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Coliftings and Gorenstein injective modules

By

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1. Introduction

Throughout this paper, R will denote a commutative noetherian ring.

We recall that an R-module M is said to be *Gorenstein injective* if and only if for any R-module Q of finite injective or projective dimension, $\mathrm{Ext}_R^i(Q, M)$, $\mathrm{Ext}_i^R(Q, M)$ for all $i \geq 1$ and $\mathrm{Ext}_R^0(Q, M)$, $\mathrm{Ext}_0^R(Q, M)$ vanish (see Enochs-Jenda [4] for equivalent definitions). We note that Gorenstein injective modules are dual to Auslander's maximal Cohen-Macaulay modules. While the latter modules are studied in the category of finitely generated modules, Gorenstein injective modules are rarely finitely generated (see [4]).

The aim of this paper is to study Maranda type of results for Gorenstein injective modules. We note that these type of results have been generalized to maximal Cohen-Macaulay modules over R-algebras where R is a complete local Gorenstein ring (see Auslander-Ding-Soldberg [1] and Ding-Soldberg [2]).

We recall that an R-module M is said to be *strongly indecomposable* if $End(M)$ is a local ring, and M is said to be *reduced* if it has no nonzero injective submodules. We note that every strongly indecomposable module is indecomposable and thus reduced.

If x is an R-regular element that is not regular on an R-module M, then $r_x(M)$ will denote the least integer r such that $x^r \cdot \mathrm{Ext}^1(\ , M) = 0$. We will show that if $r_x(M)$ is finite, $r \geq r_x(M) \geq 0$ and M is a strongly indecomposable Gorenstein injective R-module such that $\mathrm{Hom}_R(\frac{R}{x^r R}, M) \cong \mathrm{Hom}_R(\frac{R}{x^r R}, N)$ for some strongly indecomposable Gorenstein injective R-module N with $r \geq r_x(N) \geq 0$, then $M \cong N$ or $N \cong S(M) \cong S^2(N)$ (showing that N has periodic injective resolution of period 2) where $S^i(M)$ denotes the i^{th} cosyzygy of M. Furthermore, if $r_x(M)$ is finite and $r \geq r_x(M) \geq 0$ and M is strongly indecomposable, then $\mathrm{Hom}_R(\frac{R}{x^r R}, M)$ is indecomposable or $\mathrm{Hom}_R(\frac{R}{x^r R}, M) \cong L \oplus S_{R/x^r R}(L)$ for some indecomposable L (Theorem 3.3). This result is obtained without requiring the Gorenstein condition on the ring. As an easy consequence, we get that if x is R-regular, and E, E' are indecomposable injective R-modules with $\mathrm{Hom}_R(\frac{R}{xR}, E) \cong \mathrm{Hom}_R(\frac{R}{xR}, E') \neq 0$, then $E \cong E'$.

A linear map $\psi : M \to G$ where G is a Gorenstein injective R-module is said to be a *Gorenstein injective preenvelope* if $\mathrm{Hom}(G, G') \to \mathrm{Hom}(M, G') \to 0$ is exact for all

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Gorenstein injective R-modules G'. If furthermore, $f \circ \psi = \psi$ for $f \in \mathrm{Hom}(G, G)$ implies f is an automorphism, then ψ is called a *Gorenstein injective envelope.* It was shown in Enochs-Jenda-Xu [8] that if R is n-Gorenstein (that is, the injective dimension of R over itself is at most n), then every R-module has a Gorenstein injective envelope. Gorenstein injective envelopes are unique up to isomorphism. Furthermore, since every injective module is Gorenstein injective, we see that Gorenstein injective envelopes are monomorphisms. So if $\psi: M \to G$ is a Gorenstein injective envelope, we say that G, denoted $G(M)$, is the Gorenstein injective envelope of M. Likewise, $E(M)$ will denote the injective envelope of M.

A linear map $\psi: E \to M$ where E is an injective R-module is said to be an *injective precover* if $\mathrm{Hom}(E', E) \to \mathrm{Hom}(E', M) \to 0$ is exact for all injective R-modules E'. If furthermore, $\psi \circ f = \psi$ for $f \in \mathrm{Hom}(E, E)$ implies f is an automorphism, then ψ is called an *injective cover.* Injective covers were shown to exist over noetherian rings in Enochs [3]. Again, injective covers are unique but they are not surjective in general.

If $R \to S$ is a ring homomorphism and L is an S-module, then an R-module M is said to be a *colifting* of L to R if

1) $L \cong \mathrm{Hom}_R(S, M)$
2) $\mathrm{Ext}^i_R(S, M) = 0$ for all $i \geq 1$.

If L is a direct summand of $\mathrm{Hom}_R(S, M)$ satisfying (2), then M is said to be a *weak colifting* of L. We say that L is *coliftable* or *weakly coliftable* to R, respectively, if there is such an R-module M. Clearly, every coliftable S-module is weakly coliftable. But weakly coliftable modules need not be coliftable (see Example 4.3). We note that if M is Gorenstein injective, then $\mathrm{Ext}^i_R(S, M) = 0$ for all $i \geq 1$ whenever $pd_R S < \infty$. Thus if M is Gorenstein injective and $pd_R S < \infty$, then M is a colifting of an S-module L if and only if $L \cong \mathrm{Hom}_R(S, M)$.

If x is R-regular, then we will denote $\frac{R}{xR}$ by $\bar{R}$. The aim of Section 4 is to study coliftings of $\bar{R}$-modules. We show that if M is a strongly indecomposable Gorenstein injective module over an n-Gorenstein ring R and M is a colifting of a nonzero $\bar{R}$-module L such that $x \cdot \mathrm{Ext}^1(\ , M) = 0$, then $S_R(\mathrm{Hom}_R(\bar{R}, S^{-1}(M)))$ is a reduced Gorenstein injective R-module and $G_R(L) \cong M \oplus S^{-1}(M)$ where $G_R(L)$ and $S^{-1}(M)$ denote the Gorenstein injective envelope of the R-module L and the kernel of the injective cover of M, respectively (Theorem 4.2).

An R-module M is said to be an *essential colifting* of an $\bar{R}$-module L if M is a colifting of L and the R-imbedding $L \subseteq M$ is an essential extension. We also say that L is *essentially coliftable.* We argue that if R is n-Gorenstein and M is an essential colifting of an $\bar{R}$-module to R, then $\mathrm{Hom}_R(\bar{R}, G_R(M)) \cong G_{\bar{R}}(\mathrm{Hom}(\bar{R}, M))$ (Proposition 4.6). We use this Lemma to obtain an analog of Proposition 5.2 of Auslander-Ding-Soldberg [1], namely that if L is essentially coliftable to R, then $G_{\bar{R}}(L)$ and $\frac{G_{\bar{R}}(L)}{L}$ are also coliftable to R (Theorem 4.7). We then characterize essentially coliftable $\bar{R}$-modules over 2-Gorenstein rings in Theorem 4.9 giving us an analog of Proposition 4.3 of [1]. Finally, we characterize weakly coliftable

$\bar{R}$-modules in Theorem 4.11.

2. Preliminaries

We start with the following easy

Lemma 2.1. *Let $\sigma: M \to N$ be a surjective homomorphism of R-modules M and N. If σ factors through $E(M)$, then N is a direct summand of $\frac{E(M)}{\operatorname{Ker}\sigma}$.*

Proof. Since σ factors through $E(M)$, we have the following commutative diagram

$$\begin{array}{ccc} M & \subsetneq & E(M) \\ \sigma\downarrow & \swarrow_{\sigma'} & \\ N & & \end{array}$$

with $\sigma'(\operatorname{Ker}\sigma)=0$. So we have the followed induced commutative diagram

$$\begin{array}{ccc} E(M) & \xrightarrow{\tau} & \frac{E(M)}{\operatorname{Ker}\sigma} \\ \downarrow\sigma' & \swarrow_{\sigma''} & \\ N & & \end{array}$$

But then we have the following commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \ker\sigma & \to & M & \xrightarrow{\sigma} & N & \to & 0 \\ & & \| & & \downarrow & \nearrow^{\sigma'} & {}^{k}\downarrow\uparrow^{\sigma''} & & \\ 0 & \to & \ker\sigma & \to & E(M) & \xrightarrow{\tau} & \frac{E(M)}{\ker\sigma} & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & S(M) & = & S(M) & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

where $\sigma'' \circ k \circ \sigma = \sigma'' \circ \tau_{|M} = \sigma'_{|M} = \sigma$. But σ is onto. So $\sigma'' \circ k = id_N$. Thus the result follows.

Proposition 2.2 (Hilton [9]). *Let $\sigma: M \to N$ be a homomorphism of R-modules. Then the following are equivalent.*

1) *σ factors through an injective R-module.*

2) *σ factors through the injective envelope of M.*

3) *σ factors through the injective cover of N.*

4) $\sigma \cdot \mathrm{Ext}^1(\ ,\mathrm{M})=0$

5) $\sigma \cdot (0 \to M \to E(M) \to S(M) \to 0)=0$ *in* $\mathrm{Ext}^1(S(M), N)$.

Furthermore, if σ is onto and factors through an injective R-module, then $\frac{E(M)}{\mathrm{Ker}\sigma} \cong S(M) \oplus N$.

Proof. $1 \Rightarrow 2$. Let σ factor through an injective R-module E. Then the diagram

$$\begin{array}{ccc} M & \hookrightarrow & E(M) \\ {\scriptstyle\sigma}\downarrow & \searrow & \downarrow \\ N & \leftarrow & E \end{array}$$

can be completed to a commutative diagram since E is injective. So (2) follows.

$2 \Rightarrow 3$. Let $E \to N$ be the injective cover of N. Then the diagram

$$\begin{array}{ccc} M & \hookrightarrow & E(M) \\ {\scriptstyle\sigma}\downarrow & \swarrow & \downarrow \\ N & \to & E \end{array}$$

can be completed to a commutative diagram. So (3) follows.

$3 \Rightarrow 1$ is trivial.

$1 \Rightarrow 4$. Let $\sigma: M \xrightarrow{\psi} E \xrightarrow{\tau} N$ be the factorization of σ through an injective R-module E. Then we have the following induced commutative diagram

$$\begin{array}{ccc} \sigma_*: \mathrm{Ext}^1(\ ,M) & \longrightarrow & \mathrm{Ext}^1(\ ,N) \\ {\scriptstyle\psi_*}\searrow & & \nearrow{\scriptstyle\tau_*} \\ & \mathrm{Ext}^1(\ ,E)=0 & \end{array}$$

Thus $\sigma_* = \tau_* \circ \psi_* = 0$ and so $\sigma \cdot \mathrm{Ext}^1(\ M)=0$.

$4 \Rightarrow 5$. We simply note that $0 \to M \to E(M) \to S(M) \to 0 \in \mathrm{Ext}^1(S(M), M)$ and so $\sigma \cdot (0 \to M \to E(M) \to S(M) \to 0)=0$ in $\mathrm{Ext}^1(S(M), N)$ since $\sigma \cdot \mathrm{Ext}^1(\ ,M)=0$.

$5 \Rightarrow 2$. We again consider the exact sequenct $0 \to M \to E(M) \to S(M) \to 0$. We have the long exact sequence

$$0 \to \mathrm{Hom}(S(M), N) \to \mathrm{Hom}(E(M), N) \to \mathrm{Hom}(M, N) \to \mathrm{Ext}^1(S(M), N) \to \cdots$$

But $\sigma \cdot (0 \to M \to E(M) \to S(M) \to 0)=0$ in $\mathrm{Ext}^1(S(M), N)$. So the diagram

$$\begin{array}{ccc} M & \subsetneq & E(M) \\ {\scriptstyle\sigma}\downarrow & \swarrow & \\ N & & \end{array}$$

can be completed to a commutative diagram and thus (2) follows.

Now suppose $\sigma : M \to N$ is onto and factors through an injective R-module. Then σ factors through $E(M)$ by the above. So $\frac{E(M)}{\mathrm{Ker}\sigma} \cong S(M) \oplus N$ by Lemma 2.1 above.

Corollary 2.3. *Let x be R-regular and M be an x-divisible R-module. If* $x \cdot \mathrm{Ext}^1(\ , M) = 0$, then $\frac{E(M)}{\mathrm{Hom}_R(\bar{R}, M)} \cong S(M) \oplus M$.

Proof. Let multiplication by x on M be the map σ in the Proposition above. Then σ is onto since M is x-divisible. So the result follows immediately from the Proposition.

Corollary 2.4. *Let x be R-regular and M be an x-divisible R-module such that* $\mathrm{Hom}_R(\bar{R}, M) \subseteq M$ is an essential extension. If $x \cdot \mathrm{Ext}^1(\ , M) = 0$, *then* $S(\mathrm{Hom}_R(\bar{R}, M)) \cong S(M) \oplus M$.

Proof. We simply note that in this case $E(M) \cong E(\mathrm{Hom}_R(\bar{R}, M))$ and so the result follows from Corollary 2.3.

3. Gorenstein injective modules and regular elements

If x is an R-regular element and M is a Gorenstein injective R-module, then $\mathrm{Ext}^1(\bar{R}, M) = 0$ since $pd\bar{R} \leq 1$. Thus we have the exact sequence $0 \to \mathrm{Hom}_R(\bar{R}, M) \to M \xrightarrow{x} M \to 0$. In particular, we have that Gorenstein injective modules are x-divisible.

We start with the following

Lemma 3.1. *Let x be R-regular and M be a reduced Gorenstein injective R-module such that* $\mathrm{Hom}_R(\bar{R}, M) \neq 0$ *and* $x \cdot \mathrm{Ext}^1(\ , M) = 0$, *then* $\mathrm{Hom}_R(\bar{R}, M)$ *is an essential R-submodule of M.*

Proof. We first note that $E(M) \cong E(\mathrm{Hom}_R(\bar{R}, M)) \oplus E$ for some injective R-module E. And so by Corollary 2.3, we have that $S(\mathrm{Hom}_R(\bar{R}, M)) \oplus E \cong S(M) \oplus M$. If M is reduced, then $S(M)$ is reduced. It is then easy to see that $S(M) \oplus M$ is also reduced. Hence $E = 0$ and $E(M) \cong E(\mathrm{Hom}_R(\bar{R}, M))$. So the result follows.

We also need the following

Lemma 3.2. *Let x be R-regular and L be an $\bar{R}$-module. Then L is weakly coliftable to R if and only if*

$$\mathrm{Hom}(\bar{R}, S_R(L)) \cong L \oplus S_{\bar{R}}(L).$$

Proof. Let M be a weak colifting of L to R. Then we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & L & \to & M & \to & \frac{M}{L} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & L & \to & E(L) & \to & S_R(L) & \to & 0 \end{array}$$

We now apply $\mathrm{Hom}_R(\bar{R}, -)$ to the diagram to get

$$\begin{array}{ccccccccccc} \theta_L : 0 & \to & L & \xrightarrow{i} & \mathrm{Hom}_R(\bar{R}, M) & \to & \mathrm{Hom}_R(\bar{R}, \frac{M}{L}) & \to & L & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ \psi_L : 0 & \to & L & \to & \mathrm{Hom}_R(\bar{R}, E(L)) & \to & \mathrm{Hom}_R(\bar{R}, S_R L) & \to & L & \to & 0 \end{array}$$

since $\mathrm{Hom}_R(\bar{R}, L) \cong \mathrm{Ext}^1_R(\bar{R}, L) \cong L$.

But $\mathrm{Hom}_R(\bar{R}, E(L)) = E_{\bar{R}}(L)$. So $\mathrm{Ext}^1_{\bar{R}}(L, S_{\bar{R}} L)) \cong \mathrm{Ext}^2_{\bar{R}}(L, L)$. Furthermore, we note that θ_L, ψ_L represent the same elements in $\mathrm{Ext}^2_{\bar{R}}(L, L)$. But i is a split monomorphism by assumption and so θ_L is zero in $\mathrm{Ext}^2_{\bar{R}}(L, L)$. Therefore, ψ_L is zero in $\mathrm{Ext}^1_{\bar{R}}(L, S_{\bar{R}}(L))$. That is, $0 \to S_{\bar{R}}(L) \to \mathrm{Hom}_R(\bar{R}, S_R(L)) \to L \to 0$ is split exact and so the result follows.

The converse is trivial.

If x is R-regular but not regular on an R-module M, then for $r \geq 0$, we set

$$r_x(M) = \min\{r : x^r \cdot \mathrm{Ext}^1(\ , M) = 0\}.$$

If there is no such r, we set $r_x(M) = \infty$. We note that $r_x(M) = 0$ if and only if M is an injective R-module.

We are now in a position to state the following Maranda type of result.

Theorem 3.3. *Let x be R-regular and M be an x-divisible R-module such that $r_x(M)$ is finite and $r \geq r_x(M) \geq 0$. Then*

1) *If* $\mathrm{Hom}_R(\frac{R}{x^r R}, M) \cong \mathrm{Hom}_R(\frac{R}{x^r R}, N)$ *for some x-divisible R-module N with $r \geq r_x(N) \geq 0$ and* $\mathrm{Hom}_R(\frac{R}{x^r R}, M) \subseteq M$ *and* $\mathrm{Hom}_R(\frac{R}{x^r R}, N) \subseteq N$ *are essential extensions, then* $M \oplus S(M) \cong N \oplus S(N)$.

2) *If M is a strongly indecomposable Gorenstein injective R-module such that* $\mathrm{Hom}_R(\frac{R}{x^r R}, M) \cong \mathrm{Hom}_R(\frac{R}{x^r R}, N)$ *for some strongly indecomposable Gorenstein injective R-module N with $r \geq r_x(N) \geq 0$, then $M \cong N$ or $N \cong S(M) \cong S^2(N)$.*

3) *If M is a strongly indecomposable Gorenstein injective R-module, then* $\mathrm{Hom}_R(\frac{R}{x^r R}, M)$ *is indecomposable or* $\mathrm{Hom}_R(\frac{R}{x^r R}, M) \cong L \oplus S_{\frac{R}{x^r R}}(L)$ *for some in-*

decomposable $\frac{R}{x^rR}$*-module* L.

Proof. 1) easily follows from Corollary 2.4.

2) If M, N are strongly indecomposable, then $\mathrm{Hom}_R(\frac{R}{x^rR}, M) \subseteq M$ and $\mathrm{Hom}_R(\frac{R}{x^rR}, N) \subseteq N$ are essential extensions by Lemma 3.1 and so $M \oplus S(M) \cong N \oplus S(N)$ by part (1). But $S(M)$, $S(N)$ are also strongly indecomposable. So the result follows by the Krull-Remak-Schmidt-Azumaya Theorem.

3) Suppose $\mathrm{Hom}_R(\frac{R}{x^rR}, M) = L \oplus L'$. Then $S(\mathrm{Hom}_R(\frac{R}{x^rR}, M)) = S(L) \oplus S(L')$. But $S(\mathrm{Hom}_R(\frac{R}{x^rR}, M)) \cong S(M) \oplus M$ by Corollary 2.4 and Lemma 3.1 and $S(M)$ is also strongly indecomposable. So we may assume that $M \cong S(L)$. But L is weakly coliftable and so $\mathrm{Hom}_R(\frac{R}{x^rR}, M) \cong \mathrm{Hom}_R(\frac{R}{x^rR}, S(L)) \cong L \oplus S_{\frac{R}{x^rR}}(L)$ by Lemma 3.2. But M is a strongly indecomposable Gorenstein injective R-module and therefore so is $S(L)$. But then L is indecomposable.

Corollary 3.4. *Let x be R-regular and E, E' be indecomposable injective R-modules with* $\mathrm{Hom}_R(\bar{R}, E) \cong \mathrm{Hom}_R(\bar{R}, E') \neq 0$. *Then* $E \cong E'$.

We now state the following easy properties of $r_x(M)$.

Proposition 3.5. 1) *Let $0 \to M'' \to M \to M' \to 0$ be an exact sequence of R-modules such that $r_x(M'')$ and $r_x(M')$ are finite, then $r_x(M) \leq r_x(M'') + r_x(M')$.*

2) *If M is an R-module with $r_x(M) < \infty$, then $r_x(S(M)) \leq r_x(M)$.*

3) *If $M \cong \oplus_{i=1}^{s} M_i$, then $r_x(M) = \max\{r_x(M_i)\}$.*

Proof. 1) We consider the long exact sequence $\cdots \to \mathrm{Ext}^1(-, M'') \to \mathrm{Ext}^1(-, M) \to \mathrm{Ext}^1(-, M') \to \cdots$. If $r_x(M'') = m$ and $r_x(M') = n$, then $x^{m+n} \cdot \mathrm{Ext}^1(\ , M) = 0$ and so $r_x(M) \leq m + n$.

2) If $r_x(M) = m$, then $x^m \cdot \mathrm{Ext}^1(\ , M) = 0$ and so $x^m : M \to M$ factors through $E(M)$ by Proposition 2.2. So we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \to & E(M) & \to & S(M) & \to & 0 \\ & & \downarrow x^m & \swarrow & \downarrow x^m & \swarrow & \downarrow x^m & & \\ 0 & \to & M & \to & E(M) & \to & S(M) & \to & 0 \end{array}$$

Thus $x^m : S(M) \to S(M) \to$ factors through $E(M)$ and hence $x^m \cdot \mathrm{Ext}^1(\ , S(M)) = 0$. That is, $r_x(S(M)) \leq m$.

3) This follows from the fact that $\mathrm{Ext}^1(\ , M) \cong \oplus_{i=1}^{s} \mathrm{Ext}^1(\ , M_i)$.

Corollary 3.6. *Let x be R-regular but not regular on a reduced Gorenstein injective R-module M. If $r_x(M) < \infty$, then $r_x(M) = r_x(S(M)) = r_x(S^{-1}(M))$.*

Proof. If $r_x(M) < \infty$, *then* $r_x(S(M)) \le r_x(M)$ by Proposition 3.5. If $m = r_x(S(M))$, then $x^m : S(M) \to S(M)$ can be factored through the injective cover $E(M) \to S(M)$. So $x^m : M \to M$ can be factored through $E(M)$ as in the proof of the Proposition above. So $r_x(M) \le r_x(S(M))$. Thus $r_x(M) = r_x(S(M))$.

Furthermore, $M = S(S^{-1}(M))$ since M is reduced and so $r_x(M) = r_x(S^{-1}(M))$.

4. Coliftings over n-Gorenstein Rings

We start with the following result.

Lemma 4.1. *Let R be an n-Gorenstein ring, x be R-regular, and M be a reduced Gorenstein injective R-module such that x is not regular on M. Suppose $S(\mathrm{Hom}_R(\bar{R}, S^{-1}(M)))$ is a reduced R-module. Then $S(\mathrm{Hom}_R(\bar{R}, S^{-1}(M)))$ is Gorenstein injective if and only if $S(\mathrm{Hom}_R(\bar{R}, S^{-1}(M))) \cong G(\mathrm{Hom}_R(\bar{R}, M))$.*

Proof. Since M is Gorenstein injective, the injective cover $E \to M$ is surjective. So we have the exact sequence $0 \to S^{-1}(M) \to E \to M \to 0$. Thus we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}(\bar{R}, S^{-1}(M)) & \to & \mathrm{Hom}(\bar{R}, E) & \to & \mathrm{Hom}(\bar{R}, M) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}(\bar{R}, S^{-1}(M)) & \to & E(\mathrm{Hom}(\bar{R}, S^{-1}(M))) & \to & S(\mathrm{Hom}(\bar{R}, S^{-1}(M))) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & S(\mathrm{Hom}(\bar{R}, E)) & = & L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

But $id_R \mathrm{Hom}(\bar{R}, E) \le 1$ since $\mathrm{Hom}(\bar{R}, E)$ is an injective $\bar{R}$-module. So L is an injective R-module. Thus $pdL < \infty$ since R is n-Gorenstein. Consequently, if $S(\mathrm{Hom}_R(\bar{R}, S^{-1}(M)))$ is Gorenstein injective, then $\mathrm{Hom}(\bar{R}, M) \subseteq S(\mathrm{Hom}_R(\bar{R}, S^{-1}(M)))$ is a Gorenstein injective preenvelope and hence $S_R(\mathrm{Hom}_R(\bar{R}, S^{-1}(M))) \cong G_R(\mathrm{Hom}_R(\bar{R}, M)) \oplus E'$ for some injective R-module E'. But then $E' = 0$ since $S(\mathrm{Hom}_R(\bar{R}, S^{-1}(M)))$ is reduced. The converse is trivial.

Theorem 4.2. *Let R be n-Gorenstein, x be R-regular and M be a reduced Gorenstein injective R-module that is a colifting of a nonzero $\bar{R}$-module L. If $x \cdot \mathrm{Ext}^1(\ , M) = 0$, then $S_R(\mathrm{Hom}_R(\bar{R}, S^{-1}(M)))$ is a reduced Gorenstein injective R-module and $G_R(L) \cong M \oplus S^{-1}(M)$. In this case, $r_x(G_R(L)) = r_x(M)$.*

Proof. We first note that $x \cdot \mathrm{Ext}^1(\ , M) = 0$ if and only if $x \cdot \mathrm{Ext}^1(\ , S^{-1}(M)) = 0$ since $M \xrightarrow{x} M$ factors through the injective cover $E_0 \to M \to 0$ if and only if

$S^{-1}(M) \xrightarrow{x} S^{-1}(M)$ factors through the injective envelope $0 \to S^{-1}(M) \to E_0$. Furthemore, $S^{-1}(M)$ is also a reduced Gorenstein injective R-module. So $S(\mathrm{Hom}_R(\bar{R}, S^{-1}(M))) \cong M \oplus S^{-1}(M)$ by Corollary 2.4 and Lemma 3.1. Hence $S(\mathrm{Hom}_R(\bar{R}, S^{-1}(M)))$ is a reduced Gorenstein injective R-module since $M \oplus S^{-1}(M)$ is such. So $G_R(L) \cong M \oplus S^{-1}(M)$ by the Lemma above.

Example 4.3. *Let* $R = k[[x^2, x^3]]$ *with* k *a field. Then* R *is a Gorenstein local ring since* $\{0, 2, 3, 4, \cdots\} \subset N$ is symmetric (see Kunz [11]). In fact, R is a 1-Gorenstein domain.

Now let $G = k + kx^{-1} + kx^{-2} + \cdots$. Then G is a divisible R-module and thus it is Gorenstein injective since R is 1-Gorenstein. Moreover, $\mathrm{Hom}_R(G, G) \cong k[[x]]$. Hence G is a strongly indecomposable Gorenstein injective R-module. Furthermore, $\mathrm{Hom}_R(\frac{R}{x^2R}, G) = k + kx^{-1} = Socle(G)$. Also $k + kx^{-1} \subset G$ is an essential extension (where we recall that $x^2 \cdot x^{-3} = 0$). Hence G is an essential colifting of the $\frac{R}{x^2R}$-module $k + kx^{-1}$. We also note that $E(G) \cong E(k) \oplus E(k)$. So G is not an injective R-module.

We now recall from Northcott [12] that $E_R(k) = k + kx^{-2} + kx^{-3} + \cdots$ where $x^2 \cdot x^{-3} = 0$. So the imbedding $G \subset E(G)$ is given by

$$G \xrightarrow{(1,x)} E(k) \oplus E(k)$$

where $1 : G \subset E(k)$ maps x^{-1} to 0 but is an identity on $1, x^{-2}, x^{-3}, \cdots$ and $x \cdot x^{-2} = 0$.

We now consider $kx^{-1} \subset G$. We get the imbedding $\frac{G}{kx^{-1}} \subset E(k) \oplus \frac{E(k)}{k}$. But $\frac{G}{kx^{-1}} \cong E(k)$. So we have that the sequence $0 \to \frac{G}{kx^{-1}} \to E(k) \oplus \frac{E(k)}{k} \to \frac{E(k) \oplus E(k)}{G} \to 0$ is split exact. But $kx^{-1} \subset \mathrm{Hom}_R(\frac{R}{x^2R}, G) \subset G$. So the sequence

$$0 \to \frac{G}{\mathrm{Hom}_R(\frac{R}{x^2R}, G)} \to \frac{E(k) \oplus E(k)}{\mathrm{Hom}_R(\frac{R}{x^2R}, G)} \to \frac{E(k) \oplus E(k)}{G} \to 0$$

is split exact. Therefore,

$$0 \to G \to \frac{E(k) \oplus E(k)}{\mathrm{Hom}_R(\frac{R}{x^2R}, G)} \to \frac{E(k) \oplus E(k)}{G} \to 0$$

is also split exact. But then we have the following commutative diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
0 \to & \mathrm{Hom}_R(\frac{R}{x^2R}, G) & \to & G & \xrightarrow{x^2} & G & \to 0 \\
 & \| & & \downarrow & & \downarrow & \\
0 \to & \mathrm{Hom}_R(\frac{R}{x^2R}, G) & \to & E(k)\oplus E(k) & \to & \dfrac{E(k)\oplus E(k)}{\mathrm{Hom}_R(\frac{R}{x^2R}, G)} & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & \dfrac{E(k)\oplus E(k)}{G} & = & \dfrac{E(k)\oplus E(k)}{G} & \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array}$$

with the last vertical sequence split exact. Hence $x^2 \cdot \mathrm{Ext}^1(\ ,G)=0$ and so $r_{x^2}(G)=1$. Thus

$$G_R(\mathrm{Hom}_R(\frac{R}{x^2R}, G)) \cong G_R(k+kx^{-1}) \cong S_R(k+kx^{-1}) \cong G \oplus G$$

by Lemma 4.1 and Theorem 4.2 since $S(G) \cong S^{-1}(G) \cong G$. Moreover, it follows from the above and Theorem 3.3 that every R-module N with $r_x(N)=1$ and $\mathrm{Hom}_R(R/x^2R, N) \cong k+kx^{-1}$ is isomorphic to G.

We finally note that the $\frac{R}{x^2R}$-module k is weakly coliftable to R. If k were coliftable to R, then it is not hard to see that $k \cong \mathrm{Hom}(k, E(k))$ would be liftable to R. But $R=k[[x^2, x^3]]$ is not a discrete valuation ring. So k is not liftable to R (see Example 2 of Auslander-Ding-Soldberg[1, Proposition 3.2]), and thus k is weakly coliftable but not coliftable to R.

We now study properties of essential coliftings. We start with the following.

Proposition 4.4. *Let L be an R-module. Then*

1) *Every essential colifting of L is isomorphic to a submodule of $S_R(L)$.*
2) *If M is an essential colifting of L, then $S_R(M)$ is an essential colifting of $S_{\bar{R}}(L)$.*

Proof. If M is an essential colifting of L to R, then $E(M) \cong E(L)$ and so we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathrm{Ker}\, f & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \rightarrow & L & \rightarrow & E(M) & \rightarrow & S(L) & \rightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow^{f} & & \\
0 & \rightarrow & M & \rightarrow & E(M) & \rightarrow & S(M) & \rightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & M & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

with $M \cong \mathrm{Ker}\, f$ and so (1) follows.

But $\mathrm{Ext}^1_R(\bar{R}, M)=0$ by assumption. So we have an exact sequence $0 \rightarrow \mathrm{Hom}_R(\bar{R}, M) \rightarrow \mathrm{Hom}_R(\bar{R}, E(M)) \rightarrow \mathrm{Hom}_R(\bar{R}, S(M)) \rightarrow 0$. Furthermore, $E_{\bar{R}}(L) \cong \mathrm{Hom}_R(\bar{R}, E(M))$. So $S_{\bar{R}}(L) \cong \mathrm{Hom}_R(\bar{R}, S(M))$. Moreover, $\mathrm{Ext}^i_R(\bar{R}, S_R(M)) \cong \mathrm{Ext}^{i+1}_R(\bar{R}, M)=0$ for all $i>0$ and so (2) follows.

Lemma 4.5. *Let R be n-Gorenstein and M be a colifting of an $\bar{R}$-module. Then* $pd_{\bar{R}}\mathrm{Hom}_R(\bar{R}, \frac{G(M)}{M}) < \infty$.

Proof. By Iwanaga [10], $id_R \frac{G(M)}{M} < \infty$ since $pd_R \frac{G(M)}{M} < \infty$. So let $0 \rightarrow \frac{G(M)}{M} \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^r \rightarrow 0$ be an injective resolution of $\frac{G(M)}{M}$. Then the sequence $0 \rightarrow \mathrm{Hom}_R(\bar{R}, \frac{G(M)}{M}) \rightarrow \mathrm{Hom}_R(\bar{R}, E^0) \rightarrow \cdots \rightarrow \mathrm{Hom}_R(\bar{R}, E^r) \rightarrow 0$ is exact since $\mathrm{Ext}^i_R(\bar{R}, \frac{G(M)}{M}) \cong \mathrm{Ext}^{i+1}_R(\bar{R}, M)=0$ for all $i \geq 1$. But for each i, $\mathrm{Hom}_R(\bar{R}, E^i)$ is an injective $\bar{R}$-module. So $id_{\bar{R}}\mathrm{Hom}_R(\bar{R}, \frac{G(M)}{M}) < \infty$. But then the result follows since $\bar{R}$ is also Gorenstein.

Proposition 4.6. *Let R be n-Gorenstein and M be an essential colifting of an $\bar{R}$-module. Then*

$$\mathrm{Hom}_R(\bar{R}, G_R(M)) \cong G_{\bar{R}}(\mathrm{Hom}_R(\bar{R}, M)).$$

Proof. $\mathrm{Hom}_R(\bar{R}, M)$ is an essential R-submodule of M by assumption and M is a Gorenstein essential submodule of $G(M)$ by Enochs-Jenda [7, Theorem 3.3]. So it is easy to argue that $\mathrm{Hom}_R(\bar{R}, M)$ is a Gorenstein essential submodule of $G(M)$. But $\mathrm{Hom}_R(\bar{R}, M) \subseteq \mathrm{Hom}_R(\bar{R}, G(M)) \subseteq G(M)$. So $\mathrm{Hom}_R(\bar{R}, M) \subseteq \mathrm{Hom}_R(\bar{R}, G(M))$ is a Gorestein essential extension as R-modules and hence as $\bar{R}$-modules.

Since M is a colifting, we have the exact sequence

$$0 \rightarrow \mathrm{Hom}_R(\bar{R}, M) \rightarrow \mathrm{Hom}_R(\bar{R}, G(M)) \rightarrow \mathrm{Hom}_R(\bar{R}, \frac{G(M)}{M}) \rightarrow 0.$$

But $pd_R \mathrm{Hom}_R(\bar{R}, \frac{G(M)}{M}) < \infty$ by Lemma 4.5 above and $\mathrm{Hom}_R(\bar{R}, G(M))$ is a Gorenstein injective $\bar{R}$-module by Enochs-Jenda [6, Lemma 3.1]. Hence $\mathrm{Hom}_R(\bar{R}, G(M))$ is a Gorenstein essential Gorenstein injective extension of the $\bar{R}$-module $\mathrm{Hom}_R(\bar{R}, G(M))$ and so is the Gorenstein injective envelope by Theorem 3.3 of [7].

The following result is dual to Proposition 5.2 of Auslander-Ding-Soldberg [1].

Theorem 4.7. *Let R be n-Gorenstein and L be an $\bar{R}$-module. If L is essentially coliftable to R, then $G_{\bar{R}}(L)$ and $G_{\bar{R}}(L)/L$ are coliftable to R.*

Proof. If M is an essential colifting of L to R, then $G_{\bar{R}}(L) \cong \mathrm{Hom}_R(\bar{R}, G(M))$ by Proposition 4.6 above and so $G_{\bar{R}}(L)/L \cong \mathrm{Hom}_R(\bar{R}, \frac{G(M)}{M})$. But $\mathrm{Ext}^i_R(\bar{R}, \frac{G(M)}{M}) \cong \mathrm{Ext}^{i+1}_R(\bar{R}, M) = 0$ for all $i > 0$. Thus $G(M)$ and $\frac{G(M)}{M}$ are coliftings of $G_{\bar{R}}(L)$ and $\frac{G_{\bar{R}}(L)}{L}$ to R, respectively.

As a consequence, we get the following Gorenstein version of part 2 of Proposition 4.4 above.

Corollary 4.8. *Let R be n-Gorenstein and L be an $\bar{R}$-module. If L is coliftable to an R-module whose Gorenstein injective envelope is an essential extension of L, then $G_{\bar{R}}(L)$ and $\frac{G_{\bar{R}}(L)}{L}$ are essentially coliftable to R.*

To prove the converse of this corollary, we need the following easy

Lemma 4.9. *Let L be an $\bar{R}$-module. Then L is an injective $\bar{R}$-module that is essentially coliftable to a Gorenstein injective R-module if and only if $L \cong \mathrm{Hom}_R(\bar{R}, E_R(L))$.*

Proof. If L is coliftable to R, then $L \cong \mathrm{Hom}_R(\bar{R}, G)$ for some Gorenstein injective R-module G. But $id_R L = 1$ since L is an injective $\bar{R}$-module. So $\mathrm{Ext}^1_R(\frac{E(L)}{L}, G) = 0$ and thus $E(L)$ is a summand of G. But then $G \cong E(L)$ since $L \subset G$ is essential.
Conversely, L is an injective $\bar{R}$-module and $E(L)$ is an essential colifting of L.

Theorem 4.10. *Let R be 2-Gorenstein and L be an $\bar{R}$-module. Then L is coliftable to an R-module whose Gorenstein injective envelope is an essential extension of L if and only if $G_{\bar{R}}(L)$ is coliftable to a Gorenstein injective R-module that is an essential extension of L, $G_{\bar{R}}(L)/L$ is essentially coliftable to R, and the image of the lifting of the natural map $G_{\bar{R}}(L) \to \frac{G_{\bar{R}}(L)}{L}$ to R* has finite projective dimension.

Proof. The only if part is essentially Corollary 4.8 above.

We now prove the if part. If $idR = 1$, then $id_{\bar{R}} \bar{R} = 0$ and so $G_{\bar{R}}(L) = L$. Thus L is coliftable to a Gorenstein injective R-module that is an essential extension of

L by assumption. If $idR=2$, then $id_{\bar{R}}\bar{R}\leq 1$ and so $C=\frac{G_{\bar{R}}(L)}{L}$ is an injective $\bar{R}$-module by Enochs-Jenda [5, Theorem 3.3]. Hence the injective envelope of C is its colifting by the Lemma above. Now let G be a Gorenstein injective colifting of $G_{\bar{R}}(L)$ to R with $L\subseteq G$ essential. Then we have the following commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & L & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & G_{\bar{R}}(L) & \to & G & \xrightarrow{x} & G & \to & 0 \\ & & \downarrow & & \downarrow f & & \downarrow f & & \\ 0 & \to & C & \to & E(C) & \xrightarrow{x} & E(C) & \to & 0 \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

So we have an exact sequence $0\to L\to \operatorname{Ker} f\xrightarrow{x}\operatorname{Ker} f\to 0$. Thus $L\cong \operatorname{Hom}_R(\bar{R},\operatorname{Ker} f)$ and $\operatorname{Ext}^1(\bar{R},\operatorname{Ker} f)=0$. Moreover, $\operatorname{Ext}^i_R(\bar{R},\operatorname{Ker} f)=0$ for all $i\geq 2$ since $pd\bar{R}\leq 1$. So $\operatorname{Ker} f$ is a colifting of L. But $pdf(G)<\infty$ by assumption. So $G\cong G(\operatorname{Ker} f)\oplus E$ for some injective R-module E. But then $E=0$ since $L\subseteq G$ is essential. So we are done.

We now conclude the section by characterizing $\bar{R}$-modules that are weakly coliftable to Gorenstein injective R-modules.

Theorem 4.11. *The following are equivalent for an $\bar{R}$-module L.*

1) *L is weakly coliftable to a Gorenstein injective R-module.*
2) *L is a direct summand of* $\operatorname{Hom}_R(\bar{R},G_R(L))$.

Moreover, if R is 1-Gorenstein, then each of the above statements is equivalent to

3) $\operatorname{Hom}_R(\bar{R},S_R(L))\cong L\oplus S_{\bar{R}}(L)$.

Proof. $1\Rightarrow 2$. Let M be a Gorenstein injective R-module that is a weak colifting of L. Then we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & L & \to & G_R(L) & \to & \frac{G_R(L)}{L} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & L & \to & M & \to & \frac{M}{L} & \to & 0 \end{array}$$

We now apply $\operatorname{Hom}_R(\bar{R},-)$ to the diagram to get the following commutative diagram since $\operatorname{Hom}_R(\bar{R},L)\cong \operatorname{Ext}^1_R(\bar{R},L)\cong L$.

$$\begin{array}{ccccccccc} 0 & \to & L & \to & \mathrm{Hom}_R(\bar{R}, G_R(L)) & \to & \mathrm{Hom}_R(\bar{R}, \frac{G_R(L)}{L}) & \to & L & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & L & \to & \mathrm{Hom}_R(\bar{R}, M) & \to & \mathrm{Hom}_R(\bar{R}, \frac{M}{L}) & \to & L & \to & 0 \end{array}$$

But L is a direct summand of $\mathrm{Hom}(\bar{R}, M)$. So L is a direct summand of $\mathrm{Hom}_R(\bar{R}, G_R(L))$.

$2 \Rightarrow 1$ follows from the definition.

$1 \Rightarrow 3$ follows from Lemma 3.2.

$3 \Rightarrow 1$ is trivial since $S_R(L)$ is a Gorenstein injective R-module because R is 1-Gorenstein.

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