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On spherically symmetric stellar models in general relativity

By

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1. Introduction

Recently H. R. Beyer [1] and S. S. Lin [7] investigated the linearized equation for small perturbations near spherically symmetric equilibria of a self-gravitating gas in the Newtonian i. e., non-relativistic theory of stellar structure. For a long time astrophysicists had been believing that the associated differential equation reduces to a Strum-Liouville eigenvalue problem, already from the Eddington's work in 1919, without any mathematical proof. But this was not obvious, since the coefficients of the equation are quite singular. H. R. Beyer has closed this gap in [1], and independently S. S. Lin has in [7] in a wider context. The aim of this article is to carry their results to the corresponding problem in general relativity.

In order to do that, we must investigate the qualitative properties of the structure of solutions for the equation which governs the spherically symmetric static configurations of self-gravitating gas in general relativity, since the cefficients of the linearized equation for small perturbations are determined by these equilibrium configurations. The equation which governs spherically symmetric equilibria in general relativity is formulated by J. P. Oppenheimer and G. M. Volkoff in [11], 1939, and this has been familiar in astrophysical textbooks as the TOV equation. But still now no qualitative analysis of the structure of this equation can be found in mathematical literatures. Even in the non-relativistic theory, in which the equation for equilibria under the equation of state $p = Const.\rho^{\gamma}$ reduces to the wellknown Lane-Emden equation, no fully mathematically rigorous treatments were found untill in 1972/73 the work [6] by D. D. Joseph and T. S. Lundgren appeared, although many numerical computations and good intuitive analogies had been obtained as found in the Chandrasekhar's famous textbook [2]. Moreover when the relation $p = Const.\rho^{\gamma}$ holds only in asymptotic sense, for example in the equation of state for white dwarfs, there were no mathematical proof of the elementary fact that the radius of a solution obtained by the shooting methods is finite untill the work [8] by the author, 1984. In general relativity, the author did not know any mathematically rigorous study of the structure of the

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TOV equation, when he began to study the problem. However, after writing the first version of this article, the author received letter from Professor B. G. Schmidt of Max-Planck-Insitute fuer Gravitationsphysik of Potsdam, in which he kindly drew the attention of the author to the paper [13], 1990, by A. D. Rendall and B. G. Schmidt. Therefore the first part of this study shoud be regarded as another approach to the problem studied in [13].

Of course the author does not deny importance of approximate numerical computations and rough analogies using quite plausible physical intuitions. If one computes the density distribution $\rho(\mathrm{r})$ by the shooting method from $r=+0$ to the right, he could consider that he found a finite radius, say, the surface of the star, when the out-put of the value of ρ from his computer turns out to be sufficiently small or the computer downflows. Or, if $p\sim Const.\rho^\gamma$ asymptotically, one could believe that the qualitative structure of the equation should be the same to that of the case in which $p=Const.\rho^\gamma$ exactly. In fact often the radius is finite for the asymptotic case if it is for the exact case. (This analogy does hold when$\gamma>4/3$, but the situation is not so simple when $6/5<\gamma\leq 4/3$.) But in a mathematically rigorous sense these computations or analogies do not exclude the possibility that $\rho(r)$ is quite small but remains still positive for all $r\to+\infty$.

For this reason the author devotes the first section to prove logically that the radius of an equilibria is finite for many important cases. *When presented with the results of a large quantum calculation, Eugene P. Wigner once said: "It is nice to know that the computer understands the problem. But I would like to understand it, too."* (Physics Today, July 1993, p.38). The author considers that a logical proof, if possible, can provide a direct route to good understanding of the essential point of the problem.

In the following sections we will apply the argument of H. R. Beyer and S. S. Lin to the linearized theory of general-relativistic stellar pulsations. This study requires careful observation of the qualitative structure of the TOV equation, and leads us to a sufficient condition in order that the spectrum of the associated differential operator is purely discrete and a condition for the stability.

The author expects that astrophysicists consider that the present handmade article may provide a sound logical back-ground supporting their more realistic numerical simulations which become more and more largescaled day by day. Of course astrophysicists can reject this article by the reason that it does not contain abything new to them except for mathematical rigor. But the author would like to write this article for the sake of *the honor of the human spirit*.

2. Equilibrium configurations

In general relativity the spherically symmetric equilibrium of a self-gravitating gas is governed by the Tolman-Oppenheimer-Volkoff equation:

$$\frac{dm}{dr}=4\pi r^2\rho,$$
$$\frac{dp}{dr}=-\,(\rho+p/c^2)\frac{G\,(m+4\pi r^3p/c^2)}{r^2(1-2Gm/c^2r)}.$$

For the derivation of this equation, see [14], [11], or Chapter 23 of [9]. Here ρ is the total mass density including internal energy, p the pressure, and $r(>0)$ is a suitable radial coordinate. G is the gravitational constant and c the speed of light ; they are positive constants. The metric coefficients of the space-time are given by the line element

$$ds^2=e^{\nu(r)}c^2dt^2-e^{\lambda(r)}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2),$$

where

$$e^{-\lambda}=1-\frac{2Gm}{c^2r},$$
$$\frac{d\nu}{dr}=-\frac{2}{p+\rho c^2}\frac{dp}{dr}$$

In this article we assume that p is a given function of ρ. We make the assumption:

$(A0)$ $p=p(\rho)$ *is a sufficiently smooth function of* $\rho>0$ *such that* $p>0$ *and* $\frac{dp}{d\rho}>0$ *for* $\rho>0$ *and* $p\to0$ *as* $\rho\to0$.

Given the central density $\rho_0>0$ and $p_0=p(\rho_0)>0$, we must solve the intial value problem:

$$\frac{dm}{dr}=4\pi r^2\rho,$$
$$\frac{dp}{dr}=-\,(\rho+p/c^2)\frac{G\,(m+4\pi r^3p/c^2)}{r^2(1-2Gm/c^2r)},$$
$$m=0,\ p=p_0 \quad at\ r=0, \tag{1}$$

in $0<r\leq\delta$, δ being a sufficiently small positive number. Putting $q=m/r^3$, we reduce the problem (1) to the integral equation

$$q=\frac{1}{r^3}\int_0^r 4\pi r^2\rho dr,$$
$$p=p_0-\int_0^r(\rho+p/c^2)\frac{G\,(q+4\pi p/c^2)}{1-2Gqr^2/c^2}rdr. \tag{2}$$

We consider this integral equation in the functional set $\mathscr{F}=\{q\in C[0,\ \delta],\ p\in C[0,\ \delta]\,|\,0\leq q\leq\frac{4\pi}{3}\rho_0,\ p_0/2\leq p\leq p_0\}$. Denote by $\tilde q$, $\tilde p$ the right-hand side of (2) for $(q,\ p)\in\mathscr{F}$. Let $2G\frac{4\pi}{3}\rho_0\delta^2/c^2\leq1/2$. Then it is easy to see that $(\tilde q,\ \tilde p)$ belongs to $\mathscr{F}$ for any $(q,\ p)\in\mathscr{F}$, choosing δ sufficiently small. Moreover we have

$$\|\tilde{q}_1-\tilde{q}_2\|\leq\frac{4\pi}{3}\|\rho_1-\rho_2\|\leq L\|p_1-p_2\|,$$
$$\|\tilde{p}_1-\tilde{p}_2\|\leq L\delta^2(\|q_1-q_2\|+\|p_1-p_2\|)$$

for (q_2, p_2), $(q_1, p_1)\in\mathcal{F}$ with a sufficiently large L. Here $\|\cdot\|$ denotes the sup norm. Let us introduce a distance

$$d((q_1, p_1), (q_2, p_2))=\frac{1}{2L}\|q_1-q_2\|+\|p_1-p_2\|$$

Then

$$d((\tilde{q}_1, \tilde{p}_1), (\tilde{q}_2, \tilde{p}_2))\leq\frac{1}{2}\|p_1-p_2\|+L\delta^2(\|q_1-q_2\|+\|p_1-p_2\|)$$
$$\leq(\frac{1}{2}+2L^2\delta^2)d((q_1, p_1), (q_2, p_2)).$$

Put δ so small that $\frac{1}{2}+2L^2\delta^2<1$. Then the mapping $(q, p)\mapsto(\tilde{q}, \tilde{p})$ is a contraction $\mathcal{F}$ into $\mathcal{F}$ with respect to the distance d. Therefore by the fixed point theorem (2) has a unique solution in $\mathcal{F}$ which gives a solution of (1). We note that the solution $(m(r), p(r))$, $0\leq r\leq\delta$, is sufficiently smooth and

$$m=\frac{4\pi}{3}\rho_0 r^3+\mathcal{O}(r^5),$$
$$p=p_0-(p_0+p_0/c^2)G(4\pi\rho_0/3+4\pi p_0/c^2)\frac{r^2}{2}+\mathcal{O}(r^4),$$
$$\frac{dp}{dr}=-(\rho_0+p_0/c^2)G(4\pi\rho_0/3+4\pi p_0/c^2)r+\mathcal{O}(r^3) \quad (3)$$

as $r\to 0$.

Let us prolong the solution $m(r)$, $p(r)$ as long as possible in the domain $\{(r, m, p)|0<r<+\infty, 0<p<+\infty, 2Gm/c^2r<1\}$. Let $(0, R)$, where $R\leq+\infty$, be the right maximal interval of existence for this solution.

We are going to find sufficient condition for $R<+\infty$. To do that we make the following assumption:

(*A1*) $$\frac{\rho}{p}\frac{dp}{d\rho}=\gamma+\mathcal{O}(\rho^{\gamma-1})/ \textit{ as } \rho\to+0 \textit{ with } 4/3<\gamma<2.$$

Under the assumption $(A1)$, there is a positive constant a such that

$$p=a\rho^\gamma(1+\mathcal{O}(\rho^{\gamma-1})) \quad as\ \rho\to+0. \quad (4)$$

We introduce a new variable $\eta=\eta(\rho)$ by

$$\eta=\int_0^{p(\rho)}\frac{dp}{\rho+p/c}. \quad (5)$$

Then

$$\eta=\frac{a\gamma}{\gamma-1}\rho^{\gamma-1}(1+\mathcal{O}(\rho^{\gamma-1})) \quad as\ \rho\to+0. \quad (6)$$

and

$$\rho=\left(\frac{\gamma-1}{a\gamma}\right)^{\frac{1}{\gamma-1}}\eta^{\frac{1}{\gamma-1}}(1+\mathcal{O}(\eta)) \quad as\ \eta\rightarrow+0 \tag{7}$$

Now let us observe the corresponding solution $\eta=\eta(r)$, $0<r<R$, which satisfies

$$r\frac{d\eta}{dr}=-\frac{G(m+4\pi r^3 p/c^2)}{r(1-2Gm/c^2r)} \tag{8}$$

Since $dp/dr<0$, $p(r)$, $\rho(r)$, $\eta(r)$ are motone decreasing. We claim

Lemma. $\eta\rightarrow 0$ *as* $r\rightarrow R$.

Proof. We prove this by reductio ad absurdum: Suppose $\eta(r)\geq\delta>0$. Case I. Assume $R=+\infty$. Since $\rho(r)\downarrow\delta_1>0$, we see $m(r)\geq(4\pi/3)\delta_1 r^3$ for $0<r+\infty$. This is impossible, since $2Gm(r)<1$. Case II. Assume $R<+\infty$. It should be the case that $2Gm(r)/c^2r\rightarrow 1$ *as* $r\rightarrow R$, since otherwise the solution could be prolonged to the right beyond R. Then, since $1-\frac{2Gm(r)}{c^2r}\in C^1[0,R]$, we see

$$1-\frac{2Gm(r)}{c^2r}\leq C_1(R-r),$$

for $R-\delta\leq r<R$. This implies

$$\frac{d\eta}{dr}=-\frac{G(m+4\pi r^3 p/c^2)}{r^2(1-2Gm/c^2r)}\leq-\frac{C_2}{C_1}\frac{1}{R-r},$$

where C_2 is a constant near GM/R^2 with $M=m(R)=\int_0^R 4\pi\rho r^2\,dr$, so that

$$\begin{aligned}\eta(r)&\leq\eta(R-\delta)-\frac{C_2}{C_1}\int_{R-\delta}^{r}\frac{dr}{R-r}\\&=\eta(R-\delta)+\frac{C_2}{C_1}\log(R-r)-\frac{C_2}{C_1}\log\delta\\&\rightarrow-\infty\end{aligned}$$

as $r\rightarrow R$. This is a contradiction. This completes the proof.

Next, we introduce the new variables

$$x=\frac{m}{\eta r},\ y=4\pi r^2\frac{\rho^2}{p}. \tag{9}$$

Then the equation turns out to be

$$r\frac{dx}{dr}=\alpha(\eta)y-x+x\frac{G(x+\omega(\eta)y/c^2)}{1-2G\eta x/c^2},$$

$$r\frac{dy}{dr}=y\left(2-\beta(\eta)\frac{G(x+\omega(\eta)y/c^2)}{1-2G\eta x/c^2}\right),$$

$$r\frac{d\eta}{dr}=-\eta\frac{G(x+\omega(\eta)y/c^2)}{1-2G\eta x/c^2}. \tag{10}$$

Here

$$\alpha(\eta)=\frac{p}{\eta\rho}=\frac{\gamma-1}{\gamma}+\mathcal{O}(\eta),$$

$$\beta(\eta)=(\frac{2}{\eta}\frac{d\rho}{dp}-\frac{1}{p})(\rho+p/c^2)\eta=\frac{2-\gamma}{\gamma-1}+\mathcal{O}(\eta),$$

$$\omega(\eta)=\frac{p^2}{\eta\rho^2}=\mathcal{O}(\eta). \tag{11}$$

Now we are ready to prove the following theorem.

Theorem 1. *Under the assumptions* $(A0)$ $(A1)$, *there exists a* finite R *for each central density* ρ_0 *such that* $\rho(r)>0$ *for* $0\leq r<R$ *and* $\rho(r)\downarrow 0$ *as* $r\uparrow R$.

Proof. Assuming $R=+\infty$, we lead a contradiction.

At the moment suppose $x(r)\leq 1/G$ for all $r\in(0,+\infty)$. Put $\alpha_0=\inf_{0<\eta\leq\eta(0)}\alpha(\eta)$. Since $\alpha(\eta)>0$ for $\eta>0$ and $\alpha\to(\gamma-1)/\gamma$ as $\eta\to 0$, we see $\alpha_0>0$. Choose r_0 such that $\beta(\eta(r))>0$ for all $r\geq r_0$; it is possible since $\beta(\eta(r))\to(2-\gamma)/(\gamma-1)>0$ as $r\to+\infty$. Let us consider the function $V(x,y)=K+Kx-y$, where $K$is a costant such that $K>3/\alpha_0$ and $V(x(r_0),y(r_0))>0$. We are going to show that $V(x(r),y(r))>0$ for all $r\geq r_0$.

Otherwise there would exist the first $r_1>r_0$ such that $V(x(r_1),y(r_1))=0$, $V(x(r),y(r))>0$ for $r_1>r>r_0$. Then, at $r=r_1$, we have

$$\begin{aligned}r\frac{d}{dr}V(x,y)&=Kr\frac{dx}{dr}-r\frac{dy}{dr}\\&=K(\alpha y-x+x\frac{G(x+\omega y/c^2)}{1-Gx\eta/c^2})-y(2-\beta\frac{G(x+\omega y/c^2)}{1-Gx\eta/c^2})\\&>(K\alpha_0-2)y-Kx\\&=(K\alpha_0-2)y(r_1)+K-y(r_1)=(K\alpha_0-3)y(r_1)+K>0.\end{aligned}$$

This is a contradiction. Therefore $V(x(r),y(r))>0$ for all $r\geq r_0$. Since we supposed $x(r)\leq 1/G$, we know $y(r)\leq K(1+1/G)$, that is, $y(r)$ is bounded. Then we see

$$\begin{aligned}r\frac{dy}{dr}&=y(2-\beta\frac{G(x+\omega y/c^2)}{1-2G\eta x/c^2})\\&\geq\frac{2-\beta(0)}{2}y=\frac{3\gamma-4}{2(\gamma-1)}y\end{aligned}$$

for $r\geq r_1$, r_1 being sufficiently large. This implies

$$y(r)\geq y(r_1)(r/r_1)^{\frac{3\gamma-4}{2(\gamma-1)}}\to+\infty.$$

This is a contradiction. Therefore there should exist r_2 such that $x(r_2)>1/G$.

Then, by observing the differential inequality

$$r\frac{dx}{dr}\geq -x+Gx^2,$$

it is clear that $x(r)>1/G$ for all $r\geq r_2$ and that $x(r)$ blows up before r reaches $r_2\exp(1/G(x(r_2)-1/G))$, that is, $R<+\infty$. This completes the proof.

Remark. An important by-product of the above proof is the observation: *Under* $(A0)$, *if we find* R_0 *such that* $x(R_0)>1/G$, *then we can claim* R *is finite and enjoys the estimate* $R<R_0\exp[1/(Gx(R_0)-1)]$. This is convenient to use for numerical computations, since $x>1/G$ is a generic condition which can be checked by a rough computation, and we can consider $R\simeq R_0$ when $x(R_0)\gg 1$.

Now we know R is finite. Therefore $m(r)\uparrow M$ and $1-2GM/c^2r\rightarrow 1-2GM/c^2R$ as $r\uparrow R$. It is easy to show that $1-2GM/c^2R>0$. Otherwise $\eta\downarrow -\infty$ as discussed in the proof of *Lemma*. Thus the limit $d\eta\ dr\rightarrow -GM/R^2(1-2GM/c^2R)$ is finitely definite, that is, $\eta\in C^1[0,R]$. From the equation, we see

$$\begin{aligned}
\eta&=\frac{GM}{R^2(1-2GM/c^2R)}(R-r)(1+\mathcal{O}(R-r)),\\
\rho&=C(R-r)^{\frac{1}{\gamma-1}}(1+\mathcal{O}(R-r)),\\
p&=aC^{\gamma}(R-r)^{\frac{\gamma}{\gamma-1}}(1+\mathcal{O}(R-r)),\\
\frac{dp}{dr}&=-\frac{\gamma}{\gamma-1}aC^{\gamma}(R-r)^{\frac{1}{\gamma-1}}(1+\mathcal{O}(R-r)),
\end{aligned}\tag{12}$$

where

$$C=\left[\frac{\gamma-1}{a\gamma}\frac{GM}{R^2(1-2GM/c^2R)}\right]^{\frac{1}{\gamma-1}}.$$

It is clear that $m(r)\in C^1[0,R]$. Moreover, $m(r)\in C^2[0,R]$, for

$$\frac{d^2m}{dr^2}=8\pi r\rho+4\pi r^2\frac{d\rho}{dr}=\mathcal{O}(\rho)+\mathcal{O}(\rho^{\frac{2-\gamma}{\gamma-1}}).$$

In the same way we see $\eta(r)\in C^2[0,R]$.

Summing up, if we put

$$\begin{aligned}
e^{\nu}&=(1-2GM/c^2R)e^{-2\eta/c^2} && (0<r<R)\\
&=1-2GM/c^2r && (R\leq r),\\
e^{\lambda}&=\frac{1}{1-2Gm(r)/c^2r} && (0<r<R)\\
&=\frac{1}{1-2GM/c^2r} && (R\leq r),
\end{aligned}$$

these metric coefficients are twice continuously differentiable for all $r>0$. Here the metric in $r>R$ is the well-known Schwarzschild's exterior metric.

Examples. The equation of state for white dwarfs is given by

$$p=K_1c^5f(\zeta),\ \rho=K_2c^3(\zeta_2-1)^{3/2},$$

where

$$f(\zeta)=\zeta(\zeta^2-1)^{1/2}+3\log(\zeta+(\zeta^2-1)^{1/2}).$$

It is easily seen that the assumptions $(A0)$ $(A1)$ hold with $\gamma=5/3$. In fact

$$p=\frac{8}{5}K_1K_2^{-5/3}\rho^{5/3}(1+[K_2^{-2/3}\rho^{2/3}/c_1^2]_1),$$

where $[\cdot]_1$ stands for a convergent power series starting from the first order term.

The equation of statefor neutron stars is given by

$$p=K_3c^5f(\zeta),\ \rho=K_3c^3g(\zeta),$$

where

$$g(\zeta)=8\zeta(\zeta^2-1)^{3/2}-f(\zeta).$$

Then $(A0)$ $(A1)$ hold with $\gamma=5/3$ and

$$p=\frac{1}{20}K_3^{-2/3}\rho^{5/3}(1+[K_3^{-2/3}\rho^{2/3}/c^2]_1).$$

Note that $p\sim\frac{c^2}{3}\rho$ as $\rho\to+\infty$. It is said that in unpublished work at Cambridge in 1935 John von Neumann integrated the equation of equilibrium for the case $p=(c^2/3)\rho$. He should have found that $R=M=+\infty$ for this case. A proof of this fact is given in *Appendix*.

Remark Let us compare the conclusion of Theorem 1 with the result of A. D. Rendall and.B. G. Schmidt, [13].

They consider the problem under the equation of state of the form

$$p(\rho)=K\rho^\gamma(1+\varepsilon g(\rho^{\gamma-1})\rho^{\gamma-1}),$$

where K is a positive constant and g is a smooth function on R. Their result is that, for $6/5<\gamma<2$ and for a fixed central density ρ_0, the stellar model has the finite radius provided that ε and K are stufficiently small ([13], Theorem 4). They proves this result by considering the problem as a perturbation from the Newtonian model with $p=K\rho^\gamma$, which is reduced to the Lane-Emden equation. Therefore the restriction that ϵ and K are small is crucial, and this restriction can be replaced by that ρ_0 and $1/c$ are sufficiently small. If we consider the wider range $(6/5,\ 2)$ for γ, such a restriction may be inevitable. In contrast with that, we restrict γ in $(4/3,\ 2)$, but suppose no restriction on the smallness of ρ_0 and $1/c$.

3. Newtonian approximation

Let us observe the behavior of solutions for fixed ρ_0 as $c\rightarrow+\infty$, under the following assumption on the equation of state which depends upon c:

$(A1)'$. $\frac{\rho}{p}\frac{dp}{d\rho}=\gamma+\Omega(\rho^{\gamma-1}, 1/c^2)$ *as* $\rho\rightarrow+0$ *with* $4/3<\gamma<2$, *where* $\Omega(\mu, \varepsilon)$ *is continuous in* $\mu\geq0$, $\varepsilon\geq0$, $\Omega(0, \varepsilon)=0$ *and Lipschitz continuous with respect to* $\mu\geq0$ *uniformly for* $0\leq\varepsilon\leq1/c^2{}_0$.

We denote the solutions by $m^{(c)}(r)$, $p^{(c)}(r)$, etc., for each $c\geq c_0$, and $m^{(\infty)}(r)$, $p^{(\infty)}(r)$ stands for the solution of the limiting Newtonian equation

$$\frac{dm}{dr}=4\pi r^2\rho,$$
$$\frac{dp}{dr}=-\rho\frac{Gm}{r^2}.$$

Now we consider $q=m/r^3$ and $w=p/\rho$. Then the equation is

$$r\frac{dq}{dt}=4\pi\rho-3q,$$
$$r\frac{dw}{dr}=(1-\frac{p}{\rho}\frac{d\rho}{dp})(-1-\frac{w}{c^2})\frac{G(q+4\pi qw/c^2)}{1-2Gqr^2/c^2}r^2$$

The initial condition at $r=0$ is

$$q=\frac{4\pi}{3}\rho_0,\quad w=w_0=w(\rho_0).$$

We note that $\rho=(1/a)^{\frac{1}{\gamma-1}}w^{\frac{1}{\gamma-1}}(1+\Omega_1(w, 1/c^2))$ and $\frac{p}{\rho}\frac{d\rho}{dp}=\frac{1}{\gamma}+\Omega_2(w, 1/c^2)$, Ω_1, Ω_2 being functions with the same properties as Ω in $(A1)'$. Therefore the right hand side of the equation is Lipschitz continuous re $q<c^2/2Gr^2$, $0\leq w$ uniformly for $c\geq c_0$. We extend the solution beyond $R=R^{(c)}$ by putting $q^{(c)}(r)=M^{(c)}/r^3$ and $w^{(c)}(r)=-c^2[1-(\frac{1-2GM^{(c)}/c^2R^{(c)}}{1-2GM^{(c)}/c^2r})^{\frac{\gamma-1}{2\gamma}}]$ for $r\geq R^{(c)}$. By the method of construction of local solutions at $r=+0$, it is clear that for a sufficiently small δ, $q^{(c)}(r)\rightarrow q^{(\infty)}(r)$, $w^{(c)}(r)\rightarrow w^{(\infty)}(r)$ uniformy on $0\leq r\leq\delta$. Hence, by Theorem 4.3 of Chap. 1 of [3], the convergence is uniform on each bounded subinterval of $[\delta, +\infty)$.

Let us estimate $R^{(c)}$. To do that we observe the equation (10). Under the assumption $(A1)'$ the right hand side of (10) is locally Lipschitz continuous in x, y and $\eta\geq0$. We can assume $x^{(c)}(\delta)\rightarrow x^{(\infty)}(\delta)$, $y^{(c)}(\delta)\rightarrow y^{(\infty)}(\delta)$, $\eta^{(c)}(\delta)\rightarrow\eta^{(\infty)}(\delta)$. Since $x^{(\infty)}\rightarrow+\infty$ as $r\rightarrow R^{(\infty)}$, there exists $r_1\in(\delta, R^{(\infty)})$ such that $x^{(\infty)}(r_1)\geq3/G$. They by the continuity argument applied to (10) we see that $x^{(c)}\geq2/G$ for large c. Then from the differential inequality

$$r\frac{dx}{dr}\geq-x+Gx^2$$

is follows that $R^{(c)} \leq r_1 \exp[1/(Gx^{(c)}(r_1)-1)] \leq r_1 e$, that is $R^{(c)}$ is bounded as $c \to \infty$. Using this, we can see $M^{(c)} \to M^{(\infty)}$ and $R^{(c)} \to R^{(\infty)}$.

Theorem. 2. *Under the assumptions* $(A0)$ $(A1)'$, *there exists a constant* C *such that*

$$p^{(c)}/\rho^{(c)},\ r^3/m^{(c)}(r),\ m^{(c)}(r)/r \leq C \tag{13}$$

for $0 \leq r \leq R^{(c)}$ *and* $c \geq c_0$.

4. Spectrum of spherically symmetric pulsations

The linearized equation which governs a spherically symmetric perturbation from the equilibrium is

$$\frac{\partial^2 \xi}{\partial t^2} + \mathscr{A}\xi = 0, \tag{14}$$

whith

$$\mathscr{A}\xi = \frac{1}{e^{(3\lambda+\nu)/2}(\rho+p/c^2)r^4}\Big[-\frac{\partial}{\partial r}\Big(\Gamma_1 p r^4 e^{(3\nu+\lambda)/2}\frac{\partial \xi}{\partial r}\Big) + \Big\{r^3\big(4e^{(3\nu+\lambda)/2}p' - (3\Gamma_1 p e^{(3\nu+\lambda)/2})'\big)$$
$$+\frac{1}{c^2}r^4 e^{(3\nu+\lambda)/2}\Big(8\pi G e^{\lambda} p(\rho+p/c^2) - \frac{(p')^2}{\rho+p/c^2}\Big)\Big\}\xi\Big]. \tag{15}$$

Here

$$\xi = e^{-\nu/2}\frac{\Delta r}{r} \tag{16}$$

with Δr being the change of the r-coordinate of a given fluid element, and the prime stands for the derivative with respect to r. We keep the assumptions $(A0)$ $(A1)$ and fix a equilibrium configuration with $\rho(r)$, $p(r)$, $\nu(r)$, $\lambda(r)$, $0<r<R$, which evaluates the coefficients of $\mathscr{A}$. So, r is confined in the interval $(0, R)$. The quantity $\Gamma_1 = \Gamma_1(\rho)$ is defined by

$$\Gamma_1 = \frac{\rho + p/c^2}{p}\frac{dp}{d\rho}.$$

By $(A0)$ Γ_1 is a sufficiently smooth function of $\rho > 0$. For the derivation of this equation, see [10] or Chapter 26 of [9].

From the formal differential operator $\mathscr{A}$ we define the operator A_0 in the Hilbert space X of all square integrable functions on $(0, R)$ with respect to the measure $e^{(3\nu+\lambda)/2}(\rho+p/c^2)R^4 dr$, while the domain of A_0 is $C_0^2(0, R)$, the set of all compactly supported C^2-functions in $(0, R)$. In the next step we choose a self-adjoint extention A of A_0 in X. The following discussion will be carried out in the same way as in [1] and [7].

For the simplicity we make the following assumption.

$(A2)$ *As* $\rho \to 0$, *we have* $\frac{p}{\Gamma_1}\frac{d\Gamma_1}{dp}$, $p\frac{d}{dp}\left(\frac{p}{\Gamma_1}\frac{d\Gamma_1}{dp}\right)$, $p\frac{d}{dp}\left(\frac{p}{\rho}\frac{d\rho}{dp}\right) \to 0$.

Following the argument of [1], we transform the equation (14) to the standard form. This is done by the change of variables

$$h=\int_0^r \left(\frac{\rho+p/c^2}{\Gamma_1 p}\right)^{1/2} e^{(\lambda-\nu)/2} dr,$$

$$z=(\Gamma_1 p(\rho+p/c^2))^{1/4} r^2 e^{(\lambda+\nu)/2}\xi. \tag{17}$$

Then (14) reduces to the standard form

$$z_{tt}-z_{hh}+Q(h)z=0, \tag{18}$$

where $Q=Q(h)$ is given by

$$Q=\frac{\Gamma_1 p}{\rho+p/c^2}e^{\nu-\lambda}(Q_0+Q_1+Q_2),$$

$$Q_0=\frac{1}{\Gamma_1 pr}\Big[(3\Gamma_1-4+3p\frac{d\Gamma_1}{dp})(-p')-3\Gamma_1 p\frac{3\nu'+\lambda'}{2}+$$

$$+\frac{r}{c^2}(8\pi Ge^{\lambda}p(\rho+p/c^2)-\frac{(p')^2}{\rho+p/c^2})\Big],$$

$$Q_1=\frac{1}{16}\Big[\frac{8}{r}+2(\lambda'+\nu')+\frac{\rho'+p'/c^2}{\rho+p/c^2}+(1+\frac{p}{\Gamma_1}\frac{d\Gamma_1}{dp})\frac{p'}{p}\Big]\times$$

$$\times\Big[\frac{8}{r}+4\nu'-\frac{\rho'+p'/c^2}{\rho+p/c^2}+3(1+\frac{p}{\Gamma_1}\frac{d\Gamma_1}{dp})\frac{p'}{p}\Big],$$

$$Q_2=\frac{1}{4}\Big[-\frac{8}{r^2}+2(\lambda''+\nu'')+\frac{\rho''+p''/c^2}{\rho+p/c^2}-(\frac{\rho'+p'/c^2}{\rho+p/c^2})^2+$$

$$+p\frac{d}{dp}(\frac{p}{\Gamma_1}\frac{d\Gamma_1}{dp})(\frac{p'}{p})^2+(1+\frac{p}{\Gamma_1}\frac{d\Gamma_1}{dp})(\frac{p''}{p}-(\frac{p'}{p})^2)\Big].$$

Here, since

$$\left(\frac{\rho+p/c^2}{\Gamma_1 p}\right)^{1/2} e^{(\lambda-\nu)/2}\sim Const.\ \rho^{\frac{1-\gamma}{2}}\sim Const.(R-r)^{-1/2}$$

as $r\to R-0$, we see that

$$H=\int_0^R \left(\frac{\rho+p/c^2}{\Gamma_1 p}\right)^{1/2} e^{(\lambda-\nu)/2}\, dr$$

is finite and the r-interval $(0, R)$ is mapped onto the h-interval $(0, H)$.
We are going to observe the asymptotic behaviors of $Q(h)$ *as* $h\to 0$ and $h\to H$.

($h\to 0$): It is easy to see $Q_0=O(1)$, $Q_1\sim 4/r^2$, $Q^2\sim -2/r^2$. Therefore

$$Q\sim\frac{\Gamma_1(\rho_0)p_0}{\rho_0+p_0/c^2}(e^{\nu-\lambda})_{r=0}\frac{2}{r^2}.$$

On the other hand

$$h\sim\left(\frac{\rho_0+p_0/c^2}{\Gamma_1(\rho_0)p_0}\right)^{1/2}(e^{(\lambda-\nu)/2})_{r=0}r.$$

Thus we have $Q\sim\frac{2}{h^2}$. Note $2>3/4$.

$(h \to H)$: Recall $(A2)$ and note that $\Gamma_1 \to \gamma$. Moreover $p'/p \sim -\frac{\gamma}{\gamma-1}\frac{1}{R-r}$. Then we can see that $Q_0 = o((p'/p)^2)$, $Q_1 \sim \frac{1}{16}(\frac{\gamma}{1}+1)(-\frac{1}{\gamma}+3)(p'/p)^2$, $Q^2 \sim \frac{1}{4}(\frac{1}{\gamma^2}-1)(p'/p)^2$. Here we have used the following calculations:

$$\frac{p''}{p} = \left(\frac{p}{\rho}\frac{d\rho}{dp} + \mathcal{O}\left(\frac{p}{\rho}\right)\right)\left(\frac{p'}{p}\right)^2 + \mathcal{O}\left(\frac{p'}{p}\right),$$

$$\frac{\rho''}{\rho} = \left(\frac{pd}{dp}\left(\frac{p}{\rho}\frac{d\rho}{dp}\right) - \frac{p}{\rho}\frac{d\rho}{dp} + 2\left(\frac{p}{\rho}\frac{d\rho}{dp}\right)^2 + \mathcal{O}\left(\frac{p}{\rho}\right)\right)\left(\frac{p'}{p}\right)^2 + \mathcal{O}\left(\frac{p'}{p}\right)$$

Therefore

$$Q \sim \gamma\frac{p}{\rho}(e^{\nu-\lambda})_{r=R}\frac{1}{16}\frac{(3-\gamma)(1+\gamma)}{\gamma^2}\left(\frac{p'}{p}\right)^2$$

On the other hand, if $\rho \sim C(R-r)^{\frac{1}{\gamma-1}}$ and $p \sim a\rho^\gamma$, then

$$H-h \sim \frac{1}{(a\gamma)^{1/2}}C^{\frac{1-\gamma}{2}}2(e^{(\lambda-\nu)/2})_{r=R}(R-r)^{1/2}.$$

Thus we have

$$Q \sim \frac{1}{4}\frac{(3-\gamma)(1+\gamma)}{(\gamma-1)^2}\frac{1}{(H-h)^2}.$$

Note that $\frac{1}{4}\frac{(3-\gamma)(1+\gamma)}{(\gamma-1)^2} > 3/4$ for $\gamma < 2$.

As the conclusion, the operator associated with $-\frac{d^2}{dh^2}+Q$ is in the limit point case both at 0 and H by Theorem XIII. 6.23 of [4] and Theorem X.10 of [12], and is essentially self-adjoint ([12]). Furthermore by [5] it follows that the Friedrichs extension of this operator has a purely discrete spectrum. So, we have:

Theorem. 3. *Under* $(A0)$ $(A1)$ $(A2)$, *we can take* A_0^* *as* A, *which is the Friedrichs extension with a purely discrete spectum.*

5. Stability

Now we are going to discuss the stability of the equilibrium under the assumptions $(A0)$ $(A1)'$ $(A2)$.

Take $\xi(r) = 1$, which is easily seen to be in the domain of the operator $A = A_0^*$. Then we have

$$A\xi = e^{-\lambda+\nu}\frac{1}{\rho+p/c^2}\frac{p}{r}\times$$

$$\left[(3\Gamma_1 - 4 + 3p\frac{d\Gamma_1}{dp})\left(-\frac{p'}{p}\right) - 3\Gamma_1\frac{3\nu'+\lambda'}{2} + \frac{1}{c^2}r\left(8\pi Ge^\lambda(\rho+p/c^2) - \frac{(p')^2}{p(\rho+p/c^2)}\right)\right]$$

$$\geq e^{-\lambda+\nu}\frac{1}{r^3}\frac{G(m+4\pi r^3p/c^2)}{1-2Gm/c^2r}\left[(3\Gamma_1 - 4 + 3p\frac{d\Gamma_1}{dp}) - \frac{1}{c^2}F\right],$$

where

$$F=9\Gamma_1\frac{p}{\rho+p/c^2}+3\Gamma_1\frac{p\rho}{\rho+p/c^2}4\pi\frac{1}{m/r^3+4\pi p/c^2}+r^2(\rho+p/c^2)G\frac{m/r^3+4\pi p/c^2}{1-2Gm/c^2r}$$

$$\leq 9\Gamma_1\frac{p}{\rho}+3\Gamma_1 p_0 4\pi\frac{r^3}{m}+(\rho_0+p_0/c^2)G\frac{m/r+4\pi r^2 p_0/c^2}{1-2Gm/c^2r}.$$

From Theorem.2 there is a constant C such that $F=F^{(c)}(r)$ for $0\leq r\leq R^{(c)}$ and $c\leq c_0$. Thus we get the conclusion as follows. Suppose

$$(A3):\qquad \inf_{0<p\leq p_0}\,[3\Gamma_1-4+3p\frac{d\Gamma_1}{dp}]>0$$

Then for sufficiently large c we have $A\ \xi>0$ for $\xi=1$. This implies the neutral stability (see Lemma 3.7 of [7]). Thus we have

Theorem 4. *Under the assumptions* (A0) (A1)' (A2) (A3), *the equilibrium is neutrally stable in the linearized sense for large c.*

6. Appendix

We integrate the Tolman-Oppenheimer-Volkoff equation for the equation of state $p=\frac{c^2}{3}\rho$. For the sake of simplicity we take $G=c=1$. We stat with a local solution $m(r)\sim 4\pi r^2\rho_0$, $p(r)\sim\frac{1}{3}\rho_0$ at $r=+0$. Let us prolong this solution as long as possible in $\{0<\rho,\ 2m/r<1\}$, and let $(0, R)$ be the right maximal interval of existence. We shall show $R=+\infty$.

Let us introduce the variables

$$u=\frac{m}{r},\ v=4\pi r^2\rho.$$

Then the functions $\bar{u}(r)=m(r)/r$, $\bar{u}(r)=4\pi r^2\rho(r)$ satisfy the equation

$$r\frac{du}{dr}=v-u$$

$$r\frac{dv}{dr}=\frac{2v}{1-2u}(1-4u-\frac{2}{3}v).$$

Change the independent variable from r to t by

$$t=t(r)=\int_{\delta}^{r}\frac{dr}{(1-2\bar{u}(r))r}.$$

Put $T=\lim_{r\to R}t(r)$. Then $t=t(r)$, $0<r<R$, has the inverse function $r=r(t)$, $-\infty<t<T$, and the functions $u(t)=\bar{u}(r(t))$, $v(t)=\bar{v}(r(t))$ satisfy the plane autonomous system

$$\frac{du}{dt}=(v-u)(1-2u)$$

$$\frac{dv}{dt}=2v(1-4u-\frac{2}{3}v),$$

to which the geometric theory of plane dynamical system can be applied.

We are considering the path E: $(u(t), v(t))$ such that $u \sim \frac{4\pi}{3}\rho_0 e^t$, $v \sim 4\pi\rho_0 e^{2t}$ as $t \mapsto -\infty$. Now it is easily seen that the path E is confined in the bounded region $D=\{(u, v) | 0<u<1/2, 0<v<3/2\}$. Therefore $T=+\infty$. Let us determine the ω -limit set of the path E. In the closure of D there are critical points: $\mathbf{O}(0, 0)$, $\mathbf{P}(1/2, 0)$ and $\mathbf{Q}(3/14, 3/14)$. The critical points $\mathbf{O}$ and $\mathbf{P}$ are saddle points. The path E is a separatrix of $\mathbf{O}$. The characteristic roots of $\mathbf{Q}$ are $(-3\pm\sqrt{47}i)/7$, so that $\mathbf{Q}$ is a stable focus. By the theorem of Bendixson, the ω -limit set of E falls under the following categories. a) a single critical point; b) a closed path; c) a pathpolygon which is a boundary of a 2-cell containing E. It is easy to exclude the possibility of the case c), since there are no path-polygon confined in the closure of D. In fact, the stable separatrix of P, $u=1/2$, is unbounded. We go to exclude the case b) by showing that there are no closed paths in D. To do that, we introduce the variables $a = \log(1-2u)$, $b = \log v$. Then the equation reduces to

$$\frac{da}{dt}=1-e^a-2e^b$$

$$\frac{db}{dt}=-2+4e^a-\frac{4}{3}e^b.$$

We see

$$\frac{\partial}{\partial a}\left(\frac{da}{dt}\right)+\frac{\partial}{\partial b}\left(\frac{db}{dt}\right)=-e^a-\frac{4}{3}e^b<0.$$

Hence, by the criterium of Bendixson, D contains no closed paths. Thus a) is the only possible case, and we know that $(u(t), v(t))$ tends spirally to the focus $\mathbf{Q}$. Therefore $R = \exp\left[\int_0^{+\infty}(1-2u(t)\,dt\right] = +\infty$, and $\bar{u}(r)\rightarrow 3/14$, $\bar{v}(r)\rightarrow 3/14$ as $r\mapsto+\infty$. In other words, $\rho(r)$ exists to be positive for all $r>0$ and $\rho(r)\sim\frac{3}{56\pi}\frac{1}{r^2}$, $m(r)\sim\frac{3}{14}r$ as $r\mapsto+\infty$. This was to be shown.

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