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Surfaces of general type whose canonical map is composed of a pencil of genus 3 with small invariants

By

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0. Introduction

Let X be a minimal surface of general type over the complex number field. Assume that $p_g(X) \geq 3$, and $|K_X|$ is composed of a pencil. The existence of such surfaces was known as early as 1948 by Pompilij's examples. Later there have been studies by Beauville, Debarre, Xiao and others ([3], [5], [10], [12]). Refer to Section 2 of [4] for a nice survey.

Let b denote the geometric genus of the image of the canonical map and let g denote the genus of a general member of the pencil of which $|K_X|$ is composed. Assume that $g \geq 3$. Then the inequality

$$K_X^2 \geq 4p_g(X) + 4(b-1) \tag{1}$$

is valid with very few exceptions (cf. Theorem 2.3 of [4]).

In this paper we will give an example with $p_g = 3$, $b = 0$, $g = 3$ and $K^2 = 7$. Then we will prove that is the lowest possible K^2.

The other possible exception to (1) is the case $p_g = 4$ and $K_X^2 = 9$, which was proposed as an open problem in [11]. We will prove that this case does not occur, and consequently there is only one exception to (1).

1. Preliminaries

1.1. $\mathbf{P}^2$-bumdles over $\mathbf{P}^1$. First we state some basic facts about $\mathbf{P}^2$-bundles over the projective line $\mathbf{P}^1$, which will be used throughout this paper. We will use $\mathcal{O}(n)$ to denote either the invertible sheaf of degree n on $\mathbf{P}^1$ or its corresponding line bundle, depending on the context.

Let V be a vector bundle of rank 3 over $\mathbf{P}^1$. It is well-known that V can be decomposed into a direct sum of line bundles, i.e., $V \cong \mathcal{O}(k) \oplus \mathcal{O}(m) \oplus \mathcal{O}(n)$. Let $W = \mathbf{P}(V)$ be the associated $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ and let $f: W \to \mathbf{P}^1$ denote the natural map. Since $\mathbf{P}(V \otimes L) \cong \mathbf{P}(V)$ for any line bundle L, we may

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assume that $V=\mathcal{O}\oplus\mathcal{O}(m)\oplus\mathcal{O}(n)$ with $0\leq m\leq n$.

The subbundle $\mathcal{O}(m)\oplus\mathcal{O}(n)$ of V gives rise to an irreducible hypersurface E_0 inside W. Let η be a fiber of f. Then E_0 and η generate Pic (W), and $E_0^3=-m-n$, $E_0^2\eta=1$, $E_0\eta^2=\eta^3=0$. The line bundle $\mathcal{O}(n)$ gives rise to a section e of the ruled surface E_0. Obviously $e^2=m-n$ as a divisor of E_0. Let ξ be a fiber of E_0. Then

$$\mathcal{O}_{E_0}(E_0)\cong\mathcal{O}_{E_0}(e-m\xi). \tag{2}$$

The cononical divisor K_W is linearly equivalent to $-3E_0-(m+n+2)\eta$.

Lemma 1.1. *Let S be a prime divisor of the $\mathbf{P}^2$-bundle $W=\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(m)\oplus\mathcal{O}(n))$ over $\mathbf{P}^1$ with $0\leq m\leq n$. Assume that S is linearly equivalent to $4E_0+b\eta$ for some $b\in Z$. Then*

(1) $b\geq 4m$.

(2) $\chi(S)=3b-4m-4n-2$.

Proof. (1) Since S is irreducible and not equal to E_0, the linear system $|S|$ in W cuts out a non-empty subsystem of the linear system $S|_{E0}\sim 4e+(b-4m)\xi$. This implies that $b-4m\geq 0$.

(2) The short exact sequence

$$0\to\mathcal{O}_W(-4E_0-b\eta)\to\mathcal{O}_w\to\mathcal{O}_s\to 0$$

implies that

$$\chi(\mathcal{O}_s)=1-\chi(\mathcal{O}_w(-4E_0-b\eta))=1+\chi(\mathcal{O}_w(E_0+(b-m-n-2)\eta)). \tag{3}$$

Then the short exact sequences

$$0\to\mathcal{O}_w((b-m-n-2)\eta)\to\mathcal{O}_w(E_0+(b-m-n-2)\eta)\to \mathcal{O}_{E_0}(e+(b-2m-n-2)\xi)\to 0$$

and

$$0\to O_{E_0}((b-2m-n-2)\xi)\to\mathcal{O}_{E_0}(e+(b-2m-n-2)\xi)\to\mathcal{O}_e(b-m-2n-2)\to 0$$

imply that

$$\begin{aligned}
&\chi(\mathcal{O}_w(E_0+(b-m-n-2)\eta))\\
&=\chi(\mathcal{O}_w(b-m-n-2)\eta))+\chi(\mathcal{O}_{E_0}(e+(b-2m-n-2)\xi))\\
&=\chi(\mathcal{O}_w((b-m-n-2)\eta))+\chi(\mathcal{O}_{E_0}((b-2m-n-2)\xi))+\chi(\mathcal{O}_e(b-m-2n-2))\\
&=(b-m-n-1)+(b-2m-n-1)+(b-m-2n-1)\\
&=3b-4m-4n-3.
\end{aligned}$$

The result follows from Equation (3).

Let y_0, y_1, be the projective coordinates of $\mathbf{P}^1$. Let $\mathbf{P}^1=U_0\cup U_1$ be the standard open affine covering of $\mathbf{P}^1$, where $U_i=\{(y_0, y_1)\in\mathbf{P}^1|y_i\neq 0\}$. Then $z=y_1/y_0$ and $z'=y_0/y_1$ are the affine coordinates of U_0 and U_1 respectively. Let $W_i=f^{-1}(U_i)$, $i=0, 1$. Obviously $W_i\cong\mathbf{A}^1\otimes\mathbf{P}^2$ for $i=0, 1$.

Let x_0, x_1, x_2 be the fiber coordinates of $\mathcal{O}$, $\mathcal{O}(m)$ and $\mathcal{O}(n)$ over U_0 respectively. A hypersurface S_0 in W_0 is given by an equation

$$\sum_{i,j,k,r\geq 0, i+j+k=d} c_{ijkr} z^r x_0^i x_1^j x_2^k,$$

where $d\geq 0$ is a fixed integer. Let S be the closure of S_0 in W. Let $t=\max\{r+mj+nk | c_{ijkr}\neq 0\}$. Then $S\sim dE_0+t\eta$ as a divisor of W. The equation of $S\cap W_1$ is

$$\sum_{i,j,k,r\geq 0, i+j+k=d} c_{ijkr} z'^{t-r-mj-nk} x_0^i x'^j_1 x'^k_2,$$

where $z'=1/z, x'_1=z^{-m}x_1, x'_2=z^{-n}x_2$.

1.2. normal singularities of surfaces. Most results in this subsection are well-known and their proofs are omitted.

Let X be a nonsingular complete surface. Let $A_1, \ldots, A_n$ be distinct irreducible curves on X with $n\geq 1$. The set $A = \cup_{i=1}^n A_i$ is called an *exceptional set* if A is connected and the intersection matrix of these curves is negative definite. A divisor $D = \sum_{i=1}^n d_i A_i$ is called a *positive cycle* on A if every d_i is a positive integer. The integer d_i is called the coefficient of A_i in D. Let D and D' be two positive cycles on an exceptional set A. Then denote $D\leq D'$ if $D'-D$ is an effective divisor. Let I be a subset of $\{1,\ldots,n\}$ and let $D=\sum_{i=1}^n d_i A_i$ be a positive cycle on A. Assume that the set $\Delta = \cup_{i\in I} A_i$ is connected. Then we define $D|_\Delta = \sum_{i\in I} d_i A_i$.

The following two occasions of exceptional sets arise in this paper:

(1) Let $\pi : X\to Y$ be a birational morphism, where X and Y are complete surfaces and X is nonsingular. Let p be a normal singularity of Y. Then $\pi^{-1}(p)$ is an exceptional set.

(2) Let F be a fiber of a morphism $\pi : X\to C$ from a nonsingular complete surface X onto a nonsingular curve C. Let $\{A_1,\ldots,A_n\}$ be a proper subset of the set of all irreducible components of F. If the set $\cup_{i=1}^n A_i$ is connected then it is an exceptional set.

Let $A = \cup_{i=1}^n A_i$ be an exceptional set. There is a unique positive cycle $Z = \sum_{i=1}^n d_i A_i$ such that $A_i Z\leq 0$ for all i and Z is minimal with this property. (cf. [1].) This positive cycle Z is called the *fundamental cycle* of A. If A is the exceptional set of a normal singularity p, then Z is often called the fundamental cycle of p.

If every component of an exceptional set is a nonsingular rational curve with self-intersection -2, then it is the exceptional set of a rational double point (cf. [2]). Let Z be the fundamental cycle of a rational double point. Then $Z^2=-2$.

Lemma 1.2. *Let $B=\cup_{i=1}^{n}B_i$ be the exceptional set of a rational double point and let $D=\sum_{i=1}^{n}d_iB_i$ be a positive cycle. Assume that $DB_i\leq 0$ for every i and $D^2=-2$. Then D is the fundamental cycle of B.*

Proof. Let Z be the fundamental cycle of B. Then $Z\leq D$ by the definition of fundamental cycle. Let $G=D-Z$. Then $GZ\leq 0$. So $-2=D^2=Z^2+2GZ+G^2\leq -2+G$. Hence $G^2=0$, which implies $G=0$.

Lemma 1.3. *Let $B=\cup_{i=1}^{n}B_i$ be the exceptional set of a rational double point. Let Z be the fundamental cycle of B. Then $-1\leq B_iZ\leq 0$ for every i.*

Proof. Easily checked for every type of rational double points.

Lemma 1.4. *Let $B=\cup_{i=1}^{n}B_i$ be the exceptional set of a rational double point and let $D=\sum_{i=1}^{n}d_iB_i$ be a positive cycle. Assume that $DB_i\leq 0$ for every i and $D^2=-4$. Then the coefficient of B_i in D is even if $DB_i<0$.*

Proof. Let Z be the fundamental cycle of B and let $G=D-Z$. Then G is effective and $G\neq 0$. Since $-4=Z^2+2GZ+G^2$ and $G^2\leq -2$, we have $GZ=0$ and $G^2=-2$. Let B_i be a component of Supp(G). Then $B_iZ=0$. Thus $B_iG=B_iD\leq 0$. Moreover, Supp(G) is connected, for $G^2=-2$. By Lemma 1.2 G is the fundamental cycle of Supp(G).

Let B_j be a component of B such that $B_jD<0$. We claim that $d_j>1$. To prove the claim, suppose that $d_j=1$. Then B_j is not a component of G. Lemma 1.3 implies that $B_jZ=-1$, $B_jG=0$. Since $d_j=1$, the coefficient of B_j in Z is equal to 1. So the rational double point is of type A_n. We may assume that $B_iB_{i+1}=1$ for $1\leq i\leq n-1$. Then it is easy to see that $n\geq 3$, $Z=B_1+\cdots+B_n$ and $G=B_2+\cdots+B_{n-1}$. Thus there is no B_j with $B_jZ=-1$ and $B_jG=0$. This leads to a contradiction. The claim is proved.

Since $\sum_{i=1}^{n}d_iB_iD=-4$, it follows that d_i is 2 or 4 for every B_i such that $DB_i<0$.

A sequence $\{A_{i_1},\ldots,A_{i_m}\}$ of irreducible components of A is called a *computation sequence* for Z if $A_{i_k}(\sum_{j=1}^{k-1}A_{i_j})>0$ for $2\leq k\leq m$ and $\sum_{j=1}^{m}A_{i_j}=Z$.

Computation sequence always exists and i_1 can be chosen arbitrarily.

Lemma 1.5. *Let Z be the fundamental cycle of a rational double point and let $\{A_{i_1},\ldots,A_{i_m}\}$ be a computation sequence for Z. Then $A_{i_r}\sum_{j=1}^{r-1}A_{i_j}=1$ for every $r>1$.*

Proof. Easy.

Lemma 1.6. *Let X be a nonsingular complete surface with $H^1(X,\mathcal{O}_X)=0$. Let $A=\cup_{i=1}^{n}A_i$ be an exceptional set on X where $A_i\cong\mathbf{P}^1$ for every i. Let $D=\sum_{i=1}^{n}d_iA_i$ be a positive cycle on A such that $A_iD\leq 0$ for every i. Assume that $d_1=1$, $A_1(D-A_1)=2$, $D^2=A_1^2+2$, and $A_i^2=-2$ for $i>1$. Then $|K_X+D|\neq\emptyset$ and*

A_1 is not a fixed component of $|K_X+D|$.

Proof. Let $A'=\cup_{i=2}^{n}A_i$ and let $Z=D-A_1$. The equality $A_1^2+2=D^2=A_1^2+2A_1Z+Z^2$ implies that $Z^2=-2$. Hence A' is connected. For every $i>1$ we have $A_iZ\leq A_iD\leq 0$. It follows from Lemma 1.2 that Z is the fundamental cycle of A'.

Let $\{A_{i_1},\ldots,A_{i_m}\}$ be a computation sequence for Z. The short exact sequence

$$0\to\mathcal{O}_X(K_X)\to\mathcal{O}_X(K_X+A_{i_1})\to\mathcal{O}_{\mathbf{P}^1}(-2)\to 0$$

implies that $h^0(X,\mathcal{O}_X(K_X+A_{i_1}))=h^0(X,\mathcal{O}_X(K_X))$ and $H^1(X,\mathcal{O}_X(K_X+A_{i_1}))=0$.

By Lemma 1.5 we have a short exact sequence

$$0\to\mathcal{O}_X(K_X+\sum_{j=1}^{r}A_{i_j})\to\mathcal{O}_X(K_X+\sum_{j=1}^{r+1}A_{i_j})\to\mathcal{O}_{\mathbf{P}^1}(-1)\to 0$$

for every $0<r<m$. It follows from induction that $h^0(X,\mathcal{O}_X(K_X+Z))=h^0(X,\mathcal{O}_X(K_X))$ and $H^1(X,\mathcal{O}_X(K_X+Z))=0$.

Since $A_1(K_X+Z+A_1)=0$, we have a short exact sequence

$$0\to\mathcal{O}_X(K_X+Z)\to\mathcal{O}_X(K_X+Z+A_1)\to\mathcal{O}_{\mathbf{P}^1}\to 0,$$

which implies $h^0(X,\mathcal{O}_X(K_X+D))=h^0(X,\mathcal{O}_X(K_X))+1$ and A_1 is not a fixed component of $|K_X+D|$.

2. $p_g=3$, $K^2=7$

In this section we will give three different constructions of a minimal surface X of general type with the following properties:

1. $p_g(X)=3$, $K^2=7$.
2. The canonical map of X is composed of a pencil of genus 3.

2.1. as a hypersurface in a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$. Let $m=3$, $n=4$ and the equation of S_0 be

$$x_0x_2^3+(x_1^2-x_0^2)^2+z^6x_0^4+z^{12}x_0^4=0. \tag{4}$$

Then $S\sim 4E_0+12\eta$. It is easy to check that S has only two singularities p: ($z=0$, $x_0=1$, $x_1=1$, $x_2=0$) and p': ($z=0$, $x_0=1$, $x_1=-1$, $x_2=0$). Both are equivalent to the double point defined by

$$x^2+y^3+z^6=0.$$

It is well-known that this type of singularity is a minimally elliptic singularity and the exceptional curve is a nonsingular elliptic curve of self-intersection -1.

Let η_0 denote the fiber at $z=0$. Then $K_W+S\sim E_0+3\eta\sim E_0+2\eta+\eta_0$. Let $\pi: X\to S$ be the minimal resolution of S, and let $D=\pi^{-1}(p)$, $D'=\pi^{-1}(p')$.

Then $K_X \sim \pi^*(E_0+2\eta)+(\pi^*(\eta_0)-D-D')$. Since $\pi^*(\eta_0)-D-D'$ is the proper transform of $\eta_0|_S$ in X and $\dim|2\eta|=2$, we have $p_g(X)\geq 3$. From (4) it is clear that $\pi^*(E_0)=4E$ for a rational curve E on X. Let F and Z denote the proper transforms of $\eta|_S$ and $\eta_0|_S$ respectively. Then $K_X \sim 4E+2F+Z$. The self-intersection number of $4E$ in X is equal to the intersection number E_0^2S in the threefold W. Hence $16E^2=E_0^2(4E_0+12\eta)=-28+12=-16$, whence $E^2=-1$. Obviously, $F^2=FZ=0$, $EF=EZ=1$. Since $\eta_0\cap S$ is a quartic curve with two cusps, Z is a nonsingular elliptic curve. By the adjunction formula, $0=2g(Z)-2=Z(Z+K_X)=Z(4E+2F+2Z)$, which implies that $Z^2=-2$.

In order to see that $p_g(X)=3$, we need to know $\chi(\mathcal{O}_X)$. Lemma 1.1 (2) implies that $\chi(\mathcal{O}_S)=6$. Since every minimally elliptic singularity has geometric genus one, we have $\chi(\mathcal{O}_X)=\chi(\mathcal{O}_S)-2=4$. Let $\phi: X\to X'$ be the contraction of the curve E. $F'=\phi(F)$, $Z'=\phi(Z)$. Then X' is a minimal surface with $\chi(\mathcal{O}_{X'})=4$, $K_{X'}\sim 2F'+Z'$, and $K_{X'}^2=7$.

Suppose that $H^1(\mathcal{O}_{X'})\neq 0$. Then there would exist $B\in \mathrm{Div}(X')$ such that B is not linearly equivalent to 0 and $2B\sim 0$. By the Riemann-Roch Theorem, we have $h^0(\mathcal{O}_{X'}(F'+B))+h^0(\mathcal{O}_{X'}(F'+Z'-B))\geq 3$. Thus either $h^0(\mathcal{O}_{X'}(F'+B))\geq 2$ or $h^0(\mathcal{O}_{X'}(F'+Z'-B))\geq 2$. If $h^0(\mathcal{O}_{X'}(F'+B))\geq 2$, then the exact sequence

$$0\to H^0(\mathcal{O}_{X'}(B))\to H^0(\mathcal{O}_{X'}(F'+B))\to H^0(\mathcal{O}_{F'}(F'+B))$$

would imply that $H^0(\mathcal{O}_{X'}(B))>0$, which is impossible. Hence $h^0(\mathcal{O}_{X'}(F'+Z'-B))\geq 2$. As a nonsingular plane quartic curve, F is a non-hyperelliptic curve of genus 3. Hence $h^0(\mathcal{O}_{F'}(d))=1$ for every effective divisor d of degree 2 on F'. Thus $h^0(\mathcal{O}_{F'}(F'+Z'-B)=1$. The exact sequence

$$0\to H^0(\mathcal{O}_{X'}(Z'-B))\to H^0(\mathcal{O}_{X'}(F'+Z'-B))\to H^0(\mathcal{O}_{F'}(F'+Z'-B))$$

implies that $H^0(\mathcal{O}_{X'}(Z'-B))\neq 0$. Since $Z'(Z'-B)<0$, we have $H^0(\mathcal{O}_{X'}(-B))\neq 0$, which is impossible. Therefore $H^1(\mathcal{O}_{X'})=0$, which implies $p_g(X')=3$, and the canonical map of X' is composed of a pencil of genus 3.

2.2. as a Galois triple cover. A general theory for triple covers of algebraic varieties was developped by Miranda in [6]. In [7] Tan discovered a Horikawa type canonical resolution for Galois triple covers of surfaces. It is a useful tool to construct special surfaces satisfying preassigned conditions. Here we summarize some facts of triple covers that we need. For details readers may refer to [7] or [8].

Let Y be a smooth surface, L and M be divisors on Y. Assume that B, C are effective divisors such that $B\sim 2L-M$, $C\sim 2M-L$ and $B+C$ is reduced. Then the triple covering data (L,M,B,C) determines a Galois triple cover $\pi: X\to Y$ from a normal surface X to Y. The surface X is defined in the rank two vector bundle $L\oplus M$ as

$$X=\mathrm{Spec}\ \mathcal{O}_Y[z,w]/(z^2-bw,\ zw-bc,\ w^2-cz),$$

where z,w are fibre coordinates of L, M and $b \in H^0(2L-M)$, $c \in H^0(2M-L)$ whose zeros are B and C respectively. The branch locus of π is $B+C$. If $B+C$ is smooth then X is nonsingular.

There are two formulas:

$$\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-L) \oplus \mathcal{O}_Y(-M). \qquad (5)$$

$$\chi(\mathcal{O}_X) = 3\chi(\mathcal{O}_Y) + (L^2 + LK_Y)/2 + (M^2 + MK_Y)/2 \qquad (6)$$

With this preparation we start to construct our example. Let Y be a Hirzebruch surface $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-3))$. Let E denote the section of Y with $E^2 = -3$ and let η denote a fibre of Y. It is easy to see that there exists an irreducible curve $D \in |4E+12\eta|$ satisfying the following conditions:

1. D does not meet E.
2. D has two double points p, p' on a fiber η_0 and no other singularities.
3. The double points p and p' are of type A_5, i.e., they are equivalent to the double point defined by the equation

$$x^2 + y^6 = 0.$$

Next we will construct a sequence of blowingups of Y. To simplify the notation every irreducible curve and its proper transform will share the same name.

Let $\sigma_1 : Y_1 \to Y$ be the composition of blowingups of Y with centers at p and p'. Let $E_1 = \sigma_1^{-1}(p)$, $F_1 = \sigma_1^{-1}(p')$. Let $q = E_1 \cap D$ and $q' = F_1 \cap D$. Let $\sigma_2 : Y_2 \to Y$ be the blowingups of Y_1 with centers at q and q'. Let $E_2 = \sigma_2^{-1}(q)$, $F_2 = \sigma_2^{-1}(q')$. Let $s = E_2 \cap D$ and $s' = F_2 \cap D$. Let $\sigma_3 : Y_3 \to Y_2$ be the blowingups of Y_2 with centers at s and s'. Let $E_3 = \sigma_3^{-1}(s)$, $F_3 = \sigma_3^{-1}(s')$. Then D becomes a smooth curve on Y_3.

Let $r = E_1 \cap E_2$ and $r' = F_1 \cap F_2$. Let $\sigma_4 : Y_4 \to Y_3$ be the blowingups of Y_3 with centers at r and r'. Let $E_4 = \sigma_4^{-1}(r)$, $F_4 = \sigma_4^{-1}(r')$. The configurations of relevent curves on Y_4 are illustrated in Figure 1.

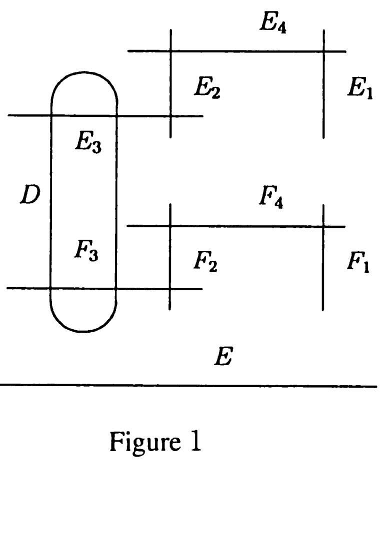

Figure 1

Let $\sigma=\sigma_1\sigma_2\sigma_3\sigma_4$ be the composition of the blowingups. Let $L=\sigma^*(3E+8\eta)-E_1-2E_2-4E_3-4E_4-F_1-2F_2-4F_3-4F_4$ and $M=\sigma^*(2E+4\eta)-E_2-2E_3-2E_4-F_2-2F_3-2F_4$. Let $B=D+E_2+F_2$ and $C=E+E_1+F_1$ as divisors on Y_4. Then $B\sim 2L-M$ and $C\sim 2M-L$.

Let π: $X\rightarrow Y_4$ be the triple cover determined by the triple covering data (L, M, B, C). Since the branch locus $B+C$ is smooth, X is a smooth surface. By (5) we have

$$\begin{aligned} p_g(X)&=h^2(X, \mathcal{O})=h^2(Y_4, \mathcal{O}(-L))+h^2(Y_4, \mathcal{O}(-M)) \\ &=h^0(Y_4, \mathcal{O}(\sigma^*(E+3\eta)-E_3-F_3))=3. \end{aligned}$$

By (6), $\chi(\mathcal{O}_X)=4$. Hence $h^1(\mathcal{O}_X)=0$.

Since π is totally ramified over $B+C$, we have

$$3K_X\sim 3\pi^*(K_{Y_4})+2\pi^*(B+C)\sim\pi^*(\sigma^*(4E+9\eta)+E_1-3E_3+F_1-3F_3).$$

Let $\bar{E}_i=\pi^{-1}(E_i)$, $\bar{F}_i=\pi^{-1}(F_i)$ for $i=1,2,3,4$. Let $\bar{E}=\pi^{-1}(E)$ and $\bar{\eta}=\pi^{-1}(\eta)$. Then $3K_X\sim 12\bar{E}+6\bar{\eta}+3\bar{\eta}_0+12\bar{E}_1+9\bar{E}_2+6\bar{E}_4+12\bar{F}_1+9\bar{F}_2+6\bar{F}_4$.

Since $h^1(\mathcal{O}_X)=0$, Pic (X) has no torsion. So $K_X\sim 4\bar{E}+2\bar{\eta}+\bar{\eta}_0+4\bar{E}_1+3\bar{E}_2+2\bar{E}_4+4\bar{F}_1+3\bar{F}_2+2\bar{F}_4$. A direct computation shows that $K_X^2=0$. Since $H^0(X, \mathcal{O}_X(2\eta))=3=p_g(X)$, $2\bar{\eta}$ is the moving part of $|K_X|$. By Hurwitz's formula the genus of $\bar{\eta}$ is 3. This shows that the canonical map of X is composed of pencil of genus 3.

Finally, one can easily see that $\bar{E}^2=\bar{E}_1^2=\bar{E}_2^2=\bar{F}_1^2=\bar{F}_2^2=-1$. and $\bar{E}_4^2=\bar{F}_4^2=-3$. So these seven curves can be contracted. Let τ: $X\rightarrow S$ be the contraction. Then S is the minimal model of X with $K_S^2=7$. The surface S is the desired surface.

2.3. as a sextic surface in $\mathbf{P}^3$. Let x_0, x_1, x_2, x_3 be the homogeneous coordinates of $\mathbf{P}^3$. Let S_0 be a sextic surface defined by the equation

$$x_1^2(x_0^2-x_1^2)^2+x_0(x_0^2-x_1^2)x_2^3+x_2^6+x_3^6=0. \tag{7}$$

It can be checked that S_0 is irreducible and has no singularities on the hyperplane $x_0=0$. Take affine coordinates $x=x_1/x_0$, $y=x_2/x_0$, $z=x_3/x_0$. Then equation (7) becomes

$$x^2(1-x^2)^2+(1-x^2)y^3+y^6+z^6=0.$$

Let $b_1=(0,0,0)$, $b_2=(1,0,0)$, $b_3=(-1,0,0)$. Then b_1, b_2, b_3 are the only singularities of S_0. The singularity at b_1 is equivalent to the one defined by $x^2+y^3+z^6$, while both b_2 and b_3 are equivalent to the one defined by $x^2+y^6+z^6$. Meanwhile b_1, b_2, b_3 are located on the line L_0: $y=0$, $z=0$.

Let ρ: $S\rightarrow S_0$ be the minimal resolution. Let $E_i=\rho^{-1}(b_i)$ for $i=1,2,3$. They are all nonsingular curves and $g(E_1)=1$, $E_1^2=-1$, $g(E_2)=g(E_3)=2$, $E_2^2=E_3^2=-2$. It is easy to see that $K_S\sim\rho^*(2H)-E_1-2E_2-2E_3$, where H is a hyperplane in $\mathbf{P}^3$. Thus $|K_S|=\{\rho^*(H_1+H_2)-E_1-2E_2-2E_3|H_1, H_2$ are planes passing through $L_0\}$. Hence the moving part of $|K_S|$ is a pencil and $p_g(S)=3$. The fixed part of $|K_S|$ is E_1. Let F be the proper transform of $H\cap S_0$ for a general hyperplane H passing through L_0. Then $K_S\sim 2F+E_1$ and $F^2=1$, $FE_1=1$. Hence $K_S^2=7$.

3. $p_g=3$, $K_2=6$

In this section we prove the non-existence of minimal surfaces of general type with $p_g=3$, $K^2=6$ whose canonical map is composed of penciles.

Let W be a smooth 3-fold, S be an irreducible surface in W. Let C be a nonsingular curve in S. If S is singular at eyery point of C then we say that C is a *singular locus* of S. Let $\mu=\min_{p\in C}\{\mu_p(S)\}$, where $\mu_p(S)$ is the multiplicity of S at p. Then μ is defined as the multiplicity of the singular locus C. The set $U=\{p\in C\,|\mu_P(S)=\mu\}$ is a non-empty open subset of C. If $\mu=2$ then C is called a *double locus* of S.

Assume that C is a double locus of S. Let σ_1: $W_1\to W$ be the blowingup of W with center at C. Let S_1 be the proper transform of S, $E_1=\sigma_1^{-1}(C)$, $C_1=S_1\cap E_1$. If C_1 is a singular locus of S_1, then it is still a double locus and is irreducible. Let σ_2: $W_2\to W_1$ be the blowingup of W_1 with center at C_1. Repeating this process of blowingup for finitely many steps, we may obtain a sequence of blowingups:

$$W_n\xrightarrow{\sigma_n}W_{n-1}\to\cdots\to W_1\xrightarrow{\sigma_1}W$$

so that C_{n-1} is a double locus of W_{n-1} while C_n is not a singular locus of W_n, although there might be isolated singularities of S_n on C_n. The number n is called the *resolution length* of the double locus C.

Lemma 3.1. *Let S be a surface in a nonsingular 3-fold W. Let C be a double locus of S. Let H be a nonsingular surface in W and $p\in H\cap C$ such that C is transversal to H at p. Assume that the curve $D=H\cap S$ on H has a double point of type A_n at p. Then the resolution length of C is less than or equal to* $[(n+1)/2]$.

Proof. Let σ_1: $W_1\to W$ be the blowingup of W with center at C, S_1 and H_1 be the proper transforms of S and H respectively. Let $E_1=\sigma_1^{-1}(C)$, $C_1=S_1\cap E_1$, $D_1=H_1\cap S_1$. Then the restriction map σ_1: $H_1\to H$ is the blowingup of H at p with D_1 as the proper transform of D.

We use the induction on n to prove our statement. If $n\leq 2$, then D_1 is smooth at $\sigma_1^{-1}(p)$. Thus S_1 is smooth at $\sigma_1^{-1}(p)$. So C_1 is not the singular locus of S_1. This implies that the resolution length of C is less that or equal to 1.

If $n>2$, then $\sigma_1^{-1}(p)$ consists of one point p_1 and this point p_1 is a double point of type A_{n-2} of the curve D_1 on H_1. By induction hypothesis the resolution length of C_1 is less than or equal to $[(n-1)/2]$. Hence the resolution length of C is less than or equal to $1+[(n-1)/2]=[(n+1)/2]$.

Lemma 3.2. *Let* π: $W=\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(m)\oplus\mathcal{O}(n))\to\mathbf{P}^1$ *be a* $\mathbf{P}^2$*-bundle over* $\mathbf{P}^1$, *where* $0<m\leq n$. *Let S be an irreducible surface in W, linearly equivalent to* $4E_0+s\eta$, *where E_0 is the divisor associated with the subbundle* $\mathcal{O}(m)\oplus\mathcal{O}(n)$ *and η is a fiber. Let* $\mathbf{P}^1=U_0\cup U_1$ *be the standard affine open covering of* $\mathbf{P}^1$ *and let* $W_i=\pi^{-1}(U_i)$ *for* $i=0,1$. *Let z be the affine coordinate of U_0 and let x_0, x_1, x_2 be the fiber coordinates of the line bundles* $\mathcal{O}$, $\mathcal{O}(m)$ *and* $\mathcal{O}(n)$ *over U_0 respectively. Assume that the equation of* $S_0=S\cap W_0$ *is*

$$ax_1^4+x_0f(x_0, x_1, x_2, z)=0,$$

where a is a non-zero constant and $f(x_0, x_1, x_2, z)$ *is homogeneous in x_0, x_1, x_2 of degree.* 3. *Let η_0 be the fiber of W_0 over the origin of U_0. Assume that* $C_0=\eta_0\cap S$ *is a nonsingular conic which is a double locus of S and there is no other singular locus. Then the resolution length of C_0 is less than or equal to* $[(s-3n)/2]$.

Proof. Since C_0 is a double locus of S, we have

$$ax_1^4+x_0f(x_0, x_1, x_2, 0)=(\alpha x_1^2+x_0\Gamma(x_0, x_1, x_2))^2,$$

where $\alpha\neq 0$ and $\Gamma(x_0, x_1, x_2)=b_0x_0+b_1x_1+b_2x_2$ is a linear form. Since C_0 is a nonsingular conic, $b_2\neq 0$. Thus the equation of S_0 can be written as

$$(\alpha x_1^2+x_0\Gamma(x_0, x_1, x_2))^2+zx_0G(x_0, x_1, x_2, z)=0,$$

where

$$G(x_0, x_1, x_2, z)=\sum_{i+j+k=3} c_{ijk}(z)x_0^ix_1^jx_2^k.$$

Since C_0 is the only singular locus of S, we have $c_{003}(z)\neq 0$, otherwise the curve defined by $x_0=x_1=0$ would be a singular locus. So the equation of S can be written as

$$(\alpha x_1^2+x_0\Gamma(x_0, x_1, x_2))^2+p(z)x_0x_2^3+zx_0\sum_{k\leq 2,\, i+j+k=3} c_{ijk}(z)x_0^ix_1^jx_2^k=0,$$

in which $p(z)=\beta z^r+\sum_{i>r}\beta_iz^i$, $\beta\neq 0$, $r\leq\deg(p(z))\leq s-3n$.

Let $V=\{(x_0, x_1, x_2, z)\in W_0 \mid x_0\neq 0\}$. Then $V\cong\mathbf{C}^3$ is an affine open subset of W_0 with $x=x_1/x_0$, $y=x_2/x_0$, z as the affine coordinates. The equation of $S\cap V$ is

$$(\alpha x^2+b_0+b_1x+b_2y)^2+p(z)y^3+z\sum_{k\leq 2,\, i+j+k=3} c_{ijk}(z)x^iy^k=0.$$

Let H be the surface in V defined by the equation $y = ux$, where u is a sufficiently general complex number. Then the equation of the curve $H \cap S \cap V$ is given by

$$\alpha^2(x-\rho_1)^2(x-\rho_2)^2+p(z)u^3x^3+z\sum_{k\le 2,\ i+j+k=3} c_{ijk}(z)u^kx^{j+k}=0,$$

where ρ_1 and ρ_2 are two roots of the quadratic equation

$$\alpha x^2+(b_2u+b_1)x+b_0=0.$$

We may assume that $\rho_1 \neq 0$. Substitute x' for $x-\rho_1$ and the equation of $H \cap S \cap V$ becomes

$$\alpha^2x'^2(x'+\rho_1-\rho_2)^2+p(z)u^3(x'+\rho_1)^3+z\sum_{k\le 2,\ i+j+k=3} c_{ijk}(z)u^k(x'+\rho_1)^{j+k}$$
$$=\alpha^2x'^2(x'+\rho_1-\rho_2)^2+\gamma(z)+\delta(z)x'+x'^2g(x', z),$$

where

$$\gamma(z)=\beta z^r u^3\rho_1^3+\sum_{i>r}\beta_i z^i u^3\rho_1^3+z\sum_{k\le 2,\ i+j+k=3} c_{ijk}(z)u^k\rho_1^{j+k},$$

$$\delta(z)=3p(z)u^3\rho_1^2+z\sum_{k\le 2,\ i+j+k=3} c_{ijk}(z)u^k(j+k)\rho_1^{j+k-1}$$

and $g(x', z)$ is some polynomial. Since u is sufficiently general, $\alpha^2(\rho_1-\rho_2)^2+g(0,0) \neq 0$ and the coefficient of z^r in $\gamma(z)$ is nonzero. Let e_1 and e_2 be the coefficients of the terms of lowest degree in $\gamma(z)$ and $\delta(z)$ respectively. Then, since u is general, $e_2^2 \neq 4e_1((\alpha^2(\rho_1-\rho_2)^2+g(0,0))^2$ and the lowest degrees of $\gamma(z)$ and $\delta(z)$ are less than or equal to r. Hence the point $x'=0$, $z=0$ is a double point of type A_q with $q \le r-1$. By Lemma 3.1 the resolution length of the double locus C_0 of S is less than or equal to $[r/2]$. Since $r \le s-3n$, the lemma is proved.

Lemma 3.3. *Let C be an irreducible quartic curve in the projective plane and L be a line. Let p be an intersecting point of C and L. Let $(C, L)_p$ denote the intersection number of C and L at p.*

(1) *If C has a double point $q \neq p$ of type A_6, i.e., equivalent to one defined by the equation $x^2+y^7=0$. Then $(C, L)_p \le 3$.*

(2) *If C has two double points q_1, q_2 of types A_m and A_n respectively with m, $n>1$. Assume that $m+n \ge 5$, $q_1 \neq p$ and $q_2 \neq p$. Then $(C, L)_p \le 3$.*

Proof. (1) The projection from the point q defines a covering of C over $\mathbf{P}^1$. Hurwitz's formula implies that there is at least one ramification point for this projection. This means that there is a line L' such that $C\cap L'=q+q'$ and $(C, L')_{q'}=2$. If $(C, L)_p=4$, then the combination of rational double points of the reduced sextic curve $C+L+L'$ would correspond to the Dynkin diagram $D_9+A_7+A_3+A_1$. This is impossible since the rank of such Dynkin diagram cannot exceed 19, ([9]). Hence $(C, L)_p\leq 3$.

(2) Assume that $(C, L)_p=4$. Let L' be the line passing through q_1 and q_2. Then the combination of rational double points of the sextic curve $C+L+L'$ corresponds to the Dynkin diagram $D_{m+3}+D_{n+3}+A_7+A_1$. So $m+n\leq 5$ for the same reason as in (1). So we may assume that q_1 and q_2 are of types A_2 and A_3 respectively. Let x_0, x_1, x_2 be the homogeneous coordinates of $\mathbf{P}^2$. With a suitable linear transformation of the coordinates we may assume that $q_1=(1,0,0)$, $q_2=(0,1,0)$ and the equation C is

$$x_0^2x_1^2+x_0x_2^3+\lambda x_0x_1x_2^2+x_2^4=0, \tag{8}$$

where λ is some constant. Let

$$ax_0+bx_1+cx_2=0 \tag{9}$$

be the equation of the line L, with coefficients a, b, c. If $(C, L)_p=4$, there would be only one solution to the simultaneous equations (8) and (9). A direct computation shows that this is possible only when $b=0$ and $c=0$. Thus $q_2\in L$, contradicting the assumption that $q_2\neq p$.

Lemma 3.4. *Let $\mathcal{F}$ be a coherent sheaf on a nonsingular curve C. Let $\mathcal{F}^{\check{}}$ denote the dual of $\mathcal{F}$. Then $h^0(C, \mathcal{F})\geq h^0(C, \mathcal{F}^{\check{}\check{}})$.*

Proof. Let $\mathcal{F}_\tau$ denote the torsion part of $\mathcal{F}$. Then we have a short exact sequence

$$0\to\mathcal{F}_\tau\to\mathcal{F}\to\bar{\mathcal{F}}\to 0, \tag{10}$$

where $\bar{\mathcal{F}}$ is torsion-free. Since every torsion-free coherent sheaf on a nonsingular curve is locally free, $\bar{\mathcal{F}}$ is locally free. Hence $\bar{\mathcal{F}}\cong\bar{\mathcal{F}}^{\check{}\check{}}\cong\mathcal{F}^{\check{}\check{}}$. Then (10) implies the exact sequence

$$0\to\mathcal{F}_\tau\to\mathcal{F}\to\mathcal{F}^{\check{}\check{}}\to 0.$$

Taking the long exact sequence we obtain

$$\cdots\to H^0(C,\mathcal{F})\to H^0(C,\mathcal{F}^{\check{}\check{}})\to H^1(C,\mathcal{F}_\tau)\to\cdots \tag{11}$$

Since $\mathcal{F}_\tau$ is supported on a proper closed subset of C, we have $H^1(C,\mathcal{F}_\tau)=0$. The result follows from (11).

Theorem 3.5. *There does not exist a minimal surface of general type X such that*

1. $p_g(X)=3$, $K_X^2=6$
2. *The canonical map of X composed of a pencil of genus greater than or*

equal to 3.

Proof. Suppose that such a surface exists.

Let Z denote the fixed part of $|K_X|$. Then $K_X \sim nF+Z$ with $n\geq 2$, where F is a member of a pencil. Since $6=K_X^2\geq 2FK_X+ZK_X\geq 2FK_X$, $F^2=2p_a(F)-2-FK_X\geq 2g(F)-5>0$. Thus $|F|$ has base points. Since $K_X^2\geq 4F^2+2FZ+K_XZ\geq 4F^2$, we havse $F^2=1$, which means that $|F|$ has exactly one base point. This implies that a general member $|F|$ is a nonsingular curve of genus $g\geq 3$. It follows that $FZ=1$, $K_XZ=0$, $n=2$ and $g=3$.

Since $K_XZ=0$, every irreducible component of Z is a (-2)-curve. We are going to show that Z is irreducible. Write $Z=\sum_{i=1}^r n_iA_i$, where each A_i is a (-2)-curve. Assume that $FA_1=1$ and $FA_i=0$ for $i\geq 2$. Then $n_1=1$. Suppose that $r>1$. The equality

$$0=A_1K_X=2A_1F+A_1Z=2-2+A_1\sum_{i=2}^r n_iA_i$$

implies that $A_1A_i=0$ for $i\geq 2$. It follows that $K(\sum_{i=2}^r n_iA_i)=(\sum_{i=2}^r n_iA_i)^2<0$, contradicting the assumption that X is a minimal surface of general type. Hence Z is a rational curve with $Z^2=-2$.

Next, we show that $H^1(X,\mathcal{O}_X)=0$. If not, there would be a divisor e such that e is not linearly equivalent to 0 but $2e$ is linearly equivalent to 0. A theorem of Xiao (cf. [10]) says that $q=h^1(X,\mathcal{O}_X)\leq 2$. Thus $\chi(\mathcal{O}_X)\geq 2$. The Riemann-Roch theorem implies that

$$h^0(X,\mathcal{O}_X(F+e))+h^0(X,\mathcal{O}_X(F+Z-e))\geq 1.$$

Hence either $h^0(X,\mathcal{O}_X(F+e))>0$ or $h^0(X,\mathcal{O}(F+Z-e))>0$. In the former case, take $D\in|F+e|$, then $2D\in|2F|$. Since dim $|2F|=2$, D is a member of $|F|$, which implies that $e\sim 0$. This is a contradiction. In the latter case $h^0(X,\mathcal{O}_X(F+Z-e))>0$. Since $Z(F+Z-e)<0$, Z is a fixed component of $|F+Z-e|$. Thus $h^0(X,\mathcal{O}(F-e))>0$. This would lead to a contradiction by the same argument as in the previous case. Hence $q=0$.

Let p be the base point of the pencil $|F|$. We discuss the following two cases.

Case 1: The point p is not on Z.

Let σ: $\bar{X}\to X$ be the blowingup of X with center at p. Let $E=\sigma^{-1}(p)$, and let $\bar{Z}$, $\bar{F}$ denote the proper transforms of Z and F respectively. Then $K_{\bar{X}}\sim 2\bar{F}+3E+\bar{Z}$. There is a natural fibration f: $\bar{X}\to\mathbf{P}^1$ such that $|\bar{F}|$ consists of fibers. Since $q=0$, we have a short exact sequence

$$0\to H^0(\bar{X},\mathcal{O}(K))\to H^0(\bar{X},\mathcal{O}(K+\bar{F}))\to H^0(\bar{F},\mathcal{O}(K_{\bar{F}}))\to 0,$$

where $\bar{F}$ is a nonsingular fiber. Thus $h^0(\bar{X},\mathcal{O}(K+\bar{F}))=6$, which implies that

$f_*(\mathcal{O}_{\bar{X}}(K)) \sim \mathcal{O}(2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Since the map $H^0(\bar{X}, \mathcal{O}(K+\bar{F})) \to H^0(\bar{F}, \mathcal{O}(K_{\bar{F}}))$ is surjective and $|K_{\bar{F}}|$ has no base point on a smooth fiber $\bar{F}$, neither E nor $\bar{Z}$ is a fixed component of $|K+\bar{F}|$. Thus the base points of $|K+\bar{F}|$ can only be located on $\bar{Z}$. Obviously we have $4 \le h^0(\bar{X}, \mathcal{O}(3\bar{F}+3E)) \le 5$.

First suppose that $h^0(\bar{X}, \mathcal{O}(3\bar{F}+3E)) = 4$. For an arbitrary nonsingular fiber $\bar{F}$, let x and y be the intersection points of $\bar{F}$ with E and $\bar{Z}$ respectively. Then $K_{\bar{F}} \sim 3x+y$. This implies that $\bar{F}$ is a non-hyperelliptic curve of genus 3. Let

$$\phi: \bar{X} \cdots \to \mathbf{P}(f_*(\mathcal{O}_{\bar{X}}(K))^{\cdot}) \cong \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(3) \oplus \mathcal{O}(3))$$

be the relative canonical map. Then ϕ is a birational map, for a general fiber of f is non-hyperelliptic. In particular, the restiction of ϕ to every nonsingular fiber is a birational morphism onto its image. We are going to show that the image of ϕ is a normal surface. It suffices to show that the restriction of ϕ to every irreducible component of any singular fiber is either a birational map or the contraction of the curve to a point.

From the exact sequence

$$0 \to H^0(\bar{X}, \mathcal{O}(3\bar{F}+3E)) \to H^0(\bar{X}, \mathcal{O}(K+\bar{F})) \to H^0(\bar{Z}, \mathcal{O}(1)) \to 0$$

we see that $|K_{\bar{X}}+\bar{F}|$ has no base points. So ϕ is a birational morphism. Let F' be a singular fiber. Let A be the irreducible component of F' with $A\bar{Z}=1$. First assume that $AE=1$. Then it is easy to see that the image of the map $H^0(\bar{X}, \mathcal{O}(K+\bar{F})) \to H^0(A, \mathcal{O}_A(K_{\bar{X}}+\bar{F}))$ has dimention 3 and A cannot be hyperelliptic. Thus the restriction of ϕ on A is a birational morphism onto its image. All the other components of F' have zero interestion with $K_{\bar{X}}$, so they are (-2)-curves and contract to points under ϕ. Next assume that $AE=0$. Since $|K_{\bar{X}}+\bar{F}|$ has no base point on A, A is a nonsingular rational curve with $A^2=-3$. From $A(K_{\bar{X}}+A)<0$. It follows that A is fixed in $|K_{\bar{X}}+A|$, so $h^0(\bar{X}, \mathcal{O}(K_{\bar{X}}+A))=3$. Write $F'=A+B$. Then the exact sequence

$$0 \to H^0(\bar{X}, \mathcal{O}(K_{\bar{X}}+A)) \to H^0(\bar{X}, \mathcal{O}(K+\bar{F})) \xrightarrow{\psi} H^0(B, \mathcal{O}(K_{\bar{X}}+\bar{F})) \to \cdots$$

implies that $\mathrm{Im}(\psi)$ has dimension 3. Thus $\phi(B)$ is a non-degenerate plane cubic curve. Write $B=B_1+C$, where B_1 is the component interesting E. Then C consists of (-2)-curves and contracts to points under ϕ. This shows that ϕ maps B_1 onto its image birationally. Note that only (-2)-curves contract to points under ϕ. Therefore $\phi(\bar{X})$ is a normal surface with rational double points as its only singularities. In particular, $\chi(\mathcal{O}_{\bar{X}}) = \chi(\mathcal{O}_{\phi(\bar{X})}) = 4$.

Let W denote the threefold $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(3) \oplus \mathcal{O}(3))$ over $\mathbf{P}^1$. Let E_0 denote the divisor $\mathbf{P}(\mathcal{O}(3)\oplus\mathcal{O}(3))$ of W, and let η denote a fiber $\mathbf{P}^2$ of W. Then as a divisor of W, $\phi(\bar{X})$ is linearly equivalent to $4E_0+n\eta$ for some $n>0$. Lemma 1.1 (2) shows that $\chi(\mathcal{O}_{\bar{X}})=3n-26$, which implies $n=10$. But this contradicts Lemma 1.1 (1). Hence $h^0(\bar{X},\mathcal{O}(3\bar{F}+3E))=4$ is impossible.

Next suppose that $h^0(\bar{X},\mathcal{O}(3\bar{F}+3E))=5$. Let D be a member of $|K_{\bar{X}}+\bar{F}|$ that does not contain $\bar{Z}$. Then D meets $\bar{Z}$ at one point x, which is the only base point of $|K_{\bar{X}}+\bar{F}|$. Let F' be the fiber of f passing through x and let A be the irreducible component of F' passing through x. Let σ_1: $X_1\to\bar{X}$ be the blowingup of $\bar{X}$ at x. Let $G=\sigma_1^{-1}(x)$ and let A_1 be the proper transform of A. The proper transform of E is still denoted by E. The linear system $|\sigma_1^*(K_{\bar{X}}+\bar{F})-G|$ does not have base points.

Let $f_1=f\sigma_1$: $X_1\to\mathbf{P}^1$ be the fibration induced from f. Let $M=f_{1*}\mathcal{O}(\sigma_1^*(K_{\bar{X}}+\bar{F})-G)$. Since $h^0(X_1,\mathcal{O}(\sigma_1^*(K_{\bar{X}}+F)-G)=6$. and $h^0(X_1,\mathcal{O}(\sigma_1^*(K_{\bar{X}})-G))=3$, the locally free sheaf M is isomorphic to $\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(3)$. The natural morphism $f_1^*M\to\mathcal{O}(\sigma_1^*(K_{\bar{X}}+\bar{F})-G)$ induces a morphism ϕ: $X_1\to\mathbf{P}(M^{\cdot})\cong\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(3)\oplus\mathcal{O}(3))$, because the sheaf $\mathcal{O}(\sigma_1^*(K_{\bar{X}}+\bar{F})-G)$ is generated by global sections.

If $AE=1$, then the restriction of $H^0(X_1,\mathcal{O}(\sigma_1^*(K_{\bar{X}}+\bar{F})-G))$ on A_1 has dimension 3, for $h^0(X_1,\mathcal{O}(\sigma_1^*(K_{\bar{X}})-G(\sigma_1^*(F')-A_1)))=3$. Thus ϕ maps A_1 to a nondegenerate cubic curve and G to a line. Using the same argument as before, we see that $\phi(X_1)$ is a normal surface with rational double points as its only singularities. This would lead to a contradiction by virtue of Lemma 1.1.

Next assume that $AE=0$. Since $A^2+1=A^2+AK_{\bar{X}}=2p_a(A)-2$, A^2 is either -1 or -3. If $A^2=-1$, then A is either a smooth elliptic curve or a rational curve with a node or cusp.

Assume that A is a smooth elliptic curve. Then the exact sequence

$$0\to H^0(\bar{X},\mathcal{O}(K_{\bar{X}}))\to H^0(\bar{X},\mathcal{O}(K_{\bar{X}}+A))\to H^0(A,\mathcal{O}_A)\to 0$$

implies that $h^0(\bar{X},\mathcal{O}(K_{\bar{X}}+A))>h^0(\bar{X},\mathcal{O}(K_{\bar{X}}))$. So A is not a fixed component of $|K_{\bar{X}}+A|$. Take $D\in|K_{\bar{X}}+A|$ which does not contain A. Then $x\notin D$, for $DA=0$. Thus $D+(F'-A)$ is a member of $|K_{\bar{X}}+\bar{F}|$ which does not pass throught x. This would lead to a contradiction. Hence A is not a smooth elliptic curve.

Assume that A is a rational curve with a node or a cups y. Let ψ: $W\to\bar{X}$ be the blowingup of $\bar{X}$ at y. Let $\Gamma=\psi^{-1}(y)$ and let A' denote the proper transform of A. Then A' is a smooth rational ccurve with $A'^2=-5$ and $A'\Gamma=2$. The exact sequence

$$0\to\mathcal{O}_W(K_W)\to\mathcal{O}_W(K_W+\Gamma)\to\mathcal{O}_\Gamma(-2)\to 0$$

implies that $h^0(W,\mathcal{O}(K_W+\Gamma))=3$ and $h^0(W,\mathcal{O}(K_W+\Gamma))=0$. Then the exact sequence

$$0\to\mathcal{O}_W(K_W+\Gamma)\to\mathcal{O}_W(K_W+\Gamma+A')\to\mathcal{O}_{A'}\to 0$$

implies that $h^0(W,\mathcal{O}(K_W+A'+\Gamma))>h^0(W,\mathcal{O}(K_W+\Gamma))=h^0(W,\mathcal{O}(K_W))$. So A' is not a fixed component of $|K_W+A'+\Gamma|$. Take $D'\in|K_W+A'+\Gamma|$ which does not contain A' and let $D=\psi(D')$. Then $D\in|K_{\bar{X}}+A|$ and $x\notin D$. Thus $D+(F'-A)$ is a member of $|K_{\bar{X}}+\bar{F}|$ which does not pass throught x. This would lead to a contradiction. Hence A could only be a smooth rational curve.

Since $h^0(X_1,\mathcal{O}(\sigma_1^*(K_{\bar{X}}))=3$, the restriction of $H^0(X_1,\mathcal{O}(\sigma_1^*(K_{\bar{X}}+\bar{F})-G))$ on $\sigma_1^*(F')-G$ has dimension 3. Thus $\phi(\sigma_1^*(F')-G)$ is a non-degenerate plane cubic curve. Write $\sigma_1^*(F')=G+A_1+B_1+R$, where B_1 is the irreducible component that intersects E, and R consists of (-2)-curves. Then ϕ maps B_1 and G birationally to a cubic curve and a line respectively. The divisor A_1+R is contracted to normal singularities under ϕ. Let $p=\phi(A_1)$ and Let $C=\cup_{i=1}^n C_i$ be the exceptional set of the normal singularity p. We may assume that $C_1=A_1$. Since $A_1^2\neq-2$ and C does not contain (-1)-curves, p is not a rational double point. As every hypersurface rational singularity is a rational double point, p is not a rational singularity. This means that $Z^2+ZK_{X_1}\geq 0$, where Z is the fundamental cycle on C. Since $ZK_{X_1}=A_1K_{X_1}=2$, we have $Z^2\geq-2$, which implies that $Z^2=-2$ because Z^2 is an even negative ingeger. In particular, this implies that $Z\neq A_1$. Let $D=(A_1+R)|_C$. Then $DC_i\leq 0$ for every i. Thus $Z\leq D$. Since $A^2=-3$, we have $A(F'-A)=3$. So $A_1(Z-A_1)\leq 3$. On the other hand, $-2=Z^2=A_1^2+(Z-A_1)^2+2A_1(Z-A_1)\leq-6+2A_1(Z-A_1)$ implies that $A_1(Z-A_1)\geq 2$.

If $A_1(Z-A_1)=2$, then A_1 is not a fixed component of $|K_{X_1}+Z|$ by Lemma 1.6. This contradicts the condition that x is a base point of $|K_X+F|$.

If $A_1(Z-A_1)=3$, let $Q=Z-A_1$. Let $\Delta_1,\cdots,\Delta_s$ be the connected components of Supp(Q) and let $Q_i=Q|_{\Delta_i}$ for $1\leq i\leq s$. Then $-4=Q^2=Q_1^2+\cdots Q_s^2$. Thus $s\leq 2$. Since $A_1Q=3$, there is a component A_j of Q such that $A_1A_j>0$ and the coefficient of A_j in Q is odd. Then $A_jQ=A_jZ-A_jA_1<0$. Lemma 1.4 implies that $s=2$. Since $A_1(Q_1+Q_2)=3$, we may assume that $A_1Q_1=2$. Then A_1 is not a fixed component of $|K_{X_1}+A_1+Q_1|$ be Lemma 1.6. This contradicts the condition that x is a base point of $|K_X+F|$.

Case 2: The point p is on Z.

Let σ: $\bar{X}\to X$ be the blowingup of X with center at p. Let $E=\sigma^{-1}(p)$, and let $\bar{Z}$, $\bar{F}$ denote the proper transforms of Z and F respectively. Then $K_{\bar{X}}\sim 2\bar{F}+4E+\bar{Z}$. There is a natural fibration f: $\bar{X}\to\mathbf{P}^1$ such that $|\bar{F}|$ consists of fibers. The complement of $\bar{Z}$ in the fiber is denoted by Z°, i.e., Z° is an effective divisor such that $\bar{F}\sim\bar{Z}+Z^\circ$.

The short exact sequence

$$0 \to H^0(\bar{X}, \mathcal{O}(K_{\bar{X}})) \to H^0(\bar{X}, \mathcal{O}(K_{\bar{X}}+\bar{F})) \xrightarrow{\phi} H^0(\bar{F}, \mathcal{O}(K_{\bar{F}})) \to 0, \tag{12}$$

where $\bar{F}$ is a nonsingular fiber, implies that $h^0(\bar{X}, \mathcal{O}(3\bar{F}+4E+\bar{Z})) = h^0(\bar{X}, \mathcal{O}(K_{\bar{X}}+\bar{F})) = 6$. Since the map ϕ in (12) is surjective, E is not a fixed component of $|3\bar{F}+4E+\bar{Z}|$. The short exact sequence

$$0 \to H^0(\bar{X}, \mathcal{O}(3\bar{F}+3E+\bar{Z})) \to H^0(\bar{X}, \mathcal{O}(K_{\bar{X}}+\bar{F})) \to H^0(E, \mathcal{O}_E) \to 0$$

implies that

$$h^0(\bar{X}, \mathcal{O}(3\bar{F}+3E+\bar{Z})) = 5. \tag{13}$$

We are going to show that E is not a fixed component of $|3\bar{F}+3E+\bar{Z}|$. Suppose E is fixed. Then $h^0(\bar{X}, \mathcal{O}(3\bar{F}+2E+\bar{Z}) = 5$. Since $h^0(\bar{X}, \mathcal{O}(2\bar{F}+2E+\bar{Z}) = 3$ and $h^0(\bar{F}, \mathcal{O}_{\bar{X}}(2E)) \leq 2$ for a general fiber $\bar{F}$, we have a short exact sequence

$$0 \to H^0(\bar{X}, \mathcal{O}(2\bar{F}+2E+\bar{Z})) \to H^0(\bar{X}, \mathcal{O}(3\bar{F}+2E+\bar{Z}))$$
$$\to H^0(\bar{F}, \mathcal{O}_{\bar{X}}(2E)) \to 0.$$

This implies that

$$f_*\mathcal{O}(2\bar{F}+2E+\bar{Z}) \cong \mathcal{O}(-1) \oplus \mathcal{O}(2).$$

It follows that $h^0(\mathbf{P}^1, f_*\mathcal{O}(2\bar{F}+2E+\bar{Z})^\vee) = 2$. The relative duality implies that the dual of $R^1f_*\mathcal{O}(2\bar{F}+2E)$ is isomorphic to $f_*\mathcal{O}(2\bar{F}+2E+\bar{Z})$. Hence $h^0(\mathbf{P}^1, R^1f_*\mathcal{O}(2\bar{F}+2E)) \geq 2$ by Lemma 3.4. But the Riemann-Roch theorem implies that $h^1(\bar{X}, \mathcal{O}(2\bar{F}+2E)) = 1$, which contradicts the Leray spectral sequence

$$0 \to H^1(f_*\mathcal{O}(2\bar{F}+2E)) \to H^1(\mathcal{O}(2\bar{F}+2E)) \to$$
$$\to H^0(R^1f_*\mathcal{O}(2\bar{F}+2E)) \to 0.$$

Therefore E cannot be a fixed component of $|3\bar{F}+3E+\bar{Z}|$.

Let G be a general member of $|3\bar{F}+3E+\bar{Z}|$ and let $\bar{F}$ be a general fiber. Let $x = \bar{F} \cap E$. Then $G \cap \bar{F} = \{x_1, x_2, x_3\}$, where x_1, x_2, x_3 are distinct from x. Thus $x_1+x_2+x_3$ is linearly equivalent to $3x$ as divisors on $\bar{F}$, which shows that $\bar{F}$ is not hyperelliptic.

Since E is not a fixed component of $|3\bar{F}+4E+\bar{Z}|$, there exists $D \in |3\bar{F}+4E+\bar{Z}|$ such that D does not contain E. Since $DE=0$, the curve D does not meet E, so $\bar{Z}$ is not a component of D. This shows that $\bar{Z}$ is not a fixed component of $|3\bar{F}+4E+\bar{Z}|$. Hence $4 \leq h^0(\bar{X}, \mathcal{O}(3\bar{F}+4E)) = h^0(\bar{X}, \mathcal{O}(3\bar{F}+3E)) \leq 5$.

We discuss the two subcases:

Case 2A: $h^0(\bar{X}, (3\bar{F}+3E)) = 5$.

It follows from (13) that

$$0\to H^0(\bar{X},\mathcal{O}(2\bar{F}+3E+\bar{Z}))\to H^0(\bar{X},\mathcal{O}(3\bar{F}+3E+\bar{Z}))\to H^0(\bar{F},\mathcal{O}(3E))\to 0$$

is exact. Thus the map $H^0(\bar{X},\mathcal{O}(4\bar{F}+3E))\to H^0(\bar{F},\mathcal{O}(3E))$ is surjective. Hence we have a short exact sequence

$$0\to H^0(\bar{X},\mathcal{O}(3\bar{F}+3E))\to H^0(\bar{X},\mathcal{O}(4\bar{F}+3E))\to H^0(\bar{F},\mathcal{O}(3E))\to 0$$

which implies that $h^0(\bar{X},\mathcal{O}(4\bar{F}+3E))=7$. Meanwhile, the short exact sequence

$$0\to H^0(\bar{X},\mathcal{O}(3\bar{F}+4E))\to H^0(\bar{X},\mathcal{O}(4\bar{F}+4E))\to H^0(\bar{F},\mathcal{O}(K_{\bar{F}}))\to 0$$

implies that $h^0(\bar{X},\mathcal{O}(4\bar{F}+4E))=8$. Hence $|4\bar{F}+4E|$ has no base points. Let ϕ denote the projective morphism determined by $|4\bar{F}+4E|$.

Let n be an arbitrary nonnegative integer. Since $\bar{F}$ is a non-hyperelliptic curve, $h^0(\bar{F},\mathcal{O}_{\bar{F}}(2E))=1$. Hence $h^0(\bar{X},\mathcal{O}((n+1)\bar{F}+2E))\leq h^0(\bar{X},\mathcal{O}(n\bar{F}+2E))+1$ by the short exact sequence

$$0\to H^0(\bar{X},\mathcal{O}(n\bar{F}+2E))\to H^0(\bar{X},\mathcal{O}((n+1)\bar{F}+2E))\to H^0(\bar{F},\mathcal{O}_{\bar{F}}(2E))$$

As $\bar{F}$ is not a fixed component of $|(n+1)\bar{F}+2E|$, we have $h^0(\bar{X},\mathcal{O}((n+1)\bar{F}+2E))=h^0(\bar{X},\mathcal{O}(n\bar{F}+2E))+1$. Hence $h^0(\bar{X},\mathcal{O}(n\bar{F}+2E))=n+1$ for all $n\geq 0$. In particular $h^0(\bar{X},\mathcal{O}(4\bar{F}+2E))=5$.

Since the image of $H^0(\bar{X},\mathcal{O}(4\bar{F}+4E))$ in $H^0(\bar{Z},\mathcal{O}(4))$ has dimension 3, $\phi(\bar{Z})$ is either a plane quartic curve or a conic. Since $h^0(\bar{X},\mathcal{O}(4\bar{F}+2E))=5$ and $h^0(\bar{X},\mathcal{O}(4\bar{F}+3E))=7$, we have a short exact sequence

$$0\to H^0(\bar{X},\mathcal{O}(4\bar{F}+2E))\to H^0(\bar{X},\mathcal{O}(4\bar{F}+3E))\to H^0(E,\mathcal{O}_E(1))\to 0.$$

Hence $|4\bar{F}+3E|$ has no base points on E, which implies that it has no base points at all. Let G be a general member of $|4\bar{F}+3E|$ and let p denote the intersection of E and $\bar{Z}$. Then $G\cap\bar{Z}=\{p_1,p_2,p_3\}$, where p_1, p_2, p_3 are distinct from p. Thus $G+E\sim 4\bar{F}+4E$, $(G+E)|_{\bar{Z}}=p+p_1+p_2+p_3$ and $4E|_{\bar{Z}}=4p$. This implies that $\phi(\bar{Z})$ is a plane quartic curve which is smooth at $\phi(p)$. Let L be the tangent line of $\phi(\bar{Z})$ at $\phi(p)$, then the intersection number $(\phi(\bar{Z}),L)_{\phi(p)}=4$.

Using the same argument as before, one can see that $Y=\phi(\bar{X})$ is a normal surface in $W=\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(3)\oplus\mathcal{O}(4))$. Let E_0 be the hypersurface of W corresponding to $\mathbf{P}(\mathcal{O}(3)\oplus\mathcal{O}(4))$, η be a general fibre of W and η_0 be the fiber containing $\phi(\bar{Z})$. Then Y has at most rational double points away from η_0. As a divisor of W, Y is linearly equivalent to $4E_0+n\eta$ for some $n>0$. Since the morphism ϕ is determined by $|4\bar{F}+4E|$, $4\bar{F}+4E\sim\phi^*(E_0+d\eta)$ for some integer d. Since $\bar{F}\sim\phi^*\eta$, we have $4E\sim\phi^*(E_0+(d-4)\eta)$. It follows from $h^0(4E)=1$ that $d=4$, so $\phi^*(E_0)=4E$. Since the canonical system of $\bar{X}$ is cut out by $K_W+Y\sim E_0+(n-9)\eta$, we have $n=12$ and $E_0+2\eta+\eta_0$ cuts out the canonical system. Lemma 1.1 (2) implies $\chi(Y)=6$. Since $\chi(\bar{X})=4=\chi(Y)-2$, either Y has one singularity of geometric genus two or Y has two singularities of geometric genus one on η_0. Note that $\eta_0\cong\mathbf{P}^2$ and it contains the singular quartic curve $\phi(\bar{Z})$. Since the multiplicity of every singularity of a plane quartic curve is less than or equal to 3, so is the multiplicity of every singularity of Y.

If Y has one triple point x, then x is a triple point of the quartic curve $\phi(\bar{Z})$. Obviously $\phi(\bar{Z})$ has no other singularities and the geometric genus of the surface singularity x is equal to two. Let π: $W_1\rightarrow W$ be the blowingup of W at x. Let $G=\pi^{-1}(x)$, and let Y_1 be the proper transform of Y. Then $K_{W_1}+Y_1\sim\pi^*(K_W+Y)-G\sim\pi^*(E_0+2\eta)+\eta_0'$, where η_0' is the proper transform of η_0. If Y_1 is not normal, then G contains a curve C such that Y_1 is singular along C. This curve C is not contained in η_0', because x is a triple point of $\phi(\bar{Z})$. Let τ: $W_2\rightarrow W_1$ be the blowingup of W_1 with center at C. Let $G_2=\tau^{-1}(C)$ Then $K_{W_2}+Y_2\sim\tau^*\pi^*(E_0+2\eta)+\tau^*(\eta_0')-(m-1)G_2$, where m is the multiplicity of a generic point of the singular locus C and Y_2 is the proper transform of Y_1. But $\tau^*\pi^*(E_0+2\eta)+\tau^*(\eta_0')-(m-1)G_2$ is not an effective divisor. This contradicts the assertion that $E_0+2\eta+\eta_0$ cuts out the canonical system of $\bar{X}$. It follows that Y_1 is normal. The surface Y_1 has an essential singularity on η'_0, for otherwise x would be an elliptic singularity of geometric genus one. This is impossible, for $\eta'_0\cap Y_1$ is a smooth rational curve. Therefore Y has no triple point.

If Y has one double point x of geometric genus two, then x is a double or triple point of the quartic curve $\phi(\bar{Z})$. Let π: $W_1\rightarrow W$ be the blowingup of W at x. Then $K_{W_1}+Y_1\sim\pi^*(E_0+2\eta)+\eta'_0+G$. The surface Y_1 has double locus along the rational curve $C=G\cap Y_1$, for otherwise x would be a rational double point of Y. The curve C is not located on η'_0, since $\phi(\bar{Z})$ has at most a triple point at x. Let τ: $W_2\rightarrow W_1$ be the blowingup of W_1 with center at C. Then the proper transform Y_2 of Y_1 is normal, for otherwise the double point x would have geometic genus greater or equal to three. For the same reason as before, Y_2 has an essential double point on the proper transform η''_0 of η'_0. But $\eta''_0\cap Y_2$ is the blowingups twice of the quartic curve $\phi(\bar{Z})$ at a double point, so $\eta''_0\cap Y_2$ has at most ordinary double points by (1) of Lemma 3.3. This implies that Y_2 has at most rational double points on η''_0, ths is a contradiction.

If Y has two essential double points x_1 and x_2, then each of these two points has geometric genus one. So they are minimally elliptic points. Let F' be the fiber containing $\bar{Z}$, then $F'=\bar{Z}+A_1+A_2$, where A_1 and A_2 are the fundamental cycles of x_1 and x_2. We have $0=F'^2=-3+2\bar{Z}(A_1+A_2)+A_1^2+A_2^2=-3+2(K_{\bar{X}}A_1+K_{\bar{X}}A_2)+A_1^2+A_2^2=-3-A_1^2-A_2^2$. So we may assume that $A_1^2=-1$ and $A_2^2=-2$. The plane quartic curve $\phi(\bar{Z})$ has two double points x_1 and x_2. Let A_m and A_n be the types of the x_1 and x_2 respectively as plane curve double points. Then $m, n>1$ since x_1 and x_2 are not rational double points as surface singularities. Since $A_2^2=-2$, we have $n>2$. This contradicts (2) of Lemma 3.3. Therefore Case 2A does not occur.

Case 2B: $h^0(\bar{X},(3\bar{F}+3E))=4$.

Since $h^0(3\bar{F}+3E)=4$ and $h^0(4\bar{F}+4E)=7$, we have $f_*\mathcal{O}_{\bar{X}}(4\bar{F}+4E)\cong\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(4)$ and $|4\bar{F}+4E|$ has no base points. Let ϕ: $\bar{X}\to W=\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(4)\oplus\mathcal{O}(4))$ be the relative morphism determined by the line boudle $\mathcal{O}_{\bar{X}}(4\bar{F}+4E)$. Then $4\bar{F}+4E\sim\phi^*(E_0+n\eta)$ for some integer n. Since $\bar{F}\sim\phi^*\eta$, we have $4E\sim\phi^*(E_0+(n-4)\eta)$. It follows from $h^0(4E)=1$ that $n=4$ and $\phi^*(E_0)=4E$.

Next we take a look at the image of $\bar{F}$ under the morphism ϕ for an arbitrary fiber $\bar{F}$. A fiber $\bar{F}$ can be written as $\bar{F}=A+B$, where A is an irreducible curve with $AE=1$ and B is an effective divisor with $BE=0$. Since the intersection matrix of the divisor B is negative definite, B is contained in the fixed part of $|3\bar{F}+4E+B|$, so $h^0(3\bar{F}+4E+B)=h^0(3\bar{F}+4E)=4$. The short exact sequence

$$0\to H^0(3\bar{F}+4E+B)\to H^0(4\bar{F}+4E)\to H^0(A,\mathcal{O}_A(4E))$$

implies that the image of A under ϕ is an irreducible plane curve not contained in a line. That means that ϕ: $A\to\phi(A)$ is either a birational morphism onto a quartic curve or a morphism of degree two onto a conic. We are going to see that there is at most one fiber whose image under ϕ is a conic.

By (13) $h^0(3\bar{F}+3E+\bar{Z})=5>h^0(3\bar{F}+3E)=h^0(3\bar{F}+2E+\bar{Z})$, so there exists $D_1\in|3\bar{F}+3E+\bar{Z}|$ which contains neither $\bar{Z}$ nor E. Let $D=D_1+Z^\circ\in|4\bar{F}+3E|$. Then there is a unique point $x\in D\cap E$ which is not in $\bar{Z}$. Let $\bar{F}$ be a fiber not containing x. Let $e=\bar{F}\cap E$. Then $4e\sim e+e_1+e_2+e_3=(E+D)|_{\bar{F}}$, where e_1, e_2, e_3 are all distinct from e, whence $\phi(\bar{F})$ is an irreducible quartic curve. This shows that only possible fiber $\bar{F}$ such that $\phi(\bar{F})$ is a conic is the fiber passing through x. In particular, $\phi(\bar{Z})$ is a quartic curve.

For an arbitrary irreducible curve B on $\bar{X}$, $\phi(B)$ is a point if and only if $B(4\bar{F}+4E)=0$. These are exactly the "vertical" curves away from E. Assume that B is such a curve and B is not a component of Z°. Then $BK_{\bar{X}}=0$, wherce B is a (-2)-curve.

Let ρ: $Y\to S$ be the normalization of $S=\phi(\bar{X})$, and let ψ: $\bar{X}\to Y$ be the morphism such that $\phi=\rho\psi$. By the above discussion, we conclude that $Y-$

$\psi(\bar{Z})$ has only rational double points as its singularities.

As a divisor in W, $\phi(\bar{X}) \sim 4E_0 + r\eta$ for some integer r. Thus

$$16 = (4\bar{F} + 4E)^2 = (E_0 + 4\eta)^2 (4E_0 + r\eta) = 4E_0^3 + r + 32 = r.$$

Hence $S = \phi(\bar{X}) \sim 4E_0 + 16\eta$, and $K_W + S \sim E_0 + 6\eta$.

If S is normal, then ϕ: $\bar{X} \to S$ is a resolution of isolated singularities. Since all singularities on $S - \phi(\bar{Z})$ are rational double points, we have $K_{\bar{X}} \sim \phi^*(E_0 + 6\eta) - \Delta \sim 4E + 6\bar{F} - \Delta$, where Δ is an effective divisor and $\mathrm{Supp}(\Delta) \subseteq \mathrm{Supp}(Z^\circ)$. This is impossible. Hence there is a fiber $\bar{F}_0$ such that $\phi(\bar{F}_0)$ is a conic C_0.

Use the same notation of local coordinates as before. Let $g(x_0, x_1, x_2, z) = 0$ be the equation of the surface $S_0 = S \cap W_0$, where x_0 is the fiber coordinate of the line bundle $\mathcal{O}$ and x_1, x_2 are the fiber coordinates over U_0 of the rank two bundle $\mathcal{O}(4) \oplus \mathcal{O}(4)$. Since $\phi^*(E_0) = 4E$, the line $E_0 \cap \eta$ intersects the quartic curve $S \cap \eta$ at a single point with contact number 4. Hence $g(x_0, x_1, x_2, z) = u(z) k(x_1, x_2)^4 + x_0 v(x_0, x_1, x_2, z)$ for some linear form $k(x_1, x_2)$. Since $\phi(\bar{F})$ is an irreducible curve of degree 4 or 2 for an arbitrary fiber $\bar{F}$, $u(z) \neq 0$ for every $z \in \mathbf{C}$. Hence $u(z)$ is a nonzero constant α. After a linear change of coordinates, we may assume that the equation of S_0 is

$$\alpha x_1^4 + x_0 v(x_0, x_1, x_2, z) = 0,$$

where $v(x_0, x_1, x_2, z)$ is homogeneous in x_0, x_1, x_2 of degree 3. Without loss of generality, we may assume that C_0 is contained in the fiber $z = 0$. By Lemma 3.2 the resolution length of the double locus C_0 is less that or equal to 2.

Let σ_1: $W_1 \to W$ be the blowingup of W with center at C_0. Let $G_1 = \sigma_1^{-1}(C_0)$. Let S_1 be the proper transform of S and let $C_1 = S_1 \cap G_1$. Then $K_{W_1} + S_1 \sim \sigma_1^*(K_W + S) - G_1 \sim \sigma_1^*(E_0 + 6\eta) - G_1$.

If the resolution length of C_0 is 1, then S_1 is normal. Let ψ: $\bar{X} \to S_1$ be the morphism such that $\phi = \sigma_1 \psi$. Then $K_{\bar{X}} \sim \phi^*(E_0 + 6\eta) - \psi^*(G_1) - \Delta \sim 4E + 5\bar{F} + \psi^*(\sigma_1^*\eta - G_1) - \Delta$, where Δ is an effective divisor and $\mathrm{Supp}(\Delta) \subseteq \mathrm{Supp}(Z^\circ)$. This is impossible, for $\sigma_1^*\eta - G_1$ is effective.

If the resolution length of C_0 is 2, let σ_2: $W_2 \to W_1$ be the blowingup of W_1 with center at C_1. Let $G_2 = \sigma_1^{-1}(C_1)$. Let S_2 be the proper transform of S_1 and let $C_2 = S_2 \cap G_2$. Then S_2 is normal and $K_{W_2} + S_2 \sim \sigma_2^*(K_{W_1} + S_1) - G_2 \sim \sigma_2^*\sigma_1^*(E_0 + 6\eta) - \sigma_2^* G_1 - G_2$. Let ψ: $\bar{X} \to S_2$ be the morphism such that $\phi = \sigma_1 \sigma_2 \psi$. Then $K_{\bar{X}} \sim \phi^*(E_0 + 6\eta) - \psi^*\sigma_2^*(G_1) - \psi^*(G_2) - \Delta \sim 4E + 4\bar{F} + \psi^*(\sigma_2^*\sigma_1^*(2\eta) - \sigma_2^*(G_1) - G_2) - \Delta$, where Δ is an effective divisor and $\mathrm{Supp}(\Delta) \subseteq \mathrm{Supp}(Z^\circ)$. However, the divisor $4E + 4\bar{F} + \psi^*(\sigma_2^*\sigma_1^*(2\eta) - \sigma_2^*(G_1) - G_2) - \Delta$ cannot be linearly equivalent to $4E + 2\bar{F} + \bar{Z}$ for $\sigma_2^*\sigma_1^*(2\eta) - \sigma_2^*(G_1) - G_2$ is effective.

4. $p_g=4$, $K_2=9$

In this section we prove the non-existence of minimal surfaces of general type with $p_g=4$, $K^2=9$ whose canonical map is composed of pencils.

Lemma 4.1. *Let X be a minimal surface of general type whose canonical map is composed of a pencil. Assume that the genus g of a general member of the pencil is greater than or equal to* 3. *If the geometric genus p_g of X is* 4, *then one of the following statements holds*:

1. $K_X^2\geq 12$.
2. $K_X^2=9$ *and* $K_X\sim 3F$ *where F is a nonsingular curve of genus* 3 *with* $F^2=1$.

Proof. Since $p_g=4$, the canonical divisor can be written as $K_X=nF+Z$ for $n\geq 3$, where Z is the fixed part of $|K_X|$, and F is a general member of the pencil of which $|K_X|$ is composed of. Consider the two cases.

Case i) $F^2=0$:

We may assume that F is a general member of the pencil. By Sard's theorem, F is a smooth curve of genus g. Hence $FZ=F(F+K_X)=2g-2\geq 4$. Then $K_X^2=nFZ+ZK_X\geq 12$.

Case ii) $F^2>0$:

In this case $n=3$. We have

$$2g-2=F^2+FK_X=4F^2+FZ\geq 4F^2, \tag{14}$$

and the equality holds if and only if $Z=0$. $K_X^2=9F^2+3FZ+ZK_X\geq F^2+2(4F^2+FZ)=F^2+4g-4$. Hence $K_X^2\geq 12$ when $g\geq 4$. If $g=3$, then (14) implies that $F^2=1$ and $Z=0$.

Lemma 4.2. *Assume that X is a surface satisfying the second condition of Lemma* 4.1. *Then* $H^1(X,\mathcal{O}_X)=0$.

Proof. (suggested by G. Xiao) Suppose $h^1(X,\mathcal{O}_X)>0$. Then there exists a divisor ε which is not linearly equivalent to zero but $3\varepsilon\sim 0$. Suppose that $H^0(X,\mathcal{O}(F+\varepsilon))\neq 0$. Let $D\in|F+\varepsilon|$. Then

$$3D\in|3F|. \tag{15}$$

Since every member of $|3F|$ is the sum of three members of $|F|$, (15) implies that $D\sim F$, whence $\varepsilon\sim 0$. This is a contradiction. Hence $H^0(X,\mathcal{O}(F+\varepsilon))=0$. For the same reason $H^0(X,\mathcal{O}(F-\varepsilon))=0$ halds too. Thus the sequence

$$0\to H^0(X,\mathcal{O}(2F-\varepsilon))\to H^0(F,\mathcal{O}_F(2F-\varepsilon)) \tag{16}$$

is exact.

Xiao's theorem implies that $q=h^1(\mathcal{O}_X)\leq 2$. The Riemann-Roch theorem implies that

$$h^0(X,\mathcal{O}(2F-\varepsilon))-h^1(X,\mathcal{O}(2F-\varepsilon))=4-q\geq 2. \tag{17}$$

Since $F(2F-\varepsilon)=2$, we have the inequality $h^0(F,\mathcal{O}_F(2F-\varepsilon))\leq 2$. It follows from (16) and (17) that $h^0(X,\mathcal{O}(2F-\varepsilon))=h^0(F,\mathcal{O}_F(2F-\varepsilon))=q=2$. This implies that F is a hyperelliptic curve with a $g^1_2=|\mathcal{O}_F(2F-E)|$. Since $\mathcal{O}_F(4F)=\mathcal{O}_F(K_X+F)=\mathcal{O}_F(K_F)$, the divisor $\mathcal{O}_F(2F)$ is also the g^1_2 of F. Hence $\mathcal{O}_F(4F+\varepsilon)=\mathcal{O}_F(4F-2\varepsilon)\cong\mathcal{O}_F(K_F))$, whence

$$h^0(F,\mathcal{O}_F(4F+\varepsilon))=3. \tag{18}$$

Since the divisor $F+\varepsilon$ is big and nef, Kawamata's vanishing theorem implies $H^1(X,\mathcal{O}(4F+\varepsilon))=0$. So we have exact sequence

$$\begin{gathered}0\to H^0(X,\mathcal{O}(3F+\varepsilon))\to H^0(X,\mathcal{O}(4F+\varepsilon))\\ \to H^0(F,\mathcal{O}_F(4F+\varepsilon))\to H^1(X,\mathcal{O}(3F+\varepsilon))\to 0.\end{gathered}$$

Hence $3=\chi(X,\mathcal{O}(3F+\varepsilon))=h^0(X,\mathcal{O}(3F+\varepsilon))-h^1(X,\mathcal{O}(3F+\varepsilon))=h^0(X,\mathcal{O}(4F+\varepsilon))-h^0(F,\mathcal{O}(4F+\varepsilon))=h^0(X,\mathcal{O}(4F+\varepsilon))-3$ by (18). Thus

$$h^0(X,\mathcal{O}(4F+\varepsilon))=6. \tag{19}$$

On the other hand, $h^0(X,\mathcal{O}(4F+\varepsilon))=\chi(X,\mathcal{O}(4F+\varepsilon))=2F^2+\chi(X,\mathcal{O})=5$, contradicting (19). Therefore $q=0$ is impossible.

Theorem 4.3. *There does not exist a minimal surface of general type X such that*

1. $p_g(X)=4$, $K_X^2=9$.
2. *The canonical map of X is composed of a pencil of genus greater than or equal to* 3.

Proof. Suppose that such a surface X exists. By Lemma 4.1 there is a nonpingular curve F of genus 3 such that $K_X\sim 3F$ and $F^2=1$. Let p denote the base point of $|F|$. We know that $q=h^1(X,\mathcal{O})=0$ by Lemma 4.2.

Let $\sigma\colon \bar{X}\to X$ be the blowingup of X with center at p. Let $E=\sigma^{-1}(p)$, and let $\bar{F}$ denote the proper transform of F. Then $K_X\sim 3\bar{F}+4E$. There is a natural fibration $f\colon \bar{X}\to\mathbf{P}^1$ such that $|\bar{F}|$ consists of fibers. We may assume that $\bar{F}$ is nonsingular.

The short exact sequence

$$0\to H^0(\bar{X},\mathcal{O}(K))\to H^0(\bar{X},\mathcal{O}(K+\bar{F}))\xrightarrow{\phi} H^0(\bar{F},\mathcal{O}(K_{\bar{F}}))\to 0$$

implies that $h^0(\bar{X},\mathcal{O}(4\bar{F}+4E))=h^0(\bar{X},\mathcal{O}(K+\bar{F}))=7$. Since the map ϕ is surjective, E is not a fixed component of $|4\bar{F}-4E|$. The short exact sequence

$$0\to H^0(\bar{X},\mathcal{O}(4\bar{F}+3E))\to H^0(\bar{X},\mathcal{O}(K_{\bar{X}}+\bar{F}))\to H^0(E,\mathcal{O}_E)\to 0$$

implies that

$$h^0(\bar{X},\mathcal{O}(4\bar{F}+3E))=6. \tag{20}$$

We are going to show that E is not a fixed component of $|4\bar{F}+3E|$.

Suppose E is fixed. Then $h^0(\bar{X},\mathcal{O}(4\bar{F}+2E))=6$. Since $h^0(\bar{X},\mathcal{O}(3F+2E))=4$ and $h^0(\bar{F},\mathcal{O}_{\bar{F}}(2E))\leq 2$ for a general fiber $\bar{F}$, we have a short exact sequence

$$0\to H^0(\bar{X},\mathcal{O}(3\bar{F}+2E))\to H^0(\bar{X},\mathcal{O}(4\bar{F}+2E))\to H(\bar{F},\mathcal{O}_{\bar{F}}(2E))\to 0.$$

This implies that

$$f_*\mathcal{O}(3\bar{F}+2E)\cong\mathcal{O}(-1)\oplus\mathcal{O}(3).$$

It follows that $h^0(\mathbf{P}^1, f_*\mathcal{O}(3\bar{F}+2E)^\vee)=2$. The ralative duality implies that the dual of $R^1f_*\mathcal{O}(2\bar{F}+2E)$ is isomorphic to $f_*\mathcal{O}(3\bar{F}+2E)$. Hence $h^0(\mathbf{P}^1,R^1f_*\mathcal{O}(2\bar{F}+2E))\geq 2$ by Lemma 3.4. But the Riemann-Roch theorem implies that $h^1(\bar{X},\mathcal{O}(2\bar{F}+2E))=1$, which contradicts the Leray spectral sequence

$$0\to H^1(f_*\mathcal{O}(2\bar{F}+2E))\to H^1(\mathcal{O}(2\bar{F}+2E))\to H^0(R^1f_*\mathcal{O}(2\bar{F}+2E))\to 0.$$

Therefore E is not a fixed component of $|4\bar{F}+3E|$. Let G be a general member of $|4\bar{F}+3E|$ and let $\bar{F}$ be a general fiber. Let $x=\bar{F}\cap E$. Then $G\cap\bar{F}=\{x_1,x_2,x_3\}$, where x_1, x_2, x_3 are distinct from x. Thus $x_1+x_2+x_3$ is linearly equivalent to $3x$ as divisors on $\bar{F}$, which shows that $\bar{F}$ is not hyperelliptic.

Since $h^0(3\bar{F}+4E)=4$ and $h^0(4\bar{F}+4E)=7$, we have $f_*\mathcal{O}_{\bar{X}}(4\bar{F}+4E)\cong\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(4)$ and $|4\bar{F}+4E|$ has no base points. Let ϕ: $\bar{X}\to W=\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(4)\oplus\mathcal{O}(4))$ be the relative morphism determined by the line bundle $\mathcal{O}_{\bar{X}}(4\bar{F}+4E)$. Then $4\bar{F}+4E\sim\phi^*(E_0+n\eta)$ for some integer n, where E_0 and η are hypersurfaces of W as defined in section 2.1. Since $\bar{F}\sim\phi^*\eta$, we have $4E\sim\phi^*(E_0+(n-4)\eta)$. It follows from $h^0(4E)=1$ that $n=4$ and $\phi^*(E_0)=4E$.

The rest of the proof is very similar to Case 2B in the proof of Theorem 3.5.

An arbitrary fiber $\bar{F}$ of f can be written as $\bar{F}=A+B$, where A is an irreducible curve with $AE=1$ and B is an effective divisor with $BE=0$. Since the intersection matrix of the divisor B is negative definite, B is contained in the fixed part of $|3\bar{F}+4E+B|$, so $h^0(3\bar{F}+4E+B)=h^0(3\bar{F}+4E)=4$. The short exact sequence

$$0\to H^0(3\bar{F}+4E+B)\to H^0(4\bar{F}+4E)\to H^0(A,\mathcal{O}_A(4E))$$

implies that the image of A under ϕ is an irreducible plane curve not contained in a line. That means that ϕ: $A\to\phi(A)$ is either a birational morphism onto a quartic curve or a morphism of degree two onto a conic. Let G be a general member of $|4\bar{F}+3E|$. Then G does not contain E and meets E at one point p. Let $\bar{F}_0$ be the fiber passing through p. The only possible fiber $\bar{F}$ such that $\phi(\bar{F})$ is a conic is $\bar{F}_0$. So there is at most one fiber whose image under ϕ is a conic.

For an arbitrary irreducible curve B on $\bar{X}$, $\phi(B)$ is a point if and only if $B(4\bar{F}+4E)=0$. These are exactly the "vertical" curves away from E. Assume that B is such a curve. Then $BK_{\bar{X}}=0$, whence B is a (-2)-curve.

Let ρ: $Y\rightarrow S$ be the normalization of $S=\phi(\bar{X})$, and let ψ: $\bar{X}\rightarrow Y$ be the morphism such that $\phi=\rho\psi$. The above discussion implies that Y has only rational double points as its singularities.

As a divisor in W, $\phi(\bar{X})\sim 4E_0+r\eta$ for some integer r. Thus

$$16=(4\bar{F}+4E)^2=(E_0+4\eta)^2(4E_0+r\eta)=4E_0^3+r+32=r.$$

Hence $S=\phi(\bar{X})\sim 4E_0+16\eta$, and $K_W+S\sim E_0+6\eta$.

If S is normal, then ϕ: $\bar{X}\rightarrow S$ is a resolution of isolated singularities. In this case all singularities on S are rational double points, so $K_{\bar{X}}\sim\phi^*(E_0+6\eta)\sim 4E+6\bar{F}$. This is obviously impossible. Hence there is a fiber $\bar{F}_0$ such that $\phi(\bar{F}_0)$ is a conic C_0.

Use the same notation of local coordinates before. Let $g(x_0,x_1,x_2,z)=0$ be the equation of the surface $S_0=S\cap W_0$, where x_0 is the fiber coordinate of the line bundle $\mathcal{O}$ and x_1, x_2 are the fiber coordinates of the rank two bundle $\mathcal{O}(4)\oplus\mathcal{O}(4)$ over U_0. Since $\phi^*(E_0)=4E$, the line $E_0\cap\eta$ intersects the quartic curve $S\cap\eta$ at a single point with contact number 4. Hence $g(x_0,x_1,x_2,z)=u(z)k(x_1,x_2)^4+x_0v(x_0,x_1,x_2,z)$ for some linear form $k(x_1,x_2)$. Since $u(z)\neq 0$ for all $z\in\mathbf{C}$, $u(z)$ is a nonzero constant α. After a linear change of coordinates, we may assume that the equation of S_0 is

$$\alpha x_1^4+x_0v(x_0,x_1,x_2,z)=0,$$

where $v(x_0,x_1,x_2,z)$ is homogeneous in x_0, x_1, x_2 of degree 3. Without loss of generality, we may assume that C_0 is contained in the fiber $z=0$. By Lemma 3.2 the resolution length of the double locus C_0 is less than or equal to 2.

Let σ_1: $W_1\rightarrow W$ be the blowingup of W with center at C_0. Let $G_1=\sigma_1^{-1}(C_0)$. Let S_1 be the proper transform of S and let $C_1=S_1\cap G_1$. Then $K_{W_1}+S_1\sim\sigma_1^*(K_W+S)-G_1\sim\sigma_1^*(E_0+6\eta)-G_1$.

If the resolution length of C_0 is 1, then S_1 is normal. Let ψ: $\bar{X}\rightarrow S_1$ be the morphism such that $\phi=\sigma_1\psi$. Then $K_{\bar{X}}\sim\phi^*((E_0+6\eta)-\psi^*(G_1)\sim 4E+5\bar{F}+\psi^*(\sigma_1^*\eta-G_1)$. This is impossible, for $\sigma_1^*\eta-G_1$ is effective.

If the resolution length of C_0 is 2, let σ_2: $W_2\rightarrow W_1$ be the blowingup of W_1 with center at C_1. Let $G_2=\sigma_1^{-1}(C_1)$. Let S_2 be the proper transform of S_1. Then S_2 is normal and $K_{W_2}+S_2\sim\sigma_2^*(K_{W_1}+S_1)-G_2\sim\sigma_2^*\sigma_1^*(E_0+6\eta)-\sigma_2^*G_1-G_2$. Let ψ: $\bar{X}\rightarrow S_2$ be the morphism such that $\phi=\sigma_2\sigma_1\psi$. Then $K_{\bar{X}}\sim\phi^*(E_0+6\eta)-\psi^*\sigma_2^*(G_1)-\psi^*(G_2)\sim 4E+4\bar{F}+\psi^*(\sigma_2^*\sigma_1^*(2\eta)-\sigma_2^*(G_1)-G_2)$. However, the divisor $4E+4\bar{F}+\psi^*(\sigma_2^*\sigma_1^*(2\eta)-\sigma_2^*(G_1)-G_2)$ cannot be linearly equivalent to $4E+3\bar{F}$ for $\sigma_2^*\sigma_1^*(2\eta)-\sigma_2^*(G_1)-G_2$ is effective. This concludes the proof of the theorem.

5. Lower bound for K^2

Theorem 5.1. *Let X be a minimal surface of general type whose canonical map is composed of a pencil. Let g denote the genus of a general member of the pencil. Assume that $g \geq 3$.*

(1) *If $p_g(X)=3$, then $K_X^2 \geq 7$.*

(2) *If $p_g(X)=4$, then $K_X^2 \geq 12$.*

(3) *If $p_g(X) \geq 5$ and $g=3$, then $K_X^2 \geq 4p_g-4$.*

Proof. (1) The canonical divisor can be written as $K_X=nF+Z$, where Z is the fixed part of $|K_X|$, and F is a general member of the pencil of which $|K_X|$ is composed of. Here $n \geq 2$. Consider the two cases.

Case i) $F^2=0$:

We may assume that F is a general member of the pencil. Since $|F|$ has no base points, F is a smooth curve of genus g. Hence $FZ=F(F+K_X)=2g-2 \geq 4$. Then $K_X^2=nFZ+ZK_X \geq 8$

Case ii) $F^2>1$:

Then $K_X^2=4F^2+2FZ+ZK_X \geq 8$.

Case iii) $F^2=1$:

Then $n=2$, and we may assume that F is a smooth curve of genus g. We have $FZ=F(F+K_X)-3=2g-5 \geq 1$. Hence $K_X^2=4F^2+2FZ+ZK_X \geq 6$. By Theorem 3.5 $K_X^2=6$ is impossible.

(2) follows from Lemma 4.1 and Theorem 4.3.

(3) In this case $K=nF+Z$ with $n \geq p_g-1$. Since $p_gF^2+FZ \leq (n+1)F^2+FZ=FK+F^2=4$, we have $F^2=0$ and $FZ=4$. This means that $|F|$ has no base points. Thus $K_X^2=nFZ+ZK_X \geq 4p_g-4$.

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