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Calculation of traces of theta series by means of the Weil representation

By

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Introduction

The purpose of this paper is to calculate the trace of a theta series associated with a lattice of a certain quadratic space over $\mathbf{Q}$ using the Weil representation. As an application, we obtain some information on the space of Siegel modular forms and the space of such theta series.

Let us explain our problems in more detail. Let n be a positive integer. Let S be a rational positive definite symmetric matrix of size $2n$ such that the determinant of S is a square of a nonzero rational number. Let $V = (\mathbf{Q}^{2n}, Q)$ be a positive definite quadratic space of rank $2n$ over $\mathbf{Q}$ associated with S.

For every integral lattice L of V, we define a theta series ϑ_L by the following formula:

$$\vartheta_L(\mathfrak{z}) = \sum_{x \in L^m} \exp(2\pi\sqrt{-1}\,\mathrm{tr}({}^t x S x \mathfrak{z}))$$

where $\mathfrak{z}$ is a point of $\mathscr{H}_m$, the Siegel upper half space of degree m. For simplicity, assume n is even in Introduction. If the level of L devides a positive integer N, the theta series ϑ_L belongs to the space $\mathscr{M}_m(n, N)$ of Siegel modular forms of weight n, degree m and level N. Let $\Theta_m(V, N)$ be the subspace of $\mathscr{M}_m(n, N)$ spanned by such theta series.

For any positive divisor N' of N we obtain the inclusion:

$$\Theta_m(V, N') \subset \Theta_m(V, N) \cap \mathscr{M}_m(n, N').$$

But the equality dose not hold in general (cf. [Böc93]).

Problem V. When does the equality

$$\Theta_m(V, N') = \Theta_m(V, N) \cap \mathscr{M}_m(n, N')$$

hold?

To attack Problem V, we can use the global trace operator $T^{(n)}_{N,N'}$ which is defined as follows. For every $\varphi \in \mathscr{M}_m(n, N)$ put

$$T^{(n)}_{N,N'}(\varphi)(\mathfrak{z}) = \sum_{\gamma} \det(c\mathfrak{z} + d)^{-n} \varphi((a\mathfrak{z} + b)(c\mathfrak{z} + d)^{-1}) \quad (\mathfrak{z} \in \mathscr{H}_m)$$

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where $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ runs over a complete set of representatives of $\Gamma_0^{(m)}(N)\backslash \Gamma_0^{(m)}(N')$. Then we have $T_{N,N'}^{(n)}(\varphi) \in \mathscr{M}_m(n, N')$. If we get

$$T_{N,N'}^{(n)}(\Theta_m(V,N)) \subset \Theta_m(V,N'),$$

the equality in Problem V holds. Thus our second problem is derived from Problem V.

Problem L. When does the global trace $T_{N,N'}^{(n)}(\vartheta_L)$ belong to $\Theta_m(n, N')$ for every integral lattice L of V of level N?

Remark. Note that, for discussing Problem L, it is sufficient to consider the case $N = N'p$ with some prime number p.

These two problems are discussed in several papers [SM89, SM91, Böc93]. The authors of these papers show the equality of Problem L in the case $m \geq \min\{n, s_1\}$ where s_1 is an integer depending only on L, N and N' (see §1.3 for more detail). Their method is global. Namely, they establish some relative commutation relations of the global trace operators and Siegel's ϕ-operators; by these relations, the above result follows from the result in singular weight case ($m > 2n$) where the theory of singular modular forms gives an affirmative answer to Problem L.

In contrast to their global method, we transform Problem L to a p-adic analogue by means of the global and local Weil representation.

We organize this paper as follows: In §1 we state our main result as Theorem V and Theorem L. In §2, we convert our classical formulation of §1 to an adelic one. We define a *local trace operator* by using the local Weil representation and formulate a p-adic analogue of Theorem L as Theorem L_p in §2.2. Theorem L_p follows immediately from Propositions 3.1, 3.2. Expressing theta series in terms of the global Weil representation (§2.3), we establish a relation of our global and local trace operators in §2.4. This relation shows that Theorem L_p implies Theorem L. In §3 we prove Propositions 3.1, 3.2. Our method is a combination of an explicit formula for local traces and of the classification of quadratic forms over a local field, its integer ring and its residue field. The local trace has so much symmetry that it can be calculated straightforwardly with no restriction of degree m or weight n.

Independently, Funke calculated the global trace in [Fun95]. He uses the above local method only in the case $m = 1$ and does not discuss the case $m > 1$ and $p \nmid N'$ (see above Remark).

The statement of Theorem L_p is not satisfactory when $p = 2$ since our method does not work well in this case because of the difference between the lattice theories over 2-adic integer ring and over other p-adic ($p > 2$) integer rings. But, if $p > 2$, Theorem L_p is better than the p-adic analogue of the result of [Böc93]. Thus Theorem L is a partial improvement of [Böc93].

The main results of this paper were announced in [Kum96].

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Notation

Let m, n be positive integers. For an commutative ring A with identity element we denote by $A^{\times}$ the group of all invertible elements and by $\mathrm{Mat}_{m,n}(A)$ the module of all $m \times n$-matrices with all entries in A; we put $A^m = \mathrm{Mat}_{m,1}(A)$, $\mathrm{Mat}_m(A) = \mathrm{Mat}_{m,m}(A)$ for simplicity. The identity and zero elements of the ring $\mathrm{Mat}_m(A)$ are denoted by 1_m and 0_m (when m needs to be stressed). The transpose of a matrix g is denoted by ${}^t g$. We denote by $\mathrm{tr}(x)$ the trace of a square matrix x. Let J be an ideal of A. We denote by $\mathrm{Sym}_m(J)$ the module of all $m \times m$-symmetric matrices with all entries in J. If all entries of a matrix $g \in \mathrm{Mat}_{m,n}(A)$ belong to J, we write $g \equiv 0 \bmod J$. We put

$$\mathbf{T} = \{\zeta \in \mathbf{C} \mid |\zeta| = 1\} \quad \mathbf{e}(c) = \exp(2\pi\sqrt{-1}c) \quad (c \in \mathbf{C}).$$

For a set E, $|E|$ denotes the cardinality of E. The characteristic function of a subset E' of E is denoted by $\mathrm{I}_{E'}$. For every locally compact Hausdorff abelian group X, we denote by $\mathscr{S}(X)$ the space of Schwartz-Bruhat functions on X.

Let ∞ and $\mathbf{h}$ be the infinite place and the set of all finite places of $\mathbf{Q}$, respectively. We identify the latter set $\mathbf{h}$ with the set of all rational primes. For any place v of $\mathbf{Q}$, We denote by $\mathbf{Q}_v$ the completion of $\mathbf{Q}$ at v. Let $\mathfrak{G}$ be an algebraic group defined over $\mathbf{Q}$. For any field k containing $\mathbf{Q}$, we denote by $\mathfrak{G}_k$ the group of k-rational points of $\mathfrak{G}$ and abbreviate $\mathfrak{G}_{\mathbf{Q}_v}$ to $\mathfrak{G}_v$ for each place v of $\mathbf{Q}$. We define the adelization $\mathfrak{G}_{\mathbf{A}}$ of $\mathfrak{G}$ and view $\mathfrak{G}_{\mathbf{Q}}$ and $\mathfrak{G}_v$ as subgroups of $\mathfrak{G}_{\mathbf{A}}$ as usual. We then denote by $\mathfrak{G}_\infty$ and $\mathfrak{G}_{\mathbf{h}}$ the infinite and the finite part of $\mathfrak{G}_{\mathbf{A}}$, respectively. For $g \in \mathfrak{G}_{\mathbf{A}}$, we denote by g_v, g_∞, and $g_{\mathbf{h}}$ its projections to $\mathfrak{G}_v$, $\mathfrak{G}_\infty$, $\mathfrak{G}_{\mathbf{h}}$.

We denote by $G^{(m)}$ the symplectic group of genus m. For a commutative ring R with identity element, we assume that the group of all R-rational points of $G_R^{(m)}$ of $G^{(m)}$ is given explicitly by

$$G_R^{(m)} = \left\{ g \in \mathrm{GL}_{2m}(R) \;\middle|\; {}^t g \begin{bmatrix} 0_m & 1_m \\ -1_m & 0_m \end{bmatrix} g = \begin{bmatrix} 0_m & 1_m \\ -1_m & 0_m \end{bmatrix} \right\}.$$

We usually denote every element g of $G_R^{(m)}$ as $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $m \times m$-matrices a, b, c, d. Let $\mathscr{H}_m$ be the Siegel upper half space of genus m. We define an action of $G_\infty^{(m)}$ on $\mathscr{H}_m$ and the factors of automorphy $j(\cdot, \cdot)$ as follows:

$$g\mathfrak{z} = (a\mathfrak{z}+b)(c\mathfrak{z}+d)^{-1}, \quad j(g,\mathfrak{z}) = \det(c\mathfrak{z}+d),$$

where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_\infty^{(m)}$ and $\mathfrak{z} \in \mathscr{H}_m$. For a positive integer N, we define a congruence subgroup $\Gamma_0^{(m)}(N)$ by

$$\Gamma_0^{(m)}(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_{2m}(\mathbf{Z}) \cap G_\infty^{(m)} \mid c \equiv 0 \bmod N\mathbf{Z} \right\}.$$

Let F be a field. We assume that the characteristic of F is not equal to 2. We denote by $(\cdot,\cdot)_F$ the Hilbert symbol of F. Let $V = (V, Q)$ be a regular quadratic space of rank n over F. We denote by B_Q the nondegenerate bilinear form associated with the quadratic form Q given by $B_Q(x,y) = Q(x+y) - Q(x) - Q(y)$ $(x, y \in V)$. For a basis $\{e_i\}$ of V there exists a regular symmetric matrix $S = [s_{ij}]$ of size $n \times n$ such that $Q(\sum_i x_i e_i) = {}^t x S x$ for any $x = [x_i] \in F^n$. We put

$$\det V = \det S \bmod (F^\times)^2 \in F^\times/(F^\times)^2.$$

It is independent of the choice of $\{e_i\}$ and S. Furthermore, assume that F is a local field. The Hasse symbol of V is denoted by $\varepsilon_F(V)$. For a certain basis of V, we can take S to be diagonal; in this case we obtain

$$\varepsilon_F(V) = \prod_{1 \le i < j \le n} (s_{ii}, s_{jj})_F.$$

Furthermore assume F is non-archimedean. Let R, ϖ and q be the maximal compact subring, a prime element and the module of F. For every lattice L of V, define the dual lattice $L^\vee$ with respect to B_Q by

$$L^\vee = \{x \in V \mid B_Q(x,y) \in R \quad (\forall y \in L)\}.$$

Then, $L^\vee$ is also a lattice of V. If L is integral ($Q(L) \subset R$) the R-sub, module of F generated by $Q(L^\vee)$ can be written as $\varpi^{-l}R$ for some non negative integer l. This number is denoted by $\mathrm{lev}_V(L)$. Represent the quadratic form Q as a symmetric matrix S' by taking some R-basis of L. We denote by $\det L$ the element $\det S' \bmod (R^\times)^2$ of $F^\times/(R^\times)^2$. It is independent of the choice of R-basis and S'.

Let $V = (V, Q)$ be a regular quadratic space of rank n over $\mathbf{Q}$. For each place v of $\mathbf{Q}$, we denote by $V_v = (V_v, Q)$ the scalar extension of V over $\mathbf{Q}_v$ as the quadratic space. We put $\varepsilon_v(V) = \varepsilon_{\mathbf{Q}_v}(V)$. For every $p \in \mathbf{h}$ and every lattice M of V_p, we put $\mathrm{lev}_p(M) = \mathrm{lev}_{V_p}(M)$. For every $p \in \mathbf{h}$ and every lattice L of V, we put $L_p = L \otimes_{\mathbf{Z}} \mathbf{Z}_p$. This module L_p becomes a lattice of V_p. If L is an integral lattice of V ($Q(L) \subset \mathbf{Z}$), L_p is also an integral lattice of V_p for every $p \in \mathbf{h}$

and $\mathrm{lev}_p(L_p) = 0$ for almost all $p \in \mathbf{h}$. We denote by level(L) the integer $\prod_{p \in \mathbf{h}} p^{\mathrm{lev}_p(L_p)}$. We define $\det V$ and $\det L$ similarly as in the local field case.

1. Main results

1.1. Preliminaries. Let N, m, n be positive integers. Fix a positive definite symmetric matrix $S \in \mathrm{Sym}_{2n}(\mathbf{Q})$ with $\det S \in (\mathbf{Q}^{\times})^2$. We obtain a regular quadratic space $V = (\mathbf{Q}^{2n}, Q)$ of rank $2n$ over $\mathbf{Q}$ by $Q(x) = {}^t xSx$ $(x \in \mathbf{Q}^{2n})$. Let X be the direct sum of m-copies of V as vector spaces. We identify this vector space X with $\mathrm{Mat}_{2n,m}(F)$. For any $x \in X$, we write $x = (x_i)$ by column vectors x_i $(1 \leq i \leq m)$.

Let (n, N) be a pair of positive integers. For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0^{(m)}(N)$, we put

$$\chi_n(\gamma) = (\det a, (-1)^n)_{\mathbf{Q}_2}.$$

From now on, we make a convention that, if n is odd, N is divisible by 4. For such a pair (n, N), χ_n is a character of $\Gamma_0^{(m)}(N)$.

For N and its positive divisor N', define a subset $P(V; N, N')$ of $\mathbf{h}$ by

$$P(V; N, N') = \{p \in \mathbf{h} \mid N_p \geq 1, N'_p = 0, (-\sqrt{-1})^{n\delta_2(p)} \varepsilon_p(V) = -1\}$$

where we write the prime factorization of N, N' as

$$N = \prod_{p \in \mathbf{h}} p^{N_p}, \quad N' = \prod_{p \in \mathbf{h}} p^{N'_p},$$

$(0 \leq N_p, N'_p \in \mathbf{Z})$ and set

$$\delta_2(p) = \begin{cases} 0 & p \neq 2 \\ 1 & p = 2 \end{cases}$$

for any $p \in \mathbf{h}$.

Since $\det V \equiv 1 \bmod (\mathbf{Q}^{\times})^2$, we can write

$$4^n \det L = \prod_{p \in \mathbf{h}} p^{2s_p(L)}$$

$(s_p(L) \in \mathbf{Z}, p \in \mathbf{h})$ for any lattice L of V.

We define an action of $\Gamma_0^{(m)}(N)$ on the space of all holomorphic functions on $\mathscr{H}_m$ as follows:

$$(f \|_n \gamma)(\mathfrak{z}) = \chi_n(\gamma) j(\gamma, \mathfrak{z})^{-n} f(\gamma \mathfrak{z})$$

where f is a holomorphic function on $\mathscr{H}_m$, $\gamma \in \Gamma_0^{(m)}(N)$, $\mathfrak{z} \in \mathscr{H}_m$. We denote by $\mathscr{M}_m(n, N)$ the space of all holomorphic functions on $\mathscr{H}_m$ satisfying the condition

$$f \|_n \gamma = f$$

for any element γ of $\Gamma_0^{(m)}(N)$.

For any integral lattice L of V, define the theta series associated with L by the following formula:

$$\vartheta_L(\mathfrak{z}) = \sum_{x \in L^m} \mathbf{e}(\mathrm{tr}({}^t xSx\mathfrak{z})) \quad (\mathfrak{z} \in \mathscr{H}_m).$$

It is well-known that if level(L) divides N then $\vartheta_L \in \mathscr{M}_m(n, N)$. Let $\Theta_m(V, N)$ be the subspace of $\mathscr{M}_m(n, N)$ generated by

$$\{\vartheta_L \mid L \text{ is an integral lattice of } V \text{ with level}(L) \text{ dividing } N\}.$$

For N and its positive divisor N', define the global trace operator $T_{N,N'}^{(n)}$ by the following formula:

$$T_{N,N'}^{(n)}(f) = \sum_{\gamma \in \Gamma_0^{(m)}(N) \backslash \Gamma_0^{(m)}(N')} f \|_n \gamma \quad (f \in \mathscr{M}_m(n, N)).$$

Then $T_{N,N'}^{(n)}$ is a well-defined $\mathbf{C}$-linear mapping of $\mathscr{M}_m(n, N)$ onto $\mathscr{M}_m(n, N')$.

1.2. Statement of main results. Now we state our main results.

Theorem V. *Let the notation be as in* §1.1. *Write the prime factorizations of* N *and its positive divisor* N' *as*

$$N = \prod_{p \in \mathbf{h}} p^{N_p}, \quad N' = \prod_{p \in \mathbf{h}} p^{N_p'}.$$

Assume

$$(1.1) \qquad \begin{cases} N_2 = N_2' \geq 2 & n \text{ odd}, \\ N_2 = N_2' \geq 2 \text{ or } 1 \geq N_2 \geq N_2' \geq 0 & n \text{ even}. \end{cases}$$

(i) *Then we have, for* $m \geq n$,

$$\Theta_m(V, N') = \Theta_m(V, N) \cap \mathscr{M}_m(n, N').$$

(ii) *Furthermore assume* $P(V; N, N')$ *is empty. Then we have, for* $m \geq 1$,

$$\Theta_m(V, N') = \Theta_m(V, N) \cap \mathscr{M}_m(n, N').$$

Theorem L. *Let the notation be as in Theorem V. Assume* (1.1) *on* N *and* N'. *Suppose* L *is an integral lattice of* V *with* level$(L) = N$. *Set*

$$s_0(L; N, N') = \begin{cases} 1 & \text{if } P(V; N, N') \text{ is empty}, \\ \max\left\{ s_p(L) - \left[\frac{N_p - 1}{2}\right] \;\middle|\; p \in P(V; N, N') \right\} & \text{otherwise}. \end{cases}$$

(i) *We have, for* $m \geq \min\{n, s_0(L; N, N')\}$,

$$T_{N,N'}^{(n)}(\vartheta_L) \in \Theta_m(V, N').$$

(ii) *Furthermore suppose* $s_0(L; N, N') > 0$. *We obtain, for* $m \geq \min\{n, s_0(L; N, N')\}$,

$$T_{N,N'}^{(n)}(\vartheta_L) = 0.$$

1.3. Comparison. Now we state the results of [Böc93] in our notation.

Theorem 1.1. *Let the notation be as in* § 1.1. *Write the prime factorizations of* N *and its positive divisor* N' *as*

$$N = \prod_{p \in \mathbf{h}} p^{N_p}, \quad N' = \prod_{p \in \mathbf{h}} p^{N'_p}.$$

(i) *Then we have, for* $m \geq n$,

$$\Theta_m(V, N') = \Theta_m(V, N) \cap \mathcal{M}_m(n, N').$$

(ii) *Furthermore assume* $P(V; N, N')$ *is empty. Then we have, for* $m \geq 1$,

$$\Theta_m(V, N') = \Theta_m(V, N) \cap \mathcal{M}_m(n, N').$$

Theorem 1.2. *Let the notation be as in Theorem* 1.1. *Suppose* L *is an integral lattice of* V *with* $\mathrm{level}(L) = N$. *Set*

$$s_1(L; N, N') = \begin{cases} 1 & \text{if } P(V; N, N') \text{ is empty,} \\ \max\{s_p(L) \mid p \in P(V; N, N')\} & \text{otherwise.} \end{cases}$$

(i) *We have, for* $m \geq \min\{n, s_1(L; N, N')\}$,

$$T^{(n)}_{N,N'}(\vartheta_L) \in \Theta_m(V, N').$$

(ii) *Furthermore suppose* $s_1(L; N, N') > 0$. *We obtain, for* $m \geq \min\{n, s_1(L; N, N')\}$,

$$T^{(n)}_{N,N'}(\vartheta_L) = 0.$$

2. Localization of global trace

2.1. Preliminaries. Let F be a local field of $\mathrm{char}(F) \neq 2$. Let $|\cdot|_F$ is the standard absolute value of F. If F is non-archimedian, it is normalized by $|\varpi|_F = q^{-1}$. Let ψ_F be a non-trivial character of F.

Fix a nonsingular symmetric matrix $S \in \mathrm{Sym}_{2n}(F) \cap \mathrm{GL}_{2n}(F)$. We obtain a regular quadratic space $V_F = (F^{2n}, Q)$ of rank $2n$ over F by $Q(x) = {}^t xSx$ $(x \in F^{2n})$. The extention $F(\sqrt{(-1)^n \det V})/F$ determines the unique character of $F^\times$ by local class field theory; this character is denoted by ω_F.

Fix a positive integer m and let X_F be the direct sum of m-copies of V_F as vector spaces. We identify this vector space X_F with $\mathrm{Mat}_{2n,m}(F)$. For any $x \in X_F$, we write $x = (x_i)$ by column vectors x_i $(1 \leq i \leq m)$.

A self-duality of X is given by $(x, y) \mapsto \psi_F(\mathrm{tr}(2{}^t xSy))$ $(x, y \in X_F)$. A map $\varphi \mapsto \varphi^\frown (\varphi \in \mathcal{S}(X_F))$ denotes the Fourier transformation on the Schwartz-Bruhat space $\mathcal{S}(X_F)$ of X_F with respect to the self dual mesure on X_F.

We have the local Weil representation π_F of $\mathrm{Sp}_m(F)$ realized on $\mathcal{S}(X_F)$; π_F is characterized by the following three conditions (cf. [Yos79]):

$$\left(\pi_F\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right)\varphi\right)(x) = \psi_F(\mathrm{tr}(b\,{}^t xSx))\,\varphi(x), \tag{2.1}$$

$$\left(\pi_F\left(\begin{bmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{bmatrix}\right)\varphi\right)(x) = \omega_F(\det a)\,|\det a|_F^n\,\varphi(xa), \tag{2.2}$$

$$\left(\pi_F\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)\varphi\right)(x) = \gamma(V_F)^m \varphi^\frown(x), \tag{2.3}$$

($\varphi \in \mathscr{S}(X_F), x \in X_F, a \in \mathrm{GL}_m(F)$ and $b \in \mathrm{Sym}_m(F)$). Here $\gamma(V_F)$ is a complex number of absolute value 1 depending only on the choice of the quadratic space V_F and of the character ψ_F (cf. [JL70, Wei64, Yos79]). The mapping

$$\mathrm{Sp}_m(F) \times \mathscr{S}(X_F) \ni (g, \varphi) \mapsto \pi_F(g)\varphi \in \mathscr{S}(X_F)$$

is continuous. If F is non-archimedian, the stabilizer of $\varphi \in \mathscr{S}(X_F)$ in $\mathrm{Sp}_m(F)$ under π_F contains an open compact subgroup of $\mathrm{Sp}_m(F)$.

We give some examples of compact subgroups of $\mathrm{Sp}_m(F)$ and of semi-invarivant vectors under the action of these subgroups.

First, suppose F is non-archimedian. Assume $\psi_F(R) = \{1\}$ and $\psi_F(q^{-1}R) \neq \{1\}$. For any non-zero element λ of R, define an open compact subgroup $D_F(\lambda)$ of $\mathrm{Sp}_m(R)$ by

$$D_F(\lambda) = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Sp}_m(R) \mid c \equiv 0 \bmod \lambda R \right\}.$$

Let L be an integral lattice of V_F with $\mathrm{lev}_{V_F}(L) = l$. Then we can easily see that

$$\ker \omega_F \supset (1 + q^l R) \cap R^\times \tag{2.4}$$

(see (2.1), (2.2), (2.3)) and that

$$\pi_F(g)\mathrm{I}_{L^m} = \begin{cases} \mathrm{I}_{L^m} & (l = 0) \\ \omega_F(\det a)\mathrm{I}_{L^m} & (l \geq 1) \end{cases} \tag{2.5}$$

for any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in D_F(q^l)$ (see (2.1), (2.2), (2.3), [Yos84]).

Next suppose $F = \mathbf{R}$. Set $\mathbf{i} = \sqrt{-1} \cdot 1_m \in \mathscr{H}_m$. Let U_F be the stabilizer of $\mathbf{i}$ under the action of $\mathrm{Sp}_m(F)$ on $\mathscr{H}_m$. We can immediately see

$$U_F = \left\{ u \in \mathrm{Sp}_m(F) \mid u = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \right\}$$

and that U_F is isomophic to $U(m)$. For every $\mathfrak{z} \in \mathscr{H}_m$, we can define an element $\varphi_{\mathfrak{z}}$ of $\mathscr{S}(X_F)$ by

$$\varphi_{\mathfrak{z}}(x) = \mathbf{e}(\mathrm{tr}({}^t xSx\mathfrak{z})) \quad (x \in X_F).$$

Then we can show (cf. [Yos84])

$$\pi_F(u)\varphi_{\mathbf{i}} = \det(A - \sqrt{-1}\,B)^n \varphi_{\mathbf{i}}$$

for any $u = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in U_F$. Therefore we obtain

$$\pi_F(g)\varphi_{\mathbf{i}} = j(g,\mathbf{i})^{-n}\varphi_{g\mathbf{i}} \tag{2.6}$$

for any $g \in \mathrm{Sp}_m(F)$.

2.2. Local trace operator. From now on in §2, let V and X be as in §1. Let v be any place of $\mathbf{Q}$. V_v (resp. X_v) denotes the localization of V (resp. X) at v. Define a nontrivial character $\psi_v : \mathbf{Q}_v \to \mathbf{T}$ by

$$\psi_v(x) = \begin{cases} \mathbf{e}(x) & \text{if } v = \infty, \\ \mathbf{e}(-\mathrm{Fr}(x)) & \text{if } v \in \mathbf{h}, \end{cases}$$

where $\mathrm{Fr}(x)$ ($x \in \mathbf{Q}_p, p \in \mathbf{h}$) is the fractional part of the p-adic expansion of x.

As we see in §2.1, we have the local Weil representation of G_v on $\mathcal{S}(X_v)$ with respect to $F = \mathbf{Q}_v$, $\psi_F = \psi_v$, $V_F = V_v$, and $X_F = X_v$. For simplicity, we put $\pi_v = \pi_F$, $\omega_v = \omega_F$, $D_v(\lambda) = D_F(\lambda)$ and $\gamma_v(V) = \gamma(V_F)$.

Under our assumptions on V, $\gamma_v(V)$ and ω_v can be easily determined [JL70, Wei64]:

$$\gamma_v(V) = \begin{cases} (\sqrt{-1})^n & \text{if } v = \infty, \\ \varepsilon_v(V) & \text{if } v \in \mathbf{h}\backslash\{2\}, \\ \varepsilon_2(V)(-\sqrt{-1})^n & \text{if } v = 2, \end{cases} \tag{2.7}$$

$$\omega_v(x) = (x, (-1)^n)_{\mathbf{Q}_v} \quad (x \in \mathbf{Q}_v^\times). \tag{2.8}$$

Notice that, for any $p \in \mathbf{h}$,

$$\ker \omega_p \supset \begin{cases} 1 + 4\mathbf{Z}_2 & p = 2, \\ \mathbf{Z}_p^\times & p \neq 2. \end{cases} \tag{2.9}$$

We define a character χ_n of a compact group $D_2(\lambda)$ ($\lambda \in \mathbf{Z}_2$ if n is even or $\lambda \in 4\mathbf{Z}_2\backslash\{0\}$ if n is odd) by

$$\chi_n(g) = \begin{cases} 1 & \text{if } n \text{ even} \\ (\det a, -1)_{\mathbf{Q}_2} & \text{if } n \text{ odd} \end{cases} \tag{2.10}$$

$\left(g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in D_2(\lambda)\right)$.

Take a non-negative integer l and a finite place $p \in \mathbf{h}$. From the assumptions on V, we have $\det V_p \equiv 1 \bmod (\mathbf{Q}_p^\times)^2$. Thus we can write

$$|4^n \det M|_p = p^{-2s_p(M)}$$

($s_p(M) \in \mathbf{Z}, p \in \mathbf{h}$) for any lattice M of V_p. Notice that, for any two lattices M, M' of V_p,

$$M \subsetneq M' \Rightarrow s_p(M) > s_p(M'). \tag{2.11}$$

Let $\mathscr{L}^{(m)}(V_p, l)$ be the subspace of $\mathscr{S}(X_p)$ spanned by all functions of the form I_{M^m} such that M is an integral lattice of V_p with $\mathrm{lev}_p(M) \leq l$. Notice that $\mathscr{L}^{(m)}(V_p, l) = 0$ if $p = 2$, n is odd and $0 \leq l \leq 1$ (see (2.4)) or if $p \in \mathbf{h}$, $\gamma_p(V_p) = -1$ and $l = 0$ (see (2.3), (2.5)).

From now on, we assume that

$$(2.12) \qquad \begin{cases} l \geq 2 & \text{if } p = 2 \text{ and } n \text{ is odd}, \\ l \geq 0 & \text{otherwise.} \end{cases}$$

Under this assumption, we define a $\mathbf{C}$-linear map $\tau_{p,l}^{(m)}$ on $\mathscr{S}(X_p)$ by the following formula: (see (2.9), (2.5), (2.10), (2.12))

$$(2.13) \qquad (\tau_{p,l}^{(m)}(\varphi))(x) = \begin{cases} \int_{D_p(p^l)} (\pi_p(u)\varphi)(x) d_{p,l}(u) & \text{if } p \neq 2, \\ \int_{D_p(p^l)} \chi_n(u)(\pi_p(u)\varphi)(x) d_{p,l}(u) & \text{if } p = 2, \end{cases}$$

($\varphi \in \mathscr{S}(X_p), x \in X_p$) where $d_{p,l}$ is the Haar measure of $D_p(p^l)$ normalized such that $\int_{D_p(p^l)} d_{p,l}(u) = 1$. The integrals on the right-hand side are essentially finite sum, and the map $\tau_{p,l}^{(m)}$ is well-defined. We call this map *local trace operator*.

By the definition of the local trace operator, we get $\tau_{p,l}^{(m)}(\varphi) = \varphi$ for $\varphi \in \mathscr{L}^{(m)}(V_p, l)$. The following theorem is a local analogue of Theorem L. We shall prove later that this theorem implies Theorem L.

Theorem $\mathrm{L_p}$. *Let the notation be as above. Let l, l' be integers with $l \geq l' \geq 0$ such that both of them satisfy* (2.12). *Take an integral lattice L_p* of V_p *with* $\mathrm{lev}_p(L_p) = l \geq 1$.

(1) *Suppose $p \neq 2$ and $l > l' > 0$. If $m \geq 1$, we have*

$$\tau_{p,l'}^{(m)}(\mathrm{I}_{L_p^{(m)}}) \in \mathscr{L}^{(m)}(V_p, l').$$

(2) *Suppose that $p \neq 2$ and $l > l' = 0$, or that $p = 2$, $l = 1 > l' = 0$ and n is even.*

(a) *If $\gamma_p(V_p) = 1$ and $m \geq 1$, we have*

$$\tau_{p,0}^{(m)}(\mathrm{I}_{L_p^{(m)}}) \in \mathscr{L}^{(m)}(V_p, 0).$$

(b) *If $\gamma_p(V_p) = -1$ and $m \geq \min\left\{n, s_p(L_p) - \left[\frac{l-1}{2}\right]\right\}$, we have*

$$\tau_{p,0}^{(m)}(\mathrm{I}_{L_p^{(m)}}) = 0.$$

(c) *If $\gamma_p(V_p) = -1$ and $m < \min\left\{n, s_p(L_p) - \left[\frac{l-1}{2}\right]\right\}$, we have*

$$\tau_{p,0}^{(m)}(\mathrm{I}_{L_p^{(m)}}) \neq 0.$$

In §3, we show Theorem $\mathrm{L_p}$ follows from Proposition 3.1 and Proposition 3.2, and prove Propositions 3.1, 3.2.

2.3. Construction of theta series via the Weil representation. The global Weil representation $\pi_{\mathbf{A}}$ of $G_{\mathbf{A}}$ realized on $\mathscr{S}(X_{\mathbf{A}})$ is defined as follows. Let φ be an element of $\mathscr{S}(X_{\mathbf{A}})$ of the form $\varphi = \prod_v \varphi_v$ such that $\varphi_p = \mathrm{I}_{\mathrm{Mat}_{2n,m}(\mathbf{Z}_p)}$ for almost all $p \in \mathbf{h}$. For any $g = (g_v) \in G_{\mathbf{A}}$, put

$$\pi_{\mathbf{A}}(g)\varphi = \prod_v \pi_v(g_v)\varphi_v.$$

This action of $G_{\mathbf{A}}$ extends by continuity to the representation $\pi_{\mathbf{A}}$ of $G_{\mathbf{A}}$ on $\mathscr{S}(X_{\mathbf{A}})$. Let f be an element of $\mathscr{S}(X_{\mathbf{h}})$. Put

$$(\varphi_{\mathbf{i}} \otimes f)(x) = \varphi_{\mathbf{i}}(x_\infty) f(x_{\mathbf{h}}) \quad x = (x_\infty, x_{\mathbf{h}}) \in X_{\mathbf{A}},$$

then we have $\varphi_{\mathbf{i}} \otimes f \in \mathscr{S}(X_{\mathbf{A}})$. For each $g \in G_{\mathbf{A}}$, set

$$\Psi(f; g) = \sum_{x \in X_{\mathbf{Q}}} \pi_{\mathbf{A}}(g)(\varphi_{\mathbf{i}} \otimes f)(x).$$

We can show (cf. [Wei64, Yos80, Yos84])

- the series in the right-hand side converges absolutely and uniformly on every compact subset of $G_{\mathbf{A}}$; hence $\Psi(f; \cdot)$ is a continuous function on $G_{\mathbf{A}}$;
- this function $\Psi(f; \cdot)$ is left $G_{\mathbf{Q}}$ invariant and right invariant under the action of some open compact subgroup of $G_{\mathbf{h}}$;
- the restriction to G_∞ determines $\Psi(f; \cdot)$ by the strong approximation theorem for $G_{\mathbf{A}}$.

For every $\mathfrak{z} \in \mathscr{H}_m$, take an element g_∞ of G_∞ such that $\mathfrak{z} = g_\infty \mathbf{i}$ and set

$$\vartheta(f; \mathfrak{z}) = j(g_\infty, \mathbf{i})^n \Psi(f; (g_\infty, 1_{\mathbf{h}})). \tag{2.14}$$

Then we have

$$\begin{aligned}\vartheta(f; \mathfrak{z}) &= \sum_{x \in X_{\mathbf{Q}}} (\varphi_{\mathfrak{z}} \otimes f)(x) \\ &= \sum_{x = (x_\infty, x_{\mathbf{h}}) \in X_{\mathbf{Q}}} \mathbf{e}(\mathrm{tr}({}^t x_\infty S x_\infty \mathfrak{z})) f(x_{\mathbf{h}}).\end{aligned}$$

Therefore we get a well-defined function $\mathfrak{z} \mapsto \vartheta(f; \mathfrak{z})$ on $\mathscr{H}_m$. Furthermore we can immediately see that

$$\vartheta(f; \gamma_\infty \mathfrak{z}) = j(\gamma_\infty, \mathbf{i})^n \vartheta(\pi_{\mathbf{h}}(\gamma_{\mathbf{h}}^{-1}) f; \mathfrak{z}) \tag{2.15}$$

for any $\gamma = (\gamma_\infty, \gamma_{\mathbf{h}}) \in G_{\mathbf{Q}}$ and any $\mathfrak{z} \in \mathscr{H}_m$. Take an integral lattice L of V and put $f_{L^m} = \prod_{p \in \mathbf{h}} \mathrm{I}_{L_p^m}$. We can easily see $f_{L^m} \in \mathscr{S}(X_{\mathbf{h}})$ and

$$\vartheta(f_{L^m}; \cdot) = \vartheta_L.$$

2.4. Relation of global and local traces. Let N be a positive integer and N' its positive divisor. We define an open compact subgroup $D_{\mathbf{h}}(N)$ of $G_{\mathbf{h}}$ by $D_{\mathbf{h}}(N) = \prod_{p \in \mathbf{h}} D_p(N)$ and set

$$D_{\mathbf{Q}}(N) = G_{\mathbf{Q}} \cap G_\infty D_{\mathbf{h}}(N).$$

By a morphism $\gamma \mapsto (\gamma,\gamma,\gamma,\dots)$, we identify $\Gamma_0^{(m)}(N)$ with $D_{\mathbf{Q}}(N)$. Since the strong approximation theorem holds for G, we can easily see that $G_\infty D_{\mathbf{Q}}(N)$ is dense in $G_\infty D_{\mathbf{h}}(N)$. Thus the image of $D_{\mathbf{Q}}(N)$ under the canonical projection $G_{\mathbf{A}} \ni g = (g_\infty, g_{\mathbf{h}}) \mapsto g_{\mathbf{h}} \in G_{\mathbf{h}}$ is dense in $D_{\mathbf{h}}(N)$. Therefore we can identify a complete set of representatives of $\Gamma_0^{(m)}(N)\backslash\Gamma_0^{(m)}(N')$ with that of $D_{\mathbf{h}}(N)\backslash D_{\mathbf{h}}(N')$.

From now on, assume (see (2.12))

$$\begin{cases} N' \mid N & \text{if } n \text{ even,} \\ 4|N'|N & \text{if } n \text{ odd.} \end{cases} \tag{2.16}$$

We regard χ_n as a character of $D_{\mathbf{h}}(N)$ via the canonical projection $D_{\mathbf{h}}(N) \to D_2(N)$. Let L be an integral lattice of V with $\mathrm{level}(L) \mid N$. We compute the global trace $T_{N,N'}^{(n)}\vartheta_L$ as follows:

$$(T_{N,N'}^{(n)}\vartheta_L)(g_\infty \mathbf{i}) = \sum_{\xi \in \Gamma_0^{(m)}(N)\backslash\Gamma_0^{(m)}(N')} \chi_n(\xi) j(\xi, g_\infty \mathbf{i})^{-n} \vartheta(f_{L^m}; \xi g_\infty \mathbf{i})$$

$$= j(g_\infty, \mathbf{i})^n \sum_{\gamma} \chi_n(\gamma_{\mathbf{h}}^{-1}) \Psi(f_{L^m}; (g_\infty, \gamma_{\mathbf{h}}^{-1})) \quad \text{(see (2.15))} \tag{2.17}$$

$$= \vartheta\left(\sum_u \chi_n(u)\pi_{\mathbf{h}}(u) f_{L^m}; (g_\infty, 1_{\mathbf{h}})\right), \quad \text{(see (2.14))} \tag{2.18}$$

where $\gamma = (\gamma_\infty, \gamma_{\mathbf{h}})$ and u extend over $D_{\mathbf{Q}}(N)\backslash D_{\mathbf{Q}}(N')$ in (2.17) and $D_{\mathbf{h}}(N')/D_{\mathbf{h}}(N)$ in (2.18), respectively. Let the prime factorization of N' be $N' = \prod_{p \in \mathbf{h}} p^{l_p(N')}$. Notice that, since $D_p(N) = D_p(N')$ for almost all $p \in \mathbf{h}$, $\prod_{p \in \mathbf{h}} \tau_{p, l_p(N')}^{(m)}(\mathrm{I}_{L_p^m})$ is an element of $\mathscr{S}(X_{\mathbf{h}})$. The last equality (2.18) shows that, up to the multiplication by an nonzero constant, the two functions

$$T_{N,N'}^{(n)}\vartheta_L \quad \text{and} \quad \vartheta\left(\prod_{p \in \mathbf{h}} \tau_{p, l_p(N')}^{(m)}(\mathrm{I}_{L_p^m}); \cdot\right)$$

are equal on $\mathscr{H}_m$. Therefore Theorem $\mathrm{L_p}$ implies Theorem L.

3. Calculation of local traces

3.1. Preliminaries. Let F be a non-archimedean local field of $\mathrm{char}(F) \neq 2$. Set $\tilde{F}$ be a finite field $R/\varpi R$ of q elements. Let ψ be an character of F. Assume $\psi(R) = \{1\}$ and $\psi(q^{-1}R) \neq \{1\}$. We keep the notation of the local Weil representation as in §2.1, but drop the suffix F for simplicity. Moreover put $\gamma = \gamma(V)$ and $B = B_Q$. Let L be an integral lattice of V with $\mathrm{lev}_V(L) = l$. We now assume $l \geq 1$ and

$$\ker \omega \supset (1 + \varpi^{l-1}R) \cap R^\times. \tag{3.1}$$

Under this assumption, it can easily be verified that

$$\chi : D(\varpi^{l-1}) \ni \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{cases} \omega(\det a) & (l \geq 2) \\ 1 & (l = 1) \end{cases} \tag{3.2}$$

is a character of $D(\varpi^{l-1})$.

For the above lattice L, we define an element $T_L^{(m)}$ of $\mathscr{S}(X)$ by

$$T_L^{(m)}(x) = \int_{D(\varpi^{l-1})} (\chi(g)\pi(g)\mathrm{I}_{L^m})(x)\, dg \quad (x \in X). \tag{3.3}$$

Here the Haar measure dg of the compact group $D(\varpi^{l-1})$ is normalized by $\int_{D(\varpi^l)} dg = 1$.

3.2. Case $l \geq 2$. In this subsection, we keep the notation and assumptions in §3.1, and furthermore we assume that

$$l = \mathrm{lev}_V(L) \geq 2. \tag{3.4}$$

We shall prove the following proposition from which Theorem $\mathrm{L_p}$ (1) follows immediately.

Proposition 3.1. *Let the notation and assumptions be as above.*

(i) *There exists a lattice K of V such that $K^\vee \subset L^\vee$ and $Q(K^\vee) \subset \varpi^{-l+1}R$; we have*

$$T_L^{(m)} = \sum_M c_M \mathrm{I}_{M^m} \quad (c_M \in \mathbf{C}),$$

where the summation on the right-hand side is taken over all the lattices M of V with the conditions $L \subset M \subset K$ and $Q(M^\vee) \subset \varpi^{-l+1}R$.

(ii) *Moreover we assume $2 \in R^\times$. Then K is an integral lattice such that*

$$l - 2 \leq \mathrm{lev}_V(K) \leq l - 1.$$

Proof. First, we prove (i). Since $l \geq 2$, the subset

$$\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \middle| \, b \in \mathrm{Sym}_m(\varpi^{l-1}R)/\mathrm{Sym}_m(\varpi^l R) \right\}$$

is a complete set of representatives of $D(\varpi^{l-1})/D(\varpi^l)$. Therefore, up to a non-zero constant multiple, the function $\pi\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) T_L^{(m)}$ on X is equal to a function

$$\begin{aligned} X \ni x \mapsto & \int_{\mathrm{Sym}_m(\varpi^{l-1}R)} (\pi(b)\mathrm{I}_{(L^\vee)^m})(x)\, db \\ &= \int_{\mathrm{Sym}_m(\varpi^{l-1}R)} (\psi(\mathrm{tr}(b\,{}^t xSx))\mathrm{I}_{(L^\vee)^m})(x)\, db \end{aligned}$$

(db is a Haar measure of $\mathrm{Sym}_m(\varpi^{l-1}R)$). Moreover, up to a non-zero constant multiple, the latter function is equal to $\mathrm{I}_{Y(m)}$, where $Y(m)$ is the open compact subset of $(L^\vee)^m$ defined by

$$Y(m) = \{x \in (L^\vee)^m \mid \psi(\mathrm{tr}(b^t xSx)) = 1 \quad \text{for all } b \in \mathrm{Sym}_m(\varpi^{l-1}R)\}.$$

We can easily show that

$$Y(m) = \{x = (x_k) \in (L^\vee)^m \mid Q(x_k) \in \varpi^{-l+1}R \ (1 \le \forall k \le m) \quad \text{and}$$
$$B(x_i, x_j) \in \varpi^{-l+1}R(1 \le \forall i < \forall j \le m)\}.$$

Define a subset $K^\vee$ of $Y(1)$ by

$$K^\vee = \{u \in Y\ (1) \mid B(u, Y(1)) \subset \varpi^{-l+1}R\}.$$

It is obvious that $K^\vee$ is an open compact R-module of V, i.e. a lattice of V. Set $K = (K^\vee)^\vee$. For any $x = (x_i) \in Y(m)$, let $(K^\vee)_x$ be a lattice in V generated by $K^\vee \cup \{x_i \mid 1 \le i \le m\}$. From the definition of $(K^\vee)_x$, we can easily see that

$$Y(m) = \bigcup_{x \in Y\ (m)} ((K^\vee)_x)^m.$$

The assertion of (i) results from this expression.

Next we prove (ii). Set $L^\flat = L + \varpi^{l-1}L^\vee$. From the definition of $L^\flat$, it is an integral lattice of V. Furthermore, we can easily see that

$$B((L^\flat)^\vee, L^\vee) \subset \varpi^{-l+1}R,$$
$$2Q((L^\flat)^\vee) \subset \varpi^{-l+1}R,$$

since $(L^\flat)^\vee = L^\vee \cap \varpi^{-l+1}L$. Therefore the assumption $2 \in R^\times$ results in $(L^\flat)^\vee \subset K^\vee$. Moreover, under the assumption $2 \in R^\times$, L and $L^\vee$ have the following orthogonal decompositions with some R-basis $\{e_i \mid 1 \le i \le m\}$ of L:

$$L = \perp_{i=1}^{2n} Re_i,$$
$$L^\vee = \perp_{i=1}^{2n} R\varpi^{-r_i}e_i.$$

Here $\{r_i \mid 1 \le i \le 2n\}$ is a set of non-negative integers with the conditions:

$$\begin{cases} Q(e_i) \in \varpi^{r_i}R^\times & (1 \le \forall i \le 2n), \\ l = r_1 = r_2 = \cdots = r_t > r_{t+1} \ge \cdots \ge r_{2n} \ge 0 & (1 \le \exists t \le 2n). \end{cases}$$

From these decompositions, we obtain

$$(L^\flat)^\vee = (\perp_{i=1}^{t} R\varpi^{-l+1}e_i) \perp (\perp_{i \ge t+1}^{2n} R\varpi^{-r_i}e_i).$$

The last conclusion of (ii) follows immediately from this expression.

3.3. Case $l = 1$. In this subsection, besides the notation and assumptions of §3.1 we assume that $l = \mathrm{lev}_V(L) = 1$. Under this assumption, we have $\gamma^2 = 1$, since $\ker\omega \supset R^\times$ (cf. (3.1)). Notice that, if $\gamma = -1$, the quadratic space V has no integral lattice with $\mathrm{lev}_V = 0$ (see (2.3), (2.5)). We write $|4^n \det L|_F = q^{-2s}$ with some positive integer s. We can easily see that

$$1 \le s \le n. \tag{3.5}$$

Proposition 3.2. *Let the notation and the assumptions as above.*

(i) *Suppose* $\gamma = 1$. *Then there exists an integral lattice* M *of* V *satisfying the following conditions*:

$$\mathrm{lev}_V(M) = 0, \tag{3.6}$$

$$L \subset M \subset L^\vee. \tag{3.7}$$

And for $m \ge 1$, *we can write*

$$T_L^{(m)} = \lambda(L, m) \sum_M \mathrm{I}_{M^m}$$

with some constant $\lambda(L, m) \in \mathbf{C}^\times$ *depending only on* L *and* m. *Here the summation on the right-hand side is taken over all the integral lattices satisfying the conditions* (3.6) *and* (3.7).

(ii) *Suppose* $\gamma = -1$. *Then,*

$$T_L^{(m)} = \begin{cases} 0 & \textit{for } m \ge s \\ \textit{a non-zero function} & \textit{for } m < s. \end{cases}$$

Theorem $\mathrm{L_p}$ (2) follows immediately from Proposition 3.2 and Theorem $\mathrm{L_p}$ (1) (see (2.11), (3.5)).

By the following discussion, we reduce the proof of Proposition 3.2 to that of Lemma 3.3 stated below.

We have $Q(L^\vee) \subset \varpi^{-1}R$, $B(L^\vee, L^\vee) \subset \varpi^{-1}R$, since $\mathrm{lev}_V(L) = 1$. Therefore, setting

$$\tilde{Q}(x \bmod L) = \varpi Q(x) \bmod \varpi R \in R/\varpi R = \tilde{F}$$

for any $x \in L^\vee$, we have a regular quadratic space $\tilde{V} = (L^\vee/L, \tilde{Q})$ over $\tilde{F}$. Let $\tilde{B}$ be the nondegenerated bilinear form associated with the quadratic form $\tilde{Q}$; $\tilde{B}(\xi, \eta) = \tilde{Q}(\xi + \eta) - \tilde{Q}(\xi) - \tilde{Q}(\eta)$ $(\xi, \eta \in \tilde{V})$. Put

$$M(V) = \{M \mid M \text{ is an integral lattice of } V \text{ satisfying (3.6) and (3.7)}\}$$

and

$$\mathscr{W}(\tilde{V}) = \{W \mid W \text{ is a maximal totally isotropic subspace of } \tilde{V} \text{ and } \dim W = \operatorname{rank} \tilde{V}/2\}.$$

It is easy to see that

$$\operatorname{rank} \tilde{V} = 2s,$$

and that

$$M(V) \ni M \mapsto M/L \in \mathscr{W}(\tilde{V}) \text{ is a bijection as sets.} \tag{3.8}$$

We can also see that

(3.9) $\gamma = 1$ if and only if $\tilde{V}$ is a hyperbolic space.

The "if" part of (3.9) follows immediately from (2.3), (2.5) and (3.8). As to the proof of the "only if" part of (3.9), see [Yos79, pp. 406–410]. After the page 408, this paper treats the case $2 \in R^{\times}$. But using the discussion of [Yos85, pp. 222–223], the similar method is applicable to the case $2 \notin R^{\times}$.

Let $\tilde{X}$ be the direct sum of m-copies of $\tilde{V}$ as vector spaces. By fixing a basis, we identify $\tilde{V}$ (resp. $\tilde{X}$) with $\tilde{F}^{2s}$ (resp. $\mathrm{Mat}_{2s,m}(\tilde{F})$) as vector spaces. For any $\xi \in \tilde{X}$, we write $\xi = (\xi_i)$ by column vectors ξ_i $(1 \leq i \leq m)$.

Define a subspace $\mathscr{S}((L^{\vee})^m, L^m)$ of $\mathscr{S}(X)$ by

$$\mathscr{S}((L^{\vee})^m, L^m) = \{\varphi \in \mathscr{S}(X) \mid \operatorname{supp} \varphi \subset (L^{\vee})^m \quad \text{and} \quad \varphi(x+y) = \varphi(x) \ (\forall x \in (L^{\vee})^m, \forall y \in L^m)\}.$$

For each $\varphi \in \mathscr{S}((L^{\vee})^m, L^m)$, define the function $\tilde{\varphi}$ on $\tilde{X}$ by

$$\tilde{\varphi}(x \bmod L^m) = \varphi(x) \quad (x \in (L^{\vee})^m),$$

then $\tilde{\varphi}$ is an element of the Schwartz-Bruhat space $\mathscr{S}(\tilde{X})$ on $\tilde{X}$. This correspondence $\varphi \mapsto \tilde{\varphi}$ is obviously a vector-space isomorphism of $\mathscr{S}((L^{\vee})^m, L^m)$ onto $\mathscr{S}(\tilde{X})$.

Define an open compact subgroup D' of $\mathrm{Sp}_m(R)$ by

$$D' = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Sp}_m(R) \mid a - 1 \equiv d - 1 \equiv b \equiv c \equiv 0 \bmod \varpi R \right\}.$$

It is easy to check that, for any $\varphi \in \mathscr{S}((L^{\vee})^m, L^m)$,

$$\begin{cases} \pi(g)\varphi \in \mathscr{S}((L^{\vee})^m, L^m) & (\forall g \in D(\varpi^0)), \\ \pi(g)\varphi = \varphi & (\forall g \in D'). \end{cases}$$

Therefore we can realize the Weil representation $\tilde{\pi}$ of $\mathrm{Sp}_m(\tilde{F}) = D(\varpi^0)/D'$ on $\mathscr{S}(\tilde{X})$ by setting

$$\tilde{\pi}(g \bmod D')\tilde{\varphi} = \widetilde{\pi(g)\varphi}$$

where $g \in D(\varpi^0)$ and $\varphi \in \mathscr{S}((L^{\vee})^m, L^m)$. It is now obvious that Proposition 3.2 results from the following lemma.

Lemma 3.3. *Let the notation and the assumptions be as above.*

(i) *Suppose* $\gamma = 1$. *Set* $\Phi = \sum_{W \in \mathscr{W}(\tilde{V})} \mathrm{I}_{W^m}$. *If* $m \geq 1$, *we obtain*

$$\widetilde{T_L^{(m)}} = \lambda(\tilde{V}, m)\Phi$$

with a constant $\lambda(\tilde{V}, m) \in \mathbf{C}^{\times}$ *depending only on the quadratic space* $\tilde{V}$ *and* m.

(ii) *Suppose* $\gamma = -1$. *Then,*

$$\widetilde{T_L^{(m)}} = \begin{cases} 0 & \text{for } m \geq s \\ \text{a non-zero function} & \text{for } m < s. \end{cases}$$

Proof. [Step 0] We introduce some notation for the statement of the proof. Let $C(m,m)$ be the set of all elements $\xi = (\xi_i)$ of $\tilde{X}$ with the conditions

$$\begin{cases} \tilde{Q}(\xi_k) = 0 & (1 \le \forall k \le m), \\ \tilde{B}(\xi_i, \xi_j) = 0 & (\forall (i,j) \quad \text{with } 1 \le i < j \le m). \end{cases}$$

For $r = 0, 1, \ldots, m$, define subsets $\tilde{X}(r)$ and $C(m,r)$ of $\tilde{X}$ by

$$\tilde{X}(r) = \{\xi = (\xi_i) \in \tilde{X} \mid \xi_i = 0 \ (1 \le \forall i \le m - r)\},$$
$$C(m,r) = \tilde{X}(r) \cap C(m,m).$$

Notice that, for any $\xi = (\xi_i) \in C(m,r) \subset \mathrm{Mat}_{2s,m}(\tilde{F})$, we have

$$\operatorname{rank} \xi \le \min(s,r)$$

since a subset $\{\xi_i \mid 1 \le i \le m\}$ of $\tilde{V}$ spans a totally isotropic subspace of $\tilde{V}$.

For each non-negative integer N and each complex number z, define a number $(z;q)_N$ by

$$(z;q)_N = \begin{cases} 1 & (\text{if } N = 0), \\ \prod_{i=0}^{N-1}(1 - q^i z) & (\text{if } N \ge 1), \end{cases}$$

and set

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}_q = \frac{(q;q)_{N_1}}{(q;q)_{N_2}(q;q)_{N_1 - N_2}}$$

for integers N_1, N_2 with $N_1 \ge N_2 \ge 0$. We can easily see the following formula

$$(3.10) \qquad (z;q)_N = \sum_{j=0}^{N} (-1)^j z^j q^{j(j-1)/2} \begin{bmatrix} N \\ j \end{bmatrix}_q \quad (N \in \mathbf{Z}, N \ge 0, z \in \mathbf{C})$$

by inducton on N.

For each non-negative integer v, let $\mathscr{H}(v)$ (resp. $\mathscr{W}(v)$) be the hyperbolic space of rank $2v$ over $\tilde{F}$ (resp. the set of all the maximal totally isotropic subspaces of $\mathscr{H}(v)$). It is easily seen that

$$|\mathscr{W}(v)| = (-1;q)_v.$$

[Step1] First, we suppose $\gamma = 1$, identify $\tilde{V}$ with $\mathscr{H}(s)$ as quadratic spaces, and calculate the value of Φ at each point on $\tilde{X}$. From the definition of Φ, we have

$$\operatorname{supp} \Phi = \bigcup_{W \in \mathscr{W}(s)} W^m.$$

By Witt's Theorem, we can easily show that the right-hand side is equal to $C(m,m)$. Now we fix $\xi = (\xi_i) \in C(m,m)$ with $t = \operatorname{rank}(\xi)$. Let U be the totally isotropic subspace spanned by $\{\xi_i \mid 1 \le i \le m\}$ in $\tilde{V}$. Notice that the dimension of U is t $(t \le \min(s,m))$. It is obvious that, for any $W \in \mathscr{W}(s)$,

$$\mathrm{I}_{W^m}(\xi) \ne 0 \quad \Leftrightarrow \quad U \subset W.$$

Thus we have

$$\Phi(\xi) = |\{W \in \mathscr{W}(s) \mid U \subset W\}|.$$

With a basis $\{e_i \mid 1 \le i \le t\}$ of U, $\tilde{V} = \mathscr{H}(s)$ has the following orthogonal decomposition:

$$\tilde{V} = (\perp_{1 \le i \le t} U_i) \perp U_0.$$

Here, for $1 \le i \le t$, U_i is a binary regular quadratic subspace of $\tilde{V}$ with a basis $\{e_i, f_i\}$ satisfying

$$\tilde{Q}(e_i) = \tilde{Q}(f_i) = 0 \quad \text{and} \quad \tilde{B}(e_i, f_i) = 1,$$

and U_0 is isomorphic to $\mathscr{H}(s-t)$ as quadratic space. For each $W \in \mathscr{W}(s)$ with $U \subset W$, the above decomposition of $\tilde{V}$ induces that of W:

$$W = U \perp (U_0 \cap W),$$

and $U_0 \cap W$ is a maximal totally isotropic subspace of $U_0 \simeq \mathscr{H}(s-t)$. Therefore we obtain

$$\Phi(\xi) = |\mathscr{W}(s-t)| = (-1; q)_{s-t}.$$

[Step2] Second, we suppose $\gamma = \pm 1$. The value of $\widetilde{T_L^{(m)}}$ at each point of $\tilde{X}$ is given by the following formulas.

$$\operatorname{supp} \widetilde{T_L^{(m)}} \subset C(m, m), \tag{3.11}$$

$$\widetilde{T_L^{(m)}}(\xi) = \sum_{t \le r \le m} (\gamma q^{-s})^r q^{r(r+1)/2} \begin{bmatrix} m-t \\ r-t \end{bmatrix}_q \quad \text{for any } \xi \in C(m,m) \text{ with } \operatorname{rank} \xi = t. \tag{3.12}$$

In this step, we shall show that Lemma 3.3 follows from the above formulas. The proof of the formulas (3.11) and (3.12) is given in the next step. Fix any $\xi \in C(m, m)$ with $\operatorname{rank} \xi = t$. Setting $j = r - t$ in the formula (3.12), we get

$$\widetilde{T_L^{(m)}}(\xi) = q^{-st} q^{t(t+1)/2} \gamma^t \times \sum_{0 \le j \le m-t} (-1)^j (-\gamma q^{-s+t+1})^j q^{j(j-1)/2} \begin{bmatrix} m-t \\ j \end{bmatrix}_q.$$

Combining this formula with (3.10), we obtain

$$\widetilde{T_L^{(m)}}(\xi) = q^{-st} q^{t(t+1)/2} \gamma^t (-\gamma q^{-s+t+1}; q)_{m-t}. \tag{3.13}$$

For any triple of non-negative integers s, m, and t with $0 \le t \le \min(s, m)$ we define a number $\lambda(\gamma, s, t, m)$ by the right-hand side of the formula (3.13). Then, from the definition of $(z; q)_N$, we can easily obtain the value of $\lambda(\gamma, s, t, m)$:

if $s \le m$ $(t \le \min(s, m) = s)$,

$$\lambda(\gamma, s, t, m) = \begin{cases} \lambda(1, s, s, m)(-1; q)_{s-t} & \text{for } \gamma = 1 \text{ and } 0 \le t \le s, \\ 0 & \text{for } \gamma = -1 \text{ and } 0 \le t < s, \end{cases}$$

if $s > m$ $(t \leq \min(s,m) = m)$,

$$\lambda(\gamma,s,t,m) = \begin{cases} \lambda(1,s,m,m)(-1;q)_{s-t}/(-1;q)_{s-m} & \text{for } \gamma = 1 \text{ and } 0 \leq t \leq m, \\ \text{a non-zero number} & \text{for } \gamma = -1 \text{ and } 0 \leq t \leq m. \end{cases}$$

Notice that, if $s \leq m$ and $\gamma = -1$, there is no element $\xi \in C(m,m)$ with $\operatorname{rank} \xi = s$ (see (3.9)). Lemma 3.3 follows immediately from this computation and the results of the previous step.

[Step 3] Now, we shall prove the formulas (3.11) and (3.12). From the definition, we get $\widetilde{\mathrm{I}_{L^m}} = \mathrm{I}_{\tilde{X}(0)}$. Put

$$B_m = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Sp}_m(\tilde{F}) \mid c = 0 \right\}.$$

Then, $D(\varpi^1)$-invariance of I_{L^m} implies B_m-invariance of $\mathrm{I}_{\tilde{X}(0)}$. Thus we have

$$\widetilde{T_L^{(m)}} = \sum_{g \in \mathrm{Sp}_m(\tilde{F})/B_m} \tilde{\pi}(g)\mathrm{I}_{\tilde{X}(0)}.$$

For $0 \leq r \leq m$, define a element w_r of $\mathrm{Sp}_m(\tilde{F})$ by

$$w_r = \left[\begin{array}{c|c|c|c} 1_{m-r} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1_r \\ \hline 0 & 0 & 1_{m-r} & 0 \\ \hline 0 & -1_r & 0 & 0 \end{array}\right].$$

From the Bruhat decomposition

$$\mathrm{Sp}_m(\tilde{F}) = \bigsqcup_{0 \leq r \leq m} B_m w_r B_m \quad \text{(disjoint union)},$$

we can write

$$\widetilde{T_L^{(m)}} = \sum_{0 \leq r \leq m} J_r,$$

$$J_r = \sum_{g \in B_m w_r B_m/B_m} \tilde{\pi}(g)\mathrm{I}_{\tilde{X}(0)}.$$

To prove the formulas (3.11) and (3.12), we shall find an explicit formula of J_r. We can take a complete set of representatives of $B_m w_r B_m/B_m$ as

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} w_r \mid a \in \mathrm{GL}_m(\tilde{F})/P_r, b \in \mathrm{Sym}_r(\tilde{F}) \right\},$$

where

$$P_r = \left\{ a \in \mathrm{GL}_m(\tilde{F}) \mid a = \left[\begin{array}{c|c} * & * \\ \hline 0_{r,m-r} & * \end{array}\right] \right\},$$

$0_{r,m-r}$ denotes the zero matrix of size $(r, m-r)$, and $\mathrm{Sym}_r(\tilde{F})$ is identified with the following set

$$\left\{ b \in \mathrm{Sym}_m(\tilde{F}) \mid b = \left[\begin{array}{c|c} 0_{m-r} & 0 \\ \hline 0 & * \end{array}\right] \right\}.$$

Thus we have

$$J_r = \sum_{a \in \mathrm{GL}_m(\tilde{F})/P_r} \tilde{\pi}\left(\begin{bmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{bmatrix}\right) \sum_{b \in \mathrm{Sym}_r(\tilde{F})} \tilde{\pi}\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right) \tilde{\pi}(w_r) \mathrm{I}_{\tilde{X}(0)}.$$

But, by the definition of the Weil representation $\tilde{\pi}$, we get

$$\left(\tilde{\pi}\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right) f\right)(\xi) = \tilde{\psi}\left(\sum_{1 \le i \le m} b_{ii} \tilde{Q}(\xi_i) + \sum_{1 \le i < j \le m} b_{ij} \tilde{B}(\xi_i, \xi_j)\right) f(\xi),$$

$$\left(\tilde{\pi}\left(\begin{bmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{bmatrix}\right) f\right)(\xi) = f(\xi a),$$

$$\left(\tilde{\pi}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) f\right)(\xi) = \gamma^m f^{\frown}(\xi),$$

for $f \in \mathcal{S}(\tilde{X})$, $\xi \in \tilde{X}$, $a \in \mathrm{GL}_m(\tilde{F})$ and $b = [b_{ij}] \in \mathrm{Sym}_m(\tilde{F})$. Here $\tilde{\psi}$ is the nontrivial character of $\tilde{F} = R/\varpi R$ defined by

$$\tilde{\psi}(\lambda \bmod \varpi R) = \psi(\varpi^{-1}\lambda) \quad (\lambda \in R),$$

and the Fourier transform $f^{\frown}$ of f is given as

$$f^{\frown}(\xi) = q^{-sr} \sum_{\eta = (\eta_i) \in \tilde{X}} f(\eta) \tilde{\psi}\left(\sum_{1 \le j \le m} \tilde{B}(\xi_i, \eta_j)\right).$$

Then we can easily prove that

$$\tilde{\pi}(w_r) \mathrm{I}_{\widetilde{X(0)}} = (\gamma q^{-s})^r \mathrm{I}_{\widetilde{X(r)}}.$$

Therefore the above expression of J_r becomes as follows:

$$J_r = (\gamma q^{-s})^r q^{r(r+1)/2} \sum_{a \in \mathrm{GL}_m(\tilde{F})/P_r} \tilde{\pi}\left(\begin{bmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{bmatrix}\right) \mathrm{I}_{C(m,r)}.$$

Since

$$C(m, r)a \subset C(m, m)$$

for $0 \le r \le m$ and $a \in \mathrm{GL}_m(\tilde{F})$, this expression of J_r implies (3.11).

Let $\xi = (\xi_i)$ be any fixed element of $C(m, m)$ ($t = \mathrm{rank}\, \xi$). To prove (3.12), we have only to show the following two properties:

(a) if $r < t$, no element a of $\mathrm{GL}_m(\tilde{F})$ satisfies $\xi a \in C(m, r)$.

(b) if $r \geq t$, the number of all the right P_r-cosets in $\mathrm{GL}_m(\tilde{F})$ which contain an element a with $\xi a \in C(m, r)$ is just $\begin{bmatrix} m-t \\ r-t \end{bmatrix}_q$.

The first property (a) holds obviously. Suppose $r \geq t$. From the expression of J_r, we may assume $\xi_i = 0$ $(i = 1, \dots, m-t)$, and that $\{\xi_i \mid i = m-t+1, \dots, m\}$ is linearly independent in $\tilde{V}$. Then, for any $a \in \mathrm{GL}_m(\tilde{F})$, we have

$$\begin{aligned} \xi a \in C(m, r) &= \tilde{X}(r) \cap C(m, m) \\ &\Leftrightarrow \xi a \in \tilde{X}(r) \\ &\Leftrightarrow a \in P_t P_r. \end{aligned}$$

Therefore the second property (b) follows from a well-known formula

$$|P_t P_r / P_r| = |P_t / (P_t \cap P_r)| = \begin{bmatrix} m-t \\ r-t \end{bmatrix}_q$$

Thus we have proved the formulas (3.11) and (3.12).

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