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Non-relativistic global limits of weak solutions of the relativistic Euler equation

By

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1. Introduction

The relativistic Euler equation for a perfect fluid in two dimensional Minkowski space-time has the form ([9], [10])

$$\begin{aligned} &\frac{\partial}{\partial t}\left\{\frac{(p+\rho c^2)}{c^2}\frac{v^2}{c^2-v^2}+\rho\right\}+\frac{\partial}{\partial x}\left\{(p+\rho c^2)\frac{v}{c^2-v^2}\right\}=0, \\ &\frac{\partial}{\partial t}\left\{(p+\rho c^2)\frac{v}{c^2-v^2}\right\}+\frac{\partial}{\partial x}\left\{(p+\rho c^2)\frac{v^2}{c^2-v^2}+p\right\}=0. \end{aligned} \tag{1.1}$$

Here $v=v(x,t)$ is the classical coordinate velocity, $\rho=\rho(x,t)$ is the mass-energy density of the fluid, $p=p(\rho)$ is the pressure and c is the speed of light. On the other hand, the non-relativistic Euler equation is

$$\begin{aligned} &\frac{\partial}{\partial t}\rho+\frac{\partial}{\partial x}(\rho v)=0, \\ &\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho v^2+p)=0. \end{aligned} \tag{1.2}$$

For the systems (1.1) and (1.2), the local existence theorems are known for the smooth solutions (see [4] and [5] for the full-dimensional case). Also, the global existence theorems are established for the one-dimensional isentropic motions $p=\rho^\gamma$, $\gamma>1$ ([1] and [7]). In the case of the isothermal motions $p=\sigma^2\rho$, where the sound speed σ is assumed to be the constant, the existence theorems with arbitrary initial data have been obtained both for (1.1) and (1.2), by J. Smoller and B. Temple [9] and by T. Nishida [6] respectively.

In physics, it is well-known that the classical mechanics reappears as the limit of the relativistic mechanics when $c\to\infty$, and in particular, it is easy to check that the relativistic Euler equation (1.1) reduces formally to the non-relativistic Euler equation (1.2) when $c\to\infty$. However, until now there are only local results for the limit of smooth solutions of the relativistic Euler equation ([5]). The aim of this paper is to discuss the convergence of weak solutions of (1.1) as $c\to\infty$. Since

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we know their global existence both for (1.1) and (1.2), it is natural to expect that the convergence is global in time. We will show that this is indeed the case.

For simplicity, we study the systems (1.1) and (1.2) for the case

$$p = \sigma^2 \rho \tag{1.3}$$

with common initial data

$$\rho(x,0) = \rho_0(x), \quad v(x,0) = v_0(x), \tag{1.4}$$

where ρ_0, v_0 are independent of c. It is not hard to see that the same conclusion holds if

$$(\rho_0^c, v_0^c) \to (\rho_0, v_0) \quad \text{as } c \to \infty, \tag{1.5}$$

strongly in L^1_{loc}.

Our main result is

Theorem 1.1. *Let $\rho_0(x) > 0$ and $v_0(x)$ satisfy*

$$\text{T.V.}\{\ln \rho_0\} < \infty, \quad \text{T.V.}\{v_0\} < \infty, \tag{1.6}$$

where T.V.$\{f\}$ *denotes the total variation of the function $f(x)$, $x \in \mathbf{R}$. Then, there exists a constant c_0 and for any $c \geq c_0$ there exists a L^∞ weak solution (ρ^c, v^c) of* (1.1), (1.4) *satisfying*

$$\text{T.V.}\{v^c(\cdot,t)\} + \text{T.V. } \{\ln(\rho^c(\cdot,t))\} \leq M, \tag{1.7}$$

for all $t \geq 0$, where M is a constant depending only on the initial data (ρ_0, v_0) but independent of $c \geq c_0$. Moreover, there exists a subsequence $\{c_k\}$, $c_k \to \infty$ $(k \to \infty)$, such that

$$\rho^{c_k} \to \rho, \quad v^{c_k} \to v, \quad \textit{strongly in } L^1_{loc}(\mathbf{R} \times \mathbf{R}^+), \tag{1.8}$$

as $k \to \infty$ and the limit (ρ, v) is a weak solution of (1.2) *and* (1.4).

To prove Theorem 1.1, we use the non-increasing property of the total variation of $\ln(\rho^c)$ in the system (1.1) given by J. Smoller and B. Temple in [9], and find out that for any fixed initial data (ρ_0, v_0) the total variations of the approximate solutions $(\rho^c_{\Delta x}, v^c_{\Delta x})$ constructed by Glimm's scheme are bounded uniformly for large c. In [9] Theorem 1, J. Smoller and B. Temple show that

$$\text{T.V.}\{\ln(\rho^c(\cdot,t))\} \leq V_0, \quad \text{T.V.}\left\{\ln\left(\frac{c + v^c(\cdot,t)}{c - v^c(\cdot,t)}\right)\right\} \leq V_1, \tag{1.9}$$

where V_0, V_1 are constants depending on T.V. $\ln(\rho_0)$ and T.V. v_0. However, their proof does not tell us anything about the c-dependency of V_0 and V_1. Moreover, even if V_0, V_1 are independent of large c, the second inequality in (1.9) becomes meaningless when $c \to \infty$ because

$$\ln\frac{c+v}{c-v} \to 0 \quad \text{as } c \to \infty, \tag{1.10}$$

for each fixed $v \in \mathbf{R}$. Therefore, in order to obtain uniform estimates for large c, we shall evaluate the total variation of v^c itself, instead of $\ln\left(\frac{c+v^c}{c-v^c}\right)$. Actually, we will prove (see Lemma 3.4 below) that

$$\text{T.V. } v^c \leq 4\sigma \exp\{\text{T.V. } \ln \rho^c\}. \tag{1.11}$$

Thus, the desired estimate will come if V_0 in (1.9) is found to be bounded uniformly for large c. To show this, then, we need to improve the estimate given in [9] for the wave strength in the approximate solution (see Lemmas 2.4 and 3.3 below).

2. Riemann problem

In this section we discuss the solution of the Riemann problem for the system (1.1). The results in this section mostly appear in [9] and our goal is to derive a sharper estimate of waves in the solution of the Riemann problem (see Lemma 2.4).

The problem of (1.1) and (1.4) is a special case of the general system of the nonlinear hyperbolic conservation laws in the sense of Lax ([3], [8]),

$$U_t + F(U)_x = 0, \tag{2.1}$$

with initial condition

$$U(x,0) = U_0(x). \tag{2.2}$$

In our case (1.1),

$$U \equiv \left(\rho\left[\frac{(\sigma^2+c^2)}{c^2}\frac{v^2}{c^2-v^2}+1\right], \rho(\sigma^2+c^2)\frac{v}{c^2-v^2}\right), \tag{2.3}$$

$$F(U) \equiv \left(\rho(\sigma^2+c^2)\frac{v}{c^2-v^2}, \rho\left[(\sigma^2+c^2)\frac{v^2}{c^2-v^2}+\sigma^2\right]\right). \tag{2.4}$$

For the mapping $(\rho, v) \to U = (u_1, u_2)$, we state the

Lemma 2.1. *The mapping $(\rho, v) \to U$ is $1-1$, and the Jacobian of this mapping is continuous and non-zero in the region $\rho > 0$, $|v| < c$. Moreover, the convergences*

$$U \to (\rho, \rho v), \quad F(U) \to (\rho v, \rho(v^2+\sigma^2)), \quad \text{as } c \to \infty, \tag{2.5}$$

are uniform in any bounded region $0 < \rho < M$, $|v| < c_0$, where M, c_0 are positive constants.

The first part of this lemma is given in [9]. The second part is easy to prove and we omit the proof.

The Riemann problem is the initial value problem when the initial data $U_0(x) \equiv U(\rho_0(x), v_0(x))$ consists of a pair of constant states $U_l \equiv U(\rho_l, v_l)$ and

$U_r \equiv U(\rho_r, v_r)$ separated by a jump discontinuity at $x = 0$,

$$U_0(x) = \begin{cases} U_l & \text{if } x < 0, \\ U_r & \text{if } x > 0. \end{cases} \tag{2.6}$$

Note that, in view of Lemma 2.1, U_l and U_r of the system (2.1) are uniquely determined by (ρ_l, v_l) and (ρ_r, v_r).

This problem can be solved in the class of functions consisting of constant states, separated by either shock waves or rarefaction waves. Shock waves are determined by the classical Rankine-Hugoniot condition

$$s[U] = [F], \tag{2.7}$$

and the Lax entropy conditions

$$s_1 < \lambda_{1l}, \quad \lambda_{1r} < s_1 < \lambda_{2r}, \quad \text{on 1-shocks}, \tag{2.8}$$

$$\lambda_{1l} < s_2 < \lambda_{2l}, \quad s_2 > \lambda_{2r}, \quad \text{on 2-shocks}. \tag{2.9}$$

Here $[f] = f(U_l) - f(U_r)$ denotes the jump of the function $f(U)$ between the left and right hand states along the curve of discontinuity in the xt plane, while $\lambda_{1l}, \lambda_{1r}, \lambda_{2l}, \lambda_{2r}$ represent the first and second eigenvalues of (2.1) on the left and right, and s_1, s_2 represent the shock speeds of 1-shock and 2-shock, respectively. Rarefaction waves are continuous solutions of form $U(x/t)$.

For the system (1.1), the eigenvalues, Riemann invariants, shock waves and rarefaction waves are given by J. Smoller and B. Temple in [9].

Lemma 2.2 ([9]). *The eigenvalues of the system* (1.1) *are real and distinct, with*

$$\lambda_1 = \frac{v - \sigma}{1 - \dfrac{\sigma v}{c^2}}, \quad \lambda_2 = \frac{v + \sigma}{1 + \dfrac{\sigma v}{c^2}}, \tag{2.10}$$

and the Riemann invariants r and s for the system (1.1) *are defined as*

$$r = \frac{1}{\sqrt{2k}} \ln\left(\frac{c+v}{c-v}\right) - \ln \rho, \tag{2.11}$$

$$s = \frac{1}{\sqrt{2k}} \ln\left(\frac{c+v}{c-v}\right) + \ln \rho, \tag{2.12}$$

where

$$k = \frac{2\sigma^2 c^2}{(\sigma^2 + c^2)^2}. \tag{2.13}$$

Here the pair of Riemann invariants r and s defined in (2.11) and (2.12) is a little different from those of [9], so that the limit of our pair (2.11) and (2.12) makes sense even when $c \to \infty$.

Lemma 2.3 ([9]). *Suppose that (ρ_l, v_l) and $(\rho, v) \equiv (\rho_r, v_r)$ satisfy the jump conditions* (2.7) *and Lax entropy conditions* (2.8), (2.9) *for the system* (1.1). *Then the shock waves are*

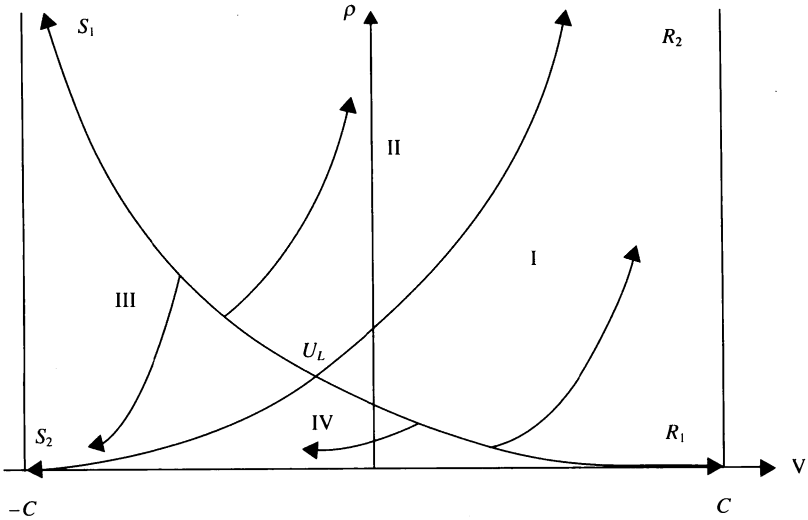

Fig. 1

(2.14) $\quad$ 1-*shock wave* $S_1 : \dfrac{\rho}{\rho_l} = f_+(\beta) = 1 + \beta + \sqrt{\beta^2 + 2\beta}, \quad (\rho > \rho_l,\ v < v_l),$

(2.15) $\quad$ 2-*shock wave* $S_2 : \dfrac{\rho}{\rho_l} = f_-(\beta) = 1 + \beta - \sqrt{\beta^2 + 2\beta}, \quad (\rho < \rho_l,\ v < v_l).$

The rarefaction waves are

(2.16) $\quad$ 1-*rarefaction wave* $R_1 : \dfrac{\rho}{\rho_l} = \left[\dfrac{(c + v_l)(c - v)}{(c - v_l)(c + v)}\right]^{1/\sqrt{2k}}, \quad (\rho < \rho_l,\ v > v_l),$

(2.17) $\quad$ 2-*rarefaction wave* $R_2 : \dfrac{\rho}{\rho_l} = \left[\dfrac{(c + v)(c - v_l)}{(c - v)(c + v_l)}\right]^{1/\sqrt{2k}}, \quad (\rho > \rho_l,\ v > v_l),$

where k *is defined in* (2.13) *and*

$$\beta = \beta(v, v_l) = \frac{(\sigma^2 + c^2)^2}{2\sigma^2} \frac{(v - v_l)^2}{(c^2 - v^2)(c^2 - v_l^2)}. \tag{2.18}$$

The wave curves are sketched in Figure 1.

Theorem 2.1 ([9]). *For any initial value* (ρ_l, v_l) *and* (ρ_r, v_r) *there exist a solution of the Riemann problem for* (1.4) *and* (2.6) *in the case of* $\rho_l, \rho_r > 0$, $-c < v_l$, $v_r < c$. *The solution* (ρ, v) *satisfies* $0 < \rho(x,t) < \infty$, $-c < v(x,t) < c$. *Moreover, the solution is given by a* 1-*wave which is followed by a* 2-*wave. The solution is unique in the class of rarefaction waves and admissible shock waves.*

Using Lemma 2.3 and Theorem 2.1 we can obtain the following inequality for the wave strength. This inequality is an improvement of that of [9], Lemma 6, and is needed to establish a uniform (in c) estimate of the wave strength at $t = 0+$. See Lemma 3.3.

Lemma 2.4. *Let (ρ_m, v_m) be the state which connects with (ρ_l, v_l) by 1-wave on the left and with (ρ_r, v_r) by 2-wave on the right. Then we have*

$$|\ln \rho_l - \ln \rho_m| + |\ln \rho_m - \ln \rho_r| \leq |\ln \rho_l - \ln \rho_r| + \left| \ln\left(\frac{c+v_l}{c-v_l}\right)^{1/\sqrt{2k}} - \ln\left(\frac{c+v_r}{c-v_r}\right)^{1/\sqrt{2k}} \right|. \tag{2.19}$$

Proof. Let I, II, III and IV be the regions depicted in Figure 1. From Theorem 2.1 we know that there are four distinct cases for the solutions according to which region U_r lies in. Now we shall discuss each case separately.

(i) The case $U_r \in I$. In this case, (ρ_m, v_m) connects with (ρ_l, v_l) by a 1-rarefaction wave and with (ρ_r, v_r) by a 2-rarefaction wave and $v_l \leq v_m \leq v_r$ holds. From (2.16) and (2.17) we then have

$$\begin{aligned} &|\ln \rho_l - \ln \rho_m| + |\ln \rho_m - \ln \rho_r| \\ &= \left| \ln\left(\frac{(c+v_m)(c-v_l)}{(c-v_m)(c+v_l)}\right)^{1/\sqrt{2k}} \right| + \left| \ln\left(\frac{(c+v_r)(c-v_m)}{(c-v_r)(c+v_m)}\right)^{1/\sqrt{2k}} \right| \\ &= \left| \ln\left(\frac{(c+v_r)(c-v_l)}{(c-v_r)(c+v_l)}\right)^{1/\sqrt{2k}} \right|. \end{aligned} \tag{2.20}$$

(ii) The case $U_r \in II$. In this case, (ρ_m, v_m) connects with (ρ_l, v_l) by a 1-shock wave and (ρ_r, v_r) by a 2-rarefaction wave and $\rho_l \leq \rho_m \leq \rho_r$ holds. Thus,

$$|\ln \rho_l - \ln \rho_m| + |\ln \rho_m - \ln \rho_r| = |\ln \rho_l - \ln \rho_r|. \tag{2.21}$$

(iii) The case $U_r \in III$. In this case, (ρ_m, v_m) connects with (ρ_l, v_l) by a 1-shock wave and (ρ_r, v_r) by a 2-shock wave and $v_l \geq v_m \geq v_r$ holds. From (2.14) and (2.15) we have

$$\begin{aligned} |\ln \rho_l - \ln \rho_m| + |\ln \rho_m - \ln \rho_r| &= \ln f_+(\beta(v_l, v_m)) + |\ln f_-(\beta(v_m, v_r))| \\ &= \ln f_+(\beta(v_l, v_m)) + \ln f_+(\beta(v_m, v_r)). \end{aligned} \tag{2.22}$$

If

$$\ln f_+(\beta(v_1, v_2)) \leq \left| \ln\left(\frac{(c+v_1)}{(c-v_1)}\right)^{1/\sqrt{2k}} - \ln\left(\frac{(c+v_2))}{(c-v_2)}\right)^{1/\sqrt{2k}} \right|, \tag{2.23}$$

then we will have

$$\text{(2.24)}\qquad \begin{aligned} &|\ln\rho_l - \ln\rho_m| + |\ln\rho_m - \ln\rho_r| \\ &\le \left|\ln\left(\frac{(c+v_l)}{(c-v_l)}\right)^{1/\sqrt{2k}} - \ln\left(\frac{(c+v_m)}{(c-v_m)}\right)^{1/\sqrt{2k}}\right| \\ &\quad + \left|\ln\left(\frac{(c+v_m)}{(c-v_m)}\right)^{1/\sqrt{2k}} - \ln\left(\frac{(c+v_r)}{(c-v_r)}\right)^{1/\sqrt{2k}}\right| \\ &= \left|\ln\left(\frac{(c+v_l)}{(c-v_l)}\right)^{1/\sqrt{2k}} - \ln\left(\frac{(c+v_r)}{(c-v_r)}\right)^{1/\sqrt{2k}}\right|, \end{aligned}$$

since $v_l \ge v_m \ge v_r$. It remains to prove (2.23). Put

(2.25)

$$g(y) = \ln\left\{\frac{1}{4k}y + 1 + \sqrt{\left(\frac{1}{4k}y+1\right)^2 - 1}\right\} - \frac{1}{\sqrt{2k}}\ln\left\{\frac{1}{2}y + 1 + \sqrt{\left(\frac{1}{2}y+1\right)^2 - 1}\right\}.$$

Then we find $g(0) = 0$, and

$$\text{(2.26)}\qquad g'(y) = \frac{1}{\sqrt{(y+4k)^2 - (4k)^2}} - \frac{1}{\sqrt{2k((y+2)^2 - 4)}}.$$

Since $2k - 1 = -\dfrac{(c^2-\sigma^2)^2}{(c^2+\sigma^2)^2} \le 0$, we obtain that $g'(y) < 0$ for $y > 0$, which means $g(y) \le 0$ for $y \ge 0$. Let $y = (\sqrt{x} - (\sqrt{x})^{-1})^2$, $x > 0$, then $\frac{1}{2}y + 1 = \frac{1}{2}\left(x + \frac{1}{x}\right)$ and

$$\text{(2.27)}\qquad \frac{1}{\sqrt{2k}}\ln\left\{\frac{1}{2}y + 1 + \sqrt{\left(\frac{1}{2}y+1\right)^2 - 1}\right\} = \frac{1}{2k}|\ln x|.$$

Put $x = (c+v_1)(c-v_2)(c-v_1)^{-1}(c+v_2)^{-1}$ to deduce

$$\begin{aligned} \frac{1}{4k}y &= \frac{1}{4k}\left(x + \frac{1}{x} - 2\right) \\ &= \frac{1}{4k}\left(\frac{(c+v_1)(c-v_2)}{(c-v_1)(c+v_2)} + \frac{(c-v_1)(c+v_2)}{(c+v_1)(c-v_2)} - 2\right) \\ &= \frac{1}{4k}\,\frac{(c+v_1)^2(c-v_2)^2 + (c-v_1)^2(c+v_2)^2 - 2(c^2-v_1^2)(c^2-v_2^2)}{(c^2-v_1^2)(c^2-v_2^2)} \\ &= \frac{1}{4k}\,\frac{4c^2(v_1-v_2)^2}{(c^2-v_1^2)(c^2-v_2^2)} = \beta(v_1, v_2), \end{aligned}$$

and

$$\ln\left\{\frac{1}{4k}y+1+\sqrt{\left(\frac{1}{4k}y+1\right)^2-1}\right\}=\ln\left(1+\beta+\sqrt{\beta^2+2\beta}\right), \tag{2.28}$$

which proves (2.23).

(iv) The case $U_r \in IV$. In this case, (ρ_m, v_m) connects with (ρ_l, v_l) by a 1-rarefaction wave and with (ρ_r, v_r) by a 2-shock wave and $\rho_l \ge \rho_m \ge \rho_r$ holds, so that we get

$$|\ln\rho_l-\ln\rho_m|+|\ln\rho_m-\ln\rho_r|=|\ln\rho_l-\ln\rho_r|. \tag{2.29}$$

The proof is now complete.

3. The difference approximation

In this section we use Glimm's scheme to construct an approximate solution $U_{\Delta x}(x,t)$ for the problem (1.1) and (1.4), and derive some estimation on $U_{\Delta x}(x,t)$ that will be used in the next section. Let Δx denote a mesh length in x and Δt a mesh length in t, and let $x_j = j\Delta x$, $t_n = n\Delta t$, denote the mesh points for the approximate solution. Let $U_0(x) = U(\rho_0(x), v_0(x))$ denote the initial data for (1.1) satisfying $\rho_0 > 0$, $-c < v_0 < c$. To start the scheme, define

$$U_{\Delta x}(x,0)=U_j^0, \quad \text{for } x_j \le x < x_{j+1}, \tag{3.1}$$

where $U_j^0 = U_0(x_j+)$. For $t_{n-1} < t < t_n$, let $U_{\Delta x}(x,t)$ be the solution of the Riemann Problem posed at time $t = t_{n-1}$. Then, define

$$U_{\Delta x}(x,t_n)=U_j^n, \quad \text{for } x_j \le x < x_{j+1}, \tag{3.2}$$

where $U_j^n = U_{\Delta x}(x_j + a_n, t_n-)$ for some $a_n \in (0,1)$, and use this as the initial data for the Riemann problem posed at $t = t_n$. Thus, $U_{\Delta x}(x,t)$ can be defined for all $x \in \mathbf{R}$ and $t \ge 0$ by induction, if the waves do not interact within one time step. In [9], it is stated that the last requirement is fulfilled if

$$\frac{\Delta x}{2\Delta t} \ge c. \tag{3.3}$$

However, this does not make sense when we consider the limit $c \to \infty$. Actually it suffices to choose $\Delta x/(2\Delta t)$ to be larger than the eigenvalues of the system (1.1).

In order to find the bounds of the eigenvalues, first, we state some estimate of $U_{\Delta x}$ which are given in [9].

Lemma 3.1 ([9]). *Let* (ρ_0, v_0) *satisfy*

$$0<\rho_0<\infty, \quad -c<v_0<c, \tag{3.4}$$

$$\text{T.V. } \ln(\rho_0)+\text{T.V. } v_0<\infty. \tag{3.5}$$

Then $(\rho_{\Delta x}, v_{\Delta x})$ *which is given by Glimm's scheme with* (3.3) *satisfies*

$$0 < \rho_{\Delta x}(x,t) < \infty, \quad -c < v_{\Delta x}(x,t) < c, \tag{3.6}$$

$$\text{T.V. } \ln(\rho_{\Delta x}(\cdot, t+)) \le \text{T.V. } \ln(\rho_{\Delta x}(\cdot, s+)), \tag{3.7}$$

whenever $0 \le s \le t$.

Here (3.3) is assumed but it is evident that the conclusion is true as long as no two waves interact. For the initial data v_0 we have the following result:

Lemma 3.2. *Let* $v_0 \in BV(\mathbf{R})$ *and* $M_0 \equiv \sup_{x \in \mathbf{R}} |v_0(x)|$. *If* $c > M_0$, *then*

$$\text{T.V. } \ln\left(\frac{c+v_0}{c-v_0}\right)^{1/\sqrt{2k}} \le \frac{\sigma(c^2+\sigma^2)}{(c-M_0)^2} \text{T.V. } v_0. \tag{3.8}$$

Proof. Let $v_2 > v_1$, $|v_1| < c$, $|v_2| < c$. Then

$$\left|\ln\left(\frac{(c+v_1)}{(c-v_1)}\right)^{1/\sqrt{2k}} - \ln\left(\frac{(c+v_2))}{(c-v_2)}\right)^{1/\sqrt{2k}}\right| = \left|\ln\left(\frac{(c+v_1)(c-v_2)}{(c-v_1)(c+v_2)}\right)^{1/\sqrt{2k}}\right|$$

$$= \frac{1}{\sqrt{2k}} \ln\left(1 + \frac{2c(v_1 - v_2)}{(c-v_1)(c+v_2)}\right).$$

Noting that $(2k)^{-1/2} = \dfrac{\sigma^2 + c^2}{2\sigma c}$ and $\ln(1+x) \le x$ for any $x \ge 0$ we have

$$\left|\ln\left(\frac{(c+v_1)}{(c-v_1)}\right)^{1/\sqrt{2k}} - \ln\left(\frac{(c+v_2)}{(c-v_2)}\right)^{1/\sqrt{2k}}\right| \le \frac{(\sigma^2+c^2)(v_2-v_1)}{\sigma(c-v_1)(c+v_2)}. \tag{3.9}$$

Then we obtain (3.8) easily from (3.9) and by

$$\frac{1}{(c-v_1)(c+v_2)} \le \frac{1}{(c-M_0)^2}. \tag{3.10}$$

Next we use Lemmas 2.4, 3.1 and 3.2 to derive bounds of T.V. $\ln(\rho_{\Delta x})$ and T.V. $v_{\Delta x}$ independent of large c, still assuming that no waves interact within one time step.

Lemma 3.3. *Let* $c > 2M_0$, *where* M_0 *is given in Lemma* 3.2. *Then*

$$\text{T.V. } \ln \rho_{\Delta x}(\cdot, t) \le \text{T.V. } \ln \rho_0 + 8\sigma \text{T.V. } v_0. \tag{3.11}$$

Proof. From Lemma 3.1 and (3.7) we know that

$$\text{T.V. } \ln \rho_{\Delta x}(\cdot, t) \le \text{T.V. } \ln \rho_{\Delta x}(\cdot, 0+). \tag{3.12}$$

Using Lemma 2.4 we obtain

$$\text{T.V. } \ln \rho_{\Delta x}(\cdot, 0+) \le \text{T.V. } \ln \rho_0 + \text{T.V. } \ln\left(\frac{c+v_0}{c-v_0}\right)^{1/\sqrt{2k}}, \tag{3.13}$$

which, combined with Lemma 3.2, yields

$$\text{T.V. } \ln \rho_{\Delta x}(\cdot, 0+) \leq \text{T.V. } \ln \rho_0 + \frac{\sigma(c^2+\sigma^2)}{(c-M_0)^2} \text{T.V. } v_0. \tag{3.14}$$

Now, (3.11) comes from (3.12) and (3.14).

Lemma 3.4. *Let* $c > 2M_0$ *and* $M_1 \equiv \text{T.V. } \ln \rho_0 + 8\sigma \text{T.V. } v_0$, *where* M_0 *is given by Lemma* 3.2. *Then*

$$\text{T.V. } v_{\Delta x} \leq 4\sigma e^{M_1}. \tag{3.15}$$

Proof. Let (ρ_i, v_i) denote the i-th constant state of $(\rho_{\Delta x}(\cdot, t), v_{\Delta x}(\cdot, t))$ for $t \neq n\Delta t$. Then

$$\text{T.V. } \ln \rho_{\Delta x}(\cdot, t) = \left(\sum_S + \sum_R\right) |\ln(\rho_i) - \ln(\rho_{i+1})| \tag{3.16}$$

where $\sum_S$ and $\sum_R$ are the sums of all shock waves case and all rarefaction waves case respectively. Since $f_+(\beta)f_-(\beta) = 1$, $f_+(\beta) \geq 1 + \sqrt{2\beta}$ where $f_\pm(\beta)$ and β are defined by (2.14), (2.15) and (2.18), we have

$$\begin{aligned} \sum_S &= \sum_S |\ln(f_+(\beta(v_i, v_{i+1})))| \\ &\geq \sum_S \ln\left(1 + \frac{(\sigma^2+c^2)}{\sigma} \frac{|v_i - v_{i+1}|}{(c+|v_i|)(c+|v_{i+1}|)}\right). \end{aligned} \tag{3.17}$$

The second term on the right hand side of (3.16) can be estimated by using (2.16), (2.17) and the inequality $(1+x)^y \geq 1 + yx$ for $y \geq 1$, $x \geq 0$ as follows:

$$\begin{aligned} \sum_R &= \sum_R \left| \ln \left[\frac{(c+v_i)(c-v_{i+1})}{(c-v_i)(c+v_{i+1})}\right]^{1/\sqrt{2k}} \right| \\ &\geq \sum_R \ln \left| \left[1 + \frac{2c|v_{i+1} - v_i|}{(c+|v_i|)(c+|v_{i+1}|)}\right]^{1/\sqrt{2k}} \right| \\ &\geq \sum_R \ln\left(1 + \frac{(\sigma^2+c^2)}{\sigma} \frac{|v_i - v_{i+1}|}{(c+|v_i|)(c+|v_{i+1}|)}\right). \end{aligned} \tag{3.18}$$

Then (3.11) and (3.16)–(3.18) yield

$$\begin{aligned} e^{M_1} &\geq \exp\{\text{T.V. } \ln \rho_{\Delta x}(\cdot, t)\} \\ &\geq \prod_{S\cup R} \left(1 + \frac{(\sigma^2+c^2)}{\sigma} \frac{|v_i - v_{i+1}|}{(c+|v_i|)(c+|v_{i+1}|)}\right) \\ &\geq \sum_{S\cup R} \frac{(\sigma^2+c^2)}{\sigma} \frac{|v_i - v_{i+1}|}{(c+|v_i|)(c+|v_{i+1}|)} \\ &\geq \sum_{S\cup R} \frac{1}{4\sigma} |v_i - v_{i+1}|. \end{aligned} \tag{3.19}$$

Here we used the inequality $\prod(1+x_i) \geq \sum x_i$ for $x_i \geq 0$ and $|v_i| < c$. Thus we get (3.15).

We summarize the results above as a theorem.

Theorem 3.1. *Let* v_0, $\ln \rho_0 \in BV(R)$ *and* $\rho_0 > 0$. *Then there exists a constant* c_0 *and for any* $c > c_0$ *we have*

$$\|v_{\Delta x}(\cdot,\cdot)\|_{L^\infty} + \|\ln(\rho_{\Delta x}(\cdot,\cdot))\|_{L^\infty} \leq M, \tag{3.20}$$

$$\text{T.V. } v_{\Delta x}(\cdot,t) + \text{T.V. } \ln \rho_{\Delta x}(\cdot,t) \leq M, \tag{3.21}$$

where M *is a constant depending only on initial data* (ρ_0, v_0).

Proof. The assumption in Theorem 3.1 implies that there exist states $\rho_\pm = \lim_{x\to\pm\infty} \rho_0(x)$ and $v_\pm = \lim_{x\to\pm\infty} v_0(x)$. From the definition of Glimm's scheme, it is easy to see that $\lim_{x\to\pm\infty} \rho_{\Delta x}(x,t) = \rho_\pm$ and $\lim_{x\to\pm\infty} v_{\Delta x}(x,t) = v_\pm$ hold good. This together with (3.21), which is proved in Lemmas 3.3 and 3.4, implies (3.20).

Recall that Lemmas 3.3, 3.4 and Theorem 3.1 were established under the assumption that no waves interact within any one time step. However, the constants M_0, M_1, M appearing there are all independent of $c > c_0$ and depend only on (ρ_0, v_0), so that we can proceed as follows. Suppose $|v_{\Delta x}| \leq M$ and $c \geq c_0$. Then the eigenvalues $\lambda_i(\rho, v)$ of the system (1.1) given in (2.10) satisfy

$$|\lambda_{1,2}| \leq \frac{M+\sigma}{1 - \dfrac{\sigma M}{c_0^2}}. \tag{3.22}$$

Now we choose

$$\frac{\Delta x}{\Delta t} \geq \frac{2(M+\sigma)}{1 - \dfrac{\sigma M}{c_0^2}}. \tag{3.23}$$

Thus our choice of $\Delta x/\Delta t$ is independent of c for $c \geq c_0$ and we see immediately that Theorem 3.1 holds also with this choice. Moreover, it allows us to show that the approximate solutions $(\rho_{\Delta x}, v_{\Delta x})$ are L_1 Lipschitz continuous in t.

Lemma 3.5. *Under the assumption of Theorem* 3.1, *it holds that for any* $0 \leq t \leq t'$ *and* $c \geq c_0$, *there exist a positive constant* M *such that*

$$\int_{-\infty}^{\infty} |v_{\Delta x}(x,t) - v_{\Delta x}(x,t')| + |\ln(\rho_{\Delta x}(x,t)) - \ln(\rho_{\Delta x}(x,t'))|\, dx \leq M|t-t'|, \tag{3.24}$$

where c_0 *is the constant given in Theorem* 3.1 *and* M *is a constant only depending on the initial data* ρ_0, v_0.

Proof. Let $\delta = \Delta x/\Delta t$ be fixed so that (3.23) holds. Then for any $0 \leq t \leq t'$, we have

$$|v_{\Delta x}(x,t) - v_{\Delta x}(x,t')| \leq \int_{x-\delta(t'-t)}^{x+\delta(t'-t)} |d_\xi v_{\Delta x}(\xi,t)|. \tag{3.25}$$

Integrating (3.25) over $\mathbf{R}$, we obtain

$$(3.26)\qquad \int_{-\infty}^{\infty} |v_{\Delta x}(x,t) - v_{\Delta x}(x,t')|\,dx \le \int_{-\infty}^{\infty}\left[\left|\int_{x-\delta(t'-t)}^{x+\delta(t'-t)} |d_\xi v_{\Delta x}(\xi,t)|\right|\right] dx$$
$$\le 2\delta(t'-t)\text{T.V. } v_{\Delta x}(\cdot,t).$$

For $\ln \rho_{\Delta x}$ the proof is the same. Thus, we complete the proof.

4. Convergence

We shall complete the proof of Theorem 1.1. First, for any initial data (ρ_0, v_0) which satisfies the assumption of Theorem 1.1, there exist positive constants c_0 and M such that for any $c > c_0$ the approximate solutions $(\rho^c_{\Delta x}, v^c_{\Delta x})$ generated by Glimm's method satisfy the following inequality (Theorem 3.1 and Lemma 3.5);

$$(4.1)\qquad \|v^c_{\Delta x}(\cdot,\cdot)\|_{L^\infty} \le M, \quad \|\ln(\rho^c_{\Delta x}(\cdot,\cdot)\|_{L^\infty} \le M,$$

$$(4.2)\qquad \text{T.V. } v^c_{\Delta x}(\cdot,t) + \text{T.V. } \ln(\rho^c_{\Delta x}(\cdot,t)) \le M,$$

$$(4.3)\qquad \|v^c_{\Delta x}(\cdot,t_1) - v^c_{\Delta x}(\cdot,t_2)\|_{L^1} \le M|t_1 - t_2|,$$

$$(4.4)\qquad \|\ln(\rho^c_{\Delta x}(\cdot,t_1)) - \ln(\rho^c_{\Delta x}(\cdot,t_2))\|_{L^1} \le M|t_1 - t_2|,$$

where M is dependent only on the initial data (v_0, ρ_0) and is independent of $c \ge c_0$. For each $c > c_0$ we apply Glimm's Theorem [2] to obtain the weak solution (ρ^c, v^c) of the system (1.1). Let $a \equiv \{a_k\} \in A$ denote a (fixed) random sequence, $0 < a_k < 1$, $1 < k < \infty$, where A denotes the infinite product of intervals $[0,1]$ endowed with Lebesgue measure.

Theorem 4.1 (Glimm, [2]). *Assume that the approximate solution $(\rho^c_{\Delta x}, v^c_{\Delta x})$ of* (1.1) *satisfies* (4.1)–(4.4). *Then there exists a subsequence of mesh lengths $\Delta x_i \to 0$ such that $(\rho^c_{\Delta x_i}, v^c_{\Delta x_i}) \to (\rho^c, v^c)$, where (ρ^c, v^c) also satisfies* (4.1), (4.2). *The approximate solutions converge pointwise a.e., and in $L_{loc}(\mathbf{R})$ at each time t, uniformly on bounded x and t sets. Moreover, there exists a set $N \subset A$ of Lebesgue measure zero such that, if $a \in A - N$, then (ρ^c, v^c) is a weak solution of the initial value problem* (1.1), (1.4).

Using this theorem we have

Lemma 4.1. *For any $c > c_0$* (4.1)–(4.4) *hold also for the limit (ρ^c, v^c) given in Theorem* 4.1, *with the same constant M.*

Proof. For any $R > 0$, we have

$$(4.5)\qquad \|v^c(\cdot,t_1) - v^c(\cdot,t_2)\|_{L^1(-R,R)} \le \|v^c(\cdot,t_1) - v^c_{\Delta x_i}(\cdot,t_1)\|_{L^1(-R,R)}$$
$$+ \|v^c(\cdot,t_2) - v^c_{\Delta x_i}(\cdot,t_2)\|_{L^1(-R,R)}$$
$$+ \|v^c_{\Delta x_i}(\cdot,t_1) - v^c_{\Delta x_i}(\cdot,t_2)\|_{L^1(-R,R)},$$

where $\{\Delta x_i\}$ is as in Theorem 4.1. Using (4.3) and Theorem 4.1 we obtain

$$\|v^c(\cdot,t_1)-v^c(\cdot,t_2)\|_{L^1(-R,R)}\leq M|t_1-t_2| \tag{4.6}$$

by taking the limit as $\Delta x_i \rightarrow 0$ in (4.5). Then passing to the limit as $R\rightarrow\infty$ in (4.6) shows that v^c also satisfies (4.3) with the same M. The proof is similar for ρ^c.

Lemma 4.2. *Let* $\{(\rho^c,v^c)\}$, $c\geq c_0$ *be a family of functions satisfying* (4.1)–(4.4). *Then there exists a subsequence* $\{c_n\}$ *such that* $\{(\rho^{c_n},v^{c_n})\}$ *converges strongly to a pair of function* (ρ,v) *pointwise a.e., in* $L^1_{loc}(\mathbf{R})$ *at each time* t *and in* $L^1_{loc}(\mathbf{R}\times\mathbf{R}^+)$. *Moreover,* (ρ,v) *also satisfies* (4.1)–(4.4) *with the same constant.*

The proof is exactly the same as the corresponding result in Chapter 16 in [8] and we omit the proof. Now it is easy to show that (ρ,v) is a weak solution of (1.2) and (1.4) since for any $t\geq 0$, it hold that as $n\rightarrow\infty$,

$$U^{c_n}(v^{c_n},\rho^{c_n})\rightarrow(\rho,\rho v) \qquad \text{in } L^1_{loc}(\mathbf{R}\times[0,\infty)), \tag{4.7}$$

$$F(U^{c_n}(\rho^{c_n},v^{c_n}))\rightarrow(\rho v,\rho v^2+\sigma^2\rho) \qquad \text{in } L^1_{loc}(\mathbf{R}\times[0,\infty)), \tag{4.8}$$

thanks to Lemmas 4.1 and 4.2. This completes the proof of Theorem 1.1.

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