J. Math. Kyoto Univ. (JMKYAZ)
38-1 (1998) 71-75

A generalization of the parallelogram equality in normed spaces

By

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Let $(X, \|\cdot\|)$ be a real normed space. Then on X^2 there always exist the functionals:

$$\tau_\pm(x, y) := \lim_{t \to \pm 0} t^{-1}(\|x+ty\| - \|x\|) \qquad (x, y \in X). \tag{1}$$

$$g(x, y) := \frac{\|x\|}{2}(\tau_-(x, y) + \tau_+(x, y)) \qquad (x, y \in X)^{1)}. \tag{2}$$

The functional g is a natural generalization of the inner product $(\cdot, \cdot)$, which follows from its properties:

$$g(x, x) = \|x\|^2 \qquad (x \in X), \tag{3}$$

$$g(\alpha x, \beta y) = \alpha\beta g(x, y) \qquad (x, y \in X; \alpha, \beta \in R), \tag{4}$$

$$g(x, x+y) = \|x\|^2 + g(x, y) \qquad (x, y \in X), \tag{5}$$

$$|g(x, y)| \leq \|x\|\|y\| \qquad (x, y \in X), \tag{6}$$

$(X, \|\cdot\|)$ is an inner product space if and only if $g(x, y)$ is an inner product of vectors x and y, for all $x, y \in X$. (7)

By use of the functional g, we may define many geometrical points in normed spaces (angle between two vectors, the projection of the vector x on the vector y, many types of orthogonalities, orthonormal system, and so on) (cf. [2] to [5]).

In an inner product space X the equality

$$\|x+y\|^4 - \|x-y\|^4 = 8(\|x\|^2 + \|y\|^2)(x, y) \qquad (x, y \in X) \tag{8}$$

holds, which is equivalent to the parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \qquad (x, y \in X). \tag{9}$$

In normed spaces, the equality

$$\|x+y\|^4 - \|x-y\|^4 = 8(\|x\|^2 g(x, y) + \|y\|^2 g(y, x)), \qquad (x, y \in X) \tag{10}$$

is a generalization of the equality (8).

Communicated by Prof. K. Ueno, November 12, 1996
Revised September 10, 1997

1) The notation g is according to the name Gâteaux.

We may put the question: Is there a normed space, which is not an inner product space, satisfying the equality (10)? The answer is yes.

Lemma 1. *There exists nontrivial normed space in which the equality* (10) *holds.*

Proof. Let us prove that the equality (10) holds true in l^4 and does not hold in l^1.

According to the definition of the functional g in the space $l^p\,(p\geq 1)$ we get

$$g(x,\,y)=\|x\|^{2-p}\sum_k|x_k|^{p-1}(\operatorname{sgn}x_k)y_k \qquad (x=(x_1,\,x_2,\cdots)\in l^p\backslash\{0\}). \quad (11)$$

Hence, with $p=4$, we have

$$\|x\|^2g(x,\,y)=\sum_k x_k^3y_k \qquad (x,\,y\in l^4).$$

From this, we get (10). But, from (11) with $p=1$, we have

$$g(x,\,y)=\|x\|\sum_k(\operatorname{sgn}x_k)y_k \qquad (x,\,y\in l^1).$$

Taking $x=(1,1,2,0,0,\cdots)\in l^1$ and $y=(1,-1,1,0,0,\cdots)\in l^1$ we readily see that the equality (10) does not hold.

Definition 1. A normed space with the equality (10) is called a quasi-inner product space (q.i.p.space).

We also use the following familiar definitions:

Definition 2. (cf. [1,p.20]) A mapping $x\mapsto f_x$ of $X\backslash\{0\}$ to $X^*\backslash\{0\}$[1] is a support mapping whenever

(i) $x\in S(X)$ implies $\|f_x\|=1=f_x(x)$[2],
(ii) $\lambda\geq 0$ implies $f_{\lambda x}=\lambda f_x$.

Definition 3. A normed space X is smooth if

$$\tau_-(x,\,y)=\tau_+(x,\,y) \qquad (x,y\in X).$$

Definition 4. A normed space X is uniformly smooth whenever given $\varepsilon>0$ there exists $\delta>0$ such that

$$\|x+y\|+\|x-y\|<2+\varepsilon\|y\|$$

if $x\in S(X)$ and $\|y\|<\delta$.

Definition 5. A normed space X is very smooth if it is smooth and its support mapping $x\mapsto f_x$ is norm to weak continuous from $S(X)$ to $S(X^*)$

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1) X^* is the topological dual of X.

2) $S(X)=\{x\in X\,|\,\|x\|=1\}$.

(cf. [1, p.31]).

Definition 6. A normed space X is strictly convex if whenever

$$\|x+y\|=\|x\|+\|y\|$$

where $x\neq 0$, $y\neq 0$, then $y=\lambda x$ for some $\lambda>0$.

Definition 7. A normed space X is uniformly convex whenever given $\varepsilon>0$ there exists $\delta>0$ such that if $x, y\in S(X)$ and

$$\|x-y\|\geq\varepsilon \text{ then } \left\|\frac{x+y}{2}\right\|\leq 1-\delta.$$

The following facts are concerning the geometry of the unit sphere $S(X)$ in q.i.p. spaces.

Theorem 1. *A q.i.p. space X is smooth.*

Proof. Let $t\in R$ and $x, y\in X$. From (10) it will then follows:

$$\|(x+ty)+y\|^4-\|(x+ty)-y\|^4=8\Big(\|x+ty\|^2 g(x+ty, y)+\|y\|^2 g(y, x+ty)\Big). \tag{12}$$

Since, in view of (5)

$g(y, x+ty)=g(y, x)+t\|y\|^2$, from (12) we have:

$$\|x+y\|^4-\|x-y\|^4=8\Big(\|x\|^2\lim_{t\to 0} g(x+ty, y)+\|y\|^2 g(y, x)\Big). \tag{13}$$

Making use of (10) once more, from (13) we get:

$$\lim_{t\to 0} g(x+ty, y)=g(x, y). \tag{14}$$

On the other hand, applying (10) for vectors $x+\frac{t}{2}y$ and $\frac{t}{2}y$, we get:

$$\|x+ty\|^4-\|x\|^4=8\left(\frac{t}{2}\left\|x+\frac{t}{2}y\right\|^2 g\left(x+\frac{t}{2}y, y\right)+\left(\frac{t}{2}\right)^3\|y\|^2 g\left(y, x+\frac{t}{2}y\right)\right)$$

Hence,

$$\frac{\|x+ty\|-\|x\|}{t}=\frac{4\left\|x+\frac{t}{2}y\right\|^2 g\left(x+\frac{t}{2}y, y\right)+t^2\|y\|^2 g(y, x)+\frac{t^3}{2}\|y\|^4}{\left(\|x+ty\|^2+\|x\|^2\right)\left(\|x+ty\|+\|x\|\right)}, \qquad (t\neq 0).$$

Therefore, in view of (14),

$$\tau_\pm(x, y)=\frac{g(x, y)}{\|x\|}, \ (x\neq 0), \text{ that is } \tau_-(x, y)=\tau_+(x, y).$$

Corollary 1. *If X is q.i.p. space, then the mapping $x\mapsto g(x, \cdot)$ is a* *support mapping.*

Proof. Since $\tau_-(x, y) = \tau_+(x, y)$, g is linear in the second variable and this gives:

$$g(x, \cdot) \in I_x \left(I_x := \left\{ f \in X^* \middle| f(x) = \|f\| \|x\|, \|f\|, = \|x\| \right\} \right).$$

This implies that the mapping $x \mapsto g(x, \cdot)$ of $X \setminus \{0\}$ to $X^* \setminus \{0\}$ has the properties:

(i) $x \in S(X)$ implies $\|g(x, \cdot)\| = 1 = g(x, x)$,
(ii) $\lambda \in R$ implies $g(\lambda x, \cdot) = \lambda g(x, \cdot)$.

Theorem 2. *A q.i.p. space X is uniformly smooth.*

Proof. It has been proved, (cf. [1, p.36]), that a normed space X is uniformly smooth if and only if there exists a support mapping $x \mapsto f_x$ which is norm-norm uniformly continuous from $S(X)$ to $S(X^*)$.

So, it suffices to show that the support mapping $x \mapsto g(x, \cdot)$ is norm-norm uniformly continuous from $S(X)$ to $S(X^*)$. For this purpose, let x, y, $t \in S(X)$.

Then we have from (10):

$$g(x, t) + g(t, x) = \frac{1}{8}\left(\|x+t\|^4 - \|x-t\|^4\right),$$
$$g(y, t) + g(t, y) = \frac{1}{8}\left(\|y+t\|^4 - \|y-t\|^4\right),$$

and hence,

$$g(x, t) - g(y, t) = \frac{1}{8}\left[\left(\|x+t\|^4 - \|y+t\|^4\right) + \left(\|y-t\|^4 - \|x-t\|^4\right)\right] - g(t, x-y). \tag{15}$$

This implies

$$|g(x, t) - g(y, t)| \leq \frac{1}{8}[32\|x-y\| + 32\|x-y\|] + \|x-y\| = 9\|x-y\|,$$

and so

$$\|g(x, \cdot) - g(y, \cdot)\| \leq 9\|x-y\|.$$

From this, we conclude that the mapping $x \mapsto g(x, \cdot)$ is norm-norm uniformly continuous from $S(X)$ to $S(X^*)$.

Corollary 2 *If X is a q.i.p. space, then the norm of X is uniformly Fréchet differentiable.*

Proof. See Theorem 1, p.36 [1].

Corollary 3. *If X is a q.i.p. space, then X^* is uniformly convex.*

Proof. See Theorem 1, p.36 [1].

Corollary 4. *A complete q.i.p. space X is reflexive.*

Proof. See Corollary 2, p.38 [1].

It is well known that uniform convexity implies strict convexity.

Theorem 3. *A q.i.p. space X is very smooth.*

Proof. From Definition 5 and Theorem 1 it suffices to prove that support mapping $x \mapsto g(x, \cdot)$ is norm to week continuous from $S(X)$ to $S(X^*)$.

Let $(x_n) \subset S(X)$ and $x_0 \in S(X)$. From (15) we have:

$$g(x_n, x) - g(x_0, x) = \frac{1}{8}\left[\left(\|x_n + x\|^4 - \|x_0 + x\|^4\right) + \left(\|x_0 - x\|^4 - \|x_n - x\|^4\right)\right] - g(x, x_n - x_0),$$

for $(x \in X)$.

Therefore, it follows that

$$|g(x_n, x) - g(x_0, x)| \leq \left[\|x\| + 8(1 + \|x\|)^3\right] \|x_n - x_0\| \qquad (x \in X).$$

By this inequality and Corollary 4, we conclude that

$$g(x_n, \cdot) \xrightarrow{w} g(x_0, \cdot).$$

Corollary 5. *If X is a complete q.i.p. space, then, for each subspace Y of X the density characters of Y and of Y^* coincide.*

Proof. See Theorem 2, p.31 [1].

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