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Herbst inequalities for supercontractive semigroups

By

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1. Introduction

We will consider a probability space (X, μ) and a symmetric contraction semigroup e^{-tA} on $L^2(X, \mu)$ which is Markovian in the sense that it takes nonnegative functions to nonnegative functions and satisfies $e^{-tA}1 = 1$ for all $t > 0$. For such a semigroup there is a rough equivalence between a knowledge of the norms, $N(t, p, q)$, of e^{-tA} as an operator from $L^p(\mu)$ to $L^q(\mu)$ and a knowledge of the function β in the family of logarithmic Sobolev inequalities

$$(1.1) \qquad \int_X f(x)^2 \log \frac{|f(x)|}{\|f\|_2} d\mu(x) \le \varepsilon \mathscr{E}(f, f) + \beta(\varepsilon)\|f\|_2^2, \quad \varepsilon > 0,\ f \in \mathscr{D}(A^{1/2}).$$

Here $\beta : (0, \infty) \to [0, \infty]$ may be taken to be a decreasing convex function. $\|f\|_2$ denotes the $L^2(\mu)$ norm and $\mathscr{E}(f, f) = \|A^{1/2}f\|_2^2$ is the Dirichlet form associated to A.

Although (1.1) will be assumed to hold throughout most of this paper for all $\varepsilon > 0$, the possibility that $\beta(\varepsilon) = \infty$ for some ε means that (1.1) may have substance only for ε in some interval $[\varepsilon_0, \infty)$ for some $\varepsilon_0 > 0$. For example if $\beta(\varepsilon) = \infty$ for $0 < \varepsilon < \varepsilon_0$ and $\beta(\varepsilon) = \beta_0$ for $\varepsilon \ge \varepsilon_0$ then (1.1) reduces to a standard logarithmic Sobolev inequality

$$(1.2) \qquad \int_X f(x)^2 \log \left(\frac{|f(x)|}{\|f\|_2} \right) d\mu(x) \le \varepsilon_0 \mathscr{E}(f, f) + \beta_0 \|f\|_2^2$$

[G1], with fixed principal coefficient ε_0 and "local norm" β_0 (also known as the "defect").

Among the semigroups of interest to us there are three classes that have been distinguished up to now. Hypercontractive semigroups have the least smoothing ability: for $1 < p < q < \infty$ there is a minimum time $t_0 = t_0(p, q) > 0$ such that $\|e^{-tA}\|_{p \to q} < \infty$ if $t \ge t_0$. These semigroups are associated with the fixed logarithmic Sobolev inequality (1.2). On the other hnad if $\beta(\varepsilon) < \infty$ for all $\varepsilon > 0$ then e^{-tA} is bounded from $L^p(\mu)$ to $L^q(\mu)$ for all p and q in $(1, \infty)$ and for all $t > 0$. Semigroups

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with this property have been called *supercontractive* [Ro]. (See also [G2].)

Among the supercontractive semigroups are those with the strongest smoothing properties: the semigroup e^{-tA} is called *ultracontractive* [DS] if $\|e^{-tA}\|_{2\to\infty}<\infty$ for all $t>0$. Sufficient conditions on the function β are known which ensure that the semigroup e^{-tA} is ultracontractive. Ultracontractive semigroups have been useful in the analysis of heat kernels [D, DS], Schrödinger operators [DS] and several problems in probability theory and statistics [Bn, CS, DSc1,2].

The question naturally arises as to which Dirichlet forms $\mathscr{E}(f,f)$ can satisfy logarithmic Sobolev inequalities such as (1.1) or (1.2). In response to a preprint of [G1], Ira Herbst wrote to the first author of the present paper showing that in the simplest case, that in which $X=\boldsymbol{R}$ and $\mathscr{E}(f,f)=\int_{\boldsymbol{R}}|f'(x)|^2d\mu(x)$ the inequality (1.2) cannot hold with $\beta_0=0$ unless μ is strongly consentrated near zero. Specifically, he showed that one must have

$$\int_{\boldsymbol{R}} e^{\alpha x^2}d\mu(x)<\infty \tag{1.3}$$

for some $\alpha>0$, [H]. This is a necessary but not sufficient condition for (1.2) to hold. Shortly hereafter such exponential integrability inequalities were discussed by R.Carmona for n-dimensional Schrödinger operators, [Ca]. Herbst's method was modified and extended in [DS]. Further exponential integrability theorems under the hypothesis that (1.2) holds have been proven in [AS, AMS, L, R]. We will refer to such exponential integrability inequalities as Herbst inequalities when they are deduced from logarithmic Sobolev inequalities. Applications of exponential integrability of functions have been made in [ASh], [Hi] and [U3].

In the present paper we will deduce Herbst inequalities from (1.1) in the supercontractive case, that is, when $\beta(\varepsilon)<\infty$ for all $\varepsilon>0$. Naturally these will be stronger than the Herbst inequalities deducible from the hypercontractive case (1.2). We will consider also the hypercontractive case (1.2), partly for the sake of example, but more importantly for discrete Dirichlet forms, in order to show that two of the natural Dirichlet forms associated to Poisson measure on Z_+ cannot satisfy a logarithmic Sobolev inequality. (See Section 5). But primarily we are interested in identifying the minimum amount of concentration of μ forced by ultracontractivity. Under standard conditions on β which ensure ultracontractivity, [D, DS], we will prove Herbst inequalities which, for the Dirichlet form $\int_{-\infty}^{\infty}f'(x)^2d\mu(x)$, reduce to an inequality of the form

$$\int_{\boldsymbol{R}} e^{\alpha x^2\log|x|}d\mu(x)<\infty. \tag{1.4}$$

Examples indicate that this is the strongest kind of integrability one can expect under the hypothesis of ultracontractivity.

Inequalities such as (1.3) have been derived from (1.2) by proving a differential inequality either for $\lambda\to E(e^{\lambda g^2})$ [H, DS, AMS, R] or for $\lambda\to E(e^{\lambda g})$ [L] or by an induction on $E(|g|^{2n})$, [R], where g is a function on X with bounded "gradient" in all cases. We will adapt Ledoux's method [L] of estimating the Laplace transform $E(e^{\lambda g})$ and combine it with a variant of a technique of [AMS] to obtain estimates of the form $E(e^{\Phi(|g|)})<\infty$ for a convex function Φ allowed by β. For the most part we will formulate our results in the context of a symmetric Markov semigroup as in [AS].

2. Notation

We will follow Aida and Stroock [AS] in their formulation of norms of generalized gradients associated to a class of Markov semigroups throughout most of this paper, but deviate from this formulation in Section 5. Let $(X,\mathscr{B},\mu)$ be a probability space and let $(t,x)\to P(t,x,\cdot)$ be a Markov transition probability function on $(X,\mathscr{B})$. We assume that the measure M_t on $X\times X$, $\mathscr{B}\times\mathscr{B}$ given by $M_t(dx\times dy)=P(t,x,dy)\mu(dx)$ is symmetric and that for each bounded measurable function f on X

$$(P_tf)(\cdot)\equiv\int_X f(y)P(t,\cdot,dy)$$

converges to f in $L^2(\mu)$ as $t\downarrow 0$. If $\bar{P}_t$ is the unique bounded extension of P_t to $L^2(\mu)$ then there is a nonnegative self-adjoint operator A on $L^2(X,\mu)$ such that $\bar{P}_t=e^{-tA}$. The Dirichlet form associated to this semigroup is by definition the bilinear form given by

$$\mathscr{E}(f,g)=(A^{1/2}f,A^{1/2}g)\qquad f,\,g\in\mathscr{D}(A^{1/2}). \tag{2.1}$$

Denoting by $B(X)$ the set of bounded measurable functions on X, the space $\mathscr{F}\equiv B(X)\cap\mathscr{D}(A^{1/2})$ is well known to be an algebra. For f and g in $\mathscr{F}$ the map

$$g\mapsto\Lambda_f(g)\equiv\mathscr{E}(gf,f)-\frac{1}{2}\mathscr{E}(g,f^2) \tag{2.2}$$

is a positive linear functional on $\mathscr{F}$. Moreover $\mathscr{F}$ is dense in $L^1(\mu)$. Define $|||f|||_\infty^2$ to be the norm of the linear functional $g\mapsto\Lambda_f(g)$ as a densely defined linear functional on $L^1(\mu)$. For details of the preceding discussion see [AS,Section 1], where it is also shown how to extend $|||\cdot|||_\infty$ to the natural domain of this norm. We will assume in the following that this extension has been made when we write $|||f|||_\infty<\infty$. We will make use of the following inequality [AS, Equ. (1.16)]. If $|||f|||_\infty<\infty$ and $\lambda\in\boldsymbol{R}$ then

$$\mathscr{E}(e^{\lambda f},e^{\lambda f})\le a|||f|||_\infty^2\lambda^2E(e^{2\lambda f})\qquad a=1\text{ or }2 \tag{2.3}$$

where $E(g)=\int_X g(x)d\mu(x)$. In Equ. (2.3) $a=1$ if $\mathscr{E}$ is local, as in Example 2.1 below. In this case (2.3) is elementary. If $\mathscr{E}$ is not local then $a=2$. See [AS] for the proof of this case.

Example 2.1. Let $X=\boldsymbol{R}^n$ and write ∇f for the gradient of f. Let $B:\boldsymbol{R}^n \to \{n\times n$ invertible matrices$\}$ be measurable and locally bounded and let $d\mu(x)=w(x)dx$ be a probability measure on $\boldsymbol{R}^n$. Assume, say, that $w>0$ on $\boldsymbol{R}^n$. Define

$$\mathscr{E}(f,f)=\int_{\boldsymbol{R}^n}(B(x)\nabla f(x),B(x)\nabla f(x))d\mu(x) \qquad f\in C_c^\infty(\boldsymbol{R}^n). \tag{2.4}$$

One can compute readily from (2.2) that in this case

$$\Lambda_f(g)=\int_{\boldsymbol{R}^n} g(x)\,|\,B(x)\nabla f(x)\,|^2 d\mu(x), \qquad f,\ g\in C_c^\infty(\boldsymbol{R}^n).$$

Hence, ignoring questions concerning the closure of the form $\mathscr{E}$, we see that

$$|||f|||_\infty=\sup\{|B(x)\nabla f(x)|:x\in\boldsymbol{R}^n\}. \tag{2.5}$$

The special case $B(x)=I$ is of particular interest for Schrödinger operators [DS]. In this case one simply has $|||f|||_\infty=\sup_x|\nabla f(x)|$.

We will not really be directly concerned in this paper with the semigroup e^{-tA} itself, whose Dirichlet form is given by the form closure of (2.4), since we will deal only with the Dirichlet form itself. For conditions on $B(\cdot)$ and μ which ensure that a unique Markov semigroup e^{-tA} is associated to (2.4) see [MR].

More generally, if μ is a probability measure on a Riemannian manifold M and we define $\mathscr{E}(f,f)=\int_M|\nabla f(x)|^2d\mu(x)$ then one has $|||f|||_\infty=\sup\{|\nabla f(x)|:x\in M\}$.

Example 2.2. *Let* $X=\boldsymbol{Z}$. Suppose that $\mu_k\geq 0$ and $\Sigma_{k\in\boldsymbol{Z}}\mu_k=1$. Let $b_k\geq 0$ for $k\in\boldsymbol{Z}$. Define

$$(Df)(k)=f(k+1)-f(k) \qquad k\in\boldsymbol{Z}. \tag{2.6}$$

Define a Dirichlet form by

$$\mathscr{E}(f,g)=\sum_{k\in\boldsymbol{Z}}(Df)(k)(Dg)(k)b_k^2\mu_k \tag{2.7}$$

for f and g of finite support in $\boldsymbol{Z}$. Two cases are of interest: Either $\mu_k>0$ for all $k\in\boldsymbol{Z}$ or else $\mu_k>0$ exactly when $k\geq 0$. In the latter case we will assume $b_k=0$ for $k<0$. In either case one can compute that (2.2) gives

$$\Lambda_f(g)=\sum_{k\in\boldsymbol{Z}}\frac{1}{2}(g(k+1)+g(k))(Df)(k)^2b_k^2\mu_k. \tag{2.8}$$

This may be rewritten

$$\Lambda_f(g)=(1/2)\sum_{k\geq 0}g(k)\{b_k^2(Df)(k)^2+(b_{k-1}^2\mu_{k-1}/\mu_k)(Df)(k-1)^2\}\mu_k \tag{2.9}$$

in the one sided case. The same expression is also correct in the two sided case if the sum is extended over all $k \in \boldsymbol{Z}$. Hence

$$\text{(2.10)} \qquad |||f|||_\infty^2 = \sup_{k \geq 0} \frac{1}{2}\{b_k^2(Df)(k)^2 + (b_{k-1}^2\mu_{k-1}/\mu_k)(Df)(k-1)^2\}$$

in the one sided case. A similar expression holds in the two sided case.

Example 2.3. Let X be a finite set. Let $K(x,y) \geq 0$ satisfy $\Sigma_y K(x,y) = 1$ for each $x \in X$. Assume that there exists a probability measure $\pi(\cdot)$ on X such that $\Sigma_x \pi(x)K(x,y) = \pi(y)$ for each $y \in X$. Assume further $\pi(x)K(x,y)$ is symmetric in x and y. So (K, π) is a symmetric Markov chain. Let

$$\text{(2.11)} \qquad \mathscr{E}(f,g) = (1/2)\sum_{x,y}(f(x) - f(y))^2\pi(x)K(x,y).$$

It is straightforward to verify that

$$\text{(2.12)} \qquad \mathscr{E}(gf,f) - (1/2)\mathscr{E}(g,f^2) = \frac{1}{2}\sum_{x,y} g(x)[f(x) - f(y)]^2\pi(x)K(x,y).$$

In the definition of $|||f|||_\infty$ above the role of μ is played here by π. It follows that

$$\text{(2.13)} \qquad |||f|||_\infty^2 = \frac{1}{2}\sup_x \sum_y K(x,y)[f(x) - f(y)]^2.$$

Define

$$\text{(2.14)} \qquad \|\nabla f\|_\infty = \sup\{|f(x) - f(y)| : K(x,y) \neq 0\}.$$

Then, since $\Sigma_y K(x,y) = 1$, (2.13) and (2.14) give

$$\text{(2.15)} \qquad |||f|||_\infty^2 \leq (1/2)\|\nabla f\|_\infty^2.$$

Finally we note that applications of moment bounds have been made to statistical mechanics in [AS].

3. Bounds on the Laplace transform

Henceforth we assume that there is a decreasing function $\beta : (0,\infty) \to [0,\infty]$ such that

$$\text{(3.1)} \qquad \int_X f(x)^2 \log\left(\frac{|f(x)|}{\|f\|_2}\right) d\mu(x) \leq \varepsilon\mathscr{E}(f,f) + \beta(\varepsilon)\|f\|_2^2 \qquad f \in \mathscr{D}(\mathscr{E}).$$

We assume $\beta(\varepsilon) < \infty$ for at least some $\varepsilon > 0$. In the following $a = 1$ if $\mathscr{E}$ is local and $a = 2$ otherwise.

Theorem 3.1. *Let g be a real valued measurable function on X such that*

(3.2) $$|||g|||_\infty \le 1.$$

Let $\alpha:(0,\infty)\to(0,\infty)$ *be continuous. Assume that* $\beta(\alpha(\tau))<\infty$ *for all* $\tau\in(0,\infty)$. *Then*

(3.3) $$E(e^{\lambda g})<\infty \quad \textit{for all} \quad \lambda\in \boldsymbol{R}$$

and

(3.4) $$E(e^{\lambda g})\le E(e^{sg})^{\lambda/s}\exp\left[\lambda\int_s^\lambda \{a\alpha(\tau)/2+2\beta(\alpha(\tau))/\tau^2\}d\tau\right] \quad 0<s<\lambda.$$

The following proof is an adaptation of Ledoux's techniques to our setting.

Lemma 3.2. *If* (3.2) *holds and in addition g is bounded then* (3.4) *holds.*

Proof. Let

$$v(\lambda)=E(e^{\lambda g}).$$

Then $v(\cdot)$ is in $C^\infty(\boldsymbol{R})$. Put $f(x)=e^{\lambda g(x)/2}$ in (3.1). Then $v(\lambda)=\|f\|_2^2$. Furthermore $v'(\lambda)=\int e^{\lambda g}g\,d\mu=(2/\lambda)\int f^2\log f\,d\mu$ while $\|f\|_2^2\log\|f\|_2=\frac{1}{2}v(\lambda)\log v(\lambda)$. By (2.3) $\mathscr{E}(f,f)\le(a\lambda^2/4)E(e^{\lambda g})=(a\lambda^2/4)v(\lambda)$. Inserting these into (3.1) gives

$$(\lambda/2)v'(\lambda)-(1/2)v(\lambda)\log v(\lambda)\le\varepsilon a(\lambda^2/4)v(\lambda)+\beta(\varepsilon)v(\lambda)$$

for any $\varepsilon>0$ and $\lambda\ne0$. Divide by $(1/2)\lambda^2v(\lambda)$ to find

$$\lambda^{-1}v'(\lambda)/v(\lambda)-\lambda^{-2}\log v(\lambda)\le\varepsilon(a/2)+2\beta(\varepsilon)/\lambda^2, \quad \lambda\ne0.$$

That is,

(3.6) $$(d/d\lambda)(\lambda^{-1}\log v(\lambda))\le\varepsilon a/2+2\beta(\varepsilon)/\lambda^2,$$

for any $\varepsilon>0$ and $\lambda\ne0$. Now choose $\varepsilon=\alpha(\lambda)$ for $\lambda>0$. Integrating (3.6) from s to λ gives

$$\lambda^{-1}\log v(\lambda)-s^{-1}\log v(s)\le\int_s^\lambda\{a\alpha(\tau)/2+2\beta(\alpha(\tau))/\tau^2\}d\tau.$$

Multiplying by λ and exponentiating gives (3.4).

The next Lemma is due to Ledoux [L]. We include his short proof for the reader's convenience.

Lemma 3.3 (Ledoux's Lemma). *Let* (X,μ) *be a probability space. Suppose that* $g:X\to\boldsymbol{R}$ *is measurable and that for some constant C*

(3.7) $$E(e^{2g}) \leq CE(e^g)^2 < \infty.$$

Choose m such that $\mu(g \geq m) \leq (4C)^{-1}$. Then

(3.8) $$E(e^g) \leq 2e^m.$$

In particular, if $g \in L^1(\mu)$ then

(3.9) $$E(e^g) \leq 2e^{4C\|g\|_1}.$$

Proof. $\mu(e^g \geq e^m) \leq (4C)^{-1}$. Hence

$$\begin{aligned}\int_X e^g d\mu &= \int_{g \geq m} e^g d\mu + \int_{g<m} e^g d\mu \\ &\leq \mu(g \geq m)^{1/2}\left(\int e^{2g} d\mu\right)^{1/2} + e^m \\ &\leq (4C)^{-1/2} C^{1/2} \int_X e^g d\mu + e^m.\end{aligned}$$

So $(1/2)E(e^g) \leq e^m$. This proves (3.8). The special case (3.9) follows by taking $m = 4C\|g\|_1$.

Proof of Theorem 3.1. Suppose that $|||g|||_\infty \leq 1$. Let $\psi_n(t) = -n$ for $t \leq -n$ and equal t for $-n < t < n$ and equal n for $t \geq n$. Then $|||\psi_n \circ g|||_\infty \leq |||g|||_\infty$ as shown in [AS, Equ. (1.15)]. Each function $\psi_n \circ g$ is bounded and satisfies $|||\psi_n \circ g|||_\infty \leq 1$. Hence (3.4) applies to the functions $\psi_n \circ g$. Let

$$C = \exp\left[2\int_1^2 \{a\alpha(\tau)/2 + 2\beta(\alpha(\tau))/\tau^2\} d\tau\right].$$

Choose m such that $\mu(g \geq m) \leq (4C)^{-1}$. Then one also has $\mu(\psi_n \circ g \geq m) \leq (4C)^{-1}$ for all n. Hence by (3.4) with $s = 1$ and $\lambda = 2$ we have $E(e^{2\psi_n \circ g}) \leq CE(e^{\psi_n \circ g})^2$. By Ledoux's Lemma it now follows that $E(e^{\psi_n \circ g}) \leq 2e^m$ for all n. By the monotone convergence theorem on $\{x : g(x) \geq 0\}$ and dominated convergence theorem elsewhere, we have $E(e^g) \leq 2e^m$. So if v is defined as in Lemma 3.2 then $v(1) < \infty$. Hence $v(s) < \infty$ for $0 \leq s \leq 1$ since one has monotonicity in s of the integrand on $\{x : g(x) \geq 0\}$. Now apply (3.4) to $\psi_n \circ g$ again, first for $0 < s \leq 1$ and $\lambda > s$, and take the limit to conclude that $v(\lambda) < \infty$ for all $\lambda > 0$. Finally apply (3.4) to $\psi_n \circ g$ again for any s and λ with $0 < s < \lambda$ and let $n \to \infty$ to see that (3.4) holds for g as stated. Since $|||-g|||_\infty = |||g|||_\infty \leq 1$ (3.3) holds for all real λ.

Example 3.4 (Hypercontractive case) [L,AMS]. Assume that (1.2) holds. Choose $\beta(\varepsilon) = \infty$ for $\varepsilon < \varepsilon_0$ and $\beta(\varepsilon) = \beta_0$ for $\varepsilon \geq \varepsilon_0$. Then (3.1) holds with this choice of β. Choose $\alpha(\tau) = \varepsilon_0$ for $\tau \in (0, \infty)$. We have $\beta(\alpha(\tau)) = \beta_0$ for $\tau \in (0, \infty)$. Inequality

(3.4) reduces to

(3.10) $$E(e^{\lambda g}) \le E(e^{sg})^{\lambda/s} \exp[\lambda\{(\lambda-s)a\varepsilon_0/2+2\beta_0[s^{-1}-\lambda^{-1}]\}] \quad \text{for } 0<s<\lambda$$

as already shown in [L]. In particular, if $\beta_0=0$ then we find $E(e^{\lambda g}) \le E(e^{sg})^{\lambda/s}\exp[\lambda(\lambda-s)a\varepsilon_0/2]$. Furthermore if g is bounded then as $s\downarrow 0 E(e^{sg})^{\lambda/s}=(1+sE(g)+0(s^2))^{\lambda/s} \to e^{\lambda E(g)}$. Hence

(3.11) $$E(e^{\lambda g}) \le e^{\lambda E(g)+\lambda^2 a\varepsilon_0/2} \quad \text{if } |||g|||_\infty \le 1 \text{ and } \lambda \ge 0$$

which one sees first for bounded g satisfying $|||g|||_\infty \le 1$ and then for all g with $|||g|||_\infty \le 1$ by the same limiting argument used in the proof of Theorem 3.1. The inequality (3.11) is especially interesting because if $X=\boldsymbol{R}$, if μ is any Gaussian measure on $\boldsymbol{R}$ with variance ε_0, if $\mathscr{E}(f,f)=\int_{\boldsymbol{R}} f'(x)^2 d\mu(x)$, and if one takes $g(x)=x$ then (1.2) holds with $\beta_0=0$ [G1], $a=1$ and (3.11) becomes an equality, showing that Theorem 3.1 is sharp in some cases. This has already been pointed out in [AMS, Remark 3.9].

The next three corollaries show that the nature of the singularity in $\beta(\varepsilon)$ as $\varepsilon\downarrow 0$ can control geometric properties of X such as its diameter.

Corollary 3.5. *Suppose that* $\alpha(\cdot)$ *can be chosen so that*

(3.12) $$K \equiv \int_1^\infty \{a\alpha(\tau)/2+2\beta(\alpha(\tau))/\tau^2\}d\tau < \infty.$$

If $|||g|||_\infty \le 1$ *then* g *is essentially bounded.*

Proof. By (3.4) with $s=1$ and (3.3) we have

$$E(e^{\lambda g}) \le \exp[\lambda\{K+\log E(e^g)\}] \quad \text{for } \lambda \ge 1.$$

Thus the $L^\lambda(\mu)$ norm of e^g, $E(e^{\lambda g})^{1/\lambda}$, remains bounded as $\lambda\to\infty$. So e^g is bounded off a set of measure zero. Similarly e^{-g} is essentially bounded.

Corollary 3.6. *Let* $0<\kappa<1$. *Suppose that* (3.1) *holds for a function* β *such that*

(3.13) $$\beta(\varepsilon) \le C_0\varepsilon^{-\kappa}+C_1 \quad \textit{for small } \varepsilon.$$

Then $|||g|||_\infty \le 1$ *implies that* g *is bounded.*

Proof. Choose $\gamma>1$ such that $\kappa\gamma<1$. Choose $b>0$ and let $\alpha(\tau)=b\tau^{-\gamma}$. Then $\beta(\alpha(\tau))/\tau^2 \le (C_0 b^{-\kappa}\tau^{\kappa\gamma}+C_1)/\tau^2$. Hence (3.12) holds.

Remark 3.7. Let Ω be an open connected set in $\boldsymbol{R}^n$. Suppose that V is a potential in Ω and that Δ denotes the Laplacian in Ω with Dirichlet boundary conditions.

Assume that the form sum, $-\Delta+V$, has a form closure, H, which is a semibounded self-adjoint operator in $L^2(\Omega, dx)$ and which has also a unique lowest normalized eigenstate φ_0. φ_0 can be taken to be strictly positive almost every-where. Let $d\mu(x)=\varphi_0(x)^2dx$. The ground state transform [J, p.71, CH p.458], $\psi(x)\rightarrow\psi(x)\varphi_0(x)^{-1}$, is a unitary map of $L^2(\Omega, dx)$ onto $L^2(\Omega,\mu)$ and carries H-(inf spectrum H) to the operator A on $L^2(\Omega,\mu)$ whose Dirichlet form is

$$\mathscr{E}(f,f)=\int_\Omega |\nabla f(x)|^2 d\mu. \tag{3.14}$$

The domain of $\mathscr{E}$ consists of those functions f in $L^2(\mu)$ whose weak gradient ∇f is also in $L^2(\mu)$.

Davies and Simon [DS] have termed the Schrödinger operator H on $L^2(\Omega, dx)$ *intrinsically ultracontractive* if the associated operator A on $L^2(\Omega,\mu)$ is ultracontractive (see the Introduction for the definition of ultracontractive). Their technique of proof of ultracontractivity e^{-tA} of relies on the following theorem, which we will quote here because it provides some of the motivation for the next section.

Theorem 3.8 (Davies and Simon [DS]). *Assume that* (3.1) *holds and that for each* $t>0$ *there exists a function* $c:[2,\infty)\rightarrow(0,\infty)$ *such that*

$$t=\int_2^\infty p^{-1}c(p)dp \tag{3.15}$$

and

$$M(t)\equiv 2\int_2^\infty p^{-2}\beta(c(p))dp<\infty. \tag{3.16}$$

Then the semigroup e^{-tA} *is ultracontractive and*

$$\|e^{-tA}\|_{2\rightarrow\infty}\le e^{M(t)}. \tag{3.17}$$

This theorem is the basis for the approach to ultracontractivity via logarithmic Sobolev inequalities. For other techniques of proving ultracontractivity see [KS], [D, Sec. 2.4], [Da],[Co] and its bibliography.

The function β is determined by both Ω and V. For example if Ω is bounded and $V=0$ then Davies and Simon show [D, page 127], [DS, Theorem 9.3] that (3.1) holds with $\beta(\varepsilon)=C_0-C_1\log\varepsilon$ for small ε provided that $\partial\Omega$ is sufficiently regular. This is a rather mild singularity for small ε. R. Banuelos [Ba] showed that if Ω is bounded then one can take $\beta(\varepsilon)=C-\frac{n}{4}\log\varepsilon+C'\varepsilon^{-\sigma}$ where σ depends on regularity properties of $\partial\Omega$. σ may be ≥ 1 or <1 depending on the regularity. On the other hand even if Ω is bounded and $V=0$ intrinsic ultracontractivity may fail. See [DS, Section 9].

In any case Corollary 3.6 implies that Ω must be bounded if the singularity in β near zero is not too strong. We have the following Corollary.

Corollary 3.9. *Take* $X=\Omega$ *and* $\mathscr{E}$ *as in Remark* 3.7. *Assume* (3.1) *holds with* β *satisfying* (3.13). *Then* Ω *is bounded.*

Proof. Fix a point x_0 in $\boldsymbol{R}^n$ and let $g(x)=$distance from x_0 to x. Then the weak gradient $|\nabla g(x)|=1$ almost everywhere. Let ψ_n be the truncating functions used in the proof of Theorem 3.1. Then $|\nabla\psi_n\circ g|\leq 1$ and $\psi_n\circ g$ is bounded. Hence $\psi_n\circ g\in\mathscr{D}(\mathscr{E})$. Since $\psi_n\circ g\to g$ in μ measure it follows that $|||g|||_\infty\leq 1$. (See the definition of $|||g|||_\infty$ in [AS].) By Corollary 3.6 g is bounded on Ω. Hence Ω is bounded.

4. Herbst inequalities

Information about the growth rate of the Laplace transform $E(e^{\lambda g})$ gives imme-diately information about the integrability of other functions of g via the following simple procedure. Suppose that $h:\boldsymbol{R}\to[0,\infty)$ has the property that

$$A\equiv\int_{-\infty}^{\infty}E(e^{\lambda g})h(\lambda)d\lambda<\infty.$$

Let

$$w(u)=\int_{-\infty}^{\infty}e^{\lambda u}h(\lambda)d\lambda.$$

Then $E(w(g))=A$ by Fubini's theorem.

Example 4.1. (Hypercontractive case). Consider first the special case $\beta_0=0$ in Example 3.4. Assume $|||g|||_\infty\leq 1$ and for simplicity assume $E(g)=0$ (or replace g by $g-E(g)$ in (3.11)). Then $E(e^{\lambda g})\leq e^{\lambda^2 a\varepsilon_0/2}$ for all real λ. Let $h(\lambda)=e^{-b\lambda^2/2}$ with $b>a\varepsilon_0$. Then

$$A\leq\int_{-\infty}^{\infty}e^{\lambda^2 a\varepsilon_0/2}h(\lambda)d\lambda=[2\pi/(b-a\varepsilon_0)]^{1/2}$$

and

$$w(u)\equiv\int_{-\infty}^{\infty}e^{\lambda u}h(\lambda)d\lambda=(2\pi/b)^{1/2}e^{u^2/2b}.$$

Hence

$$E(e^{(g-E(g))^2/(2b)})\leq[b/(b-a\varepsilon_0)]^{1/2}\quad\text{if}\quad b>a\varepsilon_0. \tag{4.1}$$

Of course, just as in Example 3.4, this is an equality when μ is a Gaussian measure on $\boldsymbol{R}$ with variance ε_0, in which case $a=1$. Cf. [AMS, Remark 3.9].

Now suppose that $\beta_0>0$ in (1.2). We cannot let $s\downarrow 0$ in (3.10). Choose $s=1$ and consider only $\lambda\geq 1$. Then (3.10) gives

$$E(e^{\lambda g})\leq \exp\{\lambda^2 a\varepsilon_0/2+B\lambda+C\} \quad \text{for } \lambda\geq 1 \tag{4.2}$$

where $B=\log E(e^g)-a\varepsilon_0/2+2\beta_0$ and $C=-2\beta_0$. Replacing g by $-g$ shows that a similar inequality holds for $\lambda<-1$. Take $h(\lambda)=e^{-b\lambda^2/2}$ for $|\lambda|\geq 1$ and $h(\lambda)=0$ othewise. We see that

$$\int_{-\infty}^{\infty} E(e^{\lambda g})h(\lambda)d\lambda<\infty \quad \text{if } b>a\varepsilon_0.$$

But for any $\delta>0$ there is a constant γ such that

$$\int_{-1}^{1} e^{\lambda u}e^{-b\lambda^2/2}d\lambda\leq \int_{-1}^{1} e^{\lambda u}d\lambda\leq \delta e^{u^2/2b}+\gamma$$

for all real u. It follows that $e^{u^2/2b}\leq$constant $\int_{-\infty}^{\infty}e^{\lambda u}h(\lambda)d\lambda$. Hence

$$E(e^{g^2/2b})<\infty \quad \text{if } b>a\varepsilon_0. \tag{4.3}$$

The inequality (4.3) was derived for large b in [L], when $\mathscr{E}$ is local (so $a=1$) and also for large b by [AS, Corollary 3.7] in the local and non-local cases. The full range of validity of (4.3), i.e., $b>\varepsilon_0$, was derived in [R] (in the local case) by two distinct methods. One method builds on Ledoux's large b result by deriving a differential inequality for $E(e^{\lambda g^2})$. The other method is based on an inductive argument forthe moments $E(g^{2n})$. The validity of (4.3) for the full range of b is also proved in [AS, Theorem 3.10] in the special case $\beta_0=0$.

Inequalities such as (4.3) have also been derived in [U1,U2] for infinite dimensional gauss measure by quite different methods and in particular without going through logarithmic Sobolev inequalities.

Remark 4.2. We wish next to derive stronger Herbst inequalities than (4.3) under the hypothesis of supercontractivity. Our goal is to derive inequalities of the form $E(e^{\Phi(|g|)})<\infty$ for some functions $\Phi(u)$ that grow faster than quadratically for large u. The proper setting seems to be that of Young's pairs [Z, Chapter 5]. Let $\varphi:[0,\infty)\to[0,\infty]$ be nondecreasing, left continuous and satisfy $\varphi(0)=0$. Denote by ψ the function which is inverse to φ in the sense that $\psi(\lambda)=\inf\{s\geq 0:\lambda \leq\varphi(s)\}$. Then the functions $\Phi(s):=\int_0^s\varphi(u)du$ and $\Psi(\lambda):=\int_0^\lambda\psi(v)dv$ from $[,\infty)$ into $[0,\infty]$ form a Young's pair. Both are convex on $[0,\infty)$, Young's inequality

$$\lambda s\leq\Phi(s)+\Psi(\lambda) \tag{4.4}$$

is satisfied, and (4.4) is an equality exactly when $\lambda=\varphi(s)$. In the cases of interest to us $\varphi(s)$ will increase near ∞ faster than linearly. The simple version of the above inverse relation, namely $\psi(\varphi(s))=s$, often fails for s in some bounded interval, in interesting cases, but holds for large s. We will assume this relation for large s.

Theorem 4.3. *Suppose that Φ and Ψ are conjugate Young's functions on $[0,\infty)$. Assume that*

(4.5) *a.* $\int_1^\infty E(e^{\lambda\delta g})e^{-\Psi(\lambda)}d\lambda<\infty$ *for* $\delta=\pm 1$,

and, in the notation of Remark 4.2,

b. for some $R>1$
i. ψ' exists and is bounded on $[R,\infty)$.
ii. φ is continuous and unbounded on $[R,\infty)$ and $\psi(\varphi(s))=s$ for $s\geq R$.

Then

(4.6) $E(e^{\Phi(|g|)})<\infty$.

Proof. Adopting the notation of Remark 4.2 the hypothesis *b.* assures that there exists $s_0\geq R$ and $\kappa<\infty$ such that whenever $s\geq s_0$ we have $\varphi(s)\geq R$ and $\Psi''(\lambda)\leq 2\kappa$ for $\lambda\in[\varphi(s),\varphi(s)+1]$. For $s\geq s_0$, the function $u(\lambda):=s\lambda-\Psi(\lambda)-\Phi(s)$ therefore satisfies $u(\varphi(s))=0$ and $u'(\varphi(s))=s-\psi(\varphi(s))=0$. Moreover $u''(\lambda)=-\psi'(\lambda)\geq-2\kappa$ on the interval $\varphi(s)\leq\lambda\leq\varphi(s)+1$. Hence $u(\lambda)\geq-\kappa$ for $\varphi(s)\leq\lambda\leq\varphi(s)+1$. It follows that

$$\int_1^\infty e^{s\lambda-\Psi(\lambda)-\Phi(s)}d\lambda\geq\int_{\varphi(s)}^{\varphi(s)+1} e^{-\kappa}\quad\text{for } s\geq s_0.$$

Hence

$$e^{\Phi(s)}\leq e^{\kappa}\int_1^\infty e^{s\lambda-\Psi(\lambda)}d\lambda\quad\text{for } s\geq s_0.$$

Therefore

$$E(e^{\Phi(g)},\ g\geq s_0)\leq e^{\kappa}\int_1^\infty E(e^{\lambda g})e^{-\Psi(\lambda)}d\lambda<\infty.$$

Similarly $E(e^{\Phi(-g)},\ g\leq -s_0)<\infty$. Since Φ is bounded on $[0,s_0]$ (4.6) now follows.

Corollary 4.4. *Suppose that g is a real valued function on X and there exist constants K and L such that*

(4.7) $$E(e^{\lambda g})\leq e^{K\lambda+L\lambda^2/(1+\log\lambda)}\quad\forall\lambda\geq 1$$

and that $-g$ satisfies the same inequality. Then

(4.8) $$E(e^{\gamma g^2\log|g|})<\infty\qquad \text{if } 0\leq\gamma<(4L)^{-1}.$$

Proof. Choose γ as in (4.8) and define $\Phi(s)=\gamma s^2 \log_+ s$. Let $\varphi(s)=0$ for $0 \le s \le 1$ and $\varphi(s)=\gamma s(1+2\log s)$ for $s>1$. Then $\Phi(s)=\int_0^s \varphi(u)du$ and Φ is a Young's function. Let ψ and Ψ be as in Remark 4.2. Then $\psi(0)=0$ and $\psi(\lambda)=1$ for $0<\lambda\le\gamma$ while ψ is C^∞ on (γ,∞). Moreover $\psi(\varphi(s))=s$ for $s>1$. Hence $\psi'(\varphi(s))\varphi'(s)=1$ for $s>1$. Thus if $s>1$ and $\lambda=\varphi(s)$ then $\Psi''(\lambda)=\psi'(\lambda)=(\varphi'(s))^{-1}=\gamma^{-1}(3+2\log s)^{-1}$. So $\Psi''(\lambda)\le(3\gamma)^{-1}$ if $\lambda>\gamma$. The hypotheses b. of Theorem 4.3 are therefore satisfied. Now if $s>1$ and $\lambda=\varphi(s)$ then $\Psi(\lambda)=s\varphi(s)-\Phi(s)$ by (4.4) at $\lambda=\varphi(s)$. Thus at $\lambda=\varphi(s)$

$$\begin{aligned}\Psi(\lambda)&=\gamma\{s^2(1+2\log s)-s^2\log s\}\\&=\gamma s^2(1+\log s)\\&=\gamma^{-1}\varphi(s)^2(1+\log s)(1+2\log s)^{-2}\\&=\gamma^{-1}(\lambda^2/\log\lambda)\{\log(\gamma s(1+2\log s))\}(1+\log s)(1+2\log s)^{-2}.\end{aligned}$$

Writing $u(\lambda)=\Psi(\lambda)/[(4\gamma)^{-1}\lambda^2/(1+\log\lambda)]$, it follows that $u(\lambda)\to 1$ *as* λ (and therefore s) goes to $+\infty$. Since $L\lambda^2/(1+\log\lambda)-\Psi(\lambda)=[\lambda^2/(1+\log\lambda)][L-(4\gamma)^{-1}u(\lambda)]$ and $L-(4\gamma)^{-1}<0$ we see that $\int_1^\infty E(e^{\lambda g})e^{-\Psi(\lambda)}d\lambda<\infty$. The same argument applies to $-g$. So the hypotheses of Theorem 4.3 are satisfied.

Similar computations can be used in the following alternate proof of Corollary 4.4 which avoids use of the concept of Young's pair: By increasing K slightly we may assume $g\ge 0$. Let $m(t)=\mu\{x:g(x)\ge t\}$. Then $E(e^{\lambda g})\ge e^{\lambda t}m(t)$. (4.7) implies

$$\lambda t+\log m(t)\le K\lambda+L\lambda^2/(1+\log\lambda).$$

So

$$\log m(t)\le\lambda(K-t)+L\lambda^2/(1+\log\lambda)\ \text{ for }\ \lambda\ge 1.$$

Take $\lambda=(2L)^{-1}t\log t$ for large t. The right side is easily seen to be asymptotic to $-(4L)^{-1}t^2\log t$. Hence

$$m(t)\le\exp[-(4L)^{-1}t^2(\log t)(1+o(1))].$$

Now

$$E(\exp[\gamma g^2\log g])=-\int_0^\infty e^{\gamma t^2\log t}dm(t)$$

and an integration by parts shows that this is finite if $\gamma<(4L)^{-1}$.

Remark 4.5. Davies and Simon [DS, Secs. 4,5,6] use Theorem 3.8 in combination with Rosen's Lemma to show that various Schrödinger operators on $\boldsymbol{R}^n$ are intrinsically ultracontractive. The equation (3.15) requires $c(p)$ to go to zero sufficiently fast at ∞ to make $\int_2^\infty p^{-1}c(p)dp<\infty$. On the other hand the singularity in $\beta(\varepsilon)$ at $\varepsilon=0$ requires $c(p)$ to go to zero sufficiently slowly to make $\int_2^\infty p^{-2}\beta(c(p))dp<\infty$. The existence of such a function $c(\cdot)$ depends therefore on the nature of the growth rate of $\beta(\varepsilon)$ as $\varepsilon \downarrow 0$.

Example 4.6. If $\beta(\varepsilon)=Ke^{(D/\varepsilon^{\kappa})}$ with $\kappa>0$ and $D>0$ take $\varepsilon(p)=C/(\log p)^{\gamma}$. Then C can be chosen such that (3.15) holds if and only if $\gamma>1$. But (3.16) then requires $\gamma\kappa\leq 1$, which forces $\kappa<1$. Davies and Simon give an example of one dimensional potentials $V_{\kappa}(x)=x^2[\log(2+|x|^2)]^{2/\kappa}$, for which intrinsic ultracontractivity holds if and only if $0<\kappa<1$. The asymptotic behavior of the ground state φ_0 is given by $-\log\varphi_0(x)\approx x^2(\log x)^{1/\kappa}$, for $\kappa>0$. This example motivates the following corollary, which is to be understood in the general context of Section 3.

Corollary 4.7. *Assume that* (3.1) *holds and that there is a function* $\alpha:[1,\infty)\to(0,\infty)$ *such that*

$$\alpha(s)\leq B(1+\log s)^{-1} \qquad s\geq 1, \tag{4.9}$$

and

$$M\equiv\int_1^{\infty}2\beta(\alpha(\tau))\tau^{-2}d\tau<\infty. \tag{4.10}$$

If $|||g|||_{\infty}\leq 1$ *then*

$$E(e^{\gamma g^2\log|g|})<\infty \qquad \textit{for } 0\leq\gamma<(2aB)^{-1}. \tag{4.11}$$

Proof. For any $\delta>0$ there is a constant C_{δ} such that

$$\int_1^{\lambda}(1+\log\tau)^{-1}d\tau\leq(1+\delta)(\lambda/(1+\log\lambda))+C_{\delta} \text{ for } \lambda\geq 1.$$

To see this one can, for example, integrate the left side by parts, obtaining $\lambda/(1+\log\lambda)-1+\int_1^{\lambda}(1+\log\tau)^{-2}d\tau$. Writing the last integral as an integral over $[1,\lambda^{1/2}]$ plus an integral over $[\lambda^{1/2},\lambda]$ and estimating the integrand by its maximum on each interval, one obtains upper bounds of $\lambda^{1/2}$ and $\lambda/(1+2^{-1}\log\lambda)^2$ respectively, which are both $o(\lambda/(1+\log\lambda))$. This yields the asserted existence of the constant C_{δ}.

Thus it now follows from (3.4) that

$$E(e^{\lambda g})\leq\exp\{\lambda\log E(e^g)+\lambda(a/2)BC_{\delta}+\lambda M+(a/2)B(1+\delta)\lambda^2/(1+\log\lambda)\}.$$

By Corollary 4.4 it follows that $E(e^{\gamma g^2\log|g|})<\infty$ if $\gamma<\{4(a/2)B(1+\delta)\}^{-1}$ for some $\delta>0$.

Example 4.8. Suppose that $X=\boldsymbol{R}$, $g(x)=x$ and $d\mu(x)=\varphi_0(x)^2dx$ where φ_0 is the ground state for the potential V_{κ} of Example 4.6. Then $\varphi_0(x)^2$ behaves like $e^{-2x^2(\log|x|)^{1/\kappa}}$ for large x. If $\kappa<1$ then (4.11) clearly holds for all $\gamma>0$. If $\kappa>1$ then (4.11) holds for no $\gamma>0$. If $\kappa=1$ then (4.11) holds only for $\gamma<2$.

Furthermore

(4.12) $$E(e^{\gamma g^2[\log(1+|g|)]^{\eta}})=\infty$$

for all small γ if $\eta>1/\kappa$. Thus if $\eta>1$ then (4.12) holds for some ultracontractive semigroup.

Remark 4.9. In view of the derivation of (4.11), the sufficient condition for ultracontractivity given by Davies and Simon's theorem (Theorem 3.8), and the various cases cited in Example 4.8, it seems reasonable to conjecture that the validity of (4.11) for some $\gamma>0$ is the "borderline" necessary condition for ultracontractivity to hold for a Markov semigroup. It should be emphasized that we dirived (4.11) from the infinitesimal version of ultracontractivity (as embodied in the use of logarithmic Sobolev inequalities via Theorem 3.8) rather than from ultracontractivity itself. These two versions of ultracontractivity are not exactly equivalent because of a loss of information involved in the interpolation required to derive (3.1) from ultracontractivity. See [D, Co] for further discussion of the circumstances of equivalence.

Next, suppose we know that $\int_0^1 \beta(\varepsilon)d\varepsilon<\infty$. This is known [DS] to imply ultracontractivity. It is a fairly strong but useful condition. We will now show

Corollary 4.10. *Assume that* (3.1) *holds. If* $\int_0^1 \beta(\varepsilon)d\varepsilon<\infty$ *and* $|||g|||_\infty\le 1$, *then for sufficiently small* θ

$$E(e^{\theta e^{2|g|/a}})<\infty.$$

Proof. Since $||||g||||_\infty\le |||g|||_\infty$ we may assume, without loss of generality, that $g\ge 0$. Putting $s=1$ and $\alpha(\tau)=\tau^{-1}$ in Equ. (3.4) we obtain

$$E(e^{\lambda g})\le E(e^g)^\lambda \exp[\frac{a}{2}\lambda\log\lambda+K'\lambda]$$

where $K'=2\int_0^1 \beta(\varepsilon)d\varepsilon$. Equivalently,

$$E(e^{\lambda g})\le \exp(\frac{a}{2}\lambda\log\lambda+K\lambda)$$

where $K=K'+\log E(e^g)$. If we now set $m(t)=\mu\{x\,|\,g(x)\ge t\}$, then since $E(e^{\lambda g})\ge e^{\lambda t}m(t)$, we have

$$\lambda t+\log m(t)\le \frac{a}{2}\lambda\log\lambda+K\lambda$$

or

$$\log m(t) \leq \frac{a}{2}\lambda \log \lambda + (K-t)\lambda.$$

Setting $\log \lambda = \frac{2}{a}(t-K) - 1$, we get $\log m(t) \leq -Ce^{2t/a}$, where $C = (a/2)e^{-\frac{2}{a}K-1}$, or

(4.13) $$m(t) \leq e^{-Ce^{2t/a}}.$$

Next, let $\theta < C$. Then

$$E(e^{\theta e^{2g/a}}) = -\int_0^\infty e^{\theta e^{2t/a}} dm(t) = -m(t)e^{\theta e^{2t/a}}\Big|_0^\infty + (2\theta/a)\int_0^\infty m(t)e^{\theta e^{2t/a}} e^{2t/a} dt.$$

Both terms are finite by (4.13).

Example 4.11. It is of interest to note that a similar conclusion holds even when $\beta(\varepsilon)$ does not have an integrable singularity at $\varepsilon = 0$. Take as an example $\beta(\varepsilon) = \varepsilon^{-1}$. By Equ. (3.6)

$$[\lambda^{-1} \log v(\lambda)]' \leq \frac{a}{2}\varepsilon + \frac{2}{\varepsilon\lambda^2}.$$

Choose $\varepsilon = \frac{2}{\sqrt{a}}\lambda^{-1}$, which minimizes the right side, to obtain

$$[\lambda^{-1} \log v(\lambda)]' \leq \frac{2\sqrt{a}}{\lambda}$$

Integrating from 1 to λ yields $\lambda^{-1}\log v(\lambda) \leq 2\sqrt{a}\log\lambda + \log v(1)$ or $\log v(\lambda) \leq 2\sqrt{a}\lambda\log\lambda + \lambda\log v(1)$.

We are now more or less in the situation described in the last corollary and readily obtain $E(e^{\theta e^{g/(2a^{1/2})}})$ is finite for

$$\theta < 2\sqrt{a}e^{(-\frac{1}{2\sqrt{a}}\log v(1) - 1)}.$$

This example suggests a general principle which may be useful in other casse. When $\beta(\varepsilon)$ is relatively simple in form, and also differentiable, the optimal procedure is to choose ε to minimize the side of the inequality (3.6) for each λ and proceed from there. Let us note here also that although we did not use the method of Theorem 4.3 in diriving Corollary 4.10, the method of Young's pairs used in Theorem 4.3 has a potential for getting more precise bounds on $E(e^{\Phi(|g|)})$.

Here is an application of Example 4.1 to the central limit theorem.

Corollary 4.12. *Let μ be a probability measure on $\boldsymbol{R}$ satisfying*

(4.14) $$\int_{\boldsymbol{R}} f(x)^2 \log(|f(x)|) d\mu(x) \leq \varepsilon_0 \int_{\boldsymbol{R}} f'(x)^2 d\mu(x) + \|f\|^2_{L^2(\mu)} \log \|f\|_{L^2(\mu)}$$

for all functions f in $C^1(\boldsymbol{R})$. *Let* $X_1, X_2, \cdots$ *be a sequence of i.i.d. random variables with distribution* μ *and mean zero. Let* $S_n = n^{-1/2}\Sigma_{i=1}^n X_i$. *Then*

$$E(\exp[S_n^2/2b]) \le [b/(b-\varepsilon_0)]^{1/2} \text{ for all } n \text{ if } b > \varepsilon_0. \tag{4.15}$$

Proof. The product measure $\mu_n(dx_1, \cdots, dx_n) = \Pi_{j=1}^n \mu(dx_j)$ on $\boldsymbol{R}^n$ satisfies (1.2) with $\beta_0 = 0$ and $\mathscr{E}(f, f) = \int_{\boldsymbol{R}^n} |\text{grad} f(x)|^2 d\mu_n(x)$ by the additivity theorem for logarithmic Sobolev inequalities [G1, Remark 3.3 or G2, Theorem 2.3]. Define $g: \boldsymbol{R}^n \to \boldsymbol{R}$ by $g(x_1, \cdots, x_n) = (x_1 + x_2 + \cdots + x_n)/n^{1/2}$. Then $|\nabla g(x_1, \cdots, x_n)| \le 1$ (and in fact equality holds.) The argument of Example 4.1 is now applicable: By (3.11) one has $E(e^{\lambda S_n}) \le e^{\lambda^2 \varepsilon_0/2}$ for all real λ. Integrating $\int_{-\infty}^{\infty} E(e^{\lambda S_n}) h(\lambda) d\lambda$ as in Example 4.1 yields (4.15).

Remark 4.13. If μ is a probability measure on $\boldsymbol{R}^k$ satisfying (4.14) with $|f'(x)|^2$ replaced by $|\nabla f(x)|^2$ and if $X_1, X_2, \cdots$ is a sequence of i.i.d. $\boldsymbol{R}^k$ valued random variables with distribution μ then Corollary 4.12 has a precise analog: Let $S_n = (X_1 + \cdots + X_n)/n^{1/2}$. Then $E(\exp[\|S_n\|^2/2b]) \le b/(b-\varepsilon_0)]^{1/2}$ for all n if $b > \varepsilon_0$. The proof given in Corollary 4.12 needs only to be modified by taking $g(x_1, \cdots, x_n) = \|x_1 + \cdots + x_n\|/n^{1/2}$ on $\boldsymbol{R}^{kn}$. One must observe also that the k-dimensional version of (4.14) must be extended to locally Lip 1 functions to accomodate use of this function g. The constants are the same as in the one dimensional case. As a result of the dimension independence of the constants the uniform integrability inequality (4.15) holds also for Banach space valued i.i.d. random variables. See [ASh] for the mechanism of formulation in this case. Let us note also that in the one dimensional case the constant in (4.15) is best possible since (4.15) becomes an equality when μ is Gaussian of mean zero and variance ε_0.

5. Discrete spaces

When the Dirichlet form is nonlocal, as in Examples 2.2 and 2.3, the chain rule, which is responsible for the key inequality (2.3) (with $a = 1$), is no longer applicable. Moreover the norm $|||f|||_\infty$, defined in Section 2, may not be useful. We will see later, in Example 5.2, how it fails to give significant information for measures on $\boldsymbol{Z}$. Instead we will use a different "gradient" norm and a different way of bounding Laplace transforms from that of Theorem 3.1. We will show that Poisson measure on $\boldsymbol{Z}_+$ cannot satisfy a logarithmic Sobolev inequality for either of the two natural Dirichlet forms. The same conclusion holds for the discrete heat kernel measure on $\boldsymbol{Z}^d$. The results of this section are a response to a question of P.Diaconis to the first named author as to whether there is a logarithmic Sobolev inequality naturally associated to Poisson measure. The method used in to derive a logarithmic Sobolev inequality for Gauss measure on the line by applying the central limit theorem to the "two point inequality" can in principle be used to derive a logarithmic Sobolev inequality for Poisson measure on $\boldsymbol{Z}_+$ by using the low density limit applied to an asymmetric two point inequality. The best logarithmic Sobolev constant in the asymmetric case was derived in [DSc1, Theorem A.2.] The tensoring method just barely fails to produce a logarithmic Sobolev inequality for Poisson measure. This section will show that the failure is intrinsic to the measure rather than the fault of the method.

First let ue observe how the previous methods may be applied in the context of EXample 2.3.

Example 5.1. Let X be a finite set and let $K(x,y)$ be a Markov transition function which is symmetric with respect to an invariant probability measure $\pi(\cdot)$. For a function $f: X \to \boldsymbol{R}$ define $\|\nabla f\|_\infty$ as in (2.14). Then by (2.15) and (2.3) with $a=2$ we have

$$\mathscr{E}(e^{\lambda g}, e^{\lambda g}) \leq \|\nabla g\|_\infty^2 \lambda^2 E(e^{2\lambda g}) \tag{5.1}$$

wherein $\mathscr{E}$ is defined as in Example 2.3. Suppose that $\|\nabla g\|_\infty \leq 1$. The key step in the proof of Lemma 3.2 is based on the inequality (2.3) which in our present context can be replaced by (5.1). The result is that the conclusion of Theorem 3.1 holds in the form

$$E(e^{\lambda g}) \leq E(e^{sg})^{\lambda/s} \exp\left[\lambda \int_s^\lambda \{\alpha(\tau)/2 + 2\beta(\alpha(\tau))\tau^{-2}\} d\tau\right] \quad 0 < s < \lambda \tag{5.2}$$

when $\|\nabla g\|_\infty \leq 1$. Since X is a finite set the potential usefulness of (5.2) lies in the quantitative bounds rather than merely the conclusion that $E(e^{\lambda g}) < \infty$, which always holds.

L. Saloff-Coste has pointed out to us that if X is infinite and π has infinite support then (1.2) cannot hold. Indeed if $\pi(a) \neq 0$ and $f(x) = \pi(a)^{-1/2}\delta_a(x)$ then one can compute easily that $\|f\|_{L^2(\pi)} = 1$ and $\mathscr{E}(f, f) = 1 - K(a, a) \leq 1$, while the left side of (1.2) is $\log \pi(a)^{-1/2}$, which is unbounded as a runs over the support of π.

Exmaple 5.2 (Poisson measure). Let $\mu_k = e^{-c}c^k/k!$ for $k \geq 0$. Adopting the notation of Example 2.2, choose $b_k = 1$ for $k \geq 0$ and $b_k = 0$ for $k < 0$. Then the Dirichlet form (2.7) is

$$\mathscr{E}(f, f) = \sum_{k \geq 0} (Df)(k)^2 \mu_k. \tag{5.3}$$

This is the principle discrete Dirichlet form that we will explore in this section. Also of special interest is the weaker Dirichlet form given by $b_k^2 = \frac{1}{k+1}$.

We will show in this section that the Poisson measure does not satisfy a logarithmic Soboley inequality (1.2) by showing that (1.2) implies a Herbst inequality which fails for Poisson measure. The norm (2.10) is not adequate for this purpose. In fact for the form (5.3) the norm (2.10) is

$$|||f|||_\infty^2 = \sup_{k\geq 0}\{(Df)(k)^2 + (k/c)(Df)(k-1)^2\}.$$

Hence if $|||g|||_\infty \leq 1$ then $|(Dg)(k-1)| \leq (c/k)^{1/2}$ for $k=1,2,3,\cdots$. By estimating a sum by an integral it follows that $|g(k)| \leq |g(0)| + 2(ck)^{1/2}$. Since $\Sigma_{k=0}^{\infty} e^{\alpha k}\mu_k < \infty$ for all $\alpha > 0$ it follows that $E(e^{g^2/2b}) < \infty$ for all $b > 0$. Therefore the inequality (4.3) holds for all $b > 0$ for Poisson measure and cannot be used to rule out a logarithmic Sobolev inequality. We will proceed differently based on use of a different gradient norm and will abandon use of the inequality (2.3).

The rest of this section is based on use of the gradient norm (5.5). We will apply the resulting Herbst inequality to rule out a logarithmic Sobolev inequality for Poisson measure. But this objective by itself can be most easily accomplished by simply inserting the function given at the end Example 5.1 into the inequality (1.2) with $\mathscr{E}$ given by (5.3). One obtains $(1/2)\log \mu_a^{-1} \leq \varepsilon_0(1 + \mu_{a-1}(\mu_a)^{-1}) + \beta_0$. For Poisson measure this clearly fails for large a, for any ε_0 and β_0, and therefore (1.2) cannot hold. The authors thank the referee for pointing this out.

Theorem 5.3. *Let μ_k be a probability density on $\boldsymbol{Z}$. Define*

$$\mathscr{E}(f,f) = \sum_{k\in \boldsymbol{Z}} (Df(k))^2 \mu_k. \tag{5.4}$$

Assume that (1.2) *holds for some* $\varepsilon_0 > 0$ *and* $\beta_0 \geq 0$. *Assume further that*

$$|||Dg|||_\infty \equiv \sup_{k\in \boldsymbol{Z}} |(Dg)(k)| \leq 1. \tag{5.5}$$

Let

$$h(\lambda) = 2\varepsilon_0 \lambda^{-2}(e^{\lambda/2} - 1)^2 + 2\beta_0\lambda^{-2}. \tag{5.6}$$

Then

$$E(e^{\lambda g}) < \infty \quad \text{for all real } \lambda \tag{5.7}$$

and

$$E(e^{\lambda g}) \leq E(e^{sg})^{\lambda/s}) \exp\left\{\lambda \int_s^\lambda h(\tau)d\tau\right\} \quad 0 < s < \lambda. \tag{5.8}$$

Proof. Just as in the proof of Theorem 3.1 it suffices to prove (5.8) in case g is bounded because the truncations $\psi_n \circ g$ used there for unbounded g behave well with respect to new norm (5.5). I.e., $\|D\psi_n \circ g\|_\infty \uparrow \|Dg\|_\infty$. We proceed then as in Lemma 3.2 with g bounded. Let $v(\lambda) = E(e^{\lambda g})$. Insert $f = e^{\lambda g/2}$ into (1.2). $\mathscr{E}(f,f)$ must be estimated differently from before because the inequality (2.3) is not valid for the norm (5.5). We have instead

$$\mathscr{E}(e^{\lambda g/2}, e^{\lambda g/2}) = \sum_k (e^{\lambda g(k+1)/2} - e^{\lambda g(k)/2})^2 \mu_k$$

$$= \sum_k e^{\lambda g(k)} (e^{\lambda[g(k+1)-g(k)]/2} - 1)^2 \mu_k$$

$$\leq (e^{\lambda/2} - 1)^2 E(e^{\lambda g})$$

because $|g(k+1) - g(k)| \leq 1$ for all k. Using this in the procedure of Lemma 3.2 we find instead of (3.5)

$$(d/d\lambda)(\lambda^{-1} \log v(\lambda)) \leq h(\lambda) \text{ for } \lambda > 0.$$

The rest of the proof is the same as that of Theorem 3.1.

Remark 5.4. If $\beta_0 = 0$ then function h (cf. Equ. (5.6)) has no singularity at $\lambda = 0$. Just as in Example 3.4 we may then let $s \downarrow 0$ in (5.8) and obtain

$$E(e^{\lambda(g - E(g))}) \leq \exp\left\{\lambda \int_0^\lambda h(\tau) d\tau\right\}, \quad \lambda \geq 0. \tag{5.9}$$

But we will focus in the following on Herbst inequalities for the general case, $\beta_0 > 0$. $\Gamma(t)$ will denote the gamma function.

Corollary 5.5. *Under the hypotheses of Theorem 5.3 we have*

$$E(b^{|g|}\Gamma(|g|+1)) < \infty \quad \textit{for all } b > 0. \tag{5.10}$$

Proof. Since $E(e^{\lambda|g|}) \leq E(e^{\lambda g}) + E(e^{-\lambda g}) < \infty$ (5.8) gives

$$E(e^{\lambda|g|}) \leq 2E(e^{|g|})^\lambda \sup\left\{\lambda \int_1^\lambda h(\tau) d\tau\right\} \quad \text{for } \lambda \geq 1. \tag{5.11}$$

It is not hard to see that

$$\lambda \int_1^\lambda h(\tau) d\tau = 0(\lambda^{-1} e^\lambda) \quad \text{as} \quad \lambda \to \infty. \tag{5.12}$$

Indeed for each of the two terms in $h(\lambda)$ containing exponentials one need only use

$$\int_1^\lambda \tau^{-2} e^{c\tau} d\tau \leq \int_1^{\lambda/2} e^{c\tau} d\tau + (\lambda/2)^{-2} \int_{\lambda/2}^\lambda e^{c\tau} d\tau \leq \text{const.}\, \lambda^{-2} e^{c\lambda}$$

for $c > 0$ and large λ. Since $E(e^{\lambda|g|}) \leq E(e^{|g|})$ for $0 \leq \lambda \leq 1$ it follows from (5.11) and (5.12) that

$$\int_0^\infty E(e^{\lambda(|g|+1)})e^{-b^{-1}e^\lambda}d\lambda<\infty \qquad \forall b>0. \tag{5.13}$$

Making the change of variables $x=b^{-1}e^\lambda$ one computes easily that

$$w(t)\equiv\int_0^\infty e^{\lambda t}e^{-b^{-1}e^\lambda}d\lambda=b^t\Gamma(t)-b^t\int_0^{b^{-1}}x^{t-1}e^{-x}dx.$$

But the last term is at most $b^t\int_0^{b^{-1}}x^{t-1}dx=t^{-1}$. Hence $b^t\Gamma(t)\le w(t)+1$ for $t\ge 1$. Replace t by $|g|+1$ and use (5.13) to get (5.10).

Corollary 5.6. *If* $\mu(k)=e^{-c}c^k/k!$ *for* $k\ge 0$ *then a logarithmic Sobolev inequality* (1.2) *for the Dirichlet form* (5.3) *cannot hold.*

Proof. Assume (1.2) holds and apply Corollary 5.5 to the function $g(k)=k$ $(k\ge 0)$. Then $(Dg)(k)=1$. So $\|Dg\|_\infty=1$. It follows from (5.10) that for all $b>0$

$$\infty>E(b^{|g|}\Gamma(g+1))=\sum_{k=0}^\infty b^k\Gamma(k+1)\mu(k)=\sum_{k=0}^\infty b^k k!\mu(k).$$

But if $b\ge 1/c$ the right side is infinite.

Remark 5.7. The no-go result of Corollary 5.5 for Poisson measure applies equally well to the class of Dirichlet forms given in Equ. (2.7), provided the coefficients b_k are bounded. For in this case the form (2.7) is dominated by a multiple of the form (5.3). In particular the Dirichlet form given by $b_k^2=(k+1)^{-1}$ for $k=0,1,2,\cdots$ fails to satisfy a logarithmic Soboley inequality for Poisson measure. This Dirichlet form is associated to the Metropolis algorithem for Poisson measure. See [DSc2] for further discussion of the usefulness of this kind of Dirichlet form.

Remark 5.8. S. Bobkov and M. Ledoux [BL] have found a family of entropy-energy inequalities somewhat in the spirit of the standard logarithmic Sobolev inequality (1.2) but which are valid for densities $(1/2)e^{-|x|}$ on the line. Of course (1.2) cannot hold for such a density because of Herbst's original inequality (1.3) (and its extension to non-zero β_0, (4.3).) S. Bobkov has kindly informed the first author of the present paper that the work in [BL] extends to discrete spaces and that the corresponding new entropy-energy inequality is satisfied by Poisson measure. In that paper the authors are able to deduce concentration inequalities from these new entropy-energy inequalities.

The *heat kernel* on $\boldsymbol{Z}$ is defined as follows. Let m be counting measure on $\boldsymbol{Z}$. Define $(Df)(k)=f(k+1)-f(k)$ as before and write D^* for its adjoint in $L^2(\boldsymbol{Z},m)$. Define

$$\Delta=-D^*D.$$

Then $(\Delta f)(k)=f(k+1)+f(k-1)-2f(k)$. Using this formula for arbitrary functions f we see that if $f_\theta(k)=e^{i\theta k}$ then $\Delta f_\theta=2(\cos\theta-1)f_\theta$. Thus if

$$f(k)=\int_{-\pi}^{\pi} e^{ik\theta}\hat{f}(\theta)d\theta$$

with $\hat{f}\in L^2(S^1,d\theta)$ then $e^{t\Delta/2}$ multiplies $\hat{f}$ by $e^{t(\cos\theta-1)}$. It is convenient to identify the heat kernel for Δ in terms of the Poisson measure μ_c on Z_+ given in Corollary 5.6. The Fourier transform of μ_c is

$$\hat{\mu}_c(\theta)\equiv\sum_{k=0}^{\infty} e^{ik\theta}e^{-c}c^k/k!=e^{c(e^{i\theta}-1)}.$$

Let $\mu_c^*(k)=\mu_c(-k)$. Then $\hat{\mu}_c^*(\theta)=e^{c(e^{-i\theta}-1)}$. Thus the convolution

$$\nu_t=\mu_{t/2}*\mu_{t/2}^* \tag{5.14}$$

has Fourier transform $\hat{\mu}_{t/2}(\theta)\hat{\mu}_{t/2}^*(\theta)=e^{t(\cos\theta-1)}$. Hence $e^{t\Delta/2}=\nu_t*$. That is, ν_t is the heat kernel for the discrete Laplacian.

Corollary 5.9. *The heat kernel ν_t for the discrete Laplacian on Z, together with the Dirichlet form $\mathscr{E}(f,f)=\int_Z|Df(k)|^2\nu_t(dk)$ does not satisfy a logarithmic Sobolev inequality of the form* (1.2).

Proof. For $k\geq 0$,

$$\nu_t(k)=\sum_{j\in Z}\mu_{t/2}(k-j)\mu_{t/2}^*(j)\geq\mu_{t/2}(k)\mu_{t/2}^*(0).$$

Hence $\nu_t(k)$ falls off no faster than the Poisson measure $\mu_{t/2}(k)$ as $k\to+\infty$. The method of proof of Corollary 5.6 applies in this case as well with $g(k)=k$, for $k\in Z$.

Remark 5.10. The same no-go theorem applies to the discrete heat kernel on Z^d. If D_j is the difference operator in the jth coordinate direction and Δ is defined as $-\Sigma_{j=1}^d D_j^*D_j$, where D_j^* is computed with respect to counting measure on Z^d, then the heat kernel for $e^{t\Delta/2}$ is the product of the heat kernels for each operator $e^{-tD_j^*D_j/2}$ on $l^2(Z)$. Taking $g((k_1,\cdots,k_d))=k_1$, the method of Corollary 5.6 applies without change. The proper notion of gradient norm is

$$\|Dg\|_\infty^2=\sup\left\{\sum_{j=1}^d|(D_jg)(k)|^2,\quad k\in Z^d\right\}.$$

For the function at hand this norm is one.

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