Bénard-Marangoni convection
with a deformable surface

By

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1. Introduction

Thermal convection is often studied by an incompressible fluid model, Boussinesq equations (see for instance [J]),

(1.1) $$\frac{1}{\text{Pr}}(u_t + u \cdot \nabla u) + \nabla p = \Delta u - \rho(T)\nabla y, \quad \text{div } u = 0,$$

(1.2) $$T_t + u \cdot \nabla T = \Delta T.$$

Here, $u = (u_1, u_2)$ is the vector field of fluid velocity, p the pressure and T the tmperature, and Pr is a constant called the Prandtl number. The density ρ is assumed to depend linearly on T, $\rho(T) = G - \text{Ra } T$; the constant Ra is called the Rayleigh number.

The simplest setting of these equations for the study of thermal convection would be: consider the equations in a strip region, say $\{-1 < y < 0\}$, with boundary conditions

$$u = 0 \text{ (non slip)} \quad \text{or} \quad p\vec{n} - (\nabla u + {}^t\nabla u) \cdot \vec{n} = 0 \text{ (stress free)}$$

on $y = 0$, -1 and

$$T = 1 \text{ on } y = -1 \text{ and } T = 0 \text{ on } y = 0.$$

This set of equations are too simple for the study of real convection of fluids, because this does not take into account complicated physical effects on the boundary. In usual setting of experiments, upper boundary of the fluid is open to the air and possible to deform from flat shape.

In this paper, we will be concerned with the equations (1.1) and (1.2) on moving two-dimensional region $\Omega(t) = \{-1 < y < \eta(x, t), \ -\infty < x < \infty\}$ with the following boundary conditions. On the bottom boundary $B = \{y = -1\}$, we consider

(1.3) $$u = 0 \quad \text{on } B,$$

(1.4) $$T = 1 \quad \text{on } B.$$

The moving boundary $\Gamma(t)$ subject to "kinematic boundary condition" : normal speed of $\Gamma(t)$ at x equals to $u \cdot \vec{n}|_\Gamma$, where $\vec{n}$ is the outward unit normal vector of $\Gamma(t)$. In terms of $\eta(x, t)$, this can be written as

$$(1.5) \qquad \eta_t = u_2 - u_1 \eta_x.$$

On the moving upper boundary $\Gamma(t) = \{y = \eta(x, t)\}$, we consider stress balance relation:

$$(1.6) \qquad (p - p_{air})\vec{n} - (\nabla u + {}^t\nabla u) \cdot \vec{n} = \sigma H \vec{n} - (\vec{t} \cdot \nabla)\sigma \vec{t} \quad \text{on} \quad \Gamma(t).$$

Here, the constant p_{air} is the pressure of the air, $\vec{t}$ is the tangential unit vectors of the boundary and H is the curvature of the deformable surface $(\frac{-\eta_x}{\sqrt{1 + \eta_x^2}})_x$. We assume that the surface stress σ is given by

$$\sigma := \sigma(T) + \mathrm{Vi}\,\vec{t} \cdot \nabla u \cdot \vec{t},$$

where the surface tension coefficient $\sigma(T)$ is assumed to depend only on the temperature. The second term represents a dissipation effect, called surface viscosity, present on the free surface. We consider heat balance on $\Gamma(t)$ given as

$$(1.7) \qquad \vec{n} \cdot \nabla T + \mathrm{Bi}\,T = -1 \quad \text{on} \quad \Gamma(t),$$

where Bi is a constant.

The equations (1.1)–(1.7) form a complete system for unknown functions η, u, T and p supplemented with initial conditions for η, u and T. These equations have an equilibrium solution

$$(1.8) \qquad \eta = 0, \ u = 0, \ T = \bar{T} := -y, \ p = \bar{p} := -\frac{\mathrm{Ra}}{2}y^2 - \mathrm{G}\,y + p_{air},$$

which represents the purely conducting state. We will be concerned with the existence of solutions of the equations (1.1)–(1.7) for initial data near this equilibrium, assuming that initial conditions are periodic in horizontal direction.

The above system of equations contains as an unknown variable the shape of deformable surface, so this is a free boundary problem. Beale proved an existence result for an incompressible fluid layer with a deformable boundary. In his paper [Beale], he used a transformation determined by the shape of the deformable boundary, which maps time dependent fluid domain to a time independent domain, and transformed the original problem with a moving boundary to a problem on a fixed domain. In section 4, we show existence of exponentially decaying solution of (1.1)–(1.7) for initial condition close to the equilibrium (1.8), using his method with minor modificartions.

The next section deals with a simpler linear system. In section 3, we will show linear stability around the equilibrium solution (1.8), when Rayleigh number and Marangoni number are small enough.

In the rest of this section, we give a weak formulation of equations. Let initial conditions, a function $\eta_0(x)$ and functions $u_0(x, y)$ and the $\theta_0(x, y)$ defined on $\Omega_0 = \{-1 < y < \eta_0(x)\}$ be given. The solution of our problem for the initial condition is $\eta(x, t)$ and functions $u(x, y, t)$ and $\theta(x, y, t)$ defind on $\{t > 0, \ -1 < y < \eta(x, t)\}$ which satisfy $\operatorname{div} u = 0$, (1.3), (1.4), (1.5) and the following two equations. (A) Momentum balance : integral form of the (1.1), (1.6):

$$(1.9) \qquad 0 = \int_{\Omega(t)} \frac{1}{\mathrm{Pr}} (u_t \cdot \Phi - u \otimes u : \nabla \Phi) + 2\boldsymbol{D} : \nabla \Phi - \Phi \cdot \nabla y \rho(T) dx dy$$

$$+ \int_{\Gamma(t)} \sigma(\boldsymbol{I} - \vec{\mathrm{n}} \otimes \vec{\mathrm{n}}) : \nabla \Phi ds \qquad (t > 0)$$

for all vector test functions Φ satisfying $\operatorname{div} \Phi = 0$ and $\Phi|_B = 0$. Here $\boldsymbol{D} = (\nabla u + {}^t\nabla u)/2$, $\boldsymbol{I}$ is the unit matrix and $\Omega(t) = \{(x, y): -1 < y < \eta(x, t)\}$, $\Gamma(t) = \{(x, y): y = \eta(x, t)\}$. (B) Energy balance : integral form of (1.2), (1.7):

$$(1.10) \qquad 0 = \int_{\Omega(t)} (T_t \Psi - Tu \cdot \nabla \Psi) + \nabla T \cdot \nabla \Psi dx dy$$

$$+ \int_{\Gamma(t)} (\mathrm{Bi}\, T + 1) \Psi ds \qquad (t > 0)$$

for all test functions Ψ satisfying $\Psi|_B = 0$.

These can be obtained from the original equations using integration by part and shown to be equivalent to them for sufficiently smooth u, T and $\Omega(t)$. We note that, in deriving (1.9), we have used a form of Stokes' formula

$$\int_{\Omega(t)} \boldsymbol{D} : \nabla \Phi - \operatorname{div} \boldsymbol{D} \cdot \Phi dx = \int_{\Gamma(t)} \vec{\mathrm{n}} \cdot \boldsymbol{D} \cdot \Phi ds$$

and

$$\int_{\Gamma(t)} (\sigma H \vec{\mathrm{n}} - (\vec{\mathrm{t}} \cdot \nabla) \sigma \vec{\mathrm{t}}) \cdot \Phi ds = \int_{\Gamma(t)} \sigma(\boldsymbol{I} - \vec{\mathrm{n}} \otimes \vec{\mathrm{n}}) : \nabla \Phi ds$$

(see [J]).

Remark. The above definition of solution does not require much regularity to the functions. This set of equations make sense, if, for example, η and η_x are continuous, u, $T \in L^2(0, \infty; H^1)$ and, when $\mathrm{Vi} > 0$, the trace of $\vec{\mathrm{t}} \cdot \nabla u \cdot \vec{\mathrm{t}}$ to $\Gamma(t)$ has meaning. The solution to be shown to exist is enough regular; η is more regular than C^1 and u and T belong to higher order Sobolev spaces.

Here, we give some notations and conventions. Ω and Γ denote the region occupied by fluid at the equilibrium (1.8), $\{(x, y): -1 < y < 0\}$ and its upper side

boundary $\{y=0\}$, respectively.   We assume that everything is periodic in horizontal direction with period L, so regions $\Omega(t), \Omega, \cdots$ must be interpreted as peridic one. $H^r(\Omega)$ and $H^r(\Gamma)$ denote the Sobolev space of periodic functions with period L.   We denote their norm by $\|\cdot\|_r$ and $\|\cdot\|_{r,\Gamma}$ respectively. $(\cdot,\cdot)$ and $(\cdot,\cdot)_\Gamma$ is the inner product of $L^2(\Omega) = H^0(\Omega)$ and $H^0(\Gamma)$.   In the following, we use an operator $\Lambda = \sqrt{1 - \Delta_h}$ and its fractional power Λ^s.

2.   Stokes System with a Deformable Surface

In this section, the dimension of the region Ω is not restricted to two.   The region Ω is a strip $\{-1 < y < 0\}$ and everything is periodic in horizontal directions $x_1, \cdots, x_{n-1}$ with period L.   We treat the vertical coordinate y as n-th coordinate x_n.   We denote u_h the horizontal part $u_h = (u_1, \cdots, u_{n-1}, 0)$ of vector u.

We will be concerned with the following linear initial-boundary value problem, the Stokes system with a deformable surface :

(2.1) $u_t + \nabla p = \Delta u + F, \quad$ in Ω,

(2.2) $\mathrm{div}\, u = 0 \quad$ in Ω,

(2.3) $T \cdot \vec{\mathrm{n}} = (-\sigma_0 \Delta_h + \mathrm{G})\eta \vec{\mathrm{n}} - \mathrm{Vi}\Delta_h u_h + f \quad$ on Γ,

(2.4) $u = 0 \quad$ on B,

(2.5) $\eta_t = u \cdot \vec{\mathrm{n}}|_\Gamma$

with homogeneous initial conditions.   Here the stress tensor of fluid is defined as $T := (pI - 2D)$ where $D := (\nabla u + {}^t\nabla u)/2$.   The constants σ_0 and G are assumed to be positive and Vi to be positive or equal to zero.   This system is different to the one trearted in [Beale] in the point that the surface viscosity term $-\mathrm{Vi}\Delta_h u_h$ is introduced in the stress balance equation.   As will be shown, this term slightly regularize the solution on the boundary.

Let $\varphi(x)$ be a vector test function which is smooth, divergence free and vanishes near B.   Taking inner product of the first equation with φ, after intergration by parts, we obtain that the solution (u, η) satisfies

$$\int_\Omega u_t \cdot \varphi + 2D : \nabla\varphi - F \cdot \varphi dx$$

$$+ \int_\Gamma \sigma_0 \nabla_h \eta \cdot \nabla_h \varphi_n + \mathrm{G}\eta\varphi_n + \mathrm{Vi}\,\nabla_h u_h : \nabla_h \varphi_h + f \cdot \varphi d\sigma = \varphi.$$

Thus, we obtain that the equations (2.1)–(2.5) are equivalent to (2.5) and

(2.6) $(u_t, \varphi) + \langle u, \varphi \rangle + b(\eta, \varphi_n|_\Gamma) + \mathrm{Vi}(\nabla_h u_h, \nabla_h \varphi_h)_\Gamma = (F, \varphi) - (f, \varphi)_\Gamma$

for all $t > 0$ and for all smooth and divergence free $\varphi(x)$ which vanishes near B.   Here

$\langle u, \varphi \rangle = \int_\Omega 2D(u):D(\varphi)dx$ and $b(\eta, \varphi_n) = \int_\Gamma (\sigma_0 \nabla_h \eta \cdot \nabla_h \varphi_n + G\eta\varphi_n)d\sigma.$ We note that Korn's inequality $\langle u, u \rangle \geq \delta \|u\|_1^2$ holds in $\{u \in H^1(\Omega): \mathrm{div}\, u = 0,\ u|_B = 0\}.$

We will treat the problem by Laplace transformation

$$\hat{u}(\lambda) = \int_0^\infty e^{-\lambda t}u(t)dt$$

in time variable as in [AV] and [Beale]. Laplace transformation transform the above equation (2.5) and (2.6) to the following boundary value problem with a complex parameter λ.

(2.7) $\qquad \lambda\hat{\eta} = \hat{u} \cdot \vec{n}\,|_\Gamma,$

(2.8) $\qquad \lambda(\hat{u}, \varphi) + \langle \hat{u}, \varphi \rangle + b(\hat{\eta}, \varphi_n) + \mathrm{Vi}(\nabla_h \hat{u}_h, \nabla_h \varphi_h)_\Gamma = (\hat{F}, \varphi) - (\hat{f}, \varphi)_\Gamma$

or, eliminating $\hat{\eta}$,

(2.9) $\qquad K(\hat{u}; \varphi) = (\hat{F}, \varphi) - (\hat{f}, \varphi)_\Gamma.$

where $K(\hat{u}, \varphi) := \lambda(\hat{u}, \varphi) + \langle \hat{u}, \varphi \rangle + \lambda^{-1}b(\hat{u}_n, \varphi_n) + \mathrm{Vi}(\nabla_h \hat{u}_h, \nabla_h \varphi_h)_\Gamma.$ (We note that, here and below $\hat{u}, \varphi, \cdots$ and inner products are suitably complexified.) We will show that this system has a unique solution when λ is in a certain region and then construct the solution of (2.1)–(2.5) with homogeneous initial conditions by transforming back through the inversion formula

(2.10) $\qquad \displaystyle u(t) = \frac{1}{2\pi} \int_{Re\,\lambda = const} \hat{u}(\lambda)e^{\lambda t}d\lambda.$

To state the result of this section, we describe some function spaces to be used. $K^r(\Omega \times I)$ denotes the space - time Sobolev space $H^0(I; H^r(\Omega)) \cap H^{r/2}(I; H^0(\Omega))$ as in [Beale] and $K^r_{-\gamma}(\Omega \times I)$ denotes its weighted version $\{f : e^{\gamma t}f \in K^r(\Omega \times I)\}$, where I is a time interval. For notational simplicity, we write K^r and $K^r_{-\gamma}$ for $K^r(\Omega \times (0, \infty))$ and $K^r_{-\gamma}(\Omega \times (0, \infty))$. $K^r_{-\gamma,(0)}$ denotes $\{f \in K^r_{-\gamma}(\Omega \times (-\infty, \infty)): f|_{t<0} = 0\}$. We can think this space as a subspace of $K^r_{-\gamma}$ and it is known that

$$K^r_{-\gamma,(0)} = \text{the closure in } K^r_{-\gamma} \text{ of } \{f \in K^r_{-\gamma}: f \text{ vanishes near } t = 0\}$$

and, when $\frac{r-1}{2}$ is not an integer,

(2.11) $\qquad \displaystyle K^r_{-\gamma,(0)} = \{f \in K^r_{-\gamma}: \frac{d^k}{dt^k}f\big|_{t=0} = 0 \quad (0 \leq k < \frac{r-1}{2})\}$

(see [LM]). Calculation shows that the norm $\|f\|_{K^r_{-\gamma}}$ is equivalent to

(2.12) $\qquad \displaystyle \int_{Re\,\lambda = -\gamma} (\|\hat{u}(\lambda)\|_r^2 + |\lambda|^r\|\hat{u}(\lambda)\|_0^2)d\lambda$

and Laplace transformation gives an isomorphism between $K^r_{-\gamma,(0)}$ and the space of $H^r(\Omega)$-valued holomorphic functions on $\{\operatorname{Re}\lambda \ge -\gamma\}$ equipped with the norm (2.12) (See section 7 of [AV]). For functions on Γ, we use similar, but a bit different, spaces $K^{r,\frac{1}{4}}(\Gamma \times I) := H^0(I; H^{r+\frac{1}{4}}(\Gamma)) \cap H^{r/2}(I; H^{\frac{1}{4}}(\Gamma))$, and $K^{r,\frac{1}{4}}_{-\gamma}(\Gamma \times I) := \{f : e^{\gamma t}f \in K^{r,\frac{1}{4}}(\Gamma \times I)\}$. When $I = R^+$, we write $K^{r,\frac{1}{4}}(\Gamma)$ and $K^{r,\frac{1}{4}}_{-\gamma}(\Gamma)$ for them. The meaning of $K^{r,\frac{1}{4}}_{-\gamma,(0)}(\Gamma)$ will be obvious from notation. Now, we state the result of this section.

Proposition 2.1. *There exists $\gamma > 0$ so that for any $r \ge 2$ and $(F, f) \in K^{r-2}_{-\gamma,(0)}$* $\times K^{r-2,\frac{1}{4}}_{-\gamma,(0)}(\Gamma)$, *the equations (2.1)–(2.5) has a solution $u \in K^r_{-\gamma,(0)}$, $\eta \in K^{r+\frac{1}{2}}_{-\gamma,(0)}(\Gamma)$ satisfying*

$$\|u\|_{K^r_{-\gamma}} + \|\eta\|_{K^{r-\frac{1}{2},\frac{1}{2}}(\Gamma)} \le C(\|F\|_{K^{r-2}_{-\gamma}} + \|f\|_{K^{r-2,\frac{1}{2}}(\Gamma)}).$$

where the constant C does not depend on F and f. When $\mathrm{Vi} > 0$, this solution satisfies $\nabla^2_h u_h|_\Gamma \in K^{r-2,\frac{1}{4}}_{-\gamma}(\Gamma)$ *and*

$$\|\nabla^2_h u_h\|_{K^{r-2,\frac{1}{2}}(\Gamma)} \le C(\|F\|_{K^{r-2}_{-\gamma}} + \|f\|_{K^{r-2,\frac{1}{2}}(\Gamma)}).$$

Remark. From (2.11), the solution in the above proposition satisfies initial conditions

$$\frac{d^k}{dt^k}u\Big|_{t=0}, \quad \frac{d^k}{dt^k}\eta\Big|_{t=0} = 0 \quad (0 \le k < \frac{r-1}{2}).$$

In the existence proof for the nonlinear problem, we will work in the range $r < 3$. In this range of r, these conditions become $u|_{t=0} = 0$ and $\eta|_{t=0} = 0$. In general, similar conditions must hold for the data F and f, and these constitute the compatibility conditions for data F and f for solvability. But, when $r < 3$, $K^{r-2}_{-\gamma,(0)} = K^{r-2}_{-\gamma}$ from (2.11), thus there is no need for compatibility conditions for the data.

For simplicity, at this point, we assume $f = 0$. The proof below works for general f without essential change. The following proposition is sufficient for the previous proposition.

Proposition 2.2. *Assume $r \ge 2$. There exists a positive constant γ determined by σ_0, G and L so that for any λ in $\{\operatorname{Re}\lambda \ge -\gamma\}$ and data $\hat{F} \in H^{r-2}$, there is a unique solution $\hat{u} \in H^r$, $\hat{\eta} \in H^{r+\frac{1}{2}}$ of (2.5)–(2.6), satisfying*

$$(2.13) \qquad \|\hat{u}\|_r + |\lambda|^{\frac{r}{2}}\|\hat{u}\|_0 + \|\hat{\eta}\|_{r+\frac{1}{2},\Gamma} + |\lambda|^{\frac{r+\frac{1}{2}}{2}}\|\hat{\eta}\|_{0,\Gamma} \le C(\|\hat{F}\|_{r-2} + |\lambda|^{\frac{r-2}{2}}\|\hat{F}\|_0).$$

Here the constant C does not depend on λ. When $\mathrm{Vi} > 0$, $\hat{u}_h|_\Gamma \in H^{r+\frac{1}{4}}(\Gamma)$ and $\|\hat{u}_h\|_{r+\frac{1}{4},\Gamma} + |\lambda|^{\frac{r-2}{2}}\|\hat{u}_h\|_{2+\frac{1}{4},\Gamma}$ *is estimated by the right hand side of the above estimate (2.13).*

The rest of this section is devoted to the proof of Proposition 2.2. In the

following of this section, we omit $\hat{}$. We assume $\mathrm{Vi} > 0$; the proof for $\mathrm{Vi} = 0$ is the obvious modification of the following.

Because Korn's inequality holds and the boundary terms $b(u_n|_\Gamma, \varphi_n|_\Gamma)$ and $(\nabla_h u_h, \nabla_h \varphi_h)_\Gamma$ are equivalent to the $H^1(\Gamma)$-inner product for functions on Γ, it is natural to consider (2.9) in the function space

$$V = \{u \in H^1(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega, \ u|_B = 0, \ u|_\Gamma \in H^1(\Gamma)\}.$$

We have

Proposition 2.3. *Assume* $F \in L^2(\Omega)$ *and* $\operatorname{Re} \lambda > 0$. *The problem* $K(u; \varphi) = (F, \varphi)$ *for all* $\varphi \in V$ *has a unique solution* $u \in V$

Proof. Setting $\varphi = u$ in (2.9), we obtain

$$(2.14) \qquad \operatorname{Re} K(u; u) = \operatorname{Re} \lambda \|u\|_0^2 + \langle u, u \rangle + \frac{\operatorname{Re}}{|\lambda|^2} b(u_n, u_n) + \mathrm{Vi} \|\nabla_h u_h\|_{0,\Gamma}^2.$$

By using Korn's inequality, when $\operatorname{Re} \lambda > 0$, we obtain

$$(2.15) \qquad K(u; u) \geq \delta \|u\|_1^2 + \operatorname{Re} \lambda \|u_n\|_{1,\Gamma}^2 + \mathrm{Vi} \|u_h\|_{1,\Gamma}^2.$$

Thus, $K(\cdot, \cdot)$ is V-coercive. We apply Lax-Milgram theorem and obtain the conclusion.

For a fixed $F(\lambda)$, from the above proposition, we obtain a solution $u(\lambda)$ for all λ in $\{\operatorname{Re} \lambda > 0\}$ and this $u(\lambda)$ is holomorphic in λ (with value, say, in V). Thus, by holomorphic continuation, the following estimates are sufficient to obtain the solution of $\{\operatorname{Re} \lambda \geq -\gamma\}$. (We note that continuation of holomorphic $u(\lambda)$ satisfying (2.9) always satisfies (2.9), because (2.9) is holomoirphic relation in λ.)

Before the estimates of the solution, we state a lemma.

Lemma 1 ([L]). *Consider a boundary value problem*:

$$\operatorname{div} \varphi = p \quad in \quad \Omega, \qquad \varphi|_{\Gamma \cup B} = b$$

for the data p, b *satisfying* $\int_{\Gamma \cup B} b \cdot \vec{n} = \int_\Omega p$ *where* $\vec{n}$ *is the outer unit normal vector. There is a solution operator* $(p, b) \mapsto \varphi$ *which satisfies*

$$\|\Lambda^s \varphi\|_1 \leq C(\|\Lambda^s p\|_0 + \|b\|_{s+\frac{1}{2}, \Gamma \cup B}).$$

Proof. We prove the lemma assuming $s = 0$. We take a function a by solving $\Delta a = p$ in Ω, $\nabla a \cdot \vec{n} = b \cdot \vec{n}$ on $\Gamma \cup B$. This a satisfies $\|\nabla a\|_1 \leq C(\|p\|_0 + \|b\|_{\frac{1}{2}, \Gamma \cup B})$, thus, we have reduced this lemma to the case $p = 0$.

We take an anti-symmetric tensor $\omega = \{\omega_{ij}\}$ vanishing on $\Gamma \cup B$ and satisfying $\omega_{in,n} = b_i$ $(1 \leq i \leq n - 1)$. Then, setting $\varphi = \operatorname{div} \omega$, $\operatorname{div} \varphi = 0$ is automatically satisfied

and, because tangential derivatives of ω vanish on $\Gamma \cup B$, φ satisfies required boundary conditions.

Proposition 2.4. *There is a small positive γ which depends only on σ_0, G and L so that, when $r \geq 2$, we have*

$$\|\Lambda^{r-1}u\|_1 + \|\eta\|_{r+\frac{1}{2},\Gamma} \leq C\|\Lambda^{r-2}F\|_0$$

for λ with $\mathrm{Re}\,\lambda \geq -\gamma$. The constant C does not depend on λ.

Proof. We take an auxiliary function a by solving

$$\mathrm{div}\,a = 0, \quad a_n|_\Gamma = u_n|_\Gamma, \quad a_h|_\Gamma = 0, \quad a|_B = 0$$

by lemma 1. Setting $\varphi = \Lambda^{2r-1}a$ in (2.9), we obtain

$$\lambda^{-1}b(\Lambda^{r-\frac{1}{2}}u_n, \Lambda^{r-\frac{1}{2}}u_n) = (\Lambda^{r-2}F, \Lambda^{r+1}a) - \langle \Lambda^{r-1}u, \Lambda^r a\rangle - \lambda(\Lambda^r u, \Lambda^{r-1}a),$$

thus

$$\|u_n\|_{r+\frac{1}{2},\Gamma}^2 \leq C|\lambda|(\|\Lambda^{r-2}F\|_0 + \|\Lambda^{r-1}u\|_1)\|\Lambda^r a\|_1 + C|\lambda|^2\|\Lambda^r u\|_0\|\Lambda^{r-1}a\|_0 .$$

$$\leq \varepsilon\|\Lambda^r a\|_1^2 + C|\lambda|^2(\|\Lambda^{r-2}F\|_0 + \|\Lambda^{r-1}u\|_1 + \|\Lambda^r u\|_0\|\Lambda^{r-1}a\|_0).$$

According to lemma 1, a satisfies $\|\Lambda^r a\|_1 \leq C\|u_n\|_{r+\frac{1}{2},\Gamma}$ and $\|\Lambda^{r-1}a\|_0 \leq C\|u_n\|_{r-2+\frac{1}{2},\Gamma}$ $\leq C\|\Lambda^{r-2}u\|_1$. Using these inequalities, we obtain

$$\|u_n\|_{r+\frac{1}{2},\Gamma}^2 \leq C|\lambda|^2(\|\Lambda^{r-2}F\|_0^2 + \|\Lambda^{r-1}u\|_1^2)$$

or

(2.16) $$\|\eta\|_{r+\frac{1}{2},\Gamma} \leq C(\|\Lambda^{r-2}F\|_0 + \|\Lambda^{r-1}u\|_1).$$

On the other hand, by setting $\varphi = \Lambda^{2(r-1)}u$ in (2.9), we obtain

(2.17) $$\sigma(\|\Lambda^{r-1}u\|_0^2 + \delta_0\|\eta\|_{r,\Gamma}^2) + \delta\|\Lambda^{r-1}u\|_1^2 \leq \|\Lambda^{r-2}F\|_0\Lambda^r u\|_0.$$

For non-negative σ, the proposition follows from (2.16) and (2.17). When σ is negative and sufficiently close to 0, (2.17) leads to

(2.18) $$\|\Lambda^{r-1}u\|_1 \leq C(|\sigma|\|\eta\|_{r,\Gamma} + \|\Lambda^{r-2}F\|_0)$$

and, combining (2.16) and (2.18), we show the proposition for sufficiently small negative σ.

Proposition 2.5. *For $\mathrm{Re}\,\lambda \geq -\gamma$, we have*

$$|\lambda|\|\Lambda^{r-2}u\|_0 \leq C\|\Lambda^{r-2}F\|_0$$

where the constant C does not depend on λ.

Proof. Since $|\lambda| \leq C(|\mathrm{Im}\,\lambda| + \mathrm{Re}\,\lambda + 1)$, it is sufficient to show

$$(2.19) \qquad |\mathrm{Im}\,\lambda| \|\Lambda^s u\|_0 \leq C\|\Lambda^s F\|_0,$$

because $\mathrm{Re}\,\lambda \|\Lambda^s u\|_0$ is estimated in (2.17). We set $\varphi = \Lambda^{2s} u$ in (2.9) and take imaginary part, then we obtain

$$\mathrm{Im}\,\lambda \|\Lambda^s u\|_0^2 = \frac{\mathrm{Im}\,\lambda}{|\lambda|^2} b(\Lambda^s u, \Lambda^s u) + \mathrm{Im}(\Lambda^s F, \Lambda^s u).$$

Thus,

$$|\mathrm{Im}\,\lambda|^2 \|\Lambda^s u\|_0^2 \leq Cb(\Lambda^s u, \Lambda^s u) + C|\mathrm{Im}\,\lambda| \|\Lambda^s F\|_0 \|\Lambda^s u\|_0.$$

Since $b(f, f) \leq C\|f\|_{1,\Gamma}$, applying Cauchy-Schwarz inequality to the last term, we obtain

$$|\mathrm{Im}\,\lambda| \|\Lambda^s u\|_0 \leq C\|u_n\|_{s+1,\Gamma} + C\|\Lambda^s F\|_0.$$

So, it is sufficient to show $\|u_n\|_{s+1,\Gamma} \leq C\|\Lambda^s F\|_0$.

From boundedness of trace, $\|u_n\|_{s+1,\Gamma} \leq C\|\Lambda^s u_n\|_2$. Here, we note that, for f satisfying $\mathrm{div}\,f = 0$, we have $f_{n,z} = -\nabla_h \cdot f_h$, thus $\|f_n\|_2 \leq C\|\Lambda f\|_1 + C\|f_{n,yy}\|_0 \leq C\|\Lambda f\|_1$. Using this fact, we obtain

$$(2.20) \qquad \|u_n\|_{s+1,\Gamma} \leq C\|\Lambda^{s+1} u\|_1.$$

Now, proposition 2.4 and the last inequality lead to the desired estimate.

Proposition 2.6. *When* $\mathrm{Vi} > 0$, *we have*

$$\|u_h\|_{r+\frac{1}{2},\Gamma} \leq C\|\Lambda^{r-2} F\|_0$$

Proof. As in the estimate of η in the proof of proposition 2.4, we solve

$$\mathrm{div}\,a = 0, \qquad a_h|_\Gamma = u_h|_\Gamma, \qquad a_n|_\Gamma = 0, \qquad a|_B = 0$$

and set $\varphi = \Lambda^{2r-1} a$ to (2.9). Then, we obtain

$$(2.21) \qquad \mathrm{Vi}\|u_h\|_{r+\frac{1}{2},\Gamma}^2 \leq C(|\lambda| \|\Lambda^{r-2} u\|_0 + \|\Lambda^{r-1} u\|_1 + \|\Lambda^{r-2} F\|_0) \|\Lambda^r a\|_0.$$

Since $\|\Lambda^r a\|_0 \leq C\|u_h\|_{r+\frac{1}{2},\Gamma}$ from Lemma 1, Proposition 2.4, 2,5 and (2.21) lead to the conclusion.

We have obtained

$$(2.22) \qquad \|\Lambda^{r-1} u\|_1 + |\lambda| \|\Lambda^{r-2} u\|_0 + \|\eta\|_{r+\frac{1}{2}} + \|u_h\|_{r+\frac{1}{2},\Gamma} \leq C\|\Lambda^{r-2} F\|_0.$$

What we have to show is the estimates of normal derivatives. We do this following the process in [SS].

First, we can obtain the corresponding pressure field p as a function satisfying $l(\varphi) = (p, \operatorname{div}\varphi)$ where $l(\cdot)$ is the functional $l(\varphi) := K(u;\varphi) - (F,\varphi)$. This p exists because $l(\cdot)$ is defined and bounded on $\{\varphi \in H^1 : \varphi|_B = 0\}$, and vanishes for divergence free φ. These p and u satisfy

$$(2.23) \qquad\qquad K(u;\varphi) - (p, \operatorname{div}\varphi) = (F,\varphi).$$

for φ not necessarily divergence free, and then, (u, p, η) satisfies equations (2.1)–(2.5) classically. For estimates for pressure p, we solve

$$\operatorname{div}\tilde{\varphi} = p \quad \text{in } \Omega, \qquad \tilde{\varphi}_h|_\Gamma = 0, \qquad \tilde{\varphi}|_B = 0$$

and set $\varphi = \Lambda^{2(r-1)}\tilde{\varphi}$ in (2.23). Then, we obtain

$$\lambda(\Lambda^{r-2}u, \Lambda^r\tilde{\varphi}) + \|\Lambda^{r-1}p\|_0^2 + \langle\Lambda^{r-1}u, \Lambda^{r-1}\tilde{\varphi}\rangle + b(\Lambda^{r-\frac{1}{2}}\eta, \Lambda^{r-1-\frac{1}{2}}\tilde{\varphi}_n) = (\Lambda^{r-2}F, \Lambda^r\tilde{\varphi}).$$

Since $\|\Lambda^{r-1}\tilde{\varphi}\|_1 + \|\tilde{\varphi}_n\|_{r-\frac{1}{2},\Gamma} \leq \|\Lambda^{r-1}p\|_0$ from Lemma 1, using (2.22), we obtain

$$\|\Lambda^{r-1}p\|_0 \leq C\|\Lambda^{r-2}F\|_0.$$

Now, because u, η and p satisfy equations classically, we can write normal derivatives of u as $u_{n,y} = -\operatorname{div}_h u_h$ and $u_{h,yy} = -\Delta_h u_h - \lambda u_h - \nabla_h p + F_h$, and we can show the needed estimates inductively for integer r. We can show the desired estimates for general r by interpolation.

Remark. *The pressure p for the solution (u, η) exists and satisfy $\|p\|_{K^{r-1}_{-y}}$ $\leq C(\|F\|_{K^{r-2}_{-y}} + \|f\|_{K^{r-2,\frac{1}{2}}_{-y}(\Gamma)})$. We did not include this in the statement of Proposition 2.1, because we do not need this in section 4.*

3. The linearized system

In this section, we are concerned with the linearization around the equilibrium (1.8):

$$\frac{1}{\Pr}u_t + \nabla q = \Delta u + \operatorname{Ra}\theta\nabla y + F, \qquad \operatorname{div}u = 0 \quad \text{in } \Omega,$$

$$\theta_t = \Delta\theta + u_2 + F_0 \quad \text{in } \Omega,$$

$$\eta_t = u_2|_\Gamma,$$
$$T\cdot\vec{n} = (-\sigma_0\Delta_h + G)\eta\vec{n} + (\operatorname{Ma}\nabla_h(\theta - \eta) - \operatorname{Vi}\Delta_h u_1)\vec{t} + f \quad \text{on } \Gamma,$$

$$\theta_y + \operatorname{Bi}(\theta - \eta) = f_0 \quad \text{on } \Gamma,$$

$$u = 0,\ \theta = 0 \quad \text{on } B.$$

Here $\Omega = \{-1 < y < 0\}$ is the domain occupied by the fluid at the equilibrium and $\Gamma = \{y = 0\}$ and $B = \{y = -1\}$ are its boundaries. The differential operators Δ_h and ∇_h which appear in the boundary conditions is $(\frac{\partial}{\partial x})^2$ and $\frac{\partial}{\partial x}$ and the vectors $\vec{n}$ and

$\vec{t}$ are normal and tangential unit vectors of Γ, $(0,1)$ and $(1,0)$. The constant σ_0 and Ma is determined as $\sigma(T) = \sigma_0 + \text{Ma}\, T + O(T^2)$.

This is the Stokes system with a deformable surface + heat equation + lower order terms. We will show that this linear system has exponentially decaying solution, when Ra and $|\text{Ma}|$ are small enough.

This linear system can be written as

(3.1)
$$\eta_t = u_2 |_\Gamma$$

(3.2)
$$K(u, 0, \eta\,;\Phi) = (F, \Phi) - (f, \Phi)_\Gamma$$

(3.3)
$$K_0(u, 0, \eta\,;\Phi) = (F_0, \Psi) - (f_0, \Psi)_\Gamma$$

for all smooth $\Phi(x)$ and $\Psi(x)$ with $\text{div}\,\Phi = 0$, $\Phi|_B = \Psi|_B = 0$, where

$$K(u, 0, \eta\,;\Phi) := \frac{1}{\text{Pr}}(u_t, \Phi) + \langle u, \Phi \rangle + b(\eta, \Phi_2) + \text{Vi}(\nabla_h u_1, \nabla_h \Phi_1)_\Gamma$$

$$- \text{Ra}(0, \Phi_2) + \text{Ma}((0 - \eta), \nabla_h \Phi_1)_\Gamma$$

$$K_0(u, 0, \eta\,;\Psi) := (0_t, \Psi) + (\nabla 0, \nabla\Psi) + \text{Bi}(0, \Psi)_\Gamma - (u_2, \Psi)$$

For this system, we prove the following result.

Proposition 3.1. *Assume* Ra *and* Ma *are sufficiently small. For data* F, $F_0 \in K^{r-2}_{-\gamma,(0)}$, f, $f_0 \in K^{r-2,\frac{1}{2}}_{-\gamma,(0)}(\Gamma)$, *the equations* (3.1)–(3.3) *has a solution* u, $0 \in K^r_{-\gamma,(0)}$, $\eta \in K^{r+\frac{1}{2}}_{-\gamma,(0)}(\Gamma)$, *which satisfies*

(3.4)
$$\|u, 0\|_{K^r_{-\gamma}} + \|\eta\|_{K^{r+\frac{1}{2}}_{-\gamma}(\Gamma)} \leq C(\|F, F_0\|_{K^{r-2}_{-\gamma}} + \|f, f_0\|_{K^{r-2,\frac{1}{2}}_{-\gamma}(\Gamma)}).$$

When Vi *is positive,* $u_1 \in K^{r-2,2+\frac{1}{4}}_{-\gamma,(0)}(\Gamma)$ *and* $\|\nabla_h^2 u_1\|_{K^{r-2,\frac{1}{2}}_{-\gamma}(\Gamma)} \leq C(\|F, F_0\|_{K^{r-2}_{-\gamma}} + \|f, f_0\|_{K^{r-2,\frac{1}{2}}_{-\gamma}(\Gamma)}).$

Proof. Existence will be obvious, once the estimates in the statement of the proposition have been shown. Applying Proposition 2.1 to (3.1)–(3.2), we obtain

$$\|u\|_{K^r_{-\gamma}} + \|\eta\|_{K^{r+\frac{1}{2}}_{-\gamma}(\Gamma)} + \|\nabla_h^2 u_1\|_{K^{r-2+\frac{1}{2}}_{-\gamma}(\Gamma)}$$

$$\leq C(\|F\|_{K^{r-2}_{-\gamma}} + \|f\|_{K^{r-\frac{3}{2}}_{-\gamma}(\Gamma)} + \text{Ra}\|0\|_{K^{r-2}_{-\gamma}} + |\text{Ma}|\|0, \eta\|_{K^{r-\frac{1}{2}}_{-\gamma}}).$$

Similar estimates can be shown for heat equation and we can obtain

$$\|0\|_{K^r_{-\gamma}} \leq C(\|F_0\|_{K^{r-2}_{-\gamma}} + \|f_0\|_{K^{r-\frac{3}{2}}_{-\gamma}(\Gamma)} + \|u\|_{K^{r-2}_{-\gamma}} + |\text{Bi}|\|\eta\|_{K^{r-\frac{3}{2}}_{-\gamma}(\Gamma)}).$$

Combining these two inequalities, we obtain

$$\|u, \sqrt{\mathrm{Ra}}\,\theta\|_{K^r_{-\gamma}} + \|\eta\|_{K^{r+\frac{1}{2}}_{-\gamma}(\Gamma)} + \|\nabla_h^2 u_1\|_{K^{r-2+\frac{1}{2}}_{-\gamma}(\Gamma)}$$

$$\leq C(\|F, \sqrt{\mathrm{Ra}}\,F_0\|_{K^{r-2}_{-\gamma}} + \|f, \sqrt{\mathrm{Ra}}\,f_0\|_{K^{r-\frac{3}{2}}_{-\gamma}(\Gamma)})$$

$$+ C\sqrt{\mathrm{Ra}}\,\|u, \sqrt{\mathrm{Ra}}\,\theta\|_{K^{r-2}_{-\gamma}} + C\,|\mathrm{Ma}|\,\|\theta, \eta\|_{K^{r-\frac{1}{2}}_{-\gamma}(\Gamma)} + C\sqrt{\mathrm{Ra}}\,|\mathrm{Bi}|\,\|\eta\|_{K^{r-\frac{3}{2}}_{-\gamma}(\Gamma)}$$

and we obtain (3.4) for Vi > 0 when Ra and $|\mathrm{Ma}|$ are small. The case Vi $= 0$ can be shown exactly in the same way.

In the next section, we need results for more general form of inhomogeneous terms as following:

(3.5) $$K(u, \theta, \eta\,; \Phi) = (F, \Phi) + (F^1, \nabla\Phi) - (f, \Phi)_\Gamma$$

(3.6) $$K_0(u, \theta, \eta\,; \Phi) = (F_0, \Psi) + (f_0^1, \nabla\Psi) - (f_0, \Psi)_\Gamma$$

where $F^1 = \{F_{ij}^1\}$, $F_0^1 = \{F_{0i}^1\}$ and $(F^1, \nabla\Phi) = \int_\Omega F_{ij}^1 \partial_i \Phi_j\, dx$.

Proposition 3.2. *Assume* Ra *and* Ma *are sufficiently small. For data* F, $F_0 \in K^{r-2}_{-\gamma,(0)}$, F^1, $F_0^1 \in K^{r-1}_{-\gamma,(0)}$, f, $f^0 \in K^{r-2,\frac{1}{2}}_{-\gamma,(0)}(\Gamma)$, *there exists a solution* u, $\theta \in K^r_{-\gamma,(0)}$, $\eta \in K^{r+\frac{1}{2}}_{-\gamma,(0)}(\Gamma)$, *which satisfies*

(3.7) $$\|u, \theta\|_{K^r_{-\gamma}} + \|\eta\|_{K^{r+\frac{1}{2}}_{-\gamma}(\Gamma)} \leq C(\|F, F_0\|_{K^{r-2}_{-\gamma}} + \|F^1, F_0^1\|_{K^{r-1}_{-\gamma}} + \|f, f_0\|_{K^{r-2,\frac{1}{2}}_{-\gamma}(\Gamma)}).$$

When Vi *is positive*, $u_1 \in K^{r-2,2+\frac{1}{4}}_{-\gamma,(0)}\Gamma'$ *and* $\|\nabla_h^2 u_1\|_{K^{r-2,\frac{1}{4}}_{-\gamma}(\Gamma)} \leq C(\|F, F_0\|_{K^{r-2}_{-\gamma}} + \|f, f_0\|_{K^{r-2,\frac{1}{2}}_{-\gamma}(\Gamma)}).$

This proposition can be reduced to the previous proposition using $\int_\Omega F^1 : \nabla\Phi\,dxdy = -\int \operatorname{div} F^1 \cdot \Phi\,dxdy + \int_\Gamma \vec{\mathbf{n}} \cdot F^1 \cdot \Phi\,dx.$

4.   Existence for the nonlinear problem

In this section, using the result in the previous section, we show an existence result for (1.1)–(1.7).

First, we write equations for perturbation (u, θ, η) to the equilibrium solution (1.8). We substitute $T = \bar{T} + \theta$ to (1.9) and (1.10). Because

$$\int_{\Omega(t)} -\Phi \cdot \nabla y \rho(T) dxdy = \int_{\Omega(t)} \Phi \cdot \nabla(Gy + \mathrm{Ra}\, y^2/2) dxdy - \mathrm{Ra} \int_{\Omega(t)} \theta\Phi_2 dxdy$$

$$= \int_{\Gamma(t)} (Gy + \mathrm{Ra}\, y^2/2)\Phi \cdot \vec{\mathbf{n}}\, ds - \mathrm{Ra} \int_{\Omega(t)} \theta\Phi_2 dxdy,$$

(1.9) becomes

(4.1)
$$\int_{\Omega(t)} \frac{1}{P_r} u_t \cdot \Phi + \text{Ra}\, \theta \Phi_2 \, dxdy + \int_{\Omega(t)} (-\frac{1}{P_r} u \otimes u + 2D) : \nabla\Phi \, dxdy$$

$$+ \int_{\Gamma(t)} (Gy + \text{Ra}\, y^2/2)\Phi \cdot \vec{n}\, ds + \int_{\Gamma(t)} \sigma(I - \vec{n} \otimes \vec{n}) : \nabla\Phi \, ds = 0.$$

(1.10) becomes

(4.2)
$$\int_{\Omega(t)} (\theta_t + u_2)\Psi \, dxdy + \int_{\Omega(t)} (-\theta u + \nabla\theta) \cdot \nabla\Psi \, dxdy$$

$$+ \int_{\Gamma(t)} \text{Bi}\,\theta\Psi\, ds + (\int_{\Omega(t)} -\Psi_y \, dxdy + \int_{\Gamma(t)} \Psi\, ds) = 0.$$

We use a transformation $t = \tilde{t}$, $x_i = X_i(\tilde{x}_\alpha, \tilde{t})$ ($i, \alpha = 1$ or 2) determined by the shape of the deformable boundary η

(4.3)
$$x = \tilde{x} + \psi_{\tilde{y}} * \eta,$$

(4.4)
$$y = \tilde{y} + (\tilde{y} + 1)\varphi_{\tilde{y}} * \eta.$$

Here, $\varphi_{\tilde{y}}$ and $\psi_{\tilde{y}}$ are $\frac{1}{\tilde{y}}\varphi(\cdot/\tilde{y})$ and $\frac{1}{\tilde{y}}\psi(\cdot/\tilde{y})$, and φ and ψ are smooth functions with compact support satisfying $\int \varphi = 1$ and $\int \psi = 0$. $*$ denotes the convolution product in $\tilde{x}$. This transformation maps the time-dependent fluid domain $\Omega(t) = \{-1 < y < \eta(x,t)\}$ to a fixed domain $\{-1 < \tilde{y} < 0\} = \Omega$.

By this transformation, the independent variables are transformed as $\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}} + (1+\tilde{y})\varphi_{\tilde{y}}*\eta_{\tilde{t}}\frac{\partial}{\partial y} + \psi_{\tilde{y}}*\eta_{\tilde{t}x}\frac{\partial}{\partial x}$, $\frac{\partial}{\partial \tilde{x}_\alpha} = X_{i,\alpha}\frac{\partial}{\partial x_i}$, and $dxdy = Jd\tilde{x}d\tilde{y} = \sqrt{1 + \eta_{\tilde{x}}^2}\, d\tilde{x}$.

As in [Beale], we define velocity field on Ω by

$$u_i = J^{-1} X_{i,\alpha} \tilde{u}_\alpha$$

where $J = \det x_{i,\alpha}$ is the Jacobian determinant of the transformation. By this choice, $\text{div}\, u = 0$ and $\eta_t = u_2 - u_1\eta_x|_{\Gamma(t)}$ are transformed to $\text{div}_{\tilde{x}}\tilde{u} = 0$ and $\eta_{\tilde{t}} = \tilde{u}_2|_\Gamma$. We transform test function Φ in the same manner. Then, the new test function $\tilde{\Phi}$ runs through $\{\text{div}\,\tilde{\Phi} = 0, \tilde{\Phi}|_B = 0\}$.

By these changes of variables, the integrals in (4.1) and (4.2) become integrals over Ω and Γ and the integrands are written in terms of $\tilde{u}$, $\tilde{\theta}$, η, $\tilde{\Phi}$, $\tilde{\Psi}$; we write obtained equations as $\tilde{K}(\tilde{z};\tilde{\Phi}) = 0$, $\tilde{K}_0(\tilde{z};\tilde{\Psi}) = 0$ where $\tilde{z} = (\tilde{u}, \tilde{\theta}, \eta)$. The difference with the left hand side of the linearized system, $\tilde{K}(\tilde{z};\tilde{\Phi}) - K(\tilde{z};\tilde{\Phi})$, can be written as $\int_\Omega (F(\tilde{z}) \cdot \tilde{\Phi} + F^1(\tilde{z}) : \tilde{\nabla}\tilde{\Phi}) dxdy - \int_\Gamma f(\tilde{z}) \cdot \tilde{\Phi} d\tilde{x}$. Similarly, we set $\tilde{K}_0(\tilde{z};\tilde{\Psi}) - K_0(\tilde{z};\tilde{\Psi}) = \int_\Omega (F_0(\tilde{z})\tilde{\Psi} + F_0^1(\tilde{z}) \cdot \tilde{\nabla}\tilde{\Psi}) dxdy - \int_\Gamma f_0(\tilde{z})\tilde{\Psi} d\tilde{x}$.

We write the linear system treated in the previous section (3.1), (3.5) and (3.6) as

$$\mathscr{L}\tilde{z} = \mathscr{F}, \quad \text{where } \tilde{z} = (\tilde{u}, \tilde{\theta}, \eta), \ \mathscr{F} = (F, F_0, F^1, F_0^1, f, f_0).$$

Then, the transformed problem can now be written as

(4.5) $\mathscr{L}\tilde{z}=\mathscr{F}(\tilde{z}),\quad \tilde{z}|_{t=0}=\tilde{z}_0$

where $\mathscr{F}(\tilde{z})=(F(\tilde{z}),F_0(\tilde{z}),F^1(\tilde{z}),F_0^1(\tilde{z}),f(\tilde{z}),f_0(\tilde{z}))$, and $\tilde{z}_0$ is determined by z_0.

We fix γ chosen according to Proposition 3.2 and consider

$$X^r=\{\tilde{z}=(\tilde{u},\tilde{\theta},\eta):$$

$$\eta_t=\tilde{u}_2|_\Gamma,\quad \operatorname{div}\tilde{u}=0 \text{ in } \Omega,\quad \tilde{u}|_B=0,\quad \tilde{\theta}|_B=0,\quad \int_\Gamma \eta=0$$

$$\tilde{u},\tilde{\theta}\in K_{-\gamma}^r,\ \eta\in K_{-\gamma}^{r+\frac{1}{2}}(\Gamma),\ \nabla_i\tilde{u}_1|_\Gamma\in K^{r,\frac{1}{2}}(\Gamma)\}$$

$$Y^r=\{\mathscr{F}=(F,F_0,F^1,F_0^1,f,f_0):$$

$$F,F_0\in K_{-\gamma}^{r-2},\ F^1,\ F_0^1\in K_{-\gamma}^{r-1},\ f,f_0\in K_{-\gamma}^{r-2,\frac{1}{2}}(\Gamma)\}$$

and X_0^r and Y_0^r be the spaces with $K_{-\gamma}^s$ replaced by $K_{-\gamma,(0)}^s$. We note that, when $2<r<3$, Y_0^r is equal to Y^r from Remark in section 2.

Let $Q=\{(x,y,t):t>0,\ -1<y<\eta(x,t)\}$ and $K^r(Q)=\{u|_Q:u\in K^r(\{y>-1\}\times \mathbf{R}^+)\}$.

Proposition 4.1. *Suppose $r>2$ and $\tilde{z}=(\tilde{u},\tilde{\theta},\eta)\in X^r$ be given. Then, corresponding u and T are well-defined and belongs to $K^r(Q)$. If $\tilde{z}\in X^r$ is a solution of (4.5), (u,T,η) is a solution of (1.1)–(1.7).*

Proof. Regularity of $\eta\in K_{-\gamma}^{r+\frac{1}{2}}$ is sufficient for $\tilde{u}\mapsto u$ and $\tilde{\theta}\mapsto \theta$ to be well-defind on the class K^r. See [Beale], especially Lemma 5.2.

If initial condition $\eta_0\in H^{r-\frac{1}{2}}(\Gamma)$ and $u_0,\theta_0\in H^{r-1}(\Omega(0))$, $u_{01}|_\Gamma\in H^{r-\frac{1}{2}}(\Gamma(0))$ is small, $\tilde{z}_0=(\tilde{u}_0,\tilde{\theta}_0,\eta_0)$ is also small and we can construct their extension $z^{(0)}\in X^r$ to $t>0$. This can be chosen to satisfy $\|z^{(0)}\|_{X^r}<C(\|\eta_0\|_{r-\frac{1}{2},\Gamma}+\|u_0\|_{r-1}+\|\theta_0\|_{r-1}+\|u_{01}\|_{r-\frac{1}{2},\Gamma})$. (This can be shown by using extension theorem for K^r. See [Beale].)

We substitute $\tilde{z}=z^{(0)}+\tilde{\tilde{z}}$ and we obtain the equation for $\tilde{\tilde{z}}$:

$$\mathscr{L}\tilde{\tilde{z}}=\mathscr{F}(z^{(0)}+\tilde{\tilde{z}})-\mathscr{L}z^{(0)},\quad \tilde{\tilde{z}}|_{t=0}=0.$$

According to previous section, the restriction of the linear part $\mathscr{L}$ to X_0^r, $\mathscr{L}_0:X_0^r\to Y_0^r$ has a bounded inverse. Thus, the problem has been reduced to a fixed point problem

$$\tilde{\tilde{z}}=\mathscr{L}_0^{-1}[\mathscr{F}(z^{(0)}+\tilde{\tilde{z}})-\mathscr{L}z^{(0)}].$$

We can show existence of solution $\tilde{z}$ for small initial data by verifying that the right hand side define contraction mapping on a small ball in X_0^r centered at the origin, if the following estimates for the nonlinear terms hold.

Proposition 4.2. *When $r>\frac{5}{2}$,*

(4.6) $\|\mathscr{F}(z)\|_{Y^r}\le C\|z\|_{X^r}^2,$

(4.7) $\|\mathcal{F}(z_1)-\mathcal{F}(z_2)\|_{Y^r}\le C\max(\|z_1\|_{X^r},\ \|z_2\|_{X^r})\|z_1-z_2\|_{X^r}.$

Transforming back $(\eta,\tilde{u},\tilde{\theta})$ to the variables defined on physical domain, we can obtain a solution (η,u,T) of the system (1.1)–(1.7).

Theorem. *We assume* $\frac{5}{2}<r<3$, *constants* Ra *and* Ma *are sufficiently small and* Vi *is zero or positive. Initial conditions* $\eta_0\in H^{r-\frac{1}{2}}$ *and* $u_0,\ T_0\in H^{r-1}(\{-1<y<\eta_0(x)\})$ *periodic in horizontal direction are given and satisfy* $\int\eta_0=0$, $\operatorname{div}u_0=0$, $u_0|_B=0$ *and* $T_0|_B=-1$. *Then, there exists* $\delta>0$ *so that for initial conditions satisfying* $\|\eta_0\|_{H^{r-\frac12}}+\|u_0\|_{H^{r-1}}+\|T_0-\bar{T}\|_{H^{r-1}}<\delta$ *(in the case* Vi>0, *moreover, satisfying* $\|u_0\cdot\vec{t}|_{y=\eta_0}\|_{H^{r-1}}<\delta$), *there exists a solution* $\eta\in K^{r+\frac12}_{-\gamma}(\Gamma\times(0,\infty))$, $u,\ T\in K^r_{-\gamma}(Q)$ *periodic in* x *with the same period of initial condition. When* Vi>0, *moreover, we have* $u\cdot\vec{t}|_{y=\eta}\in K^{r+\frac12}_{-\gamma}(\Gamma\times(0,\infty))$.

Remark. Even when Ra and Ma are not small, we can show Proposition 3.1 with some positive γ and, working in spaces like $K^r(\Omega\times(0,T))$, obtain an existence of the solution in finite time for general Ra and Ma.

Proof of Proposition 4.2. We don't present full detail. We describe the proof for (4.6);(4.7) can be shown similarly. For estimates of these nonlinear terms, we use the following ineqqulities.

(4.8) $\|fg\|_{K^s}\le C\|f\|_{K^r}\|g\|_{K^s}$ $(f\in K^r,\ g\in K^s,\ r>2,\ 0\le s\le r),$

(4.9) $\|fg\|_{K^{s,\frac12}(\Gamma)}\le C\|f\|_{K^{r,\frac12}(\Gamma)}\|g\|_{K^{s,\frac12}(\Gamma)}$

$$(f\in K^{r,\frac12}(\Gamma),\ g\in K^{s,\frac12}(\Gamma),\ r>\frac{3}{2},\ 0\le s\le r).$$

((4.8) is proved in [Beale], Lemma 5.1. (4.9) can be proved by the same argument.) Using (4.8), when $r>2$, we obtain

(4.10) $\|F\|_{K^{r-2}_{-\gamma}}\le C\|\tilde{\nabla}X-I\|_{K^r}(\|\tilde{u}_{\tilde{t}}\|,\ \tilde{\nabla}\tilde{u}\|_{K^{r-2}_{-\gamma}}+\|\tilde{\theta}\|_{K^{r-2}_{-\gamma}}),$

(4.11) $\|F_0\|_{K^{r-2}_{-\gamma}}\le C\|\tilde{\nabla}X-I\|_{K^r}(\|\tilde{\theta}_{\tilde{t}}\|,\ \tilde{\nabla}\tilde{\theta}\|_{K^{r-2}_{-\gamma}}+\|\tilde{u}\|_{K^{r-2}_{-\gamma}}),$

(4.12) $\|F^1\|_{K^{r-1}_{-\gamma}}\le C\|\tilde{\nabla}X-I\|_{K^r}(\|\tilde{\nabla}\tilde{u}\|_{K^{r-1}_{-\gamma}}+C\|\tilde{u}\|_{K^r}\|\tilde{u}\|_{K^{r-1}_{-\gamma}}),$

(4.13) $\|F_0^1\|_{K^{r-1}_{-\gamma}}\le C\|\tilde{\nabla}X-I\|_{K^r}(\|\tilde{\nabla}\tilde{\theta}\|_{K^{r-1}_{-\gamma}}+C\|\tilde{u}\|_{K^r}\|\tilde{\theta}\|_{K^{r-1}_{-\gamma}}),$

and, we obtain the required estimates from $\|\tilde{\nabla}X-I\|_{K^r}\le C\|\eta\|_{K^{r+\frac12}_{-\gamma}(\Gamma)}$, which can be shown from the definition of X (see [Beale]).

The last integral in (4.1) becomes

$$\int_{\Gamma(t)} \sigma_0 H\Phi \cdot \vec{n} - Ma(\vec{t}\cdot\nabla)(\theta-\eta)\Phi\cdot\vec{t} + Ma(\theta-\eta)H\Phi\cdot\vec{n} + O(\theta-\eta)^2)\vec{t}\cdot\nabla\Phi\cdot\vec{t}$$

$$+ Vi(\vec{t}\cdot\nabla u\cdot\vec{t})(\vec{t}\cdot\nabla\Phi\cdot\vec{t})\,ds.$$

Because $H = (-1 + O(\eta_{\tilde{x}}))\eta_{\tilde{x}\tilde{x}}$, $\vec{t}\cdot\nabla u\cdot\vec{t} = (-1 + O(\eta_{\tilde{x}}))\tilde{u}_{1,\tilde{x}} + O(\eta_{\tilde{x}}\eta_{\tilde{x}\tilde{x}}\tilde{u})$ and $ds = \sqrt{1+\eta_{\tilde{x}}^2}\,d\tilde{x}$, recalling the definition of $f(\tilde{z})$, when $r > \frac{5}{2}$,

$$\|f\|_{K_{-\gamma}^{r-2,\frac{1}{2}}(\Gamma)} \le C\|\eta_{\tilde{x}}\|_{K_{-\gamma}^{r-1,\frac{1}{2}}(\Gamma)}(\|\eta\|_{K_{-\frac{1}{2}}^{r}(\Gamma)} + \|\tilde{\theta} - \eta\|_{K_{-\gamma}^{r-1,\frac{1}{2}}(\Gamma)} + Vi\|\tilde{u}_{1,\tilde{x}\tilde{x}}\|_{K_{-\gamma}^{r-2,\frac{1}{2}}(\Gamma)})$$

$$+ C\|\tilde{\theta}-\eta\|_{K_{-\gamma}^{r-1,\frac{1}{2}}(\Gamma)}\|\tilde{\theta}-\eta\|_{K_{-\gamma}^{r-2,\frac{1}{2}}(\Gamma)}$$

and, with $\|\tilde{\theta}\|_{K_{-\gamma}^{r-1,\frac{1}{2}}(\Gamma)} \le C\|\theta\|_{K_{-\gamma}^{r}}$, this leads to the desired estimates for $f(\tilde{z})$. By similar computation, we can obtain $\|f_0(\tilde{z})\|_{K_{-\gamma}^{r-2,\frac{1}{2}}(\Gamma)} \le C\|\eta\|_{K_{-\frac{1}{2}}^{r}(\Gamma)}(\|\eta\|_{K_{-\frac{1}{2}}^{r}(\Gamma)} + \|\tilde{\theta}\|_{K_{-\gamma}^{r-2,\frac{1}{2}}(\Gamma)})$ and we have completed the proof.

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