ is the identity operator on $\tilde{\mathfrak{H}}$. We have $\lambda(u\gamma u^{-1}) = \lambda(\gamma)$ for any $\gamma \in F$ and $u \in K_0$.*

Proof. Since $\mathfrak{H}(\delta)\not\subset\mathcal{K}$, there exists a K_0-irreducible subspace $W\subset\mathfrak{H}(\delta)$ such that $W\not\subset\mathcal{K}$. We take any non-zero vector $w_0\in W$. It is easily checked that $T(\gamma)T(u)=T(u)T(\gamma)$ $(\gamma\in F,\ u\in K_0)$ on the subspace $\mathcal{H}(\mathfrak{H}(\delta))$, so we have $T(\gamma)T(u)w_0=T(u)T(\gamma)w_0$.

Denote by $\tilde{w}_0$ the image of w_0 under the natural map of $\mathfrak{H}$ onto $\tilde{\mathfrak{H}}=\mathfrak{H}/\mathcal{K}$. The fundamental subspace $(\tilde{\mathfrak{H}})_0$ of $\tilde{\mathfrak{H}}$ is given by

$$(\tilde{\mathfrak{H}})_0=\{\tilde{T}(h)\tilde{w}_0\mid h\in C_c(G_0)\}.$$

From the relation $\tilde{T}(\gamma)\tilde{T}(u)\tilde{w}_0=\tilde{T}(u)\tilde{T}(\gamma)\tilde{w}_0$ we know that $\tilde{T}(\gamma)$ commutes with all $\tilde{T}(y)$ $(y\in G_0)$. Therefore $\tilde{T}(\gamma)=\lambda(\gamma)I_{\tilde{\mathfrak{H}}}(\gamma\in F)$ for some $\lambda\in\hat{F}$.

The second statement is clear.

Since $\ker\lambda$ is a K_0-invariant subgroup of S, K_0 naturally acts on $S/\ker\lambda=\mathbf{Z}^n(F/\ker\lambda)\mathbf{R}^m$. In this sense we may write $G_0/\ker\lambda=\{\mathbf{Z}^n(F/\ker\lambda)\mathbf{R}^m\}\rtimes K_0$, and consider $\{\tilde{\mathfrak{H}},\tilde{T}(y)\}$ as a $\hat{K}_0$-finite topologically irreducible representation of $G_0/\ker\lambda$. By the way, the group $F/\ker\lambda$ is either a finite group or isomorphic to the 1-dimensional torus $\mathbf{T}$. Therefore, our original problem, that is, to prove that the representation $\{\mathfrak{H},T(x)\}$ of G is essentially a Banach representation, is reduced to the same one for following two types of locally compact unimodular groups $G=S\rtimes K$. One is when $S=\mathbf{Z}^n\mathbf{T}\mathbf{R}^m$ (direct product) and K acts on $\mathbf{T}$ trivially, and another is when $S=Z\mathbf{R}^m$ (direct product) where Z is a finitely generated discrete abelian group.

2.2. Second reduction. To proceed the second reduction we shall deal with the above two types of locally compact groups $G=S\rtimes K$ at the same time. So, we assume here that $S=ZT\mathbf{R}^m$ (direct product) where Z is a finitely generated discrete abelian group, and T is $\{e\}$ or 1-dimensional torus $\mathbf{T}$ on which K acts trivially.

Since Z is discrete it holds that $k\mathbf{R}^m k^{-1}\subset T\mathbf{R}^m$. We now put

$$krk^{-1}=\tau(k,r)\bar{k}(r),$$

where $r\in\mathbf{R}^m$, $\tau(k,r)\in T$, and $\bar{k}(r)\in\mathbf{R}^m$. Because, for each $z\in Z$, the subgroup $\{k\in K\mid kzk^{-1}\in zT\mathbf{R}^m\}$ is open in K, and because Z is finitely generated, the subgroup $K_0=\{k\in K\mid kzk^{-1}\in zT\mathbf{R}^m \text{ for all } z\in Z\}$ is an open subgroup of K. By Theorem 4 we have only to do our work for the group $G_0=S\rtimes K_0$. Thus we may assume from the begining that

$$kzk^{-1}\in zT\mathbf{R}^m \quad \text{for all } z\in Z.$$

Let $z_1,\dots,z_{n'}$ be the generators of Z. We define continuous $\mathbf{R}^m$-valued functions $r_i(k)$ and T-valued functions $\tau_i(k)$ on K by

$$kz_ik^{-1}=z_i\tau_i(k)r_i(k).$$

It follows from

$$k_1k_2z_ik_2^{-1}k_1^{-1}=k_1z_i\tau_i(k_2)r_i(k_2)k_1^{-1}$$
$$=z_i\tau_i(k_1)r_i(k_1)\tau_i(k_2)\tau(k_1, r_i(k_2))\bar{k}_1(r_i(k_2))$$
$$=z_i\tau(k_1, r_i(k_2))\tau_i(k_1)\tau_i(k_2)\bar{k}_1(r_i(k_2))r_i(k_1)$$

that

$$r_i(k_1k_2)=\bar{k}_1(r_i(k_2))r_i(k_1)$$

for $k_1, k_2\in K$. Now if we put

$$\bar{r}_i=\int_K r_i(k)d_K(k),$$

we clearly have $\bar{r}_i=\bar{k}(\bar{r}_i)r_i(k)$, and therefore

$$kz_i\bar{r}_ik^{-1}=z_i\tau(k, \bar{r}_i)\tau_i(k)\bar{k}(\bar{r}_i)r_i(k)=z_i\tau(k, \bar{r}_i)\tau_i(k)\bar{r}_i\in z_i\bar{r}_iT$$

for $1\leq i\leq n'$. Now we denote by Z' the discrete subgroup of S generated by $\{z_i\bar{r}_i|1\leq i\leq n'\}$. Then Z' is isomorphic to Z and $S=ZT\mathbf{R}^m=Z'T\mathbf{R}^m$ (direct product). Here we note that $kz'k^{-1}\in z'T$ for all $k\in K$ and $z'\in Z'$. Therefore our group may be considered as one of the following two types.

(a) $G=S\rtimes K$ where $S=\mathbf{Z}^n\mathbf{T}\mathbf{R}^m$ (direct product). $\mathbf{T}$ is contained in the center of G, and $kzk^{-1}\in z\mathbf{T}$, $k\mathbf{R}^mk^{-1}\subset\mathbf{T}\mathbf{R}^m$ for all $k\in K$, $z\in\mathbf{Z}^n$.

(b) $G=S\rtimes K$ where $S=Z\mathbf{R}^m$ (direct product). Z is a discrete subgroup contained in the center of G, and $k\mathbf{R}^mk^{-1}\subset\mathbf{R}^m$.

Now assume G is as in the case (b). Let $\{\mathfrak{H}, T(x)\}$ be a $\hat{K}$-finite topologically irreducible representation of G. Then $T(z)$, $z\in Z$, is a scalar multiple of the identity operator on $\mathfrak{H}$. This implies that, taking y only in the subgroup $\mathbf{R}^m\rtimes K$, $\{\mathfrak{H}, T(y)\}$ is $\hat{K}$-finite and topologically irreducible as a representation of $\mathbf{R}^m\rtimes K$, and that it is essentially a Banach representation if and only if the representation $\{\mathfrak{H}, T(x)\}$ of G is. Therefore, in our position, we may assume that $G=\mathbf{R}^m\rtimes K$. If $\{\mathfrak{H}, T(x)\}$ is a $\hat{K}$-finite topologically irreducible representation of $G=\mathbf{R}^m\rtimes K$, then it is considered as that of the group $G'=\mathbf{Z}^n\mathbf{T}G$ (direct product) by defining $T(zt)=I_{\mathfrak{H}}$ for any $zt\in\mathbf{Z}^n\mathbf{T}$. It is clear that the former is essentially a Banach representation of G if and only if the latter is also essentially a Banach representation of G'. Since G' is a special case of type (a), our problem remains only in the case (a).

§3. Algebras on the group $G=(\mathbf{Z}^n\mathbf{T}\mathbf{R}^m)\rtimes K$

By the two steps of reductions in §2, we may assume that our group is $G=S\rtimes K$ where $S=\mathbf{Z}^n\mathbf{T}\mathbf{R}^m$ (direct product), and that

$$kzk^{-1}\in z\mathbf{T} \qquad (\forall z\in\mathbf{Z}^n),$$

$$kt = tk \qquad (\forall t \in \mathbf{T}),$$
$$k\mathbf{R}^m k^{-1} \subset \mathbf{TR}^m$$

for any $k \in K$. For the sake of simplicity of notations, we put $Z = \mathbf{Z}^n$ and $R = \mathbf{TR}^m$.

3.1. Algebra $C^\infty(G)$. For any $k \in K$ and $r \in \mathbf{R}^m$, we define elements $\tau(k, r) \in \mathbf{T}$ and $\bar{k}(r) \in \mathbf{R}^m$ by

$$krk^{-1} = \tau(k, r)\bar{k}(r),$$

and, for any $k \in K$, $z \in Z$, we also define an element $\tau(k, z) \in \mathbf{T}$ by

$$kzk^{-1} = z\tau(k, z).$$

Then $\bar{k}$ is a linear transformation on the vector space $\mathbf{R}^m$, and, for any fixed k, the map $r \mapsto \tau(k, r)$ is a continuous character on $\mathbf{R}^m$. Since the action of K on $\mathbf{T}$ is trivial, the transformation $r \mapsto krk^{-1}$ on the Lie group $R = \mathbf{TR}^m$ is infinitely differentiable. The automorphism on the Lie algebra $\mathfrak{r}$ of R corresponding to $r \mapsto krk^{-1}$ is denoted by $\mathrm{Ad}(k)$, which can be naturally extended to an automorphism on the universal enveloping algebra $\mathcal{U}(\mathfrak{r}_\mathbf{C})$ of the complexification $\mathfrak{r}_\mathbf{C}$ of $\mathfrak{r}$.

When we consider $\alpha \in \mathcal{U}(\mathfrak{r}_\mathbf{C})$ as a right (and left) invariant differential operator on R, αf denotes the image of $f \in C^\infty(R)$ by α. On the other hand we may regard α as a distribution $f \mapsto \alpha f(e)$ on R. In this case we write

$$\alpha(f) = \int_R f(r)\, d\alpha(r) = \alpha f(e).$$

The element $\check{\alpha} \in \mathcal{U}(\mathfrak{r}_\mathbf{C})$ is defined by the equality

$$\int_R f(r)\, d\check{\alpha}(r) = \int_R \check{f}(r)\, d\alpha(r) \qquad (\check{f}(r) = f(r^{-1})).$$

Defnition. We shall denote by $C^\infty(G)$ the space of functions f on G such that, for any $z \in Z$ and $k \in K$, $r \mapsto f(zrk)$ are infinitely differentiable on R. The subspace of functions in $C^\infty(G)$ with compact supports will be denoted by $C_c^\infty(G)$.

A function f on G is in $C^\infty(G)$ if and only if, for any $x \in G$, $r \mapsto f(rx)$ is infinitely differentiable on R, and if and only if $r \mapsto f(xr)$ is. Put $\check{f}(x) = f(x^{-1})$, then $f \in C^\infty(G)$ if and only if $\check{f} \in C^\infty(G)$. Following notations will be used when convenient.

$$f * \alpha(x) = \int_R f(xr^{-1})\, d\alpha(r) = \int_R f(xr)\, d\check{\alpha}(r),$$
$$\alpha * f(x) = \int_R f(r^{-1}x)\, d\alpha(r) = \int_R f(rx)\, d\check{\alpha}(r),$$
$$\alpha f(x) = \int_R f(rx)\, d\alpha(r) = \check{\alpha} * f(x).$$

When we compute integrals, we often need the following Lemma 10. We will omit the proof which is quite elementally.

Lemma 10. (i) *For any $f\in C^\infty(G)$ and $\alpha\in\mathcal{U}(\mathfrak{r}_C)$, both $f*\alpha$* and $\alpha*f$ *are in $C^\infty(G)$.*

(ii) *If at least one of $f\in C^\infty(G)$ and $g\in C(G)$ is compactly supported, then both $f*g$ and $g*f$ are functions in $C^\infty(G)$, and for any $\alpha\in\mathcal{U}(\mathfrak{r}_C)$ we have*

$$\int_R f*g\,(x_1rx_2)\,d\alpha(r)=\int_G g(y)\,d_G(y)\int_R f(x_1rx_2y^{-1})\,d\alpha(r),$$
$$\int_R g*f(x_1rx_2)\,d\alpha(r)=\int_G g(y)\,d_G(y)\int_R f(y^{-1}x_1rx_2)\,d\alpha(r).$$

(iii) *If f is in $C^\infty(G)$, then f° belongs to $C^\infty(G)$, and for any $\alpha\in\mathcal{U}(\mathfrak{r}_C)$ we have*

$$\int_R f^\circ(x_1rx_2)\,d\alpha(r)=\int_K d_K(k)\int_R f(kx_1rx_2k^{-1})\,d\alpha(r).$$

By Lemma 10 we know that $C_c^\infty(G)$ is an algebra over $\mathbf{C}$ with the convolution product. For every $\delta\in\widehat{K}$, the subspaces $C_c^\infty(G)*\overline{\chi_\delta}$ and $C_{c,\delta}^\infty(G)=\overline{\chi_\delta}*C_c^\infty(G)*\overline{\chi_\delta}$ are subalgebras of $C_c^\infty(G)$, and $I_{c,\delta}^\infty(G)=I_{c,\delta}(G)\cap C^\infty(G)$ is a dense subalgabra of $I_{c,\delta}(G)$. Here we consider the usual inductive limit topology in $C_c(G)$, that is, $C_c(G)$ is the inductive limit of Banach spaces, with the supremum norm, of continuous functions on G, whose supports are contained in arbitrarily given compact subsets of G.

Now we shall denote by $\mathcal{U}^\circ(\mathfrak{r}_C)$ the subalgebra of $\mathcal{U}(\mathfrak{r}_C)$ consisting of elements $\alpha=\alpha^\circ$ where

$$\alpha^\circ=\int_K \mathrm{Ad}(k)\,\alpha\,d_K(k).$$

We take a basis $X_0, X_1,\ldots, X_m$ of $\mathfrak{r}$. Then $\mathrm{Ad}(k)$ can be represented by a real matrix with respect to this basis. From this, it is easily checked that

$$\Delta=(X_0^2+\cdots+X_m^2)^\circ=\sum_{i,j=0}^m q_{ij}X_iX_j$$

for some positive definite matrix $Q=(q_{ij})$.

Lemma 11. *Let $\Delta\in\mathcal{U}^\circ(\mathfrak{r}_C)$ be as above. If a function $f\in C^\infty(R)$ is a solution of the differential equation $(\Delta-\lambda)^l f=0$ for some integer $l>0$ and $\lambda\in\mathbf{C}$, then f is analytic on R.*

Proof. By the general theory on elliptic differential operators, we know that $(\Delta-\lambda)^{l-1}f$ is analytic on R since $(\Delta-\lambda)(\Delta-\lambda)^{l-1}f=(\Delta-\lambda)^l f=0$. This means that $(\Delta-\lambda)^{l-2}f$ is analytic since $(\Delta-\lambda)(\Delta-\lambda)^{l-2}f=(\Delta-\lambda)^{l-1}f$. Repeating this procedure we know that f is analytic on R.

3.2. Algebra $\mathscr{L}_\mu(G)$. We consider the vector space $\mathscr{L}(G) = C_c(Z \times K) \otimes_{\mathbf{C}} \mathscr{U}(\mathfrak{r}_{\mathbf{C}})$, where $C_c(Z \times K)$ is the vector space of compactly supported continuous functions on the product space $Z \times K$. We define transformations $\mathrm{Ad}(k)$, $k \in K$, on $\mathscr{L}(G)$ as

$$\mathrm{Ad}(k)\Big(\sum_i \varphi_i \otimes \alpha_i\Big) = \sum_i \varphi_i \otimes \mathrm{Ad}(k)\,\alpha_i.$$

When we consider the element $\xi = \sum_i \varphi_i \otimes \alpha_i \in \mathscr{L}(G)$ as a $\mathscr{U}(\mathfrak{r}_{\mathbf{C}})$-valued function defined on the product space $Z \times K$, we write

$$\xi(z, k) = \sum_i \varphi_i(z, k)\,\alpha_i.$$

Therefore,

$$[\mathrm{Ad}(k_1)\,\xi](z, k) = \sum_i \varphi_i(z, k)\,\mathrm{Ad}(k_1)\,\alpha_i = \mathrm{Ad}(k_1)\,(\xi(z, k)).$$

Now we fix a character $\mu \in \hat{\mathbf{T}}$ and define

$$[\xi * \eta](z, k) = \int_{Z \times K} \mu(\tau(k_1, z_1^{-1}z))\,\xi(z_1, k_1)\,[\mathrm{Ad}(k_1)\,\eta](z_1^{-1}z, k_1^{-1}k)\,d_Z(z_1)\,d_K(k_1)$$

for $\xi, \eta \in \mathscr{L}(G)$, where $d_Z(z)$ is the Haar measure on Z normalized in such a way that each point has volume one. Then it is easy to see that $\xi * \eta \in \mathscr{L}(G)$. When we want to emphasize the product $*$ in $\mathscr{L}(G)$, we write $\mathscr{L}_\mu(G)$ instead of $\mathscr{L}(G)$.

Lemma 12. *$\mathscr{L}_\mu(G)$ is an associative algebra over* $\mathbf{C}$.

Proof. We prove the associativity. For ξ, η, and ζ in $\mathscr{L}_\mu(G)$, we have

$$\begin{aligned}
&[(\xi * \eta) * \zeta](z, k) \\
&= \int_{Z \times K} \mu(\tau(k_2, z_2^{-1} z))\,(\xi * \eta)(z_2, k_2)\,[\mathrm{Ad}(k_2)\,\zeta](z_2^{-1}z, k_2^{-1}k)\,d_Z(z_2)\,d_K(k_2) \\
&= \int_{Z \times Z \times K \times K} \mu(\tau(k_2, z_2^{-1}z))\,\mu(\tau(k_1, z_1^{-1}z_2))\,\xi(z_1, k_1)\,[\mathrm{Ad}(k_1)\,\eta](z_1^{-1}z_2, k_1^{-1}k_2) \\
&\qquad\qquad \times [\mathrm{Ad}(k_2)\,\zeta](z_2^{-1}z, k_2^{-1}k)\,d_Z(z_1)\,d_Z(z_2)\,d_K(k_1)\,d_K(k_2) \\
&= \int_{Z \times Z \times K \times K} \mu(\tau(k_1k_2, z_2^{-1}z_1^{-1}z))\,\mu(\tau(k_1, z_2))\,\xi(z_1, k_1)\,[\mathrm{Ad}(k_1)\,\eta](z_2, k_2) \\
&\qquad\qquad \times [\mathrm{Ad}(k_1k_2)\,\zeta](z_2^{-1}z_1^{-1}z, k_2^{-1}k_1^{-1}k)\,d_Z(z_1)\,d_Z(z_2)\,d_K(k_1)\,d_K(k_2) \\
&= \int_{Z \times Z \times K \times K} \mu(\tau(k_1, z_1^{-1}z))\,\xi(z_1, k_1) \\
&\qquad \times \mathrm{Ad}(k_1)\,\{\mu(\tau(k_2, z_2^{-1}z_1^{-1}z))\,\eta(z_2, k_2)\,[\mathrm{Ad}(k_2)\,\zeta](z_2^{-1}z_1^{-1}z, k_2^{-1}k_1^{-1}k)\} \\
&\qquad\qquad d_Z(z_1)\,d_Z(z_2)\,d_K(k_1)\,d_K(k_2) \\
&= \int_{Z \times K} \mu(\tau(k_1, z_1^{-1}z))\,\xi(z_1, k_1)\,[\mathrm{Ad}(k_1)\,(\eta * \zeta)](z_1^{-1}z, k_1^{-1}k)\,d_Z(z_1)\,d_K(k_1) \\
&= [\xi * (\eta * \zeta)](z, k)
\end{aligned}$$

which is the associativity of the convolution.

Extend any function $\varphi\in C(K)$ to a function in $C_c(Z\times K)$ in such a way that $\varphi(e,k)=\varphi(k)$ and $\varphi(z,k)=0\ (z\neq e)$, then the mapping $\varphi\mapsto\varphi\otimes 1$ is an isomorphism of the convolution algebra $C(K)$ into $\mathscr{L}_\mu(G)$. From this point of view we may consider $C(K)$ as a subalgebra of $\mathscr{L}_\mu(G)$. So, for any $\delta\in\widehat{K}$, regarding $\overline{\chi_\delta}$ as an element in $\mathscr{L}(G)=\mathscr{L}_\mu(G)$, both $\mathscr{L}_\mu(G)*\overline{\chi_\delta}$ and $\overline{\chi_\delta}*\mathscr{L}_\mu(G)*\overline{\chi_\delta}$ are subalgebras of $\mathscr{L}_\mu(G)$.

For any element $\xi\in\mathscr{L}_\mu(G)$, we define

$$\xi^\circ(z,k)=\int_K\mu(\tau(h,z))\,[\mathrm{Ad}(h)\xi]\,(z,h^{-1}kh)\,d_K(h),$$

then $\xi\mapsto\xi^\circ$ is a projection of $\mathscr{L}_\mu(G)$ onto $\mathscr{L}_\mu^\circ(G)=\{\xi^\circ|\xi\in\mathscr{L}_\mu(G)\}$. It is clear that ξ belongs to $\mathscr{L}_\mu^\circ(G)$ if and only if

$$\mu(\tau(h,z))\,[\mathrm{Ad}(h)\xi]\,(z,h^{-1}kh)=\xi(z,k)$$

for all $h,k\in K$. $\mathscr{L}_\mu^\circ(G)$ is a subalgebra of $\mathscr{L}_\mu(G)$ since direct calculations show

$$(\xi*\eta^\circ)^\circ=(\xi^\circ*\eta)^\circ=\xi^\circ*\eta^\circ$$

for any $\xi,\eta\in\mathscr{L}_\mu(G)$. We also have a subalgebra

$$\mathscr{L}_{\mu,\delta}^\circ(G)=\mathscr{L}_\mu^\circ(G)\cap\overline{\chi_\delta}*\mathscr{L}_\mu(G)*\overline{\chi_\delta}$$

which plays an important role in this paper.

We put $\mu*C^\infty(G)=\{\mu*f\,|\,f\in C^\infty(G)\}$ where

$$\mu*f(x)=\int_{\mathbf{T}}\mu(t)f(t^{-1}x)\,d_{\mathbf{T}}(t),$$

$d_{\mathbf{T}}(t)$ is the normalized Haar measure on $\mathbf{T}$. For $\xi=\sum_i\varphi_i\otimes\alpha_i\in\mathscr{L}_\mu(G)$ and $f\in\mu*C^\infty(G)$, we define

$$\int_G f(x)\,d\xi(x)=\sum_i\int_{Z\times R\times K}f(zrk)\,\varphi_i(z,k)\,d_Z(z)\,d\alpha_i(r)\,d_K(k),$$

then it holds that

$$\int_G f(x)\,d(\xi*\eta)(x)=\int_{G\times G}f(xy)\,d\xi(x)\,d\eta(y),$$

$$\int_G f^\circ(x)\,d\xi(x)=\int_G f(x)\,d\xi^\circ(x).$$

For any function f in $\bar{\mu}*C^\infty(G)$ and $\xi\in\mathscr{L}_\mu(G)$ we define

$$\xi*f(x)=\int_G f(y^{-1}x)\,d\xi(y),$$

which is a new element in $\bar{\mu}*C^\infty(G)$. We easily have

$$(\xi*\eta)*f=\xi*(\eta*f)$$

for ξ, $\eta \in \mathscr{L}_\mu(G)$ and $f \in \bar{\mu} * C^\infty(G)$. It is also easy to see that $\xi * f \in \bar{\mu} * C_c^\infty(G)$ if f belongs to $\bar{\mu} * C_c^\infty(G)$.

3.3. Representations of $\mathscr{L}_\mu(G)$ obtained from those of G. Let $\{\mathfrak{H}, T(x)\}$ be a representation of G such that $T(t) = \mu(t) I_{\mathfrak{H}}$ ($\forall\, t \in \mathbf{T}$), where $I_{\mathfrak{H}}$ denotes the identity operator on $\mathfrak{H}$. The space $\mathfrak{H}$ is, as in §1, a locally convex Hausdoff topological vector space. We shall denote by $\mathfrak{H}'$ the dual space of $\mathfrak{H}$. The pairing on $\mathfrak{H} \times \mathfrak{H}'$ will be denoted by $\langle \cdot, \cdot \rangle$.

Definition. The linear subspace generated by all vectors $T(f)v$, where $f \in \bar{\mu} * C_c^\infty(G)$ and $v \in \mathfrak{H}$, is denoted by $\mathfrak{H}^\infty$.

Lemma 13. *The function $\phi_{v,w}(x) = \langle T(x) v, w \rangle$ for $v \in \mathfrak{H}^\infty$ and $w \in \mathfrak{H}'$ belongs to $\mu * C^\infty(G)$.*

Proof. For $v = T(f) v_1 \in \mathfrak{H}^\infty$ where $f \in \bar{\mu} * C_c^\infty(G)$ and $v_1 \in \mathfrak{H}$, the equality

$$\begin{aligned}\phi_{v,w}(x) &= \langle T(x) T(f) v_1, w \rangle = \int_G \langle T(xy) v_1, w \rangle f(y) d_G(y) \\ &= \phi_{v_1,w} * \check{f}(x)\end{aligned}$$

implies that $\phi_{v,w}$ belongs to $\mu * C^\infty(G)$ by Lemma 10.

For $T(f)v \in \mathfrak{H}^\infty (f \in \bar{\mu} * C_c^\infty(G))$, $w \in \mathfrak{H}'$, and $\alpha \in \mathscr{U}(\mathfrak{r}_C)$, we have

$$\begin{aligned}&\int_R \langle T(r) T(f) v, w \rangle d\alpha(r) = \int_R \phi_{v,w} * \check{f}(r) d\alpha(r) \\ &= \int_G \phi_{v,w}(x) d_G(x) \int_R \check{f}(x^{-1} r) d\alpha(r) = \langle T(\alpha * f) v, w \rangle.\end{aligned}$$

Since $\alpha * f$ is in $\bar{\mu} * C_c^\infty(G)$, we can define a linear operator $T(\alpha)$ on $\mathfrak{H}^\infty$ which satisfies

$$T(\alpha) T(f) v = T(\alpha * f) v$$

for all $f \in \bar{\mu} * C_c^\infty(G)$ and $v \in \mathfrak{H}$. It is easy to see that $T(\alpha\beta) = T(\alpha) T(\beta)$. For $\xi \in \mathscr{L}_\mu(G)$ we also define a linear operator on $\mathfrak{H}^\infty$ by

$$\langle T(\xi) T(f) v, w \rangle = \int_G \langle T(x) T(f) v, w \rangle d\xi(x).$$

Then the following lemma is clear.

Lemma 14. *$\{\mathfrak{H}^\infty, T(\xi)\}$ is a representation of the algebra $\mathscr{L}_\mu(G)$.*

Now we assume that the representation $\{\mathfrak{H}, T(x)\}$ is δ-finite for some $\delta \in \widehat{K}$, i.e., $0 < \dim \mathfrak{H}(\delta) < +\infty$. Since $\overline{\chi_\delta} * C_c^\infty(G) \subset C_c^\infty(G)$, the dense subspace $\mathfrak{H}^\infty$ of $\mathfrak{H}$ is invariant under $E(\mathfrak{H}, \delta)$, and hence we know $\mathfrak{H}(\delta) = E(\mathfrak{H}, \delta) \mathfrak{H} = E(\mathfrak{H}, \delta) \mathfrak{H}^\infty \subset \mathfrak{H}^\infty$. Thus the subspace $\mathfrak{H}(\delta)$ lies in the domain of $T(\Delta)$, where $\Delta \in \mathscr{U}(\mathfrak{r}_C)$ was given in Lemma 11, and is invariant under $T(\Delta)$. Then we can find a constant $\lambda \in \mathbf{C}$, a positive integer l, and a non-zero vector $v \in \mathfrak{H}(\delta)$ such that $(T(\Delta)-\lambda)^l v=0$. Now we know that the infinitely differentiable function $\phi_{v,w}(r)$, $w\in\mathfrak{H}'$, is analytic on R since

$$[(\Delta-\lambda)^l\phi_{v,w}](r)=\langle T(r)(T(\Delta)-\lambda)^l v,\ w\rangle=0$$

(see Lemma 11). It follows from this observation that $\phi_{v,w}(r)$ are analytic for all $v\in\mathfrak{H}(\delta)$ and $w\in\mathfrak{H}'$.

Theorem 5. *Let $\{\mathfrak{H},\ T(x)\}$ be a δ-finite representation of G for some $\delta\in\widehat{K}$, such that $T(t)=\mu(t)I_{\mathfrak{H}}$ for $t\in\mathbf{T}$. Then the function $\phi_{v,w}(x)=\langle T(x)v,\ w\rangle$, for any $v\in\mathfrak{H}(\delta)$ and $w\in\mathfrak{H}'$, is analytic with respect to $r\in R$.*

Proof. The equality $T(k)T(z)=\mu(\tau(k,\ z))T(z)T(k)$ means that $T(z)\mathfrak{H}(\delta)$ is invariant under $T(k)$. Thus there exist some vectors $v_i\in\mathfrak{H}(\sigma_i)$, $\sigma_i\in\widehat{K}$ for $i=1,\dots,l$, such that $T(z)T(k)v=v_1+\cdots+v_l$. Therefore the function

$$\phi_{v,w}(zrk)=\langle T(z)T(r)T(k)v,\ w\rangle=\sum_{i=1}^{l}\phi_{v_i,w}(r)$$

is analytic with respect to r.

Let $\{\mathfrak{H},\ T(x)\}$ be a δ-finite topologically irreducible representation of G such that $T(t)=\mu(t)I_{\mathfrak{H}}$ for $t\in\mathbf{T}$. Then it is clear that the subspace $\mathfrak{H}(\delta)$ is invariant and irreducible under the action of $\overline{\chi_\delta}*\mathscr{L}_\mu(G)*\overline{\chi_\delta}$. Any operator on $\mathfrak{H}(\delta)$ which commutes with all $T(k)|_{\mathfrak{H}(\delta)}$ $(k\in K)$ is equal to some $T(\xi)|_{\mathfrak{H}(\delta)}$ where ξ is an element in $\mathscr{L}^\circ_{\mu,\delta}(G)$.

Theorem 6. *Let $U_\delta(x)$ be a spherical matrix function of type δ of height p. Then all matrix elements of $U_\delta(x)$ are analytic on R if they are considered as functions on $Z\times R\times K$. If we put*

$$U_\delta(\xi)=\int_G U_\delta(x)\,d\xi(x),$$

then $\xi\mapsto U_\delta(\xi)$ is an irreducible representation of the algebra $\mathscr{L}^\circ_{\mu,\delta}(G)$.

Conversely, let $U(x)$ be an $M(p,\ \mathbf{C})$-valued function on G whose matrix elements are analytic on R when they are considered to be defined on $Z\times R\times K$. If it satisfies $U^\circ(x)=U(x)$ and if $\xi\mapsto U(\xi)$ is a p-dimensional irreducible representation of $\mathscr{L}^\circ_{\mu,\delta}(G)$, then $U(x)$ is a spherical matrix function of type δ.

Proof. The first half is clear by the definition of spherical matrix function. So we will prove the second half. Since $\overline{\chi_\delta}$ is in $\mathscr{L}^\circ_{\mu,\delta}(G)$ and $\overline{\chi_\delta}*\xi=\xi$ for every $\xi\in\mathscr{L}^\circ_{\mu,\delta}(G)$, we have $U(\overline{\chi_\delta})U(\xi)=U(\overline{\chi_\delta}*\xi)=U(\xi)$ which implies that $U(\overline{\chi_\delta})$ is the unit matrix. Thus, for any $\xi\in\mathscr{L}_\mu(G)$, it follows that

$$\int_G U*\chi_\delta(x)\,d\xi(x)=\int_{K\times G}U(xk^{-1})\chi_\delta(k)\,d_K(k)\,d\xi(x)$$
$$=\int_{K\times G}U(x)\,d(\xi*\overline{\chi_\delta})(x)=U((\xi*\overline{\chi_\delta})^\circ)=U(\xi^\circ*\overline{\chi_\delta})$$

$$=U(\xi^\circ)U(\overline{\chi_\delta})=U(\xi).$$

From this, together with the analyticity of $U*\chi_\delta(zrk)$ and $U(zrk)$ as functions of r, it follows that $U*\chi_\delta=U$.

Next we take an arbitrary $\xi\in\mathscr{L}_\mu(G)$ and put

$$V(x)=\int_G U(xy)d\xi^\circ(y),$$

then $V(zrk)$ is analytic with respect to r. By the equality

$$V(\eta)=U(\eta*\xi^\circ)=U(\overline{\chi_\delta}*\eta^\circ*\xi^\circ*\overline{\chi_\delta})=U(\overline{\chi_\delta}*\eta^\circ)U(\xi^\circ*\overline{\chi_\delta})=U(\eta)V(e)$$

for any $\eta\in\mathscr{L}_\mu(G)$ we know $V(x)=U(x)V(e)$, i.e.,

$$\int_G U(xy)d\xi^\circ(y)=U(x)\int_G U(y)d\xi^\circ(y)=U(x)U(\xi).$$

Since this equality holds for all $\xi\in\mathscr{L}_\mu(G)$ we obtain

$$\int_K U(xkyk^{-1})d_K(k)=U(x)U(y).$$

Moreover, it is clear that $\{U(x)\mid x\in G\}$ is an irreducible family of matrices, hence $U(x)$ is a spherical matrix function of type δ.

Definition. For a $\widehat{K}$-finite representation $\{\mathfrak{H}, T(x)\}$ of G such that $T(t)=\mu(t)I_{\mathfrak{H}}$ $(\forall t\in\mathbf{T})$, we put

$$\mathfrak{H}^K=\oplus\sum_{\delta\in\widehat{K}}\mathfrak{H}(\delta) \qquad \text{(algebraic direct sum)}.$$

The subspace $\mathfrak{H}^K$ is contained in $\mathfrak{H}^\infty$ because $\mathfrak{H}(\delta)\subset\mathfrak{H}^\infty$ for all $\delta\in\widehat{K}$. So it is in the domain of the operator $T(\xi)$ for $\xi\in\mathscr{L}_\mu(G)$.

Lemma 15. *For any $\widehat{K}$-finite representation $\{\mathfrak{H}, T(x)\}$ of G which satisfies $T(t)=\mu(t)I_{\mathfrak{H}}$ for $t\in\mathbf{T}$, the subspace $\mathfrak{H}^K$ is invariant under $T(\xi)$ for all $\xi\in\mathscr{L}_\mu(G)$.*

Proof. First we will show that $\mathfrak{H}^K$ is invariant under

$$T(\varphi)=\int_{Z\times K}\varphi(z,k)T(zk)d_Z(z)d_K(k)$$

for any $\varphi\in C_c(Z\times K)$. Let $\{z_1, z_2,\dots, z_q\}$ be all distinct elements in Z such that $\varphi(z_i,k)$ are not identically zero as functions on K. Then the calculation

$$T(k)T(\varphi)=\int_{Z\times K}\varphi(z,k_1)T(kzk_1)d_Z(z)d_K(k_1)$$
$$=\sum_{i=1}^q\int_K\varphi(z_i,k_1)\mu(\tau(k,z_i))T(z_ikk_1)d_K(k_1)$$

shows that $T(k)T(\varphi)\mathfrak{H}(\delta)$ is contained in the finite-dimensional subspace $\sum_{i=1}^{q}T(z_i)\mathfrak{H}(\delta)$ for all $k\in K$. This means $T(\varphi)\mathfrak{H}(\delta)\subset\mathfrak{H}^K$, or, $T(\varphi)\mathfrak{H}^K\subset\mathfrak{H}^K$.

Next we will show that $\mathfrak{H}^K$ is invariant under $T(\alpha)$ for all $\alpha\in\mathcal{U}(\mathfrak{r}_C)$. We take an arbitrary vector v in $\mathfrak{H}(\delta)$. Since $\mathfrak{H}(\delta)\subset\mathfrak{H}^\infty$, there exists a function $f\in\bar{\mu}*C_c^\infty(G)$ and a vector $v_1\in\mathfrak{H}$ such that $v=T(f)v_1$. Then, putting $f^k(x)=f(k^{-1}xk)$, the equality

$$\begin{aligned}T(k)T(\alpha)v&=T(k)T(\alpha)T(f)v_1=T(k)T(\alpha*f)v_1=T((\mathrm{Ad}(k)\alpha)*f^k)T(k)v_1\\&=T(\mathrm{Ad}(k)\alpha)T(k)T(f)v_1=T(\mathrm{Ad}(k)\alpha)T(k)v\end{aligned}$$

holds. Since $\mathrm{Ad}(k)\alpha$ remains in a finite-dimensional subspace of $\mathcal{U}(\mathfrak{r}_C)$ independent of $k\in K$, this equality shows that $T(\alpha)\mathfrak{H}(\delta)\subset\mathfrak{H}^K$.

Theorem 7. *Let $\{\mathfrak{H}, T(x)\}$ be a $\widehat{K}$-finite topologically irreducible representation of G such that $T(t)=\mu(t)I_{\mathfrak{H}}$ ($\forall\, t\in\mathsf{T}$). Then $\{\mathfrak{H}^K, T(\xi)\}$ is an algebraically irreducible representation of $\mathcal{L}_\mu(G)$.*

Proof. At first we prove that the subspace $\mathcal{H}_v=\{T(\xi)v|\xi\in\mathcal{L}_\mu(G)\}$, for any non-zero vector $v\in\mathfrak{H}^K$, is dense in $\mathfrak{H}$. Assume $w\in\mathfrak{H}'$ be a vector such that $\langle T(\xi)v, w\rangle=0$ for all $\xi\in\mathcal{L}_\mu(G)$. Then, for all $\varphi\in C_c(Z\times K)$ and $\alpha\in\mathcal{U}(\mathfrak{r}_c)$, we have

$$\int_R\langle T(r)T(\varphi)v, w\rangle d\alpha(r)=0.$$

This implies that $\phi_{T(\varphi)v,w}(r)=0$ because the function $\phi_{T(\varphi)v,w}(r)$ is analytic (see the proof of Lemma 15, and Theorem 5). Since this is true for all $\varphi\in C_c(Z\times K)$, we get $\phi_{v,w}(x)=0$ for all $x\in G$. Then the topological irreducibility of $\{\mathfrak{H}, T(x)\}$ implies $w=0$. Thus we know that $\mathcal{H}_v$ is dense in $\mathfrak{H}$.

The dense subspace $\mathcal{H}_v$ is invariant under $E(\mathfrak{H},\delta)$ since $\overline{\chi_\delta}*\xi$ belongs to $\mathcal{L}_\mu(G)$. Hence $\mathfrak{H}(\delta)\subset\mathcal{H}_v$ for all $\delta\in\widehat{K}$. This implies $\mathcal{H}_v=\mathfrak{H}^K$.

3.4. Linear map Ψ. We take a $\delta\in\widehat{K}$, denote by d its degree, and fix an irreducible unitary matrix representation $k\mapsto D(k)$ of K which belongs to δ.

We shall denote by $\mathcal{A}$ the space of all $d\times d$ matrices whose matrix elements are in the algebra $C_c(Z)\otimes_C\mathcal{U}(\mathfrak{r}_C)$. When we consider an element $A\in\mathcal{A}$ a function on Z, with its value in the set of $d\times d$ matrices whose matrix elements are in $\mathcal{U}(\mathfrak{r}_C)$, we write $A(z)=(a_{ij}(z))$. For $A(z)=(a_{ij}(z))$ and $B(z)=(b_{ij}(z))$ in $\mathcal{A}$, we give the convolution product by

$$A*B(z)=\int_Z A(z_1)B(z_1^{-1}z)\,d_Z(z_1),$$

here, $A(z_1)B(z_1^{-1}z)$ is the formal matrix product of $A(z_1)$ and $B(z_1^{-1}z)$.

Definition. A linear map Ψ: $\mathcal{L}_\mu(G)*\overline{\chi_\delta}\to\mathcal{A}$ is defined by

$$\Psi(\xi)=\int_K \xi(z,k)\,{}^tD(k)\,d_K(k)=\sum_i\Big(\int_K \varphi_i(z,k)\,{}^tD(k)\,d_K(k)\Big)\alpha_i$$

for $\xi=\sum_i \varphi_i\otimes\alpha_i\in\mathscr{L}_\mu(G)*\overline{\chi_\delta}$.

It is easy to see that the linear map Ψ is bijective and its inverse is given by

$$[\Psi^{-1}(A)](z,k)=d\cdot\operatorname{trace}[\overline{D(k)}A(z)].$$

Lemma 16. *For* $\xi\in\mathscr{L}_\mu(G)*\overline{\chi_\delta}$ and $\eta\in\mathscr{L}^\circ_{\mu,\delta}(G)$ *we have* $\Psi(\xi*\eta)=\Psi(\xi)*\Psi(\eta)$.

Proof. First we calculate $\xi*\eta$ for $\xi\in\mathscr{L}_\mu(G)*\overline{\chi_\delta}$ and $\eta\in\mathscr{L}^\circ_{\mu,\delta}(G)$;

$$\begin{aligned}
&\xi*\eta(z,k)\\
&=\int_{Z\times K}\mu(\tau(k_1,z_1^{-1}z))\,\xi(z_1,k_1)\,[\mathrm{Ad}(k_1)\eta]\,(z_1^{-1}z,k_1^{-1}k)\,d_Z(z_1)\,d_K(k_1)\\
&=\int_{Z\times K}\mu(\tau(k_1,z_1^{-1}z))\,\overline{\mu(\tau(k_1,z_1^{-1}z))}\;\xi(z_1,k_1)\,\eta(z_1^{-1}z,kk_1^{-1})\,d_Z(z_1)\,d_K(k_1)\\
&=\int_{Z\times K}\xi(z_1,k_1)\,\eta(z_1^{-1}z,kk_1^{-1})\,d_Z(z_1)\,d_K(k_1).
\end{aligned}$$

So, it follows that

$$\begin{aligned}
&[\Psi(\xi*\eta)](z)\\
&=\int_{Z\times K\times K}\xi(z_1,k_1)\,\eta(z_1^{-1}z,kk_1^{-1})\,{}^tD(k)\,d_Z(z_1)\,d_K(k_1)\,d_K(k)\\
&=\int_{Z\times K\times K}\xi(z_1,k_1)\,\eta(z_1^{-1}z,k)\,{}^tD(k_1)\,{}^tD(k)\,d_Z(z_1)\,d_K(k_1)\,d_K(k)\\
&=\int_Z\Big\{\int_K\xi(z_1,k_1)\,{}^tD(k_1)\,d_K(k_1)\Big\}\Big\{\int_K\eta(z_1^{-1}z,k)\,{}^tD(k)\,d_K(k)\Big\}d_Z(z)\\
&=[\Psi(\xi)*\Psi(\eta)](z),
\end{aligned}$$

which is the equality we wanted.

§4. Proof of the main theorem

We are now in a position to prove the following main

Theorem 8. *Let $G=S\rtimes K$ be a locally compact group which is a semi-direct product of a compactly generated abelian group S and a compact group K acting on S. Then every $\widehat{K}$-finite topologically irreducible representation $\{\mathfrak{H},T(x)\}$ of G is essentially a Banach representation.*

To prove Theorem 8 we may assume that S is just the one given in §3, and we keep any other notations in §3.

Let $\{\mathfrak{H},T(x)\}$ be δ-finite ($\delta\in\widehat{K}$), and $T(t)=\mu(t)I_{\mathfrak{H}}$ ($\forall\, t\in\mathbf{T}$). Let $U_\delta(x)$ be the corresponding spherical matrix function of type δ of height p. By Theorem 6 we know that $U_\delta(zrk)$ is analytic with respect to $r\in R$. It is clear that $U_\delta(tx)=U_\delta(xt)=\mu(t)U_\delta(x)$ for all $t\in\mathbf{T}$.

Now we put

$$\tilde{U}_\delta(A)=U_\delta(\Psi^{-1}(A))$$

for all $A\in\mathscr{A}$. By Lemma 16 the linear map $A\mapsto\tilde{U}_\delta(A)$ gives a p-dimensional representation of the algebra $\mathscr{A}^\circ=\Psi(\mathscr{L}^\circ_{\mu,\delta}(G))$. Moreover it is irreducible because the representation $\xi\mapsto U_\delta(\xi)$ of $\mathscr{L}^\circ_{\mu,\delta}(G)$ is irreducible (Theorem 6). Note that the algebra $\mathscr{A}$ has the unit $E=\Psi(\overline{\chi_\delta}\otimes 1)$ and it is in $\mathscr{A}^\circ$. Here we should understand that $\overline{\chi_\delta}$ is the extended function on $Z\times K$ as in §3, and that E is the function on Z taking the $d\times d$ unit matrix as its value at $z=e$ and 0 at $z\neq e$. The mapping

$$a\mapsto a*E(z)=\int_Z a(z_1)E(z_1^{-1}z)\,d_Z(z_1)=\begin{bmatrix} a(z) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a(z)\end{bmatrix}$$

of $C_c(Z)\otimes_{\mathbf{C}}\mathscr{U}(\mathfrak{r}_{\mathbf{C}})$ into $\mathscr{A}$ is an injection.

To aviod confusion let us denote by ${}^t(\mathbf{C}^p)$ the vector space of p-dimensional column vectors with components in $\mathbf{C}$. Now we take an arbitrary non-zero vector $v\in{}^t(\mathbf{C}^p)$ and put

$$\mathfrak{A}_v=\{A\in\mathscr{A}^\circ \mid \tilde{U}_\delta(A)v=0\},$$

then $\mathfrak{A}_v$ is a maximal left ideal in $\mathscr{A}^\circ$. Moreover we put

$$\mathfrak{M}_v=\{A\in\mathscr{A} \mid \tilde{U}_\delta(BA)v=0 \text{ for all } B\in\mathscr{A}\}.$$

Lemma 17. *$\mathfrak{M}_v$ is a left ideal in $\mathscr{A}$ such that $\mathfrak{M}_v\cap\mathscr{A}^\circ=\mathfrak{A}_v$.*

Proof. If A is in $\mathfrak{M}_v\cap\mathscr{A}^\circ$, then $\tilde{U}_\delta(A)v=\tilde{U}_\delta(I_dA)v=0$. Thus A is in $\mathfrak{A}_v$.

Conversely, let $A=\Psi(\xi)$ be an arbitrary element in $\mathfrak{A}_v$ where $\xi\in\mathscr{L}^\circ_{\mu,\delta}(G)$. Then, for any $B=\Psi(\eta)$, we have $\tilde{U}_\delta(BA)v=U_\delta(\eta*\xi)v=U^\circ_\delta(\eta*\xi)v=U_\delta(\eta^\circ*\xi)v=U_\delta(\eta)U_\delta(\xi)v=0$. Therefore A belongs to $\mathfrak{M}_v\cap\mathscr{A}^\circ$.

Let $\mathfrak{M}$ be an arbitrary maximal left ideal in $\mathscr{A}$ containing $\mathfrak{M}_v$. It is easy to see $\mathfrak{A}_v\subset\mathfrak{M}\cap\mathscr{A}^\circ\subsetneqq\mathscr{A}^\circ$, which implies $\mathfrak{A}_v=\mathfrak{M}\cap\mathscr{A}^\circ$. Therefore, we can regard $\mathscr{A}^\circ/\mathfrak{A}_v$ as a subspace of $\mathscr{A}/\mathfrak{M}$. We denote by $\Pi(A)$ the natural action of $A\in\mathscr{A}$ on $\mathscr{A}/\mathfrak{M}$. Then the subrepresentation $A\mapsto\Pi(A)$ of $\mathscr{A}^\circ$ on $\mathscr{A}^\circ/\mathfrak{A}_v$ is irreducible, and is equivalent to $A\mapsto\tilde{U}_\delta(A)$.

We shall denote by E_{ij} the element in $\mathscr{A}$ such that $E_{ij}(e)$ is the $d\times d$-matrix whose (i,j)-matrix element is 1 and the others are 0, and that $E_{ij}(z)=0$ for $z\neq e$. Then we have a direct sum decomposition

$$\mathscr{A}/\mathfrak{M} = \Pi(E_{11})(\mathscr{A}/\mathfrak{M}) \oplus \cdots \oplus \Pi(E_{dd})(\mathscr{A}/\mathfrak{M}).$$

Since each subspace $\Pi(E_{ii})(\mathscr{A}/\mathfrak{M})$ is invariant under $\Pi(a * E)$ for any element $a \in C_c(Z) \otimes_{\mathbf{C}} \mathscr{U}(\mathfrak{r}_{\mathbf{C}})$, we get d representations of $C_c(Z) \otimes_{\mathbf{C}} \mathscr{U}(\mathfrak{r}_{\mathbf{C}})$ on $\Pi(E_{ii})(\mathscr{A}/\mathfrak{M})$ $(1 \leq i \leq d)$.

Lemma 18. *The representations of $C_c(Z) \otimes_{\mathbf{C}} \mathscr{U}(\mathfrak{r}_{\mathbf{C}})$ on $\Pi(E_{ii})(\mathscr{A}/\mathfrak{M})$ are algebraically irreducible and mutually equivalent.*

Proof. The operator $\Pi(E_{ij})$ is clearly an intertwing operator of $\Pi(E_{jj})(\mathscr{A}/\mathfrak{M})$ onto $\Pi(E_{ii})(\mathscr{A}/\mathfrak{M})$. So we have only to show the irreducibility. Assume that $\Pi(E_{11})(\mathscr{A}/\mathfrak{M})$ be reducible, then there exists a non-trivial invariant subspace H_1 of $\Pi(E_{11})(\mathscr{A}/\mathfrak{M})$. We put

$$H = H_1 \oplus \Pi(E_{21}) H_1 \oplus \cdots \oplus (E_{d1}) H_1.$$

Then, for any $A = (a_{ij}) = \mathscr{A}$, we have

$$\Pi(A) H = \sum_{i,j=1}^{d} \Pi(a_{ij} * E_{ij}) H = \sum_{i,j=1}^{d} \Pi(a_{ij} * E) \Pi(E_{ij}) H$$

$$= \sum_{i,j=1}^{d} \Pi(a_{ij} * E) \Pi(E_{i1}) H_1 \subset \sum_{i=1}^{d} \Pi(E_{i1}) H_1 = H,$$

which contradicts the irreducibility of the natural representation $A \mapsto \Pi(A)$ of $\mathscr{A}$ on $\mathscr{A}/\mathfrak{M}$.

Now we must find out how an algebraically irreducible representation of the algebra $C_c(Z) \otimes_{\mathbf{C}} \mathscr{U}(\mathfrak{r}_{\mathbf{C}})$ can be given. To do this, we find another algebra which is isomorphic to $C_c(Z) \otimes_{\mathbf{C}} \mathscr{U}(\mathfrak{r}_{\mathbf{C}})$.

Let X_0 be a basis of the Lie algebra of $\mathbf{T}$, and $X_1, \ldots, X_m$ a basis of the Lie algebra of $\mathbf{R}^m$. Here we can take $X_i (1 \leq i \leq m)$ as the partial differential operator with respect to the i-th coordinate on $\mathbf{R}^m$. Then $X_0, X_1, \ldots, X_m$ form a basis of the Lie algebra $\mathfrak{r}$ of $R = \mathbf{TR}^m$, and $\mathscr{U}(\mathfrak{r}_{\mathbf{C}})$ is the algebra of polynomials $\mathfrak{P} = \mathbf{C}[X_0, \ldots, X_m]$. When we emphasize an element $\alpha \in \mathscr{U}(\mathfrak{r}_{\mathbf{C}})$ is a polynomial of $X_0, \ldots, X_m$, we write $\alpha = \alpha(X_0, \ldots, X_m)$.

On the other hand, we take n variables $Y_1, \ldots, Y_n$ and put

$$\mathfrak{Q} = \bigcup_{j_1, \ldots, j_n \geq 0} Y_1^{-j_1} \cdots Y_n^{-j_n} \mathbf{C}[Y_1, \ldots, Y_n],$$

where $j_1, \ldots, j_n$ are non-negative integers. Then $\mathfrak{Q}$ is an algebra with the obvious operations of sum and product. The algebra of polynomials of $Y_1, \ldots, Y_n$, X_0, $X_1, \ldots, X_m$ will be denoted by $\mathbf{C}[Y, X]$ for short.

Since a function $a \in C_c(Z)$ is defined on $Z = \mathbf{Z}^n$, we write $a = a(z) = a(z_1, \ldots, z_n)$ where $z = (z_1, \ldots, z_n) \in \mathbf{Z}^n$. We now define an isomorphism $\wedge$ of the algebra $C_c(Z) \otimes_{\mathbf{C}} \mathscr{U}(\mathfrak{r}_{\mathbf{C}})$ onto $\mathfrak{Q} \otimes_{\mathbf{C}} \mathfrak{P}$ by

$$(a \otimes \alpha)^{\wedge}(Y_1,\dots,Y_n,\ X_0,\dots,X_m)$$
$$=\sum_{z\in Z} a(z_1,\dots,z_n)\, Y_1^{z_1}\cdots Y_n^{z_n}\alpha(X_0,\dots,X_m).$$

Lemma 19. *Algebraically irreducible representations of $C_c(Z)\otimes_{\mathbf{C}}\mathfrak{U}(\mathfrak{r}_{\mathbf{C}})$ are one-dimensional. They are parametrized by $\theta=(\theta_1,\dots,\theta_n)\in(\mathbf{C}^{\times})^n$ $(\mathbf{C}^{\times}=\mathbf{C}-\{0\})$ and $\nu=(\nu_0,\ \nu_1,\dots,\nu_m)\in\mathbf{C}^{m+1}$, and given by*

$$\Lambda^{\theta,\nu}\Big(\sum_i a_i\otimes\alpha_i\Big)=\sum_i (a_i\otimes\alpha_i)^{\wedge}(\theta_1,\dots,\theta_n,\ \nu_0,\ \nu_1,\dots,\nu_m).$$

Proof. By the above observation it is enough to give all algebraically irreducible representations of $\mathfrak{Q}\otimes_{\mathbf{C}}\mathfrak{P}$, or equivalently, all proper maximal ideals in it.

Let $\mathfrak{X}$ be a maximal ideal in $\mathfrak{Q}\otimes_{\mathbf{C}}\mathfrak{P}$. Then $\mathfrak{X}_0=\mathfrak{X}\cap\mathbf{C}[Y,X]$ is an ideal in $\mathbf{C}[Y,X]$, and it is easy to see

$$\mathfrak{X}=\bigcup_{j_1,\dots,j_n\geq 0} Y_1^{-j_1}\cdots Y_n^{-j_n}\mathfrak{X}_0.$$

Suppose there exists an ideal $\mathfrak{Y}_0$ in $\mathbf{C}[Y,X]$ such that $\mathfrak{X}_0\subsetneqq\mathfrak{Y}_0\subset\mathbf{C}[Y,X]$, then

$$\mathfrak{Y}=\bigcup_{j_1,\dots,j_n\geq 0} Y_1^{-j_1}\cdots Y_n^{-j_n}\mathfrak{Y}_0$$

is an ideal in $\mathfrak{Q}\otimes_{\mathbf{C}}\mathfrak{P}$, and satisfies $\mathfrak{X}\subsetneqq\mathfrak{Y}$. This implies $\mathfrak{Y}=\mathfrak{Q}\otimes_{\mathbf{C}}\mathfrak{P}$. Hence it follows that $\mathfrak{Y}_0=\mathfrak{Y}\cap\mathbf{C}[Y,X]=\mathbf{C}[Y,X]$. Therefore, we know that $\mathfrak{X}_0$ is a maximal ideal in $\mathbf{C}[Y,X]$. As is well known, there exists $\theta=(\theta_1,\dots,\theta_n)\in\mathbf{C}^n$ and $\nu=(\nu_0,\ \nu_1,\dots,\nu_m)\in\mathbf{C}^{m+1}$ such that

$$\mathfrak{X}_0=\{P\in\mathbf{C}[Y,X]\,|\,P(\theta,\ \nu)=0\}.$$

If $\theta_i=0$, then $Y_i\in\mathfrak{X}_0$. But this means that $1=Y_i^{-1}Y_i\in\mathfrak{X}$, a contradiction. So, θ must belong to $(\mathbf{C}^{\times})^n$. Conversely, if θ belongs to $(\mathbf{C}^{\times})^n$, then $\mathfrak{X}$ is a proper ideal since it can not contain 1. As a result, we have proved that

$$\mathfrak{X}=\{P\in\mathfrak{Q}\otimes_{\mathbf{C}}\mathfrak{P}\,|\,P(\theta,\ \nu)=0\},$$

and that every maximal ideal is given in this way.

Thus every algebraically irreducible representation of $\mathfrak{Q}\otimes_{\mathbf{C}}\mathfrak{P}$ is given by $P\mapsto P(\theta,\ \nu)\in\mathbf{C}$ for some pair $(\theta,\ \nu)\in(\mathbf{C}^{\times})^n\times\mathbf{C}^{m+1}$. Since $(a\otimes\alpha)\mapsto(a\otimes\alpha)^{\wedge}$ is an isomorphism of $C_c(Z)\otimes_{\mathbf{C}}\mathfrak{U}(\mathfrak{r}_{\mathbf{C}})$ onto $\mathfrak{Q}\otimes_{\mathbf{C}}\mathfrak{P}$, this proves the statement of the lemma.

By Lemmas 18 and 19, the natural representation $A\mapsto\Pi(A)$ of $\mathscr{A}$ on $\mathscr{A}/\mathfrak{M}$ is equivalent to the irreducible representation on ${}^t(\mathbf{C}^d)$ given by

$$\pi(A)\begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix} = \begin{bmatrix} \Lambda^{\theta,\nu}(a_{11}) & \cdots & \Lambda^{\theta,\nu}(a_{1d}) \\ \vdots & \ddots & \vdots \\ \Lambda^{\theta,\nu}(a_{d1}) & \cdots & \Lambda^{\theta,\nu}(a_{dd}) \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{d} \Lambda^{\theta,\nu}(a_{1i}) v_i \\ \vdots \\ \sum_{i=1}^{d} \Lambda^{\theta,\nu}(a_{di}) v_i \end{bmatrix}$$

for $A=(a_{ij})\in\mathscr{A}$ with $a_{ij}\in C_c(Z)\otimes_{\mathbf{C}}\mathscr{U}(\mathfrak{r}_{\mathbf{C}})$.

Lemma 20. *There exists a p-dimensional subspace $\mathscr{H}$ of ${}^t(\mathbf{C}^d)$, invariant under $\pi(A)$ ($\forall\ A\in\mathscr{A}^\circ$), such that the representation $A\mapsto\pi(A)|_{\mathscr{H}}$ of $\mathscr{A}^\circ$ is equivalent to $A\mapsto\tilde{U}_\delta(A)$.*

Proof. Since the representation $A\mapsto\tilde{U}_\delta(A)$ of $\mathscr{A}^\circ$ was equivalent to $A\to\Pi(A)$ realized on the subspace $\mathscr{A}^\circ/\mathfrak{A}_v$ of $\mathscr{A}/\mathfrak{M}$, our statement is clear by the definition of π.

Recall $\overline{\chi_\delta}$ as an element in $C_c(Z\times K)$ and put $A_0=(\overline{\chi_\delta}\otimes X_0)I_d$. Then A_0 is an element in $\mathscr{A}$, and it satisfies

$$\begin{aligned}\tilde{U}_\delta(A_0) &= U_\delta(\Psi^{-1}(A_0)) = U_\delta(\overline{\chi_\delta}\otimes X_0)\\ &= \int_{Z\times\mathbf{T}\times K} U_\delta(ztk)\overline{\chi_\delta}(z,k)\,d_Z(z)\,dX_0(t)\,d_K(k)\\ &= \int_{\mathbf{T}\times K}\mu(t)\,U_\delta(k)\overline{\chi_\delta}(k)\,dX_0(t)\,d_K(k)\\ &= \mu(X_0)I_d,\end{aligned}$$

where $\mu(X_0)=\frac{d}{dt}\mu(\exp tX_0)|_{t=0}$. From this we obtain

$$\begin{bmatrix} \mu(X_0) & & 0 \\ & \ddots & \\ 0 & & \mu(X_0) \end{bmatrix} = \pi(A_0) = \begin{bmatrix} \Lambda^{\theta,\nu}(\overline{\chi_\delta}\otimes X_0) & & 0 \\ & \ddots & \\ 0 & & \Lambda^{\theta,\nu}(\overline{\chi_\delta}\otimes X_0) \end{bmatrix},$$

that is, $\mu(X_0)=\Lambda^{\theta,\nu}(\overline{\chi_\delta}\otimes X_0)=\nu_0$. So, $\nu_0=\mu(X_0)$ is determined by the representation $\{\mathfrak{H},\ T(x)\}$ (and X_0). Now we pick up a 1-dimensional representation

$$\Lambda^{\theta,\nu}(ztr)=\theta_1^{z_1}\cdots\theta_n^{z_n}\mu(t)\,e^{r_1\nu_1+\cdots+r_m\nu_m}$$

of the group $ZR=\mathbf{Z}^n\mathbf{T}\mathbf{R}^m$, where $z=(z_1,\dots,z_n)\in\mathbf{Z}^n$, $t\in\mathbf{T}$, $r=(r_1,\dots,r_m)\in\mathbf{R}^m$. Then the above 1-dimensional representation $\Lambda^{\theta,\nu}$ of the algebra $C_c(Z)\otimes_{\mathbf{C}}\mathscr{U}(\mathfrak{r}_{\mathbf{C}})$ is obtained by

$$\Lambda^{\theta,\nu}(a\otimes\alpha)=\int_{Z\times R}\Lambda^{\theta,\nu}(zr)a(z)d_Z(z)d\alpha(r).$$

We put $W(zrk)=\Lambda^{\theta,\nu}(zr)\,{}^tD(k)$ and define

$$W^\circ(zrk)=\int_K W(hzrkh^{-1})d_K(h)=\int_K\Lambda^{\theta,\nu}(hzrh^{-1})\,{}^tD(hkh^{-1})d_K(h).$$

For any $\xi\in\mathscr{L}^\circ_{\mu,\delta}(G)$ put $A=\Psi(\xi)=(a_{ij})\in\mathscr{A}^\circ$, then

$$W^\circ(\xi)=W(\xi)=\int_G W(x)d\xi(x)=\int_{Z\times R\times K}\Lambda^{\theta,\nu}(zr)\,{}^tD(k)d\xi(zrk)$$
$$=\begin{bmatrix}\Lambda^{\theta,\nu}(a_{11}) & \cdots & \Lambda^{\theta,\nu}(a_{1d})\\ \vdots & \ddots & \vdots\\ \Lambda^{\theta,\nu}(a_{d1}) & \cdots & \Lambda^{\theta,\nu}(a_{dd})\end{bmatrix}=\pi(A).$$

From this equality and Lemma 20, it follows that the p-dimensional irreducible representation $\xi\mapsto U_\delta(\xi)$ of $\mathscr{L}^\circ_{\mu,\delta}(G)$ is equivalent to $\xi\mapsto W^\circ(\xi)|_{\mathscr{H}}$, where $\mathscr{H}$ is the p-dimensional subspace of ${}^t(\mathbf{C}^d)$ stated in Lemma 20.

The fact that $\mathscr{H}$ is invariant under $W^\circ(x)$ is easy to prove, and we give the proof for the sake of completeness. Let w be any element in the dual space of ${}^t(\mathbf{C}^d)$ such that $\langle v, w\rangle=0$ for all $v\in\mathscr{H}$. Then, for any $v\in\mathscr{H}$ and $\xi\in\mathscr{L}_\mu(G)$, we have

$$\int_G\langle W^\circ(x)v, w\rangle d\xi(x)=\int_G\langle W(x)v, w\rangle d\xi^\circ(x)$$
$$=\int_{G\times K}\langle W(xk^{-1})v, w\rangle\chi_\delta(k)d\xi^\circ(x)d_K(k)=\int_G\langle W(x)v, w\rangle d(\xi^\circ*\overline{\chi_\delta})(x)$$
$$=\langle\pi(\Psi(\xi^\circ*\overline{\chi_\delta}))v, w\rangle=0,$$

which means $\langle W^\circ(x)v, w\rangle=0$ because $\langle W^\circ(x)v, w\rangle$ is analytic with respect to $r\in R$. Thus we get $W^\circ(x)v\in\mathscr{H}$.

Since $U_\delta(e)$ and $W^\circ(e)|_{\mathscr{H}}$ are both the identity operator, $U_\delta(x)$ is equivalent to $W^\circ(x)|_{\mathscr{H}}$, that is, there exists an invertible operator P on $\mathscr{H}$ such that $U_\delta(x)=P^{-1}(W^\circ(x)|_{\mathscr{H}})P$. The function $W^\circ(x)|_{\mathscr{H}}$ is clearly quasi-bounded, and hence $U_\delta(x)$ is also quasi-bounded. In other words, the representation $\{\mathfrak{H}, T(x)\}$ is SF-equivalent to a subquotient of the Banach representation induced from 1-dimensional representation $\Lambda^{\theta,\nu}$ of the subgroup $S=ZR=\mathbf{Z}^n\mathbf{T}\mathbf{R}^m$ (cf. [12]). Therefore, the proof of Theorem 8 is now completed.

Added in proof. The proof of Lemma 19 is not valid, because the equality $\mathfrak{Y}_0=\mathfrak{Y}\cap\mathbf{C}[Y, X]$ is not always true. So, we show here how to find $\theta=(\theta_1,\dots,\theta_n)\in(\mathbf{C}^\times)^n$ and $\nu=(\nu_0,\nu_1,\dots,\nu_m)\in\mathbf{C}^{m+1}$.

Let $\mathbf{C}[Y, Y', X]$ be the algebra of polynomials of $Y_1,\dots, Y_n$, $Y'_1,\dots, Y'_n$, and X_0, $X_1,\dots, X_m$ with complex coefficients. We denote by I the ideal in $\mathbf{C}[Y, Y', X]$ generated by $Y_1Y'_1-1,\dots, Y_nY'_n-1$. Then the homomorphism φ: $\mathbf{C}[Y, Y', X]\to\mathfrak{O}\otimes_{\mathbf{C}}\mathfrak{P}$ such that $\varphi(Y_i)=Y_i$, $\varphi(Y'_i)=Y_i^{-1}(1\leq i\leq n)$, $\varphi(X_j)=X_j(0\leq j\leq m)$

naturally induces an isomorphism of $\mathbf{C}[Y, Y', X]/I$ onto $\mathfrak{Q}\otimes_{\mathbf{C}}\mathfrak{P}$. So, any maximal ideal $\mathfrak{X}$ in $\mathfrak{Q}\otimes_{\mathbf{C}}\mathfrak{P}$ corresponds bijectively to a maximal ideal $\mathfrak{Y}=\varphi^{-1}(\mathfrak{X})$ in $\mathbf{C}[Y, Y', X]$ containing I. Now there exist uniquely $\theta=(\theta_1,\ldots, \theta_n)\in\mathbf{C}^n$, $\theta'=(\theta'_1,\ldots, \theta'_n)\in\mathbf{C}^n$, and $\nu=(\nu_0, \nu_1,\ldots, \nu_m)\in\mathbf{C}^{m+1}$ such that

$$\mathfrak{Y}=\{P\in\mathbf{C}[Y, Y', X] \mid P(\theta, \theta', \nu)=0\}.$$

Since $Y_iY'_i-1$ $(1\leq i\leq n)$ are in $\mathfrak{Y}$, we have $\theta_i\theta'_i=1$ $(1\leq i\leq n)$. Therefore $\theta\in(\mathbf{C}^\times)^n$ and we clearly have

$$\mathfrak{X}=\{P\in\mathfrak{Q}\otimes_{\mathbf{C}}\mathfrak{P} \mid P(\theta, \nu)=0\}.$$

Conversely, it is easy to see that, for every $(\theta, \nu)\in(\mathbf{C}^\times)^n\times\mathbf{C}^{m+1}$, the right hand side is a proper maximal ideal in $\mathfrak{Q}\otimes_{\mathbf{C}}\mathfrak{P}$.

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