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Global deformations of $\mathbf{P}^2$-bundles over $\mathbf{P}^1$

By

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§0. Introduction

In the present article we study complex analytic global deformations of $\mathbf{P}^2$-bundles over $\mathbf{P}^1$. In the two dimensional case there are two homeomorphism classes of $\mathbf{P}^1$-bundles over $\mathbf{P}^1$, each class being stable (or closed) and transitive under global deformation. In the three dimensional case there are exactly three homeomorphism classes of $\mathbf{P}^2$-bundles over $\mathbf{P}^1$, that is, first of all those with the first Chern class divisible by three, secondly those homeomorphic to $\mathbf{P}^1 \times \mathbf{P}^2$, and the rest. We note that no $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ with the first Chern class divisible by three is homeomorphic to $\mathbf{P}^1 \times \mathbf{P}^2$. Any $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ is something like a Fano threefold of index greater than 3 but less than 4, though its anti-canonical line bundle may not be ample. Using this Fano-like character of $\mathbf{P}^2$-bundles over $\mathbf{P}^1$, we prove the following

Theorem 0.1. *The set consisting of all $\mathbf{P}^2$-bundles over $\mathbf{P}^1$ with the first Chern class divisible by three is closed and transitive under global deformation.*

Theorem 0.2. *The set consisting of all $\mathbf{P}^2$-bundles over $\mathbf{P}^1$ whose first Chern class is indivisible by three and which are not homeomorphic to $\mathbf{P}^1 \times \mathbf{P}^2$ is closed and transitive under global deformation.*

Theorem 0.3. *The set consisting of all $\mathbf{P}^2$-bundles over $\mathbf{P}^1$ homeomorphic to $\mathbf{P}^1 \times \mathbf{P}^2$ and of all $\mathbf{P}^1$-bundles over $\mathbf{P}^2$ homeomorphic to $\mathbf{P}^1 \times \mathbf{P}^2$ is closed and transitive under global deformation.*

See Theorems 2.3 and 4.1. See also Kollár [Ko], Peternell [P1] [P2], Siu [S1] [S2] and Nakamura [N1] [N2] [N3] [N5] [N6] [N7] for the related topics.

We note $\mathbf{P}^1 \times \mathbf{P}^2$ can be deformed both as a $\mathbf{P}^1$-bundle over $\mathbf{P}^2$ and as a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$. This is the reason why $\mathbf{P}^2$-bundles over $\mathbf{P}^1$ appear in Theorem 0.3.

The present article is organized as follows. In section one we recall the structures of $\mathbf{P}^2$-bundles over $\mathbf{P}^1$. We prepare a few lemmas. In sections two and three, we prove Theorems 0.1 and 0.2. In section 3, we show that there are infinitely many non-isomorphic $\mathbf{P}^1$-bundles over $\mathbf{P}^2$ homeomorphic to $\mathbf{P}^1 \times \mathbf{P}^2$, which arise from topologically trivial unstable rank two bundles over $\mathbf{P}^2$. We prove that they are global deformations of $\mathbf{P}^1 \times \mathbf{P}^2$.

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In section 4 we study global deformations of $\mathbf{P}^1 \times \mathbf{P}^2$ and settle the remaining case of the study in section 2 so as to prove Theorem 0.3. The major part of the results of the present article was announced in [N2].

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Notation.

$\mathrm{Bs}\lvert L\rvert$	the scheme-theoretic base locus of $\lvert L\rvert$
$c(E)$	the total Chern class $\sum_{i\in\mathbf{Z}} c_i(E)$ of a vector bundle E
$c_i(E)$	the i-th Chern class of a vector bundle E
$c_i(X)$	the i-th Chern class of X
$\mathrm{disc}(E)$	the discriminant of a vector bundle E on $\mathbf{P}^2$, (3.2)
$E(C, L, \phi)$	(3.7)
$\mathcal{F}(a, b, c)$	$O_{\mathbf{P}^1}(a) \oplus O_{\mathbf{P}^1}(b) \oplus O_{\mathbf{P}^1}(c)$
$\mathbf{F}_b$	$\mathrm{Proj}(O_{\mathbf{P}^1}(b) \oplus O_{\mathbf{P}^1})$
$g^*\lvert L\rvert$	$\{g^*D;\ D \in \lvert L\rvert\}$
$h^q(X, F)$	$\dim H^q(X, F)$ for a coherent sheaf F
$N_{C/X}$	the normal bundle of C in X
O_X, O_S, O_Z	the structure sheaf of X, S, Z respectively
$\widehat{O}_X$	the formal completion of O_X
$\mathbf{P}(\mathcal{F}(a, b, c))$	$\mathrm{Proj}(\mathcal{F}(a, b, c))$
$\mathrm{sp}^+(E)$	the spectrum of a vector bundle E on $\mathbf{P}^2$, (6.2)
$\chi(X, F)$	$\sum_{q\in\mathbf{Z}} (-1)^q h^q(X, F)$
$(\ )_S$, $(\ )_X$	the intersection numbers on S, X
$\equiv$	the linear equivalence
$(p.\ q)$	Theorem $p.\ q$, or Lemma $p.\ q$, or Proposition $p.\ q$ Paragraph or Equation $(p.\ q)$

§1. $\mathbf{P}^2$-bundles over $\mathbf{P}^1$

(1.1) The structure of $\mathbf{P}^2$ bundles. First we review $\mathbf{P}^2$-bundles over $\mathbf{P}^2$. Let $k=0$, 1 or 2. Choose integers $a \geq b \geq 0$ such that $a+b-k$ is divisible by 3. Let $3n = a+b-k \geq 0$. Let $\mathcal{F} := \mathcal{F}(a, b, 0) = O_{\mathbf{P}^1}(a) \oplus O_{\mathbf{P}^1}(b) \oplus O_{\mathbf{P}^1}$, $X = \mathbf{P}(\mathcal{F})$ and let π: $X \to \mathbf{P}^1$ be the natural projection. Let H be a tautological line bundle of X with $\pi_* H \simeq \mathcal{F}$. Then the canonical sheaf K_X of X is given by the formula,

$$K_X = -3H + \pi^*(\det \mathcal{F} + K_{\mathbf{P}^1}) = -3H + (a+b-2)F$$

where F is a fiber of π. Letting $L := L(\mathcal{F}) = H - nF$, we have $K_X = -3L - (2-k)F$, $L^3 = \deg \pi_* L = k$. Since $\pi_* L \simeq \mathcal{F} \otimes O_{\mathbf{P}^1}(-n)$, and $R^q \pi_* L = 0\ (q \geq 1)$, we have

$$H^q(X, L) \simeq H^q(\mathcal{F} \otimes O_{\mathbf{P}^1}(-n)) \quad (q \geq 0).$$

We see that $R^q \pi_*(-pL) = 0$ $(q \geq 0,\ p = 1, 2)$, whence $H^q(X, -pL) = 0$ for the same values of q and p. There are 3 cases.

Case 1. $n=0, a\geq b\geq 0$.
Case 2. $a\geq b\geq n\geq 1$.
Case 3. $a\geq n>b\geq 0$.

Case 1-1. Assume that $k=2$ and $a=b=1$. Then $h^0(X, L)=5$ and $\mathrm{Bs}|L|=\emptyset$. The morphism $\rho_L: X\to \mathbf{P}^4$ associated with $|L|$ has a hyperquadric W with Hessian-rank 4 as its image. In fact, we can choose elements x_0, x_1 (resp. x_2, x_3) from $H^0(O_{\mathbf{P}^1}(a-n)\oplus 0\oplus 0)$ (resp. $H^0(0\oplus O_{\mathbf{P}^1}(b-n)\oplus 0))$ such that $x_0x_3=x_1x_2$. ρ_L is a small resolution of W whose exceptional set is $\mathbf{P}(O_{\mathbf{P}^1})\simeq \mathbf{P}^1$ with normal bundle $\simeq O_{\mathbf{P}^1}(-1)\oplus O_{\mathbf{P}^1}(-1)$.
Case 1-2. Assume that $k=2$, $a=2$, $b=0$. Then $h^0(X, L)=5$ and $\mathrm{Bs}\ |L|=\emptyset$. The morphism ρ_L: $X\to \mathbf{P}^4$ associated with $|L|$ has a hyperquadric W with Hessian-rank 3 as its image. In fact, we can choose elements x_0, x_1 and x_2 from $H^0(O_{\mathbf{P}^1}(2)\oplus 0\oplus 0)$ such that $x_1^2=x_0x_2$. ρ_L is a divisorial contraction whose exceptional set is $E:=\mathbf{P}(O_{\mathbf{P}^1}(b)\oplus O_{\mathbf{P}^1})\simeq \mathbf{P}^1\times \mathbf{P}^1$. The restriction map $\rho_{L|E}$: $E\to \mathbf{P}^1$ is a $\mathbf{P}^1$-bundle whose arbitrary fiber C has the normal bundle $N_{C/X}\simeq O_{\mathbf{P}^1}\oplus O_{\mathbf{P}^1}(-2)$.
Case 1-3. Assume that $k=a=1$ and $b=0$. Then $h^0(X, L)=4$ and $\mathrm{Bs}|L|=\emptyset$. The morphism ρ_L: $X\to \mathbf{P}^3$ associated with $|L|$ is a divisorial contraction whose exceptional set is $\mathbf{P}(O_{\mathbf{P}^1}(b)\oplus O_{\mathbf{P}^1})\simeq \mathbf{P}^1\times \mathbf{P}^1$. The morphism ρ_L is a monoidal transfomation of $\mathbf{P}^3$ with a line center. This is seen as follows. Let l be a line of $\mathbf{P}^3$, and p: $Y\to \mathbf{P}^3$ the monoidal transform of $\mathbf{P}^3$ with l center. Let L be the pull back of the hyperplane bundle of $\mathbf{P}^3$ by p, $E:=p^{-1}(l)$. Then $E\simeq \mathbf{P}(N^\vee_{l/\mathbf{P}^3})\simeq \mathbf{P}^1\times \mathbf{P}^1$. Since $h^0(Y, L-E)=2$ and $\mathrm{Bs}\ |L-E|=\emptyset$, we have a surjective morphism π: $Y\to \mathbf{P}^1$ with any fiber $\simeq \mathbf{P}^2$. Defining $\mathcal{F}:=\pi_*(L)$, then we have $Y\simeq \mathbf{P}(\mathcal{F})$. Let $\mathcal{F}\simeq \mathcal{F}(a', b', c')$ $(a'\geq b'\geq c')$. Then $L^3=a'+b'+c'=1$ and $a'\geq b'\geq c'\geq 0$ because $\mathrm{Bs}\ |L|=\emptyset$. Hence $a'=1$, $b'=c'=0$. Hence $X\simeq Y$.
Case 1-4. If $k=a=b=0$, then $x\simeq \mathbf{P}^1\times \mathbf{P}^2$, $h^0(X, L)=3$ and $\mathrm{Bs}\ |L|=\emptyset$.
Case 2. In this case, $h^0(X, L)=n+k+2$, $B:=\mathrm{Bs}\ |L|\simeq \mathbf{P}(O_{\mathbf{P}^1})\simeq \mathbf{P}^1$. Since $\pi_*L\simeq \mathcal{F}\otimes O_{\mathbf{P}^1}(-n)$, any element of $H^0(X, L)$ is written as $s_0(x)y_0+s_1(x)y_1+s_2(x)y_2$ for some $(s_0, s_1, s_2)\in H^0(\mathcal{F}\otimes O_{\mathbf{P}^1}(-n))$ and suitable homogeneous coordinates y_i of fibers $(\simeq \mathbf{P}^2)$. In particular, $h^0(X, L)=a+b-2n+2=n+k+2$. Since $s_2(x)\equiv 0$, $B:=\mathrm{Bs}\ |L|=\{y_0=y_1=0\}\simeq \mathbf{P}^1$ and $N_{B/X}\simeq O_B(-a)\oplus O_B(-b)$.

Let f: $Y\to X$ be the blowing-up of X with B center, E the total transform of B, and $N:=f^*L-E$. We see also that $E\simeq \mathbf{P}(N^\vee_{B/X})\simeq \mathbf{F}_{a-b}(a-b\geq 0)$.

Let $N_E:=N\otimes O_E$, and e_0 (resp. e_∞ or f_0) a section (resp. a section or a fiber) of $f_{|E}$: $E\to B$ with $(e_0^2)_E=a-b$ (resp. $(e_\infty^2)_E=-a+b$). Then we see

$$(f^*L)_E\simeq f^*(L_B)\simeq -nf_0,\ N_E\simeq e_0+(b-n)f_0,\ E_E\simeq -e_0-bf_0,$$
$$(N_E^2)_E=n+k,\ \mathrm{Bs}\ |N|=\emptyset,\ H^0(X, L)\simeq H^0(Y, N)\simeq H^0(E, N_E).$$

Let C_w be a line in $F(\simeq \mathbf{P}^2)$, $\widehat{C}_w$ a proper transform of C_w by f. Since C_w intersects B transversally at one point, we have $(E\widehat{C}_w)=1$ and $(N\widehat{C}_w)_Y=0$.

Hence the morphism $g: Y \to \mathbf{P}^{n+k+1}$ associated with $|N|$ has an image $g(Y) \simeq g(E)$. Since $(N_E^2)_E = n+k$ and $h^0(E, N_E) = n+k+2 \geq 5$, the image $g(E)$ is a cone over a smooth variety of minimal degree. In fact, if $b > n$, then $g(E) \simeq E \simeq \mathbf{F}_{a-b}$ and Y is a $\mathbf{P}^1$-bundle over $g(E)$. If $b = n$, then $g_{|E}$ contracts e_∞ so that $g(E)$ is a cone over a smooth rational curve $g(e_0)$ of degree $n+k$ with $g(e_\infty)$ its vertex.
Case 3. In this case, $h^0(X, L) = a - n + 1 \, (\geq n+k+2)$, $B := \mathrm{Bs}\,|L| \simeq \mathbf{P}(O_{\mathbf{P}^1}(b) \oplus O_{\mathbf{P}^1}) \simeq \mathbf{F}_b$ and $|L| = |(a-n)F| + B$. The image of the morphism ρ_L is $\mathbf{P}^1$. The natural morphism π is the same as that associated with $|F|$.

(1.2) Topological types of $\mathbf{P}^2$ bundles Topological types of $\mathbf{P}^2$-bundles over $\mathbf{P}^1$ are classified by $\pi_1(PGL(3, \mathbf{C}))$ $(\simeq \mathbf{Z}/3\mathbf{Z})$. Each equivalence class (homeomorphism class) is represented by $M_k := \mathbf{P}(\mathcal{F}(k, 0, 0))$ $(0 \leq k \leq 2)$. Let L_k (resp. F_k) be the tautological line bundle (resp. a fiber over $\mathbf{P}^1$). Then we see

$$(1.2.k) \qquad H^2(M_k, \mathbf{Z}) \simeq \mathbf{Z}L_k \oplus \mathbf{Z}F_k, \; L_k^3 = k, \; L_k^2 F_k = 1, \; F_k^2 = 0.$$

Any homeomorphism σ of M_k keeps F_k invariant up to sign, $\sigma^*(F_k) = \pm F_k$. Then it is easy to see that $\sigma^*(F_k) = F_k$, and $\sigma^*(L_k) = \pm L_k + aF_k$ for some integer a. Since the rational Pontrjagin class $p_1(M_k) := 3L_k^2 - 2kL_kF_k$ is a topological invariant, we have $\sigma^*(L_k) = L_k$ if $k \neq 0$, while $\sigma^*(L_k) = \pm L_k$ if $k = 0$. Hence $\sigma^*(L_k)^3 = L_k^3 = k$ mod 3. Thus L_k^3 mod 3 determines the homeomorphism class of M_k uniquely.

Lemma 1.3. *Let X be a Moishezon 3-fold with $H^*(X, \mathbf{Z}) \simeq H^*(M_k, \mathbf{Z})$ for some k $(k = 0, 1, 2)$. Then we have*
(1.3.1) *$H^q(X, O_X) = 0$ for $q > 0$.*
(1.3.2) *There exist line bundles L and F on X such that $L^3 = k$, $L^2F = 1$, $F^2 = 0$ and $H^2(X, \mathbf{Z}) \simeq \mathbf{Z}L + \mathbf{Z}F$, $H^4(X, \mathbf{Z}) \simeq \mathbf{Z}L^2 + \mathbf{Z}LF$. The line bundle L and F on X with $L^3 = k$, $Ll^2 F = 1$, $F^2 = 0$ are uniquely determined if $k = 1$ or 2, while $\pm L$ and F are the only ones satisfying $L^3 = 0$, $L^2 F = 1$, $F^2 = 1$ for $k = 0$.*

Proof. By [U], the Hodge spectral sequence of X degenerates at E_1-terms, and Hodge duality $h^{p,q} = h^{q,p}$ is true. Since $b_1 = b_3 = 0$, we have $h^{p,q} = 0$ if $p+q = 1$ or 3. Moreover $h^{1,1} + 2h^{2,0} = b_2 = 2$, so that $h^{1,1} = 2$ and $h^{2,0} = h^{0,2} = 0$. This prover (1.3.1). (1.3.2) follows from (1.3.1)readily. See also (1.2)

Definition 1.4. Let $k = 0, 1, 2$. *A fake $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ of type k* is a Moishezon threefold X which has a pair of line bundles L and F such that

$$(1.4.k) \qquad \begin{aligned} &H^4(X, \mathbf{Z}) \simeq \mathbf{Z}L^2 \oplus \mathbf{Z}LF, \; L^3 = k, \; L^2F = 1, \; F^2 = 0, \\ &c_1(X) = 3L + (2-k)F, \; c_2(X) = 3L^2 + (6-2k)LF. \end{aligned}$$

Roughly speaking a fake $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ is a Moishezon threehold X which has the same cohomology ring over $\mathbf{Z}$ and the same Chern classes as a

$\mathbf{P}^2$-bundle over $\mathbf{P}^1$. We call the pair L and F *canonical generators* of Pic X. We call *a fake* $\mathbf{P}^2$*-bundle over* $\mathbf{P}^1$ *of type* 0 *a fake* $\mathbf{P}^1\times\mathbf{P}^2$ simply.

Lemma 1.5. *Let X be a Moishezon 3-fold, L and F line bundles on X. Assume $H^2(X, \mathbf{Z})\simeq\mathbf{Z}^{\oplus 2}$ and that $L^2F=1$, $F^2=0$. If $HH'=0$ for two nontrivial line bundles H, H' on X, then $H\equiv bF$ and $H'\equiv b'F$ for some b and b'.*

Proof. It is easy to see that $H^2(X, \mathbf{Z})\simeq\mathbf{Z}L\oplus\mathbf{Z}F$. Let $H\equiv aL+bF$ and $H'\equiv a'L+b'F$. Then by the assumption, we have $0=HH'=aa'L^2+(ab'+a'b)LF$, whence $aa'=ab'+a'b=0$. Hence $a=a'=0$.

§2. Global deformations of $\mathbf{P}(\mathcal{F}(a, b, 0))$ with $a+b\equiv 1$ or 2 mod 3

Lemma 2.1. *Let X be a fake $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ of type k, L and F canonical generators of* Pic X. *Assume $h^0(X, L-F)\geq 1$ and $h^0(X, F)\geq 2$. If the linear system $|F|$ has no fixed components, then $X\simeq\mathbf{P}(\mathcal{F}(a, b, 0))$ for some $a\geq b\geq 0$ $(a+b\equiv k$ mod $3)$.*

Proof. Let X be a fake $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ of type k. Let F, F' be two distinct general members of $|F|$. Since $F^2=0$, F'_F is a topologically trivial effective divisor of F. Since F is an algebraic surface, this implies $F\cap F'=\emptyset$ so that $h^0(X, F)=2$. It follows that any general member Z of $|F|$ is smooth and $K_Z\simeq-3L_Z$. We note that L_Z is effective by $h^0(X, L)\geq 1$.

Let π: $X\to\mathbf{P}^1$ be the morphism associated with $|F|$.

We assume $c_1(L_Z)=0$ to derive a contradiction. If $c_1(L_Z)=0$, then $c_1(K_Z)=0$. Then by [Ka] deg $12\pi_*(\omega_{X/\mathbf{P}^1})\geq 0$. Therefore we have

$$h^0(X, -3L+kF)=h^0(X, K_X+2F)=h^0(\mathbf{P}^1, \pi_*(\omega_{X/\mathbf{P}^1}))\geq 1,$$

with contradicts $h^0(X, 3L-kF)\geq h^0(X, 3L-3F)\geq h^0(X, L-F)\geq 1$. Hence $c_1(L_Z)\neq 0$, whence Z is $\mathbf{P}^2$ or Z has a pencil of smooth rational curves f_t with $(f_t^2)_Z=0$. Clearly the second case is impossible. Hence $Z\simeq\mathbf{P}^2$.

We prove that any fiber Z' of π is isomorphic to $\mathbf{P}^2$. Let $Z'=\sum_{i=0}^{q} m_iZ_i$ be the decomposition of Z' into irreducible components. By the upper semi-continuity, we have for any positive integer m,

$$h^0(Z', mL_{Z'})\geq h^0(\mathbf{P}^2, O_{\mathbf{P}^2}(3m)),$$

whence there is an irreducible component Z_0 of Z' such that $\kappa(Z_0, L_{Z_0})=2$.

Let h: $S_0\to Z_0$ the minimal resolution of the normalization of Z_0. Then the canonical bundle of S_0 is given by $K_{S_0}=h^*(K_X+Z_0)-P_0$ for some effective divisor P_0 of S_0. Hence we have

$$m_0K_{S_0}=-((3r-1)m_0h^*A+3m_0h^*(F^*)+m_0P_0)-\sum_{i\neq 0}m_ih^*(Z_i).$$

Therefore S_0 is either $\mathbf{P}^2$ or a ruled surface. If S_0 has a pencil of smooth rational curves f_t with $(f_t^2)_{S_0}=0$, then we have

$$2=-(K_{S_0}f_t)_{S_0}\geq 3(h^*(L)f_t)_{S_0},$$

whence $(h^*(L)f_t)_{S_0}=0$. This contradicts $\kappa(S_0, h^*(L))=\kappa(Z_0, L_{Z_0})=2$. Hence $S_0\simeq\mathbf{P}^2$. Since $S_0\simeq\mathbf{P}^2$, we have $P_0=0$, whence $S_0\simeq Z_0$ by the same argument as above. Hence $Z_0\simeq\mathbf{P}^2$ and Z_0 is a connected component of Z', whence $Z'\simeq Z_0$. Therefore X is a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$, which is isomorphic to $\mathbf{P}(\pi_*(L))$.

Theorem 2.2. *The set of all $\mathbf{P}^2$-bundles $\mathbf{P}(\mathcal{F}(a, b, 0))$ over $\mathbf{P}^1$ with $a+b\equiv 1 \bmod 3$ is stable and transitive under global deformation.*

Proof. We prove the following

CLAIM. *Let $k=0$, or 1. Let X be a fake $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ of type k, L and F canonical generators of* Pic X. *Assume $h^0(X, L-F)\geq 1$ and $h^0(X, F)\geq 2$. Then $|F|$ has no fixed components.*

Proof. Let $Z_1+\cdots+Z_r+G^*$ be a general member of $|F|$, Z_i movable components and G^* the fixed components. Let $Z:=Z_1$, $\nu: Y\to Z$ be the normalization of Z, $f: S\to Y$ the minimal resolution of Y, $g=\nu\cdot f$. Then we have $K_S=g^*(K_X+Z)-E-G$ where E and G are effective divisor of S such that E is finite over $f(E)$, while $g_*(G)=0$. Since $h^0(L-F)\geq 1$, there exists an effective divisor H on X such that $L\equiv F+H\equiv rZ+H+G^*$. Hence we have

$$\begin{aligned}K_S&=-(g^*(3L+(2-k)F)-Z+E+G)\\&=-((5r-kr-1)g^*Z'+3g^*H+(5-k)g^*G^*+E+G),\end{aligned}$$

where Z' is another movable component of $|F|$.

If $g^*(G^*)\neq 0$, then $\kappa(S)=-\infty$. Therefore $S\simeq\mathbf{P}^2$ or S has a pencil $F_S\simeq\mathbf{P}^1$ with $F_S^2=0$. However since $5r-kr-1\geq 4r-1\geq 3$, and $supp(E+G)\cap g^{-1}(supp(Z'+G^*)\cap Z)$, whence the coefficient of any component of $E+G$ is at least 4. Moreover the coefficient of $g^*(G^*)$ is $5-k\geq 4$. Therefore if S has a pencil of $F_S\simeq\mathbf{P}^1$ with $F_S^2=0$, then $-K_SF_S\geq 3$, a contradiction. Hence $S\simeq\mathbf{P}^2$. However then $g^*(G^*)=0$ by $5-k\geq 4$. Then $G^*Z=0$ in $H^4(X, \mathbf{Z})$. By (1.5), $G^*\in|b^*F|$ and $Z\in|bF|$, whence $G^*=0$ by $h^0(X, F)\geq 2$. This shows that $|F|$ has no fixed componets.

The remainder of the present section is devoted to proving

Theorem 2.3. *The set of all $\mathbf{P}^2$-bundles $\mathbf{P}(\mathcal{F}\ a, b, 0))$ over $\mathbf{P}^1$ with $a+b\equiv 2 \bmod 3$ is stable and transitive under global deformation.*

Our proof of (2.3) will be given in (2.5)-(2.9).

Corollary 2.4. *Let $k=1$ or 2. Any jumping-deformation of $\mathbf{P}(\mathcal{F}(a, b, 0))$ with $a\geq b\geq 0$ and $a+b=3n+k$ is isomorphic to $\mathbf{P}(\mathcal{F}(c, d, 0))$ for some c, d with $c\geq d\geq 0$, $c+d=3m+k$ and $c-a\geq m-n\geq 0$.*

We call X *a jumping-deformation* of Y if $X_0\simeq X$, and if $X_t\simeq Y$ for any $t\neq 0$ for a smooth family $X_t(t\in\Delta)$ of complex manifolds over a disc Δ.

Proof of (2.4). In fact, this is a corollary to the proof of (2.5). In view of (2.2) and (2.3) any global deformation of $\mathbf{P}(\mathscr{F}(a, b, 0))$ with $a \geq b \geq 0$, $n \geq 0$ and $a+b=3n+2$ is isomorphic to $\mathbf{P}(\mathscr{F}(c, d, 0))$ for some c, d with $c \geq d \geq 0$ and $c+d=3m+2$. Therefore it is sufficient to prove the following

CLAIM. *Let $k=0$, 1 or 2. $\mathbf{P}(\mathscr{F}(a, b, 0))$ with $a \geq b \geq 0$, $a+b=3n+k$ is a small deformation of $\mathbf{P}(\mathscr{F}(c, d, 0))$ with $c \geq d \geq 0$, $c+d=3m+k$ if and only if $c-a \geq m-n \geq 0$.*

Proof of Claim. Let $\{X_t\}_{t \in \Delta}$ be a complex analytic family over a disc Δ such that $X_0 \simeq \mathbf{P}(\mathscr{F}(c, d, 0))$, $X_t \simeq \mathbf{P}(\mathscr{F}(a, b, 0))$ for $t \neq 0$ small. Since X_t satisfies the condition in (2.1), we have unique canonical generators L_t and F_t of Pic X_t. By the proof of (2.5), the linear system $|F_t|$ defines a morphism π_t: $X_t \to \mathbf{P}^1$ with any fiber $\simeq \mathbf{P}^2$. Then by (1.1) we have $(\pi_0)_*(L_0) \simeq \mathscr{F}(c-m, d-m, -m)$ and $(\pi_t)_*(L_t) \simeq \mathscr{F}(a-n, b-n, -n)$ for $t(\neq 0)$ small. We also see that $F_t \simeq \pi_t^* O_{\mathbf{P}^1}(1)$. Let $A_t := L_t - (c-m+1)F_t$ and $B_t := L_t + (m-1)F_t$. Then we have $h^0(X_t, A_t) \leq h^0(X_0, A_0) = 0$, whence $c-m \geq a-n$. Similarly by $h^1(X_t, B_t) \leq h^1(X_0, B_0) = 0$, we have $m \geq n$.

Conversely if $c-a \geq m-n \geq 0$, it is easy to construct a flat family of vector bundles $\mathscr{F}_t (t \in \Delta)$ such that $\mathscr{F}_0 \simeq \mathscr{F}(c-m, d-m, -m)$ and $\mathscr{F}_t \simeq \mathscr{F}(a-n, b-n, -n)$ for $t \neq 0$. Then the family $\mathbf{P}(\mathscr{F}_t)$ $(t \in \Delta)$ is a smooth family of 3-folds. This completes the proof of the Claim, hence of (2.4).

A $\mathbf{P}^2$-bundle $\mathbf{P}(\mathscr{F}(a, b, 0))$ with $a \geq b \geq 0$ and $a+b \equiv 2 \bmod 3$ is a global deformation (a smooth limit) of $\mathbf{P}(\mathscr{F}(1, 1, 0))$. Clearly $\mathbf{P}(\mathscr{F}(a, b, 0))$ is homeomorphic to $\mathbf{P}(\mathscr{F}(1, 1, 0))$.

It is clear that any global deformation of $\mathbf{P}(\mathscr{F}(a, b, 0))$ $(a+b \equiv 2 \bmod 3)$ is a fake $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ of type 2 whose canonical generators L and F satisfy the conditions $h^0(X, L-F) \geq 1$ and $h^0(X, F) \geq 2$. Therefore for the proof of (2.3) we need only to verify

Lemma 2.5. *Let X be a fake $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ of type 2, L and F canonical generators of* Pic X. *If $h^0(X, L-F) \geq 1$ and $h^0(X, F) \geq 2$, then $X \simeq \mathbf{P}(\mathscr{F}(a, b, 0))$ for some $a \geq b \geq 0$, $a+b \equiv 2$* mod 3.

The rest of the section is devoted to proving (2.5).

(2.6) *Plan of the proof of* (2.5). Let X be a fake $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ of type 2, L and F canonical generators of Pic X. By the Poincaré duality we have

(2.6.1) $$H^4(X, \mathbf{Z}) \simeq \mathbf{Z}L^2 \oplus \mathbf{Z}LF,$$

Since $K_X = -3L$ and $h^0(X, L) \geq 2$ by the conditions in (2.5), we have $h^3(X, O_X) = 0$. Also $h^1(X, O_X) = 0$. Since $h^2(X, O_X) = \chi(X, OX) - 1$, we have

(2.6.2) $$\chi(X, O_X) = \frac{1}{24} c_1(X) c_2(X) = 1, \ h^q(X, O_X) = 0 \ (q \geq 1),$$

We use (2.6.2) frequently without mentioning in the subsequent proofs. We see also

$$(2.6.3) \qquad \chi(X, pL+qF) = \frac{1}{6}(p+1)(p+2)(2p+3q+3).$$

We note that $h^0(X, L) \geq 2$ by $h^0(X, L-F) \geq 1$ and $h^0(X, F) \geq 2$.

Let D be a general member of $|L|$. Let $D = Z_1 + \cdots + Z_r + F^*$ be the decomposition of D into irreducible components, Z_i movable components ($1 \leq i \leq r$), F^* the fixed components. Since $h^1(X, O_X) = 0$, any Z_i is linearly equivalent, so we have $D \equiv rZ + F^*$ where we set $Z = Z_1$. Let $A := O_X(Z) \in \mathrm{Pic}\, X$. Let ν: $Y \to Z$ be the normalization of Z, f: $S \to Y$ the minimal resolution of Y, $g = \nu \cdot f$. Then there exist by [N3, (2.A)] effective Cartier divisors E and G on S with no components in common such that the canonical bundle K_S of S is given by

$$K_S = g^*(K_X + A) - E - G$$

where $f_*(G) = 0$ and E is finite over $f(E)$. Let $\Sigma := E \cup g^{-1}(\mathrm{Sing}\, Z)$. Then Σ contains $supp(E+G)$ and $g_{|S \setminus \Sigma}$ is an isomorphism. We also note that the base locus Bs $g^*|L|$ contains $supp$ $(E+G)$ if D is sufficiently general. Since $h^0(X, Z) \geq 2$, $g^*(A)$ is effective. Let $g^*(A) = M + N$ be a general member of $g^*|A|$, M (resp. N) the movable part (resp. the fixed part) of $g^*|A|$. Then

$$K_S = -((3r-1)M + (3r-1)N + 3g^*(F^*) + E + G)$$

whence S is either $\mathbf{P}^2$ or a ruled surface.

Case 1. $S \simeq \mathbf{P}^2$

Case 2. ρ: $S \to \mathbf{P}^1$ is a surjective morphism with general fiber $F_s \simeq \mathbf{P}^1$.

We discuss *Case* 1 in (2.7), and *Case* 2 in (2.8) - (2.9). In any case we prove $X \simeq \mathbf{P}(\mathscr{F}(a, b, 0))$ with $a + b \equiv 2$ mod 3. The indices a and b are given as follows.

	S	dim W	(a, b)	
Case 1.	$\mathbf{P}^2$	1	$a \geq n > b \geq 0, a+b = 3n+2$	(2.7)
Case 2-a	ruled	2	$a \geq b \geq n \geq 1, a+b = 3n+2$	(2.8.3)
Case 2-b	ruled	3	(2, 0) or (1, 1)	(2.8.4)

where W is the image of X by the rational map ρ_L.

Lemma 2.7. (*Case* 1) $X \simeq \mathbf{P}(\mathscr{F}(a, b, 0))$ *for some* a, b ($a \geq n > b \geq 0$, $a + b = 3n + 2$).

Proof. By the assumption $S \simeq \mathbf{P}^2$ under the notation as in (2.6). Then $G = 0$. We prove that $M = N = E = 0$ and $g^*F^* \in |O_S(1)|$. Assume $g^*(F^*) = 0$. If moreover $N = 0$, then $E = 0$ by $E_{\mathrm{red}} \leq N_{\mathrm{red}}$. Hence $-K_S = (3r-1)M$, a contradiction. Therefore $N \neq 0$, $E \neq 0$ and $M = 0$, whence $N = E \in |O_S(1)|$. It follows from the subadjunction formula [N3, (2.A)] that Z is singular generically along $g(E)$ with

$$e(Q''_V, E_U) - e(Q'_V, E_U) = 1$$

where V is a suitable Zariski open subset of Z, U the inverse image of V in S and $E_U := E \cap U \neq \phi$. Meanwhile $(Lg_*(E))_X = (g^*(L)E)_S = (NE)_S = 1$, which shows deg $(g_{|E}) = 1$. However if deg $(g_{|E}) = 1$, then $e(Q'_V, E_U) - e(Q''_V, E_U) \geq 2$ by [N3, (2.A) and (2.6)], a contradiction. Hence we have $M = N = E = 0$ and $g^*(F^*) = 0 \in |O_S(1)|$. It follows from $E = 0$ that Sing Z is isolated, whence Z is normal. Therefore $S \simeq Y \simeq Z \simeq \mathbf{P}^2$. From now we identify S with Z, g with the identity of Z.

Since $A_Z \simeq O_Z$, we have $h^0(X, A) = 2$ and Bs $|A| = \phi$ by $h^1(X, O_X) = 0$. Let π: $X \to \mathbf{P}^1$ be the morphism associated with $|A|$. Then by the same argument as in (2.1) we see that any fiber Z' of π is isomorphic to $\mathbf{P}^2$. Therefore X is a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$, which is isomorphic to $\mathbf{P}(\pi_*(L))$.

The direct image $\pi_*(L)$ is a locally free sheaf of rank 3 over $\mathbf{P}^1$, so that $\pi_*(L) \simeq \mathcal{F}(a', b', c')$ for some $a' \geq b' \geq c'$ by a theorem of Grothendieck. Let $a := a' - c'$, $b := b' - c'$ and $n := -c'$. Then $a + b = 3n + 2$ because $a' + b' + c' =$ deg $\pi_*(L) = \chi(\mathbf{P}^1, \pi_*(L)) - 3 = \chi(X, L) - 3 = 2$ by (2.6.3). Since dim Bs $|L| = 2$, we have $a' \geq 0$, $b' < 0$, $c' < 0$, whence $a \geq n > b \geq 0$.

(2.8) *Case* 2. Now we come back to (2.6). We have settled (2.6) *Case* 1 in (2.7). Here we consider (2.6) *Case* 2. Let F_S be a general fiber of ρ. Under the notation in (2.6) we have

$$2 = -K_S F_s = ((3r-1)M + (3r-1)N + 3g^*(F^*) + E + G)F_s.$$

We recall *supp* $(E+G) \subset$ *supp* (N) by Bertini's theorem. Hence if $(E+G) F_s \geq 1$, then $(3r-1)NF_s \geq 2$, which leads to a contradiction $-K_S F_s \geq 3$. Therefore $EF_s = 0$, $GF_s = 0$. Hence $MF_s = 1$ or $NF_s = 1$ and in either case we have $r = 1$, $g^*(L)F_s = 1$ and $g^*(F^*)F_s = 0$.

Lemma 2.8.1. *Let h: $X \to \mathbf{P}^m$ be the rational map associated with $|L|$, W the closure of the image of $X \backslash$ Bs $|L|$ and $m = h^0(X, L) - 1$. Then*
(2.8.1.1) $r = 1$, $EF_s = GF_s = g^*(F^*)F_s = 0$ *and* $g^*(L)F_s = 1$.
(2.8.1.2) dim $W \geq 2$ *if and only if* $m \geq 2$.
(2.8.1.3) *If* dim $W = 3$, *then any general M is a* smooth rational curve and $MF_s = 1$, $NF_s = 0$, $M^2 = 2$, $MN = Mg^*(F^*) = ME = MG = 0$.

Proof. (2.8.1.1) was proved above. If dim $W = 1$, then r is divisible by d: $=$deg W, whence $d = 1$, $W \simeq \mathbf{P}^1$ and $m = 1$. This proves (2.8.1.2).

Next we assume dim $W = 3$. Then $M \neq \emptyset$. If $NF_s = 1$, then $MF_s = 0$ so that $M \in |aF_s|$ for some $a \geq 1$. Then since $h \cdot g(M)$ is a point by $M^2 = 0$, whence dim $W = 2$, a contradiction. Therefore $NF_s = 0$ and $MF_s = 1$. Hence there is a unique irreducible component Γ of M such that $\Gamma F_s = 1$. Since M is general, we have $M = \Gamma$. Then we have $0 \leq \Gamma^2 \leq 2$. In fact,

$$2 - 2g = -(K_S + \Gamma)\Gamma = \Gamma^2 + (2N + 3g^*(F^*) + E + G)\Gamma \geq \Gamma^2,$$

where q is the virtual genus of Γ, whence $\Gamma^2 \leq 2$. We also see

$$\Gamma^2 = g^*(L)\Gamma - N\Gamma \geq g^*(L)\Gamma - \deg \mathrm{Bs}\, g^*|L|_\Gamma \geq \deg(h \cdot g)_{|\Gamma} \cdot \deg(h \cdot g)(\Gamma) \geq 1,$$

whence $1 \leq \Gamma^2 \leq 2$ and $N\Gamma = 0$. Hence $E\Gamma = G\Gamma = 0$ by *supp* $(E+G) \subset$ *supp* (N). Clearly $g^*(F^*)\Gamma = 0$ so that $\Gamma^2 = 2$, $g = 0$ and $\Gamma \simeq \mathbf{P}^1$.

Lemma 2.8.2. dim $W \geq 2$ *and* $h^0(X, L-F) \geq 2$.

Proof. By (2.8.1) it suffices to prove $h^0(X, L-F) \geq 2$. Assume the contrary. Hence $h^0(X, L-F) = 1$ by the assumption in (2.2). With the notation in (2.8.1) we have $g^*(Z)F_s = (M+N)F_s = 1$, whence $g_*(F_s) \neq 0$. If $g^*(F)F_s = 0$, then $F^* \equiv qF$ for some $q \geq 1$ in view of (1.5.2) because $g^*(F^*)F_s = 0$. This contradicts $h^0(X, F) \geq 2$, because F^* is the fixed part of $|L|$. Since F_s is movable, $g^*(F)F_s \geq 1$. Similarly $g^*(L-F)F_s \geq 0$ by $h^0(X, L-F) \geq 1$, whence $g^*(F)F_s = 1$, $g^*(L-F)F_s = 0$ by $g^*(L)F_s = 1$. Let $H \equiv L-F$ and $F^* \equiv pL + qF$ for some p, q. Then $p+q = 0$ by $g^*(F^*)F_s = 0$. Therefore $p \geq 0$ and $F^* \equiv pH$, whence $F^* = pH$ as effective divisors. If $p = 0$, then the liniear system $|L|$ has no fixed components so that $X \simeq \mathbf{P}(\mathcal{F}(a, b, 0))$ for some $a \geq b \geq 0$, $a+b \equiv 2 \bmod 3$ by Appendix (A.1). However then $h^0(X, L-F) \geq 2$, a contradiction. Therefore $p \geq 1$.

Since Z is irreducible reduced, we have $h^q(X, -Z) = 0$ for $q = 0, 1$, while $h^3(X, -Z) = h^0(X, -2L - F^*) = 0$. Therefore by (2.6.3)

$$h^2(X, -Z) = \chi(X, -Z) = \chi(X, (p-1)L - pF) = \frac{1}{6}p(p+1)(1-p),$$

whence $p = 1$, $Z \in |F|$. In particular, any general member of $|F|$ is irreducible reduced and F has no fixed components.

Let F, F' be two distinct general members of $|F|$. Since $F^2 = 0$, F'_F is a topologically trivial effective divisor of F. Since F is an algebraic surface, this implies $F \cap F' = \emptyset$ so that $h^0(X, F) = 2$. It follows that any general member Z of $|F|$ is smooth and $K_Z \simeq -3L_Z$, whence $Z \simeq \mathbf{P}^2$. This contradicts the assumption of *Case* 2.

Lemma 2.8.3. (*Case* 2-a) *If* dim $W = 2$, *then* $X \simeq \mathbf{P}(\mathcal{F}(a, b, 0))$ *for some* $a \geq b \geq n \geq 1$ $(a+b = 3n+2)$.

A proof of (2.8.3) is given in (2.9).

Lemma 2.8.4. (*Case* 2-b) *If* dim $W = 3$, *then* $X \simeq \mathbf{P}(\mathcal{F}(1, 1, 0))$ *or* $\mathbf{P}(\mathcal{F}(2, 0, 0))$.

Proof. We keep the notation in (2.6) and (2.8.1). We apply the results and the arguments in [N4] and [N5], some of which are reviewed in the appendix. We note that most of the arguments in [N5, §1-§3] can be applied to X. The image $C := g(M)$ of M is an irreducible component outside Bs $|L|$ of $Z \cap Z'$ for some $Z' \in |L|$ with $LC = Lg_*(M) = g^*(L)M = 2$. Moreover by [N5, Lemma

2.1], C is a smooth rational curve, which is a connected component of $Z\cap Z'$. Since $2=LC=\deg \mathrm{Bs}\,|L|_C+\deg(h_{|C})\deg W$, we have $\deg W=1$ or 2.

If $\deg W=1$, then $h^0(X, L)=4$ and we can prove by the arguments in [N5, Lemma 4.3.2] that Bs $|L|$ consists of a single point. Hence by (A. 1), $X\simeq \mathbf{P}(\mathcal{F}(a, b, 0))$ for some a, b. See Appendix. However there are no cases in (A. 3) with dim Bs $|L|=0$. Hence $h^0(X, L)=4$ is impossible. Therefore by the argument in [N5, Lemma 3.2] $h^0(X, L)=5$ and W is a hyperquadric in $\mathbf{P}^4$. We can prove Bs $|L|=\emptyset$ by applying the arguments in [N5, Lemmas 3.6-3.7]. If W is smooth, then $X\simeq \mathbf{Q}^3$ by (A.2), which contradicts $b_2(\mathbf{Q}^3)=1$. If W is singular, then $X\simeq \mathbf{P}(\mathcal{F}(1, 1, 0))$ or $\mathbf{P}(\mathcal{F}(2, 0, 0))$ by (A.2).

(2.9) *Proof of* (2.8.3). We keep the notation in (2.6) and (2.8.1). The proof is divided into several steps.

Step 1. By (2.1) we may assume that the linear system $|F|$ has a fixed component. Further we assume $h^0(X, L-2F)\geq 1$. For any general $F\in|F|$ there exists an effective divisor H such that $L\equiv 2F+H$. Therefore

$$K_S=-(4g^*F+2g^*H+3g^*F^*+E+G).$$

Since $K_SF_s=-2$, we have $g^*(H)F_s=1$, $g^*(F^*)F_s=0$, $g^*(F)F_s=0$ by (2.8.1).

Since L and F span $H^2(X, \mathbf{Z})$, we have $F^*=aL+bF$ for some integers a, b. Then as $(F_sg^*L)_S=1$, we have $0=F_sg^*F^*=aF_sg^*L+bF_sg^*F=a$, whence $F^*=bF$. Since F^* is the fixed part of $|L|$, we have $b=0$. Consequently $|L|$ has no fixed components and dim Bs $|L|\leq 1$. By (A.3), $X\simeq \mathbf{P}(\mathcal{F}(a, b, 0))$ for some $a\geq b\geq 0$. Since dim $W=2$, we have $a\geq b\geq n\geq 1$, $a+b=3n+2$ for some n.

Step 2. We assume $h^0(X, L-2F)=0$ and that the linear system $|F|$ has a fixed component. We prove that it is impossible.

We note that $h^0(X, L-F)\geq 2$ and $h^0(X, F)\geq 2$ by the assumption in (2.2) and (2.8.2). Let $Z_1'+\cdots+Z_p'+F_1^*$ (resp. $Z_1''+\cdots+Z_q''+F_2^*$) be a general member of $|L-F|$ (resp. $|F|$) where F_1^* (resp. F_2^*) is the fixed part of $|L-F|$ (resp. $|F|$). Let $Z':=Z_1'$ and $Z'':=Z_1''$. Let $g':S'\to Z'$ (resp. $g'':S''\to Z''$) be the minimal resolution of the normalization of Z' (resp. Z''). Let M' (resp. M'') be the movable part of $g'^*(Z')$ (resp. $g''^*(Z'')$) and let N' (resp. N'') be the fixed part of $g'^*(Z')$ (resp. $g''^*(Z'')$). Then we have

$$K_{S'}=-(3p-1)(M'+N')-g'^*(3qZ''+3F_1^*+3F_2^*)-(E'+G'),$$
$$K_{S''}=-(3q-1)(M''+N'')-g''^*(3pZ'+3F_1^*+3F_2^*)-(E''+G'')$$

for some effective divisors E', G'', E'' and G''' as in (2.6). There are three cases.

Case 2-1. $S'\simeq\mathbf{P}^2$.

Case 2-2. $S''\simeq\mathbf{P}^2$.

Case 2-3. S' and S'' have a morphism onto a curve with general fiber $\simeq \mathbf{P}^1$.

Case 2-1. By the assumption, $F_2^*\neq 0$ and $Z''\neq 0$. If $g'^*(Z'')=0$, then $Z'=b'F$

and $Z''=b''F$ for some $b'\geq 1$ and $b''\geq 1$ by (2.8.1). Hence $(pb'-1)F+F_1^*\in |L-2F|$, which contradicts $h^0(X, L-2F)=0$. Therefore $q=1$, $g'^*(Z'')\neq 0$, $M'=N'=g'^*(F_i^*)=E'=G'=0$. Hence $F_1^*=b'''F$, whence $(pb'+b''')F=L-F$, a contradiction

Case 2-2. The same as in *Case* 2-1.

Case 2-3. Let ρ': $S'\to B'$ (resp. ρ'': $S''\to B''$) be a morphism onto a curve with general fiber $F'_s\simeq \mathbf{P}^1$ (resp. $F''_s\simeq \mathbf{P}^1$). By the same argument as in (2.8) we see that $p=q=1$, $(M'+N')F'_s=1$ and $(M''+N'')F''_s=1$.

There exists an irreducible component Γ' of $M'+N'$ with $\Gamma' F'_s=1$. We prove that Γ' is a component of N'. Assume the contrary. Then $M'=\Gamma'$ and $(\Gamma')^2\geq 0$. Let $K_{S'}=-2\Gamma'-D'$ for an effective D'. Then $(\Gamma')^2=-(K_{S'}+\Gamma')\Gamma'-D'\Gamma'\leq 2-D'\Gamma'\leq 2$, whence $0\leq(\Gamma')^2\leq 2$.

Case 2-3-1. Assume $(\Gamma')^2=2$. Then $\Gamma'\simeq\mathbf{P}^1$, $h^1(S', O_{S'})=0$, whence S' is a rational surface and $\rho'_*O_{S'}(\Gamma')$ is a locally free $O_{\mathbf{P}^1}$-module of rank two. Let $\rho'_*O_{S'}(\Gamma')\simeq O_{\mathbf{P}^1}(c)\oplus O_{\mathbf{P}^1}(d)$. Then $c+d=2$ by $(\Gamma')^2=2$. Moreover since $h^0(S', \Gamma')=4$ and Bs $|\Gamma'|=\emptyset$, we have $(c, d)=(2, 0)$ or (1.1). In either case we have a birational morphism h': $S'\to W':=\mathbf{P}(\rho'_*O_{S'}(\Gamma'))$ ($\simeq\mathbf{F}_2$ or $\mathbf{F}_0$). We note $\Gamma'\simeq h'(\Gamma')$ and $K_{W'}\simeq h'_*(K_{S'})\simeq -2h'_*(\Gamma')$. S' is obtained from W' by repeating blowing-ups. Any rational curve C with $C^2=-1$ at any intermediate step of blowing downs is contained in the image of *supp* D' because any irreducible component of D' has the coefficient≥ 2. Therefore if S' is not isomorphic to W', then at least a blowing up is performed at a point of $h'(\Gamma')$, whence $(\Gamma')^2<h'_*(\Gamma')^2=2$. However $(\Gamma')^2=h'_*(\Gamma')^2=2$ by the assumption, which shows that $S'\simeq W'$. Hence $g'^*(Z'')=0$. Therefore by (2.8.1), we have $Z'=b'F$, $Z''=b''F$and $(pb'-1)F+F_1^*\in|L-2F|$, which contradicts $h^0(X, L-2F)=0$.

Case 2-3-2. If $(\Gamma')^2=1$, then $K_S\Gamma'+(\Gamma')^2=-(\Gamma')^2-D'\Gamma'\leq -1$. Hence $\Gamma'\simeq\mathbf{P}^1$, and $K_{S'}\Gamma'=-3$ and $D'\Gamma'=1$. However any irreducible component of D' has the coefficient≥ 2 because *supp* $(E'+G')\subset$ *supp* N'. Since $\Gamma'\not\subset D'$, we have $\Gamma'D'\geq 2$, a contradiction.

Case 2-3-3. Assume $(\Gamma')^2=0$. There is $\Gamma^*(\neq\Gamma')\in|\Gamma'|$. Hence $\Gamma'\Gamma^*=0$, $O_{\Gamma'}(\Gamma')\simeq O_{\Gamma'}$, whence Bs $|\Gamma'|=\emptyset$. Therefore any general $\Gamma'\in|\Gamma'|$ is smooth. If $K_{S'}\Gamma'=0$ (resp. $K_{S'}\Gamma'=-2$), then Γ' is a smooth elliptic curve (resp. a smooth rational curve). We have a morphism $\rho_{|\Gamma'|}$: $S'\to\mathbf{P}^1$ associated with the linear system $|\Gamma'|$. Since $\Gamma' F'_s=1$, we have a birational morphism $h':=\rho_{|\Gamma'|}\times\rho'$: $S'\to\mathbf{P}^1\times\Gamma'$ $(=: W')$. S' is obtained from W' by repeating blowing-ups. Note that $K_{W'}\simeq h'_*(K_{S'})\simeq -2h'_*(F'_s)$ (resp. $-2(h'_*(\Gamma')+h'_*(F'_s))$) if Γ' is elliptic (resp. rational). Since $(\Gamma')^2=0$ and $(F'_s)^2=0$, the centers of blowing-ups are chosen from the outside of $h'(F'_s)$ (resp. $h'(\Gamma')$ and $h'(F'_s)$). Hence it follows from the form of canonical bundles of S' and W' that $S'\simeq W'$. Hence we derive a contradiction in the same manner as in *Case* 2-3-1.

Thus we see that Γ' is an irreducible component of N'. Similarly the unique irreducible component Γ'' of $M''+N''$ with $\Gamma'' F''_s=1$ is contained in N''.

Step 3. Next we show that $g'(\Gamma')$ is a curve on X. Since $(E'+G')F'_s=0$, Γ' is not contained in *supp* $(E' + G')$. Therefore if $g'(\Gamma')$ is a point p_0, the normalization of (Z', p_0) is a Du Val singularity. Hence $(\Gamma')^2 = -2$, $K_{S'}\Gamma' = 0$, $\Gamma' \simeq \mathbf{P}^1$. On the other hand movable components Z' of $|L-F|$ (resp. Z'' of $|F|$) sweep out an open subset of X so that $g'^*(Z'')$ has a nontrivial movable component. Since $g'^*(Z'')\ F'_s = 0$, $g'^*(Z') = bF'_s$ for some $b \geq 1$. As $M'F'_s = 0$, we have $M' = aF'_s$ for some $a \geq 1$. Hence we have

$$-K_{S'}\Gamma' = 2(\Gamma')^2 + 2a + 3b + 3g'^*(F_1^* + F_2^*)\Gamma + (E' + G')\Gamma \geq 1,$$

a contradiction. Therefore $g'(\Gamma')$ is a curve on X. Similarly $g''(\Gamma'')$ is a curve on X.

Step 4. Let $Z', W' \in |Z'|$ and $Z'', W'' \in |Z''|$ be general members, and let $D_1 = Z' + W'' + F_1^* + F_2^*$ and $D_2 = W' + Z'' + F_1^* + F_2^*$. Then the intersection $l := D_1 \cap D_2$ is one-dimensional outside $F_1^* + F_2^*$. The curves $g'(\Gamma')$, $g''(\Gamma'')$ and $Z' \cap Z''$ are curve-components of l outside $F_1^* + F_2^*$. $Z' \cap Z''$ contains $g'(F'_s)$ and $g''(F''_s)$ as movable components. Note that $g'(\Gamma') \subset Z' \cap W'$ and $g''(\Gamma'') \subset Z'' \cap W''$. By [N5, Lemma 2.1] $g'(\Gamma')$ (and $g''(\Gamma'')$) is the unique irreducible component of l intersecting movable components of $Z' \cap Z''$. Therefore $g'(\Gamma') = g''(\Gamma'')$, whence it is a subset of $Z' \cap Z''$. However $g'(\Gamma') \not\subset Z''$ by $g'^*(Z'')F'_s = 0$. This is a contradiction. Thus we complete the proof of (2.8.3).

§3. Unstable rank two vector bundles over $\mathbf{P}^2$

In the present section we show that there are many Moishezon 3-folds homeomorphic to $\mathbf{P}^1 \times \mathbf{P}^2$ other than $\mathbf{P}(\mathcal{F}(a, b, 0))$ with $a + b \equiv 0 \bmod 3$. We also prove that any of them is a global deformation of $\mathbf{P}^1 \times \mathbf{P}^2$. See (3.10).

Proposition 3.1. *Let $\mathcal{E}$ be a rank two vector bundle over $\mathbf{P}^2$. Then the following conditions are equivalent.*

(3.1.1) *$\mathbf{P}(\mathcal{E})$ is homeomorphic to $\mathbf{P}^1 \times \mathbf{P}^2$.*

(3.1.2) *$c_1(\mathcal{E})^2 = 4c_2(\mathcal{E})$.*

(3.1.3) *There exists a rank two vector bundle $\mathcal{G}$ with $c_j(\mathcal{G}) = 0$ $(j = 1, 2)$ over $\mathbf{P}^2$ such that $\mathcal{E} \simeq \mathcal{G} \otimes O_{\mathbf{P}^2}(p)$ for some integer p.*

Proof. The equivalence of (3.1.2) and (3.1.3) is clear. We prove the equivalence of (3.1.1) and (3.1.2).

Let $X := \mathbf{P}(\mathcal{E})$, $S := \mathbf{P}^2$, $\alpha := c_1(O_S(1))$, $\pi : X \to S$ the natural projection, and H the tautological line bundle on X with $\pi_*(H) = \mathcal{E}$, $L := \pi^* O_S(1)$. Let $c_1(\mathcal{E}) = p\alpha$ and $c_2(\mathcal{E}) = q\alpha^2$. We have

$$\pi^* c_2(\mathcal{E}) - \pi^* c_1(\mathcal{E}) c_1(H) + c_1(H)^2 = 0.$$

See Grothendieck [G]. From this we infer

$$H^2(X, \mathbf{Z}) \simeq \mathbf{Z}H \oplus \mathbf{Z}L, \ H^4(X, \mathbf{Z}) \simeq \mathbf{Z}HL \oplus \mathbf{Z}L^2,$$
$$H^2 = pHL - qL^2, \ H^3 = p^2 - q, \ H^2L = p, \ HL^2 = 1, \ L^3 = 0.$$

On the other hand, we let $Y:=\mathbf{P}^1\times\mathbf{P}^2$, and let $A:=$ (a point) $\times\mathbf{P}^2$ and $B:=\mathbf{P}^1\times$ (a line). Then we have

$$H^2(Y,\mathbf{Z})\simeq\mathbf{Z}A\oplus\mathbf{Z}B,\ H^4(Y,\mathbf{Z})\simeq\mathbf{Z}AB\oplus\mathbf{Z}B^2,$$
$$A^2=0,\ AB^2=1,\ B^3=0.$$

Assume (3.1.1), that is, X is homeomorphic to Y. Let i: $X\to Y$ be a homeomorphism. Let $i^*(A)=aH+bL$ for some integers a and b. We note that a and b are mutually prime. Since $A^2=0$, we have $qa^2=b^2$, $pa^2+2ab=0$. Hence $p^2=4q$ and $pa+2b=0$.

Let $\mathcal{G}:=\mathcal{E}\otimes O_S\left(-\frac{p}{2}L\right)$. Then $X\simeq\mathbf{P}(\mathcal{G})$ and $c_j(\mathcal{G})=0$ $(j=1,2)$. Hence (3.1.3) follows.

Conversely if $c_j(\mathcal{G})=0$ $(j=1,2)$, then $\mathcal{G}$ is topologically trivial, whence X is homeomorphic to $\mathbf{P}^1\times\mathbf{P}^2$. Thus we see the equivalence of (3.1.1) and (3.1.2). See also [OSS, p.144] [T].

Proposition-Definition 3.2. *Let $\mathcal{G}$ be a rank two vector bundle over $\mathbf{P}^2$ with $c_j(\mathcal{G})=0$ $(j=1,2)$.*
(3.2.1) *If $\mathcal{G}$ is semi-stable, then $\mathcal{G}\simeq O_{\mathbf{P}^2}^{\oplus 2}$.*
(3.2.2) *If $\mathcal{G}$ is unstable, then there exists a positive integer p and an ideal sheaf I of $O_{\mathbf{P}^2}$ defining a 0-dimensional locally complete intersection subscheme Σ of $\mathbf{P}^2$ with $O_\Sigma:=O_{\mathbf{P}^2}/I$ such that $h^0(O_\Sigma)=p^2$ and the following sequence is exact.*

$$0\to O_{\mathbf{P}^2}(p)\to\mathcal{G}\to IO_{\mathbf{P}^2}(-p)\to 0.$$

We define $\mathrm{sp}^+(\mathcal{G}):=p$ *and call it the (reduced) spectrum of $\mathcal{G}$. We set* $\mathrm{sp}^+(\mathcal{G})=0$ *if $\mathcal{G}\simeq O_S^{\oplus 2}$. We also denote $\Sigma:=\mathrm{disc}(\mathcal{G})$ and call it the discriminant of $\mathcal{G}$.*

Proof. Let $S:=\mathbf{P}^2$. If $\mathcal{G}$ is semi-stable, then $\mathcal{G}$ is represented by a complex called a monad [OSS, p. 251]. Indeed, $\mathcal{G}$ is the cohomology of the following complex

$$H^1(S,\mathcal{G}(-2))\otimes O_S(-1)\to H^1(S,\mathcal{G}\otimes\Omega_S^1)\otimes O_S\to H^1(S,\mathcal{G}(-1))\otimes O_S(1).$$

If $c_j(\mathcal{G})=0$, then $H^1(S,\mathcal{G}(-2))=H^1(S,\mathcal{G}(-1))=0$, whence $\mathcal{G}\simeq O_S^{\oplus 2}$.

Next we prove (3.2.2). Since $\mathcal{G}$ is unstable, $\mathcal{G}$ has a rank one subsheaf E with positive degree $p\geq 1$. We may assume that E is saturated. Hence E is reflexive, so that E is locally free. Therefore $E\simeq O_S(p)$ for some $p\geq 1$. Let $F:=\mathcal{G}/E$. Since F is torsion free, there exist an integer q and an ideal sheaf I of O_S such that $F\simeq IO_S(q)$ with dim *supp* $O_S/I=0$. As $\mathcal{G}$ is locally free, I is spanned by a system of two parameters. We define a subscheme Σ by $O_\Sigma:=O_S/I$. Then Σ is locally a complete intersection. Since $c_j(\mathcal{G})=0$, we have $q=-p$. Moreover we see that the following sequence is exact,

$$0\to E\otimes F^\vee\to E\otimes\mathcal{G}^\vee\to O_S\to O_S/I\to 0,$$

where $F^\vee\simeq O_S(p)$. It follows that $h^0(O_\Sigma)=\chi(O_S(2p))-2\chi(O_S(p))+1=p^2$.

(3.3) The stucture of P $(\mathcal{E})$. Let $\mathcal{E}$ be a topologically trivial rank two vector bundle over $\mathbf{P}^2$ and $\pi(\mathcal{E})$ the natural projection of $\mathbf{P}(\mathcal{E})$ onto $\mathbf{P}^2$. Let $L(\mathcal{E}):=\pi(\mathcal{E})^*O_{\mathbf{P}^2}(1)$ and $F(\mathcal{E})$ the tautological line bundle with $\pi(\mathcal{E})_*(F(\mathcal{E}))\simeq\mathcal{E}$. Then we see

$$K_{\mathbf{P}(\mathcal{E})}\simeq -2F(\mathcal{E})+\pi(\mathcal{E})^*(K_{\mathbf{P}^2}+\det\mathcal{E})\simeq -2F(\mathcal{E})-3L(\mathcal{E}).$$

We also have $H^0(\mathbf{P}(\mathcal{E}), L(\mathcal{E}))\simeq H^0(\mathbf{P}^2, O_{\mathbf{P}^2}(1))$. Let $p:=\mathrm{sp}^+(\mathcal{E})$ and $\Sigma:=$ disc $(\mathcal{E})$, and I the ideal of $O_{\mathbf{P}^2}$ defining Σ. Assume $p\geq 1$. Then the following sequence is exact,

$$0\to O_{\mathbf{P}^2}(p)\to\mathcal{E}\to IO_{\mathbf{P}^2}(-p)\to 0,$$

whence $H^0(\mathbf{P}(\mathcal{E}), F(\mathcal{E}))\simeq H^0(\mathbf{P}^2, O_{\mathbf{P}^2}(p))$. Let G^* be the fixed component of the linear system $|F(\mathcal{E})|$. Then we have $|F(\mathcal{E})|=|pL(\mathcal{E})|+G^*$ and G^* is defined by the ideal generated by $H^0(\mathbf{P}^2, O_{\mathbf{P}^2}(p))$, hence by the subsheaf $O_S(p)$ of $\mathcal{E}$. Therefore $G^*\simeq\mathbf{P}(IO_{\mathbf{P}^2}(-p))\simeq\mathbf{P}(I)$, which is the blowing-up of $\mathbf{P}^2$ with Σ center.

If $\mathrm{sp}^+(\mathcal{E})=0$, then $\mathcal{E}\simeq O_{\mathbf{P}^2}^{\oplus 2}$ and $\mathbf{P}(\mathcal{E})\simeq\mathbf{P}^1\times\mathbf{P}^2$.

(3.4) Some unstable bundle over $\mathbf{P}^2$. Let $S:=\mathbf{P}^2$, p_0 a point of S, and let $\sigma: W\to S$ be the blowing-up of S with p_0 center. Let $C:=\sigma^{-1}(p_0)\simeq\mathbf{P}^1$. For any integer $p>0$, we choose a nontrivial extension of locally free O_C-modules

(3.4.1) $$0\to O_C(-p)\to O_C^{\oplus 2}\to O_C(p)\to 0.$$

Then [OSS, pp. 120-122] shows there exists a rank two vector bundle $\mathcal{F}$ over W such that $\mathcal{F}\simeq O_W^{\oplus 2}$ near C, and

(3.4.2) $\mathcal{F}$ is a nontrivial extension given by the exact sequence,

$$0\to O_W(pC)\otimes\sigma^*O_S(p)\xrightarrow{\xi}\mathcal{F}\xrightarrow{\eta}O_W(-pC)\otimes\sigma^*O_S(-p)\to 0$$

whose restricition to C gives (3.4.1)

Then the sheaf $\sigma_*(\mathcal{F})$ is a rank two vector bundle over S with $c_j(\sigma_*(\mathcal{F}))=0\ (j=1, 2)$. See [OSS, chapter I, §6] for the detail. The extension (3.4.1) is given by two homogeneous polynomials $f_1(x_0, x_1)$ and $f_2(x_0, x_1)$ of degree p having no zeroes on $\mathbf{P}^1$ in common. The sheaf $\sigma_*(\mathcal{F})$ fits in the exact sequence,

(3.4.3) $$0\to O_S(p)\xrightarrow{\sigma_*(\xi)}\sigma_*(\mathcal{F})\xrightarrow{\sigma_*(\eta)}m^pO_S(-p)\to\mathbf{C}^{\oplus p(p-1)/2}\to 0.$$

where m is the maximal ideal of O_S defining p_0. Let x and y be a local coordinate at p_0. Then there exists a germ of holomorphic function $F_i(x, y)$ at p_0 such that $F_i(x, y)\equiv f_i(x, y)$ mod m^{p+1} and $\sigma_*(\xi)$ is locally given by the pair (F_1, F_2) at p_0. Defining an ideal I of O_S by $I:=O_SF_1(x, y)+O_SF_2(x, y)$ at p_0 and $I:=O_S$ elsewhere, we have Im $\sigma_*(\eta)=IO_S(-p)$. Thus we have the exact sequence

(3.4.4) $$0 \to O_S(p) \to \mathcal{G} \to IO_S(-p) \to 0.$$

Lemma 3.5. *Let* σ: $W \to Y$ *be a blowing-*up of a surface Y with $p_0 \in Y$ *center, E the exceptional curve of* σ, *L a line bundle on Y and I an ideal sheaf of* O_Y *with* dim *supp* $(O_Y/I)=0$. *Suppose that we are given a rank two vector bundle F over Y such that*

(3.5.1) $$0 \to L \overset{\xi}{\to} F \overset{\eta}{\to} IL^{-1} \to 0$$

is exact. Let $a := \min\{\mathrm{mult}_{p_0} f; f \in I\}$ *and* $N := 0_W(aE) \otimes \sigma^*(L)$. Then there exist a rank two vector bundle $G := \sigma^*(F)$ *on W and an ideal sheaf J of* O_W *with dim supp* $(O_W/J) \leq 0$ *such that*

(3.5.2) $h^0(O_W/J) = h^0(O_Y/I) - a^2$,

(3.5.3) $0 \to N \overset{\xi'}{\to} G \overset{\eta'}{\to} JN^{-1} \to 0$ *is exact,* and

(3.5.4) *the direct image of* (3.5.3) *by* σ_* *induces* (3.5.1).

Proof. The homomorphism ξ is given by a pair (s_1, s_2) of germs of functions locally at p_0 by trivialising F and L, say, $\xi(u) = (us_2, -us_1)$ and $\eta(v_1, v_2) = s_1v_1 + s_2v_2$. Let $t=0$ be a local equation of E at a point $q \in E$, $\sigma_{i,q} := t^{-a}\sigma^* s_i$. We define $J := O_W\sigma_{1,q} + O_W\sigma_{2,q}$, and the homomorphisms ξ': $N \to G$ and η': $G \to N^{-1}$ at q by

$$\xi'(u') := (u'\sigma_{2,q}, -u'\sigma_{1,q}),\ \eta'(v_1', v_2') = \sigma_{1,q}v_1' + \sigma_{2,q}v_2'.$$

It is easy to see that ξ' and η' are globally well defined. Let C_i be a local curve defined by $s_i = 0$ at p_0, and $C_i' := \sigma^*(C_i) - aE$. Then I is the ideal defining the complete intersection $C_1 \cap C_2$ at p_0. Let J be the ideal defining $C_1' \cap C_2'$ along E and $J = \sigma^*(I)$ elsewhere. We prove (3.5.2). We have

$$h^0(S, O_S/I) - h^0(W, O_W/J) = h^0(U, O_S/I) - h^0(V, O_W/J) = C_1C_2 - C_1'C_2' = a^2,$$

where U (resp. V) are sufficiently small open neighborhoods of p_0 (resp. E).

The condition (3.5.3) is clear from the definitions.

Finally we prove (3.5.4). By taking the direct image of (3.5.3) by σ_*, we obtain an exact sequence

$$0 \to \sigma_*(N) \overset{\sigma_*(\xi')}{\to} \sigma_*(G) \overset{\sigma_*(\eta')}{\to} \sigma_*(JN^{-1})\ (\subset \sigma_*(N^{-1})) \to 0$$

where $\sigma_*(N) \simeq L$, $\sigma_*(G) \simeq F$ and $\sigma_*(\xi') = 0$. Moreover since $\sigma_*(JN^{-1})$ is canonically a subsheaf of L^{-1}, the homomorphism $\sigma_*(\eta')$ can be viewed as a homomorphism of F into L^{-1}, which coincides with η. This is what we claim in (3.5.4).

Corollary 3.6. *Let* $\mathcal{G}$ *be an unstable rank two vector bundle over* $\mathbf{P}^2$ *with* $c_j(\mathcal{G}) = 0$ $(j=1, 2)$. *Then there exists a modification* σ: $W \to \mathbf{P}^2$, *a rank two vector bundle G and a line bundle N on W such that*

(3.6.1) *G is an extension with* $0 \to N \to G \to N^{-1} \to 0$ exact,
(3.6.2) *$\mathcal{G} \simeq \sigma_*(G)$ and the direct image of* (3.6.1) *induces the sequence in* (3.2.2).

The minimal modification σ and N are uniquely determined by the ideal $I := I_{\mathrm{disc}(\mathcal{G})}$.

Proof. Clear from (3.5).

Next we show that (at least) some of the 3-folds $\mathbf{P}(\mathcal{G})$ can be deformed into $\mathbf{P}^1 \times \mathbf{P}^2$ by deforming the vector bundle $\mathcal{G}$. The following lemmas (3.7) and (3.8) were suggested (in fact given for $\mathrm{sp}^+(\mathcal{G}) = 1$ by Maruyama.

Lemma 3.7. *Let $\mathcal{G}$ be a rank two vector bundle over $\mathbf{P}^2$. Then the following conditions are equivalent.*
(3.7.1) *$\mathcal{G}$ is an unstable bundle with $c_j(\mathcal{G}) = 0$ ($j = 1, 2$) such that $\mathrm{sp}^+(\mathcal{G}) = p$ and $\mathrm{disc}(\mathcal{G})$ is a complete intersection of two curves of degree p.*
(3.7.2) *There exist a (possibly reducible nonreduced) curve C of degree $4p$ and a surjective homomorphism $\phi(2p): O_S^{\oplus 2}(2p) \to O_S(3p) \otimes O_C (=: O_C(3p))$ such that $\mathcal{G} \simeq \mathrm{Ker}\ \phi(2p)$.*

Proof of (3.7). *Step* 1. (Maruyama) Let $S = \mathbf{P}^2$ and $O_S(1)$ a hyperplane bundle S. Let C be any (possibly nonreduced) irreducible curve of degree $4p$ in S, L a line bundle on C such that $\deg L = 4p^2$ and $\mathrm{Bs}\,|L| = \emptyset$. Suppose that we are given a surjective homomorphism ϕ: $O_S^{\oplus 2} \to L \otimes O_C$ as $\phi(a_1 \oplus a_2) = a_1\bar{s}_1 + a_2\bar{s}_2$ with two global sections s_i of L. By the syzyzy theorem (see [AK, Chapter III (5.7), (5.8), (5.19)], $\mathrm{Ker}\ \phi$ is locally O_S-free of rank two. Let $\phi(k) := \phi \otimes O_S(k)$ and $E := E(C, L, \phi) = \mathrm{Ker}\ \phi(2p)$. Then $c_j(E) = 0$ for $j = 1, 2$. Consider the exact sequence

$$(3.7.3) \qquad 0 \to E(-p) \to O_S(p)^{\oplus 2} \xrightarrow{\phi(p)} L \otimes O_C(p) \to 0.$$

Assume that $L \simeq O_C(p)$. Then since $H^0(O_S(2p)) \simeq H^0(O_C(2p))$ and ϕ is surjective, ϕ is given by two homogeneous polynomials s_1 and s_2 of degree p with no irreducible factors in common. We also have $h^0(E(-p)) = \dim \mathrm{Ker}\ H^0(\phi(p)) = 1$. In fact, $H^0(O_S(2p)) \simeq H^0(O_C(2p))$ so that Ker $H^0((\phi)(p))$ $H^0(\phi(p))$ is generated by the pair $(s_2, -s_1)$. Similarly we have $h^0(E(-p-1)) = 0$. It follows that we have an injective homomorphism ι: $O_S(p) \to E$, which yields an exact sequence

$$(3.7.4) \qquad 0 \to O_S(p) \xrightarrow{\iota} E \to IO_S(-p) \to 0$$

where $I = s_1O_S + s_2O_S$ is an ideal of O_S. This shows that E is an unstable rank two bundle with $\mathrm{sp}^+(E) = p$. Clearly $\mathrm{disc}(E)$ is a complete intersection defined by the ideal I.

Step 2. We prove that (3.7.1) implies (3.7.2). Let $p := \mathrm{sp}^+(\mathcal{G})$. We start with recalling the exact sequence

(3.7.5) $$0 \to O_S(p) \xrightarrow{\xi} \mathcal{G} \xrightarrow{\eta} IO_S(-p) \to 0$$

where $p := \mathrm{sp}^+(\mathcal{G})$. Tensoring the dual of (3.7.5) with $O_S(2p)$, we obtain an exact sequence

$$0 \to O_S(3P) \xrightarrow{\eta^\vee(2p)} \mathcal{G}^\vee(2p) \xrightarrow{\xi^\vee(2p)} IO_S(p)\ (\subset O_S(p)) \to 0.$$

On the other hand, since disc $(\mathcal{G})$ is a complete intersection, we have an exact sequence

$$0 \to O_S(-2p) \to O_S(-p)^{\oplus 2} \xrightarrow{\tau} I \to 0,$$

whence we have $h^0(IO_S(p)) = 2h^0(O_S) = 2$. Therefore we have two sections σ_i ($i = 1, 2$) of $\mathcal{G}^\vee(2p)$ such that $s_i := H^0(\xi^\vee(2p))(\sigma_i)$ generate $H^0(IO_S(p))$. Using σ_i, we define a homomorphism Ψ: $\mathcal{G} \to O_S(2p)^{\oplus 2}$ by $\Psi(a) := (a\sigma_2, -a\sigma_1))$. We consider the following commutative diagram with exact rows and columns. The nine lemma shows that $Q :=$ Coker $\Psi \simeq$ Coker ψ.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & O_S(p) & \xrightarrow{\xi} & \mathcal{G} & \xrightarrow{\eta} & IO_S(-p) & \longrightarrow & 0 \\
 & & \downarrow \simeq & & \downarrow \Psi & & \downarrow \psi & & \\
0 & \longrightarrow & O_S(p) & \xrightarrow{(s_2, -s_1)} & O_S(2p)^{\oplus 2} & \xrightarrow{\tau \otimes O_S(3p)} & IO_S(3p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \Lambda & & \downarrow \lambda & & \\
0 & \longrightarrow & 0 & \longrightarrow & Q & \xrightarrow{\simeq} & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Moreover we see

CLAIM 3.7.6 $\underline{Hom}(IO_S(-p), IO_S(3p)) \simeq O_S(4p)$.

Proof of (3.7.6). Let $\Sigma := \mathrm{disc}(\mathcal{G})$. Since Σ is a complete intersection, we have a locally free resolution of O_Σ as follows,

$$0 \to O_S(-2p) \xrightarrow{(s_2, -s_1)} O_S(-p)^{\oplus 2} \xrightarrow{\tau} O_S \to O_\Sigma \to 0.$$

Hence $\underline{Ext}^q(O_\Sigma, O_S)$ is the q-th cohomology of the complex of O_S-modules

$$\underline{Hom}(O_S, O_S) \to \underline{Hom}(O_S(-p), O_S) \to \underline{Hom}(O_S(-2p), O_S),$$

whence $\underline{Ext}^q(O_\Sigma, O_S) = 0$ ($q = 0, 1$). Now we consider the exact sequence,

$$0 \to I \to O_S \to O_\Sigma \to 0,$$

from which we infer $\underline{Hom}\ (I, O_S) \simeq \underline{Hom}\ (O_S, O_S) \simeq O_S$. We note that the isomorphsim is induced from the natural inclusion of I into O_S. Consequently we see $\underline{Hom}\,(I, I) \simeq \underline{Hom}\,(I, O_S) \simeq O_S$, whence (3.7.6).

Now we complete the proof of (3.7). By the proof of (3.7.6) we see that the homomorphism ψ is just the multiplication by a homogeneous polynomial h of degree $4p$. Let ψ_0 be the homomorphism of $O_S(-p)$ into $O_S(3p)$ defined by the multiplication by h. Let C be a curve defined by $h=0$ and $O_C := O_S/hO_S$. Then there is a natural homomorphism j of $Q\,(\simeq \text{Coker}\ \psi)$ into $O_C\,(3p)$ ($\simeq$ Coker ψ_0). Since depth $Q=0$, j is injective so that $Q \simeq IO_C\,(3p)$. We show that $Q \simeq O_C\,(3p)$. Let $m := \dim O_C/IO_C$. Then we have

$$c\,(\mathscr{G}) = c\,(O_S\,(2p))^2 c\,(Q)^{-1} = c\,(O_S\,(2p))^2 c\,(O_C\,(3p))^{-1} c\,(O_C/IO_C) = 1 + mH^2$$

where H is a hyperplane of S. Hence $m=0$, which shows $Q \simeq IO_C\,(3p) \simeq O_C\,(3p)$. It follows that $C \cap \text{disc}\,(\mathscr{G}) = \emptyset$. This proves (3.7.2).

Lemma 3.8 (Maruyama). *Let $\mathscr{G}$ be an unstable vector bundle over* $\mathbf{P}^2$ *of rank two with $c_j(\mathscr{G}) = 0$ ($j = 1, 2$). Assume* $\text{sp}^+(\mathscr{G}) = p$ *and that* disc $(\mathscr{G})$ *is a complete intersection of curves of degree p. Then there exists a flat $O_{\mathbf{P}^2 \times D}$-module $\mathscr{F}$ such that $\mathscr{F}_0 \simeq \mathscr{G}$ and* $\text{sp}^+(\mathscr{F}_t) \leq p-1$ $(t \neq 0)$ *where D is a connected curve and $\mathscr{F}_t := \mathscr{F} \otimes O_{\mathbf{P}^2 \times \{t\}}$.*

Proof. We keep the notation in (3.7). Let $E := E\,(C, L, \phi)$. Note that $c_j(E) = 0$. Since $H^0(O_S\,(p)) \simeq H^0(O_C\,(p))$, we have $H^0(E\,(-p)) \simeq \text{Ker}\ H^0(\phi\,(p) \simeq \text{Ker}\ H^0(\phi\,(p)_{|C})$. On the other hand by the exact sequence

$$0 \to O_C\,(p) \otimes L^{-1} \to O_C\,(p)^{\oplus 2} \xrightarrow{\phi(p)_{|C}} O_C\,(p) \otimes L \to 0,$$

we have Ker $\phi\,(p)_{|C} \simeq H^0(O_C\,(p) \otimes L^{-1})$. Hence $H^0(E\,(-p)) \simeq H^0(O_C\,(p) \otimes L^{-1})$. Therefore $h^0(E\,(-p)) \geq 1$ if and only if $L \simeq O_C\,(p)$ because deg $L =$ deg $O_C\,(p) = 4p^2$. If L is not $O_C\,(p)$, then $E \simeq O_S^{\oplus 2}$ or E is unstable with $\text{sp}^+(E) \simeq p-1$ by (3.2). Thus we have a desired that $O_{\mathbf{P}^2 \times D}$-module $\mathscr{F}$ parametrized by a curve D in Pic C.

Lemma 3.9. *Any unstable rank two bundle $\mathscr{G}$ over* $\mathbf{P}^2$ *with $c_j(\mathscr{G}) = 0$ ($j=1$, 2) can be deformed into the trivial vector bundle $O_{\mathbf{P}^2}^{\oplus 2}$ (under flat deformation).*

Proof. Any unstable rank two bundle E over $S := \mathbf{P}^2$ is given as an extension of $O_S\,(p)$ by $IO_S\,(-p)$ for some positive integer p and a locally complete intersection ideal I of O_S. The extension class $\delta\,(E)$ belongs to

$$\text{Ext}^1(IO_S\,(-p), O_S\,(p)) \simeq \text{Ext}^1(I, O_S\,(2p)) \simeq O_\Sigma$$

where $\Sigma := \text{disc}\,(E)$. Now we consider a flat deformation of O_Σ with $\text{sp}^+(E)$ constant. In other words, we choose a point q of *supp* (Σ) and a local generator f and g of the stalk I_q. Then we choose a pertubation $F\,(t)$ and $G\,(t)$ with $F\,(0) = f$ and $G\,(0) = g$. We let Δ be the unit disc, $\mathscr{S} := S \times \Delta$, $\mathscr{I} := (F, G)$ the ideal of

$O_{\mathcal{S}}$ generated by F and G. Then we have

$$\mathrm{Ext}^1(\mathcal{J}O_{\mathcal{S}}(-p), O_{\mathcal{S}}(p)) \simeq \mathrm{Ext}^1(\mathcal{J}, O_{\mathcal{S}}(2p)) \simeq O_{\mathcal{S}}/(F, G)$$

where $O_{\mathcal{S}}(k) := O_S(k) \boxtimes O_{\Delta}$. We choose an extension $\mathcal{E}$ whose extension class is $\delta(\mathcal{E}) \in O_{\mathcal{S}}/(F, G)$ with $\delta(\mathcal{E})_{|t=0} = \delta(E)$. Then we have an exact sequence

$$0 \to O_S(p) \to \mathcal{E} \to \mathcal{J}O_{\mathcal{S}}(-p) \to 0.$$

Therefore $\mathcal{E}$ is a coherent O_S-Module, whence $\mathcal{E}$ is a locally free O_S-Module of rank two by shrinking Δ if necessary because E is locally free. Let $E_t := \mathcal{E} \otimes O_{S\times t}$. Then it is clear $h^0(S, E_t) = h^0(S, O_S(p))$, whence $\mathrm{sp}^+(E_t) = p$ and $I_{\mathrm{disc}(E_t)} = \mathcal{J}O_{S\times t}$.

If we choose a sufficiently general F and G at any point of *supp* (Σ), we have reduced $\mathrm{disc}(E_t)$, that is, a union of distinct p^2 points. The set of p^2 distinct points in suitable position is a complete intersection of two curves on S of degree p. Then $E_t \simeq E(C, L, \phi)$ for some triplet C, L and ϕ by (3.7). Then by (3.8) E_t can be deformed into an unstable E' with $c_j(E') = 0$ $(j=1, 2)$ and $\mathrm{sp}^+(E') \leq p-1$. It follows from the induction on sp^+ that any unstable E with $c_j(E) = 0$ $(j=1, 2)$ can be deformed into the trivial bundle $O_S^{\oplus 2}$.

From (3.9), we infer

Proposition 3.10. *Let $\mathcal{G}$ be an unstable rank two bundle over $\mathbf{P}^2$ with $c_j(\mathcal{G}) = 0$ $(j=1, 2)$. Then $\mathbf{P}(\mathcal{G})$ is a global deformation of $\mathbf{P}^1 \times \mathbf{P}^2$.*

§4. Global deformations of $\mathbf{P}(\mathcal{F}(a, b, 0))$ with $a+b \equiv 0 \bmod 3$

The main purpose of this section is to prove

Theorem 4.1. *The set of all $\mathbf{P}^2$-bundles $\mathbf{P}(\mathcal{F}(a, b, 0))$ over $\mathbf{P}^1$ with $a+b \equiv 0 \bmod 3$ and of all $\mathbf{P}^1$-bundles $\mathbf{P}(\mathcal{E})$ over $\mathbf{P}^2$ with $\mathcal{E}$ topologically trivial rank two vector bundles is stable and transitive under global deformation.*

(4.2) Conditions. Let X be a fake $\mathbf{P}^1 \times \mathbf{P}^2$, L and F canonical generators of Pic X. We consider the following conditions

(4.2.1) $$h^0(X, L) \geq 3,\ h^0(X, L-F) = 0,\ h^0(X, F) \geq 2.$$

It is easy to derive from (1.4.0)

(4.2.2) $$\chi(X, pL+qF) = \frac{1}{2}(p+1)(p+2)(q+1).$$

Lemma 4.3. *Let X be a fake $\mathbf{P}^1 \times \mathbf{P}^2$, L and F canonical generators of* Pic X. *If $h^0(X, L) \geq 3$, $h^0(X, F) \geq 2$, then $X \simeq \mathbf{P}(\mathcal{F}(a, b, 0))$ or $X \simeq \mathbf{P}(\mathcal{E})$ where $a \geq b \geq 0$, $a+b \equiv 0 \bmod 3$, while $\mathcal{E}$ is a rank two vector bundle over $\mathbf{P}^2$ with $c_j(\mathcal{E}) = 0$ $(j=1, 2)$.*

(4.4) Proof of (4.3) —Start. First we consider the simplest case.

Lemma 4.4.1. *Let X be a fake $\mathbf{P}^1\times\mathbf{P}^2$, L and F canonical generators of* Pic X. *Assume* (4.2.1) *and that $|F|$ has no fixed components. Then $X\simeq\mathbf{P}^1\times\mathbf{P}^2$.*

Proof. We can prove in the same manner as in (2.1) that $F_F\simeq O_F$, $h^0(X, F)=2$ and Bs $|F|=\emptyset$. Let F be a general member of $|F|$. Then Bs $|F|=\emptyset$, F is smooth and irreducible. Since $K_F=-3L_F$, we have $F\simeq\mathbf{P}^2$ and $L_F\in|O_{\mathbf{P}_2}(1)|$. Let $\pi:=\rho_F: X\to\mathbf{P}^1$ be the morphism associated with $|F|$. Then it is easy to see that π is a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$. We see $X\simeq\mathbf{P}(\pi_*L)$ and $\pi_*L\simeq O_{\mathbf{P}^1}(a')\oplus O_{\mathbf{P}^1}(c')$ for some $a'\geq b'\geq c'$. Since $h^0(X, L-F)=0$, we have $a'\leq 0$, while $a'+b'+c'=0$. Hence $a'=b'=c'=0$ and $X\simeq\mathbf{P}^1\times\mathbf{P}^2$.

In view of (2.2) Claim and (4.2) we may assume $h^0(X, L-F)=0$. We also assume in what follows in (4.4) and (4.5) that X is not isomorphic to $\mathbf{P}^1\times\mathbf{P}^2$. By (4.4.1) $|F|$ has fixed components.

Lemma 4.4.2.
(4.4.2.1) *The linear system $|L|$ has no fixed components.*
(4.4.2.2) *Any general member Z of $|L|$ is irreducible and reduced.*

Proof. First we prove (4.4.2.1). Assume that $|L|$ has fixed components. Let $V_1+\cdots+V_r+F^*\in|L|$ be a general member of $|L|$, V_j movable components and F^* fixed components. Let $V=V_1$ and $g:=S\to V$ be the minimal resolution of the normalization of V. Then the canonical line bundle of S is given by $K_S=-g^*((3r-1)V+3F^*+2F)-(E+G)$ as in the proof of (2.1). We note that *supp* $(E+G)\subset$ *supp* (g^*V') for general V' linearly equivalent to V.

Since $-K_S$ is effective, $S\simeq\mathbf{P}^2$ or S has a morphism $\pi: S\to C$ onto a curve with general fiber $F_s\simeq\mathbf{P}^1$. If $S\simeq\mathbf{P}^2$, then $F_V^*\simeq O_V$ or $F_V\simeq O_V$. In either case $V\in|aF|$ for some $a\geq 1$ by (1.5). Hence $h^0(X, L-F)\geq 1$, which contradicts (4.2.1). Therefore S has a morphism $\pi: S\to C$ onto a curve with general fiber $F_s\simeq\mathbf{P}^1$. Then we have

$$2=-K_SF_s=g^*((3r-1)V+3F^*+2F)F_s+(E+G)F_s.$$

It follows that $F^*F_s=0$ and that $VF_s=1$ or $FF_s=1$. If $VF_s=1$, then $r=1$ and $LF_s=1$, $FF_s=0$. Let $F^*\equiv pL+qF$. Then $p=F^*F_s=0$, whence $q\geq 1$ and $h^0(X, L-F)\geq 1$, a contradiction. If $FF_s=1$, then $VF_s=LF_s=0$. Let $F^*\equiv pL+qF$. Then $q=F^*F_s=0$, whence $p\geq 1$ and $F^*\in|pL|$, a contradiction.

Next we prove (4.4.2.2). Let $D=Z_1+\cdots+Z_r$ be a general member of $|L|$, Z_i movable by (4.4.1). Then we have $r^2Z^2F=L^2F=1$, whence $r=1$.

Lemma 4.4.3. *Let Z and Z' be general members of $|L|$, and $l:=Z\cap Z'$. Then*
(4.4.3.1) $h^0(O_Z)=1$, $h^q(O_Z)=0\,(q\geq 1)$.
(4.4.3.2) $h^q(O_Z(-L))=0\,(q\geq 0)$.
(4.4.3.3) $h^q(O_Z(-2L))=0\,(q\neq 1)$, $h^1(O_Z(-2L))=1$.
(4.4.3.4) $h^0(O_l(-pL))=1$, $h^1(O_l(-pL))=0\,(p=0, 1)$.

Proof. We see $h^2(X, -3L)=1$ and $h^q(X, -pL)=0$ $(1\leq p\leq 3\ ;0\leq q\leq 3)$ except for $(p, q)=(3, 2)$. In fact, since Z is irreducible, we have $h^1(X, -pL)=0$ for $p\leq 1$. We also see $h^0(X, -pL)=h^3(X, -pL)=0$ for $1\leq p\leq 3$. Hence we have $h^2(X, -3L)=\chi(X, -3L)=1$, while $h^2(X, -pL)=\chi(X, -pL)=0$ for $p=1, 2$. (4.4.3) follows from it readily.

Lemma 4.4.4. *Let* $m:=h^0(X, L)-1$ *and* $\rho_L: X\to \mathbf{P}^m$ *the rational map associated with* $|L|$. *Then* $\dim \operatorname{Im} \rho_L\geq 2$.

Proof. Let $B:=\mathrm{BS}\,|L|$, W the closure of $\rho_L(X\backslash B)$ and $d:=\deg W$. Assume $\dim W=1$. Then d is equal to the number of irreducible components of a general member of $|L|$, whence $d=1$ by (4.4.2). Hence $m=1$, which contradicts $h^0(X, L)\geq 3$.

Lemma 4.4.5. *Let* Z *and* Z' *be general members of* $|L|$, *and* $l:=Z\cap Z'$. *Then* l *is a smooth rational curve with* $Ll=0$ *and* $Fl=1$.

Proof. Step 1. In view of (4.4.4), l has movable irreducible components. Let $C_i (1\leq i\leq r)$ be movable components of l. Then $LC_i\geq 0$ and $FC_i\geq 0$. Let $C=C_1$, $\alpha:=LC\geq 0$ and $\beta:=FC\geq 0$. By (4.4.3.4) we have $h^1(O_C)=0$, whence C is a smooth rational curve. We also see $h^1(O_C(-L))=0$ by (4.4.3.4), whence $0\leq LC\leq 1$.

We set $I_C/I_C^2\simeq O_C(a)\oplus O_C(b)$ for some integers $a\geq b$. It follows $a+b=K_XC+2=-(3\alpha+2\beta)+2$. Then since l is reduced generically along C, we have an injective homomorphism

$$\phi: (I_l/I_l^2)\otimes O_C(\simeq O_C(-\alpha)\oplus O_C(-\alpha)\to I_C/I_C^2(\simeq O_C(a)\oplus O_C(b)),$$

whence $\alpha+2\beta\leq 2$. It follows that $(\alpha, \beta)=(1, 0)$, or $\alpha=0$, $0\leq\beta\leq 1$.
Step 2. First we assume $LC=1$. Then by *Step* 1, $FC=0$. Let $V_1+\cdots+V_s+G^*$ be a general member of $|F|$, V_j (resp. G^*) a movable component (resp. the fixed components) and $V:=V_1\equiv V_j$. Then since C is movable and $FC=0$, we have $VC=G^*C=0$. Let $V:\equiv pL+qF$ for some integers p and q. Then $p=VC=0$ so that $V\in|qF|$. Similarly $G^*\in|q^*F|$ for some q^*, whence $sq+q^*=1$. It follows from $h^0(X, F)\geq 2$ that $s=q=1$, $q^*=0$. Thus any general member of $|F|$ is irreducible and reduced. Hence $|F|$ has no fixed components. Therefore $X\simeq\mathbf{P}^1\times\mathbf{P}^2$ by (4.3). However in this case $0=L^3=Ll=LC=1$, a contradiction.
Step 3. By *Step* 2, $LC=0$. Since l is general, Sing l is contained Bs $|L|$. If C intersects Sing l, then C is contained in Bs $|L|$, a contradiction. Therefore C is a connected component of l. By (4.4.3.4), l is connected so that $l\simeq C$ and $r=1$. It follows that Bs $|L|=\emptyset$ and that $FC=L^2F=1$.

Lemma 4.4.6. Bs $|L|=\emptyset$, $L\otimes O_l\simeq O_l$ *and* $h^0(X, L)=3$.

Proof. Bs $|L|=\emptyset$ and $h^0(X, L)=3$ are clear from (4.4.3.4) and the proof of (4.4.4). Hence there exists a third member Z' of $|L|$ such that Z' does not contain l. Since Z, Z' and Z' are pull-backs of hyperplanes of $\mathbf{P}^2$ by ρ_L, the intersection $Z \cap Z' \cap Z''$ is empty, whence $L \otimes O_l \simeq Z'' \otimes O_l \simeq O_l$.

Lemma 4.4.7. *Any member of $|L|$ is irreducible reduced.*

Proof. Let $Z_1 + \cdots + Z_r \in |L|$, Z_i irreducible components. Let $Z_i \equiv p_i L + q_i F$. Let Z and Z' be general members of $|L|$, $C := Z \cap Z'$. Then C is a smooth rational curve with $LC = 0$ and $FC = 1$ by (4.4.5). Since $q_i = Z_i C \geq 0$, we have $q_i = 0$ by $q_1 + \cdots + q_r = 0$. Hence $Z_1 \equiv p_i L$, $p_1 + \cdots + p_r = 1$ so that $r = p_1 = 1$. Therefore any member of $|L|$ is irreducible. and reduced

(4.5) *Proof of* (4.3)-Completion.

Lemma 4.5.1. *Let Z, Z' be general members of $|L|$, and $C := Z \cap Z'$. Let $V + G^*$ be a general member of $|F|$, V movable and G^* fixed parts respectively. Then*
(4.5.1.1) *$V \in |pL|$ for some $p \geq 1$ and $VC = 0$, $G^* C = 1$.*
(4.5.1.2) *G^* and V are irreducible and reduced.*

Proof. By (4.4.5) we have $C \simeq \mathbf{P}^1$, $LC = 0$ and $FC = 1$. With the notation in (4.4.5) let $V_1 + \cdots + V_s + G^*$ be a general member of $|F|$, and $V := V_1 \equiv V_j \equiv pL + qF$. Then since C is movable and $FC = 1$, there are two cases.

Case 1. $VC = 0$, $G^* C = 1$,
Case 2. $VC = 1$, $G^* C = 0$, $s = p = 1$

Case 1. We have $L^3 = 0$ and $L^2 G^* = G^* C = 1$. Let $V \equiv pL + qF$. Then $q = 0$ by (4.4.5) so that $V \in |pL|$and $p \geq 1$ by $h^0(X, F) \geq 2$. By (4.4.4) and (4.4.6), any general member of $|spL|$ is irreducible by Bertini's theorem. Hence $s = 1$.

Let G_0^* be the unique irreducible component of G^* with $G_0^* C = 1$, G_j^* other irreducible components of G^*. Since $G_j^* C = 0$, $G_j^* \in |p_j L|$, whence $p_j = 0$ and $G_j^* = 0$ by (4.4.6). Therefore $G^* = G_0^*$.
Case 2. Let $V \equiv pL + qF$ and $G^* = rL + tF$. Then $G^* \in |rL|$ by $t = G^* C = 0$, whence $r = 0$ and $G^* = 0$. Hence $p = 0$, $s = q = 1$ and any general $V \in |F|$ is irreducible and reduced. Therefore $|F|$ has no fixed components. Hence $X \simeq \mathbf{P}^1 \times \mathbf{P}^2$ by (1.6), which contradicts the assumption in (4.4).

Lemma 4.5.2. *Let Z and Z' be any pair of distinct members of $|L|$,* and *$l := Z \cap Z'$. Then*
(4.5.2.1) $h^q(X, -rL - F) = 0 \quad (0 \leq r \leq 2;\ 0 \leq q \leq 3)$
(4.5.2.2) $h^q(O_Z(-rL - F)) = 0 \quad (r = 0, 1; 0 \leq q \leq 2)$
(4.5.2.3) $h^q(O_l(-F)) = 0 \quad (q = 0, 1)$.

Proof. By (4.5.1) any general member of $|F|$ is reduced and connected. Hence we have $h^1(X, -rL - F) = 0$ for any $r \geq 0$. Since $K_X = -3L - 2F$, we have $h^3(X, -rL - F) = 0$ for $r \leq 3$. By (4.3.2), we have $h^2(X, -rL - F) = \chi(X, -rL - F) = 0$ for $0 \leq r \leq 3$, which proves (4.5.2.1). The rest follows readily.

Lemma 4.5.3. *Let Z and Z' be any pair of distinct members of $|L|$, and $l := Z \cap Z'$. Then l is a smooth rational curve with $Ll = 0$, $Fl = 1$.*

Proof. *Step* 1. Since $Fl=1$, there is an irreducible component C of l with $FC\geq 1$. Then by (4.4.5) $LC=0$, while $C\simeq \mathbf{P}^1$ by (4.4.3). Let $I_C/I_C^2\simeq O_C(a)\oplus O_C(b)$ $(a\geq b)$ and $s:=a+b=K_XC+2=-2FC+2\leq 0$. Since $h^1(O_C(-F))=0$ by (4.5.2) we have $FC\leq 1$, whence $FC=1$ and $s=0$. Note that $\chi((O_X/I_C^n)(-F))=0$ for any $n\geq 1$.

Step 2. By *Step* 1, $a+b=0$. Assume $a\geq 1$ and $I_l\subset I_C^2$. Then consider a (possibly identically zero) homomorphism

$$\phi: (I_l/I_l^2)\otimes O_C(\simeq O_C^{\oplus 2})\rightarrow I_C^2/I_C^3(\simeq O_C(2a)\oplus O_C(a+b))\oplus O_C(2b)).$$

Let $I:=O_C(2a)\oplus O_C(a+b)+I_C^3$. Since $2b\leq -2$, Im $\phi\subset O_C(2a)\oplus O_C(a+b)$ whence $I_l\subset I$. Hence $h^1((O_X/I)(-F))=0$ by (4.5.2.3) so that

$$0\leq\chi((O_X/I)(-F))=\chi((O_X/I_C^2)(-F))+\chi((I_C^2/I)(-F)=2b,$$

a contradiction. Hence $I_l\not\subset I_C^2$. Therefore we have the nontrivial homomorphism $\phi: (I_l/I_l^2)\otimes O_C\rightarrow I_C/I_C^2$. If $a\geq 1$, then $I_l\subset I:=O_C(a)+I_C^2$. Hence

$$0\leq\chi((O_X/I)(-F))=\chi((O_X/I_C)(-F)+\chi((I_C/I)(-F))=b,$$

a contradiction. Hence $a=b=0$.

Step 3. Let $g:=\rho_{L|G^*}:G^*\rightarrow \mathbf{P}^2$ be the restriction of ρ_L to G^*. Then g is a birational morphism because any general fiber $\rho_L^{-1}(\mathrm{p})$ $(p\in\mathbf{P}^2)$ is a smooth rational curve with $G^*\rho_L^{-1}(p)=1$ by (4.4.5). Hence there exists a proper analytic subset Σ of $\mathbf{P}^2$ such that g is an isomorphism of $G^*\backslash q^{-1}(\Sigma)$ onto $\mathbf{P}^2\backslash\Sigma$. Let p be a point outside Σ. Then $\rho_L^{-1}(p)$ has an irreducible component C with $G^*C=1$ along which $\sigma:=\rho_L^{-1}(p)$ is reduced generically. By *Step* 2, $I_C/I_C^2\simeq O_C^{\oplus 2}$, whence $(I_\sigma/I_\sigma^2)\otimes O_C\simeq I_C/I_C^2$. This shows that $\sigma\simeq C$. Therefore $\rho_L^{-1}(p)$ is a smooth rational curve if $p\notin\Sigma$.

Step 4. Let $C(\simeq\mathbf{P}^1)$ be an irreducible component of l with $LC=0$ and $FC=1$. Since $LC=0$, $\rho_L(C)=0$ is a point of $\mathbf{P}^2$. By *Step* 3, we may assume $\rho_L(C)\in\Sigma$. We may also assume that $\rho_L^{-1}(p)$ is a smooth rational curve for general $p\neq h(0)$ if p is close to $h(0)$. Meanwhile by *Step* 2, $N_{C/X}\simeq O_C^{\oplus 2}$. Hence there are a proper smooth family τ: $\mathscr{C}\rightarrow\Delta$ (a versal family of displacements of C in X) over a two dimensional disc Δ with $\tau^{-1}(0)\simeq C$ and a morphism j: $\mathscr{C}\rightarrow\Delta\times X$ such that $C_t:=j(\tau^{-1}(t))\simeq\mathbf{P}^1$ is a displacement of C in X. Since $LC_t=0$, $\rho_L(C_t)$ is one point of $\mathbf{P}^2$. Therefore we have a morphism h of Δ into $\mathbf{P}^2$ such that $C_t=\rho_L^{-1}(h(t))$ for $t\neq 0$. By the versality of the family $\mathscr{C}$, $h(\Delta)$ is an open subset of $\mathbf{P}^2$ containing $h(0)$.

This implies that $\rho_L^{-1}(h(\Delta\backslash\{0\}))=j(\mathscr{C}\backslash\tau^{-1}(0))$, whence $\rho_L^{-1}(h(\Delta))=j(\mathscr{C})$, which is the interior of the closure of $j(\mathscr{C}\backslash\tau^{-1}(0))$. Therefore $l_{\mathrm{red}}\simeq C$. Since $Fl=FC=1$, l is reduced generically along C. Since $a=b=0$, the natural homomorphism $\phi: (I_l/I_l^2)\otimes O_C\rightarrow I_C/I_C^2$ is an isomorphism. Hence $l\simeq C$.

Lemma 4.5.4 *If $|F|$ has a fixed component, then $X \simeq \mathbf{P}(\mathscr{E})$ for a topologically trivial rank two vector bundle $\mathscr{E}$ over $\mathbf{P}^2$ with* $\mathrm{sp}^+(\mathscr{E}) \geq 1$.

Proof. Let $\pi := \rho_L$, $\mathscr{E} := \pi_*(F)$ and $l := \pi^{-1}(p)$ for a point $p \in \mathbf{P}^2$. Then since $\mathrm{Bs}\,|F \otimes O_l| = \emptyset$ and $h^0(F \otimes O_l) = 2$ by (4.5.3), $\mathscr{E}$ is a locally free sheaf of rank two over $\mathbf{P}^2$. Let $\alpha := c_1(O_S(1))$, $c_1(\mathscr{E}) = p\alpha$ and $c_2(\mathscr{E}) = q\alpha^2$. Then by the proof of (3.1) we have $F^2 = pFL - qL^2$, whence $p = q = 0$ by $F^2 = 0$. Hence $\mathscr{E}$ is topologically trivial. We have a natural surjective *morphism* $h: X \to \mathbf{P}(\mathscr{E})$. Since $L \simeq \pi^* O_{\mathbf{P}^2}(1) \simeq h^*\pi(\mathscr{E})^* O_{\mathbf{P}^2}(1)$ and $F = h^*F(\mathscr{E})$, we have $K_X \simeq h^* K_{\mathbf{P}(\mathscr{E})}$ by (3.3). Hence h is an isomorphism. Note that $\mathrm{sp}^+(\mathscr{E}) \geq 1$ because $|F|$ has fixed components.

Thus we complete the proof of (4.3).

Appendix. Threefolds with $c_1(X) = 3c_1(L)$

We recall from [N1] and [N4] some results on threefolds with $c_1(X) = 3c_1(L)$.

Theorem A.1. *Let X be a Moishezon 3-fold and L a line bundle on X. Assume that $h^1(X, O_X) = 0$, $c_1(X) = 3\,c_1(L)$, $h^0(X, L) \geq 2$,* and $\dim \mathrm{Bs}\,|L| \leq 1$. *Then* $X \simeq \mathbf{Q}^3$ *or* $\mathbf{P}(\mathscr{F}(a, b, 0))$ $(a \geq b \geq n \geq 0,\ a + b = 3n + 2)$.

Our proof of (A.1) in [N4] consists of a series of lemmas as follows.

Lemma A.2 *Assume $B := \mathrm{Bs}\,|L| = \emptyset$. Let $h: X \to \mathbf{P}^4$ be a morphism associated with $|L|$, $W := h(X)$. Then W is a hyperquadric and h is birational.*

(1) *If W is smooth, then $X \simeq W \simeq \mathbf{Q}^3$.*

(2) *If $B = \emptyset$ and if* $\dim \mathrm{Sing}\, W = 0$, *then* $X \simeq \mathbf{P}(\mathscr{F}(1, 1, 0))$.

(3) *If $B = \phi$ and if* $\dim \mathrm{Sing}\, W = 1$, *then* $X \simeq \mathbf{P}(\mathscr{F}(2, 0, 0))$.

Lemma A.3. *If $B \neq \emptyset$ and if $\dim B \leq 1$, then $B \simeq \mathbf{P}^1$ and $X \simeq \mathbf{P}(\mathscr{F}(a, b, 0))$ $(a \geq b \geq n \geq 1,\ a + b = 3n + 2)$.*

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