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On a property of Nirenberg type operator

By

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§ 1. Introduction

Let X be a nowhere-zero C^∞ complex vector field in R^n. Let $S^X = \{f \in C^\infty(R^n);\ Xu = f$ has a C^1 solution near the origin.$\}$ and $S_X = \{f \in C^\infty(R^n);\ Xu = f$ has a C^1 solution near the origin such that $du(0) \neq 0\}$.

The following facts are classically well known:

(1) $\mathscr{A} \subset S_X$ if X is real-analytic, where $\mathscr{A}$ denotes the set of real-analytic functions in R^n.

(2) $S_X = C^\infty(R^2)$ if $n = 2$, and $X(0)$, $\bar{X}(0)$ are C-linearly independent (In this case, X is an elliptic operator).

And we can easily obtain the following fact owing to Hölmander [1] and Treves [4]:

(3) $S_X = C^\infty(R^n)$ if X is a solvable operator at the origin.

Though it is trivial, we also know the following fact:

(4) $\mathscr{A} \subset S_X \subsetneq C^\infty(R^n)$ if X is a non-solvable operator at the origin and real-analytic.

We thus see $S_X = S^X$ in each case of the above. Does there exist a non-solvable vector field X such that $S_X \subsetneq S^X$?

This paper aims at showing that the answer is "Yes". We shall give such vector fields L_α, which we call Nirenberg type:

Let $\alpha(t,x)$ be a real-valued $C^\infty(R^2)$ function satisfying the following conditions:

(A.1) *$\alpha(t,x) \geq 0$ in a neighborhood ω of the origin.*

(A.2) *There exist positive constants c, d, and a monotonously increasing sequence $\{p_n\}$ of positive integers such that*

$$\iint_{D(p_k)} \alpha(t,x)\, dt dx > \frac{9}{(p_k + d)(p_k + c)}$$

for every sufficiently large k, where $D(p_k) = \left(0, \frac{1}{p_k}\right) \times \left(0, \frac{1}{p_k}\right)$.

We shall define L_α in the following manner:

$$L_\alpha = \partial_t + i(2t + \alpha)\partial_x.$$

Then we assert the following

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Theorem A.

$$S_{L_\alpha} \subsetneq S^{L_\alpha}.$$

Example *(This is obtained by modifying an example of Nirenberg [3] (p. 8)).* *Let* $a_{n,p} = \dfrac{1}{(n+p-1)(n+p)}$ $(n, p = 1, 2, \dots)$ *and* $\{B_{n,p}\}$ *the set of non-overlapping open discs in the* (t, x) *plane satisfying the following conditions*:

(i) *The ordinate of the center of* $B_{n,p}$ *equals* $\dfrac{1}{p+1} + \dfrac{1}{2p(p+1)}$.

(ii) *The abscissa of the center of* $B_{n,p}$ *equals* $\dfrac{1}{p} - \left(a_{1,p} + a_{2,p} + \cdots + a_{n-1,p} + \dfrac{a_{n,p}}{2}\right)$.

(iii) *The radius of* $B_{n,p}$ *equals* $\dfrac{a_{n,p}}{2}$.

Next let $\{f_{n,p}\}$ *be the set of* C^∞ *functions having the following properties* $(n, p = 1, 2, \dots)$:

(i) $0 \leq f_{n,p} \leq \dfrac{64 \cdot 18}{\pi(n+p+1)^2}$.

(ii) $f_{n,p}$ *vanishes outside of* $B_{n,p}$ *and equals* $\dfrac{64 \cdot 18}{\pi(n+p+1)^2}$ *inside of the closed disc* $C_{n,p}$ *with radius* $\dfrac{a_{n,p}}{4}$, *where the ordinate of the center of* $C_{n,p}$ *equals that of* $B_{n,p}$ *and the abscissa of the center of* $C_{n,p}$ *equals that of* $B_{n,p}$.

Next we define a C^∞ *function* $r(t, x)$ *as follows*:

(i) $r(-t, x) = r(t, x)$.

(ii) $r(t, x) = f_{n,p}$ *in* $B_{n,p}$.

(iii) $r(t, x)$ *vanishes outside of the union of all the* $B_{n,p}$.

Finally we define $\alpha(t, x)$ *by* $\alpha(t, x) = r(t, x)$.

Then we can check that the conditions (A.1) and (A.2) are satisfied. The proof is given in §4.

Now, to prove Theorem A, we first derive a necessary condition on $f(t, x)$ for $\partial_t u + ia\partial_x u = f(t, x)$ to have a C^1 solution near the origin such that $u_x(0) \neq 0$ under the following assumption:

(a.1) $a = a(t, x)$ *is a real-valued* $C^\infty(R^2)$ *function.*

(a.2) $a(0, x)$ *vanishes identically.*

(a.3) *There is a neighborhood* ω *of the origin such that*

(a.3.1) $t\{a(t, x) - a(-t, x)\} > 0$ *in* $\{t \neq 0\} \cap \omega$

and

(a.3.2) $a(t, x) + a(-t, x) \geq 0$ *in* ω.

Hereafter for a function $F(t, x)$ we shall denote by F_e and F_o the even part of $F(t, x)$ with respect to t and the odd one.

Now we have the following

Lemma 1 *([2]). Assume (a.1) and (a.3.1). Then there exist a neighborhood Ω_w of the origin and a function $w(t,x) \in C^1(\Omega_w)$ such that*

$$\min(\inf_{\Omega_w} \mathrm{Re}\, w_x, \inf_{\Omega_w} \mathrm{Im}\, w_x) > 0 \quad and \quad (\partial_t + ia_o(t,x)\partial_x)w = 0 \quad in\ \Omega_w.$$

Hereafter we shall set $m(w, \Omega_w) = \min(\inf_{\Omega_w} \mathrm{Re}\, w_x, \inf_{\Omega_w} \mathrm{Im}\, w_x)$. Then, we obtain the following

Theorem B. *Assume (a.1), (a.2), and (a.3). Let w and Ω_w be any one of the couples of a function and a neighborhood satisfying Lemma 1. Let a $C^\infty(R^2)$ function $f(t,x)$ be given. Assume that*

$$L_a u \equiv \partial_t u + ia\partial_x u = f(t,x)$$

has a C^1 solution near the origin such that $u_x(0) \neq 0$. Then, there exist positive constants C_1, N, and T_0, where T_0 is independent of w and Ω_w, such that, for any simply connected domain D contained in $(0,T_0) \times (-T_0,T_0) \cap \Omega_w$ with piecewise smooth boundary ∂D, the following holds:

(i) *In case of $f_e(0) \neq 0$,*

$$\iint_D a_e\, dtdx + C_1 \iint_D \Big\{ \mathrm{Re}\, f_e(0)\, \mathrm{Re}\, f_e + \mathrm{Im}\, f_e(0)\, \mathrm{Im}\, f_e + \mathrm{Re}\, f_e(0)\, \mathrm{Im}\, f_e$$

$$- \mathrm{Im}\, f_e(0)\, \mathrm{Re}\, f_e - |f_e(0)|^2 + \frac{2}{N} \Big\} dtdx \leq \frac{\sup_{\partial D}|w| \cdot |\partial D|}{m(w, \Omega_w)}.$$

(ii) *In case of $f_e(0) = 0$,*

$$\iint_D a_e\, dtdx + C_1 \iint_D \left(\frac{2}{N} + \mathrm{Re}\, f_e + \mathrm{Im}\, f_e \right) dtdx \leq \frac{\sup_{\partial D}|w| \cdot |\partial D|}{m(w, \Omega_w)},$$

where the N can be replaced with ∞ and the T_0 is independent of N when $f \equiv 0$.

This is proved in §2 and by making use of the estimate in Theorem B, Theorem A is proved in §3.

§2. Proof of Theorem B

CASE $f_e(0) \neq 0$. We shall set $u^I = -\bar{f}_e(0)u$. Multiplying u^I by a suitable constant $e^{i\theta}$, where θ is a real number, we can assume that $\mathrm{Re}(e^{i\theta}u_e^I)_x(0,0)$ and $\mathrm{Im}(e^{i\theta}u_e^I)_x(0,0)$ are positive, so from beginning we can assume that $\mathrm{Re}\,\partial_x u_e^I(0,0) \equiv \alpha$ and $\mathrm{Im}\,\partial_x u_e^I(0,0) \equiv \beta$ are positive. Let us set $\delta = \min(\alpha, \beta)$. Let N be a positive constant. Then, since

$$\operatorname{Re} f_e(0) \operatorname{Re} f_e + \operatorname{Im} f_e(0) \operatorname{Im} f_e - |f_e(0)|^2 + \frac{1}{N}, \quad \operatorname{Re} f_e(0) \operatorname{Im} f_e - \operatorname{Im} f_e(0) \operatorname{Re} f_e + \frac{1}{N}$$

are positive at the origin, we take a positive constant T_1 small such that

$$\operatorname{Re} f_e(0) \operatorname{Re} f_e + \operatorname{Im} f_e(0) \operatorname{Im} f_e - |f_e(0)|^2 + \frac{1}{N}$$

and

$$\operatorname{Re} f_e(0) \operatorname{Im} f_e - \operatorname{Im} f_e(0) \operatorname{Re} f_e + \frac{1}{N}$$

are positive in $(-T_1, T_1) \times (-T_1, T_1)$

Next we take a positive constant T_2 such that

$$L_a u^I = -\bar{f}_e(0) f \quad \text{in} \quad U_{T_2} = (-T_2, T_2) \times (-T_2, T_2),$$

$$\operatorname{Re} \partial_x u_e^I > \frac{\delta}{2}, \ \operatorname{Im} \partial_x u_e^I > \frac{\delta}{2} \quad \text{in} \quad U_{T_2} = (-T_2, T_2) \times (-T_2, T_2).$$

Then we take a positive constant T_0 such that $T_0 < \min(T_1, T_2)$. By setting $u^{II} = u^I + \left(|f_e(0)|^2 - \frac{1+i}{N} \right) t$, it follows that

$$L_a u^{II} = -\bar{f}_e(0) f + |f_e(0)|^2 - \frac{1+i}{N}.$$

Then setting $v = (2u^{II})/\delta$, we see $\inf_{U_{T_0}} \operatorname{Re} \partial_x v_e \geq \frac{\delta}{2} \cdot \frac{2}{\delta} = 1$ and $\inf_{U_{T_0}} \operatorname{Im} \partial_x v_e \geq \frac{\delta}{2} \cdot \frac{2}{\delta} = 1$.

Now we remark $\partial_x v_o(0, x) = 0$. And also, from

$$L_a v = (2/\delta) L_a u^{II} = (2/\delta) \left(-\bar{f}_e(0) f + |f_e(0)|^2 - \frac{1+i}{N} \right),$$

we have

$$(2.1) \qquad (\partial_t + i a_o \partial_x) v_o = -i a_e \partial_x v_e + (2/\delta) \left\{ -\bar{f}_e(0) f_e + |f_e(0)|^2 - \frac{1+i}{N} \right\}.$$

So we see

$$\partial_t v_o(0, x) = \frac{-2(1+i)}{N\delta}.$$

Here taking N sufficiently large and T_0 sufficiently small, we can assume that

$$M = \max \left(\sup_{U_{T_0}} |\partial_t v_o|, \sup_{U_{T_0}} |\partial_x v_o| \right) \leq \frac{1}{2}.$$

As it has been remarked,

$$\inf_{U_{T_0}} \operatorname{Re} \partial_x v_e \geq 1, \quad \inf_{U_{T_0}} \operatorname{Im} \partial_x v_e \geq 1.$$

Now we obtain the following

Lemma 2. *For any simply connected domain D contained in* $(0, T_0) \times (-T_0, T_0) \cap \Omega_w$ *with piecewise smooth boundary,*

$$(2.2) \qquad i \iint_D a_e w_x \partial_x v_e \, \mathrm{d}t\mathrm{d}x + \iint_D (2/\delta) \left\{ \bar{f}_e(0) f_e - |f_e(0)|^2 + \frac{1+i}{N} \right\} w_x \, \mathrm{d}t\mathrm{d}x$$
$$= \int_{\partial D} w \partial_t v_o \, \mathrm{d}t + w \partial_x v_o \, \mathrm{d}x.$$

Proof. From (2.1),

$$-w_x\{(\partial_t + ia_o\partial_x)v_o\} = ia_e w_x \partial_x v_e + (2/\delta)\left\{\bar{f}_e(0) f_e - |f_e(0)|^2 + \frac{1+i}{N}\right\} w_x.$$

And hence we have

$$\iint_D -w_x\{(\partial_t + ia_o\partial_x)v_o\} \, \mathrm{d}t\mathrm{d}x$$
$$= \iint_D ia_e w_x \partial_x v_e \, \mathrm{d}t\mathrm{d}x + \iint_D (2/\delta)\left\{\bar{f}_e(0) f_e - |f_e(0)|^2 + \frac{1+i}{N}\right\} w_x \, \mathrm{d}t\mathrm{d}x.$$

The left-hand side above =

$$\iint_D -\{w_x \partial_t v_o - w_t \partial_x v_o\} \, \mathrm{d}t\mathrm{d}x = \iint_D \mathrm{d}\{w(t, x) \, \mathrm{d}v_o(t, x)\} = \int_{\partial D} w \partial_t v_o \, \mathrm{d}t + w \partial_x v_o \, \mathrm{d}x,$$

ending the proof of Lemma 2.

From this lemma we have, by setting $C_1 = 2/\delta$:

$$(2.3) \qquad \iint_D [a_e\{\operatorname{Re} \partial_x v_e \operatorname{Im} w_x + \operatorname{Im} \partial_x v_e \operatorname{Re} w_x\}] \, \mathrm{d}t\mathrm{d}x$$
$$+ C_1 \iint_D \left[\left\{ \operatorname{Re} f_e(0) \operatorname{Re} f_e + \operatorname{Im} f_e(0) \operatorname{Im} f_e - |f_e(0)|^2 + \frac{1}{N} \right\} \operatorname{Im} w_x \right.$$
$$\left. + \left\{ \operatorname{Re} f_e(0) \operatorname{Im} f_e - \operatorname{Im} f_e(0) \operatorname{Re} f_e + \frac{1}{N} \right\} \operatorname{Re} w_x \right] \mathrm{d}t\mathrm{d}x$$
$$\leq \int_{\partial D} |w \partial_t v_o \, \mathrm{d}t + w \partial_x v_o \, \mathrm{d}x|.$$

Denoting $\min(\inf_{U_{T_0}} \operatorname{Re} \partial_x v_e, \inf_{U_{T_0}} \operatorname{Im} \partial_x v_e)$ by m_0, from (2.3) we have

$$(2.4) \quad m(w, \Omega_w)\left[m_0 \iint_D a_e(t,x)\, dt dx + C_1 \iint_D \left\{ \operatorname{Re} f_e(0) \operatorname{Re} f_e + \operatorname{Im} f_e(0) \operatorname{Im} f_e \right.\right.$$

$$\left.\left. + \operatorname{Re} f_e(0) \operatorname{Im} f_e - \operatorname{Im} f_e(0) \operatorname{Re} f_e - |f_e(0)|^2 + \frac{2}{N} \right\} dt dx \right]$$

$$\leq \int_{\partial D} |w \partial_t v_o \, dt + w \partial_x v_o \, dx|.$$

Since $m_0 \geq 1$, and $M = \max(\sup_{U_{T_0}} |\partial_x v_0|, \sup_{U_{T_0}} |\partial_x v_0|) \leq \frac{1}{2}$, we obtain the following inequality:

$$\iint_D a_e \, dt dx + C_1 \iint_D \left\{ \operatorname{Re} f_e(0) \operatorname{Re} f_e + \operatorname{Im} f_e(0) \operatorname{Im} f_e \right.$$

$$\left. + \operatorname{Re} f_e(0) \operatorname{Im} f_e - \operatorname{Im} f_e(0) \operatorname{Re} f_e - |f_e(0)|^2 + \frac{2}{N} \right\} dt dx$$

$$\leq \frac{\sup_{\partial D} |w| \cdot |\partial D|}{m(w, \Omega_w)},$$

which gives the assertion (i).

Case $f_e(0) = 0$. The reasoning is nearly same: First we may assume that

$$\operatorname{Re} \partial_x u_e(0,0) \equiv \alpha > 0 \quad \operatorname{Im} \partial_x u_e(0,0) \equiv \beta > 0.$$

Let us set $\delta = \min(\alpha, \beta)$. Let N be a positive constant. Since

$$\frac{1}{N} + \operatorname{Re} f_e, \quad \frac{1}{N} + \operatorname{Im} f_e$$

are positive at the origin, we take a positive constant T_1 small such that $\frac{1}{N} + \operatorname{Re} f_e$ and $\frac{1}{N} + \operatorname{Im} f_e$ are positive in $(-T_1, T_1) \times (-T_1, T_1)$. We shall set $u^* = -u - \frac{(1+i)t}{N}$. Then we take a positive constant T_2 such that

$$L_a u^* = -f - \frac{1+i}{N} \quad \text{in} \quad U_{T_2} = (-T_2, T_2) \times (-T_2, T_2),$$

$$\operatorname{Re} \partial_x u_e^* > \frac{\delta}{2}, \ \operatorname{Im} \partial_x u_e^* > \frac{\delta}{2} \quad \text{in} \quad U_{T_2} = (-T_2, T_2) \times (-T_2, T_2).$$

Setting $v = \frac{2u^*}{\delta}$, we have

$$(\partial_t + i a_o \partial_x) v_o = -i a_e \partial_x v_e + (2/\delta)\left(-f_e - \frac{1+i}{N}\right).$$

So,

$$\partial_t v_o(0,x) = \frac{-2(1+i)}{N\delta}.$$

Then we take a positive constant T_0 such that $T_0 < \min(T_1, T_2)$. By the same reasoning as in the preceding proof, we find the following:

We can take positive constants T_0 (which is independent of w and Ω_w), and N such that

$$M = \max\left(\sup_{U_{T_0}} |\partial_t v_o|, \sup_{U_{T_0}} |\partial_x v_0|\right) \leq \frac{1}{2},$$

$$\inf_{U_{T_0}} \operatorname{Re} \partial_x v_e \geq 1, \quad \inf_{U_{T_0}} \operatorname{Im} \partial_x v_e \geq 1,$$

and for any simply connected domain D contained in $(0, T_0) \times (-T_0, T_0) \cap \Omega_w$ with piecewise smooth boundary,

$$i \iint_D a_e w_x \partial_x v_e \,\mathrm{d}t\mathrm{d}x + \iint_D (2/\delta)\left(f_e + \frac{1+i}{N}\right) w_x \,\mathrm{d}t\mathrm{d}x = \int_{\partial D} w\partial_t v_o \,\mathrm{d}t + w\partial_x v_o \,\mathrm{d}x.$$

Thus we obtain the asserion (ii). When $f \equiv 0$, the conclusion stated in the last part of the assertion (ii) is easily obtained, completing the proof.

§ 3. Proof of Theorem A

Assume

$$S_{L_\alpha} = S^{L_\alpha}.$$

Since $S^{L_\alpha} \ni 0$, $L_\alpha u = 0$ has a C^1 solution near the origin such that $u_x(0) \neq 0$. Setting $a_o(t,x) = 2t$, we easily find that

$$w = (1-i)(t^2 + ix) \quad \text{and} \quad \Omega_w = R^2$$

satisfy Lemma 1; in this case we see $|w| = \{2(t^4 + x^2)\}^{1/2}$ and $m(w, \Omega_w) = 1$. Taking a positive integer N_0 such that $N_0^{-1} < T_0$, for every integer p such that $p > N_0$, from Theorem B we get the following:

$$\iint_D \alpha(t,x)\,\mathrm{d}t\mathrm{d}x \leq 8p^{-2},$$

by taking $D = \left(0, \frac{1}{p}\right) \times \left(0, \frac{1}{p}\right)$. But this contradicts our assumption (A.2), ending the proof of Theorem A.

§ 4. Proof of Example

We have only to prove that the $\alpha(t,x)$ satisfies the condition (A.2). First we shall set $c = 1$, $d = 2$, and $p_k = 1, 2, \ldots$. By putting $p_k = p$, the left-hand side of the inequality of (A.2)

$$\geq \sum_{n=1}^{\infty} \iint_{C_{n,p}} \alpha\, \mathrm{d}t\mathrm{d}x + \sum_{k=p+1}^{\infty} \iint_{C_{1,k}} \alpha\, \mathrm{d}t\mathrm{d}x$$

$$= \sum_{n=1}^{\infty} \frac{\pi a_{n,p}^2}{16} \cdot \frac{64 \cdot 18}{\pi(n+p+1)^2} + \sum_{k=p+1}^{\infty} \frac{\pi a_{1,k}^2}{16} \cdot \frac{64 \cdot 18}{\pi(k+2)^2}$$

$$= \sum_{n=1}^{\infty} 18 \cdot \left[\frac{2}{(n+p-1)(n+p)(n+p+1)}\right]^2 + \sum_{k=p+1}^{\infty} 18 \cdot \left[\frac{2}{k(k+1)(k+2)}\right]^2.$$

Since

$$\frac{2}{(n+p-1)(n+p)(n+p+1)} = \frac{1}{(n+p-1)(n+p)} - \frac{1}{(n+p+1)(n+p)}$$

$$= \frac{1}{n+p-1} - \frac{1}{n+p} - \left\{\frac{1}{n+p} - \frac{1}{n+p+1}\right\}$$

$$= \frac{1}{n+p-1} - \frac{2}{n+p} + \frac{1}{n+p+1}$$

and

$$\frac{2}{k(k+1)(k+2)} = \frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2},$$

we have

$$\sum_{n=1}^{\infty} 18 \cdot \left[\frac{2}{(n+p-1)(n+p)(n+p+1)}\right]^2 + \sum_{k=p+1}^{\infty} 18 \cdot \left[\frac{2}{k(k+1)(k+2)}\right]^2$$

$$= 18 \sum_{n=1}^{\infty} \left[\frac{1}{(n+p-1)^2} + \frac{4}{(n+p)^2} + \frac{1}{(n+p+1)^2} - 4\left\{\frac{1}{n+p-1} - \frac{1}{n+p}\right\}\right.$$

$$\left. + \left\{\frac{1}{n+p-1} - \frac{1}{n+p+1}\right\} - 4\left\{\frac{1}{n+p} - \frac{1}{n+p+1}\right\}\right]$$

$$+ 18 \sum_{k=p+1}^{\infty} \left[\frac{1}{k^2} + \frac{4}{(k+1)^2} + \frac{1}{(k+2)^2} - 4\left\{\frac{1}{k} - \frac{1}{k+1}\right\}\right.$$

$$\left. - 4\left\{\frac{1}{k+1} - \frac{1}{k+2}\right\} + \left\{\frac{1}{k} - \frac{1}{k+2}\right\}\right]$$

$$= 18\left[\sum_{n=1}^{\infty}\left\{\frac{1}{(n+p-1)^2}+\frac{4}{(n+p)^2}+\frac{1}{(n+p+1)^2}\right\}\right.$$

$$\left.-\frac{4}{p}+\left\{\frac{1}{p}+\frac{1}{p+1}\right\}-\frac{4}{p+1}\right]$$

$$+18\left[\sum_{k=p+1}^{\infty}\left\{\frac{1}{k^2}+\frac{4}{(k+1)^2}+\frac{1}{(k+2)^2}\right\}-\frac{4}{p+1}-\frac{4}{p+2}+\frac{1}{p+1}+\frac{1}{p+2}\right]$$

$$= 18\left[\sum_{n=1}^{\infty}\left[\left\{\frac{1}{(n+p-1)^2}+\frac{4}{(n+p)^2}+\frac{1}{(n+p+1)^2}\right\}-\frac{3}{p}-\frac{3}{p+1}\right]\right.$$

$$\left.+18\left[\sum_{k=p+1}^{\infty}\left\{\frac{1}{k^2}+\frac{4}{(k+1)^2}+\frac{1}{(k+2)^2}\right\}-\frac{3}{p+1}-\frac{3}{p+2}\right]\right]$$

$$= 18\left[12\sum_{k=p+2}^{\infty}\frac{1}{n^2}+\frac{1}{p^2}+\frac{6}{(p+1)^2}-\frac{1}{(p+2)^2}-\frac{3}{p}-\frac{6}{p+1}-\frac{3}{p+2}\right].$$

So we have only to prove that, for sufficiently large p,

$$12\sum_{n=p+2}^{\infty}\frac{1}{n^2}+\frac{1}{p^2}+\frac{6}{(p+1)^2}-\frac{1}{(p+2)^2}-\frac{3}{p}-\frac{6}{p+1}-\frac{3}{p+2}$$

$$\equiv S(p)$$

$$\geq \frac{1}{2(p+1)(p+2)}.$$

Now we see the following Lemma 3 holds, which shows that the above statement is valid, ending the proof.

Lemma 3. *For every positive integer* p,

$$S(p) > \frac{1}{2(p+1)(p+2)}.$$

Proof. We shall show this by mathematical induction. First,

$$S(1) = 12(3^{-2}+4^{-2}+5^{-2}+\cdots\cdots)+1+\frac{3}{2}-\frac{1}{9}-3-3-1$$

$$= 2[6\{(1^{-2}+2^{-2}+3^{-2}+4^{-2}+\cdots\cdots)-1^{-2}-2^{-2}\}]-\frac{9}{2}-\frac{1}{9}$$

$$= 2\left[6\left(\frac{\pi^2}{6}-1-\frac{1}{4}\right)\right]-\frac{9}{2}-\frac{1}{9}$$

$$= 2\left(\pi^2-6-\frac{3}{2}-\frac{9}{4}\right)-\frac{1}{9}$$

$$= 2(9.86960440\ldots\ldots-9.75)-0.1111\ldots\ldots$$

$$= 0.1196\ldots\ldots.$$

On the otherhand $\frac{1}{12} = 0.083 \ldots\ldots$ And so surely,

$$S(1) > \frac{1}{2 \cdot 2 \cdot 3}.$$

Next assume $S(p) > \dfrac{1}{2(p+1)(p+2)}$. Then

$$\begin{aligned}
S(p+1) - \frac{1}{2(p+2)(p+3)} &= 12 \sum_{n=p+3}^{\infty} \frac{1}{n^2} + \frac{1}{(p+1)^2} + \frac{6}{(p+2)^2} - \frac{1}{(p+3)^2} \\
&\quad - \frac{3}{p+1} - \frac{6}{p+2} - \frac{3}{p+3} - \frac{1}{2(p+2)(p+3)} \\
&= 12 \left[\sum_{n=p+2}^{\infty} \frac{1}{n^2} - \frac{1}{(p+3)^2} \right] + \frac{1}{(p+1)^2} + \frac{6}{(p+2)^2} \\
&\quad - \frac{1}{(p+3)^2} - \frac{3}{p+1} - \frac{6}{p+2} - \frac{3}{p+3} - \frac{1}{2(p+2)(p+3)} \\
&> \frac{1}{2(p+1)(p+2)} + \frac{3}{p} + \frac{6}{p+1} + \frac{3}{p+2} + \frac{1}{(p+2)^2} \\
&\quad - \frac{6}{(p+1)^2} - \frac{1}{p^2} - \frac{12}{(p+3)^2} + \frac{1}{(p+1)^2} \\
&\quad + \frac{6}{(p+2)^2} - \frac{1}{(p+3)^2} - \frac{3}{p+1} - \frac{6}{p+2} \\
&\quad - \frac{3}{p+3} - \frac{1}{2(p+2)(p+3)} \\
&= \frac{1}{(p+1)(p+2)(p+3)} + \frac{3}{p(p+1)} + \frac{6}{(p+1)(p+2)} \\
&\quad + \frac{3}{(p+2)(p+3)} + \frac{7}{(p+2)^2} - \frac{5}{(p+1)^2} - \frac{1}{p^2} - \frac{13}{(p+3)^2} \\
&= \frac{A_1}{p(p+1)(p+2)(p+3)} + \frac{A_2}{[p(p+1)(p+2)(p+3)]^2} \\
&\equiv S,
\end{aligned}$$

where

$$A_1 \equiv p + 3(p+2)(p+3) + 6p(p+3) + 3p(p+1) = 12p^2 + 37p + 18$$

and

$$\begin{aligned}
A_2 &\equiv 7\{p(p+1)(p+3)\}^2 - 5\{p(p+2)(p+3)\}^2 \\
&\quad - \{(p+1)(p+2)(p+3)\}^2 - 13\{p(p+1)(p+2)\}^2 \\
&= 7p^2(p^2+4p+3)^2 - 5p^2(p^2+5p+6)^2 \\
&\quad - 13p^2(p^2+3p+2)^2 - (p^3+6p^2+11p+6)^2 \\
&= 7p^2(p^4+8p^3+22p^2+24p+9) \\
&\quad - 5p^2(p^4+10p^3+37p^2+60p+36) \\
&\quad - 13p^2(p^4+6p^3+13p^2+12p+4) \\
&\quad - (p^6+12p^5+58p^4+144p^3+193p^2+132p+36) \\
&= (7p^6+56p^5+154p^4+168p^3+63p^2) \\
&\quad - (5p^6+50p^5+185p^4+300p^3+180p^2) \\
&\quad - (13p^6+78p^5+169p^4+156p^3+52p^2) \\
&\quad - (p^6+12p^5+58p^4+144p^3+193p^2+132p+36) \\
&= -(12p^6+84p^5+258p^4+432p^3+362p^2+132p+36).
\end{aligned}$$

And so

$$\begin{aligned}
S = [&(12p^2+37p+18)p(p+1)(p+2)(p+3) \\
&- (12p^6+84p^5+258p^4+432p^3+362p^2 \\
&+ 132p+36)]/[p(p+1)(p+2)(p+3)]^2.
\end{aligned}$$

And the numerator

$$\begin{aligned}
&= [(12p^2+37p+18)p(p^3+6p^2+11p+6) \\
&\quad - (12p^6+84p^5+258p^4+432p^3+362p^2+132p+36) \\
&= p(12p^5+72p^4+132p^3+72p^2+37p^4+222p^3+407p^2+222p \\
&\quad + 18p^3+108p^2+198p+108) \\
&\quad - (12p^6+84p^5+258p^4+432p^3+362p^2+132p+36) \\
&= 25p^5+114p^4+155p^3+58p^2-24p-36 \\
&\geq 292,
\end{aligned}$$

completing the proof.

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