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K3 surfaces with order five automorphisms

By

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Introduction

Let T be a normal projective algebraic surface over $\mathbf{C}$ with at worst quotient singular points (= Kawamata log terminal singular points in the sense of [Ka, Ko]). T is called a *log Enriques surface* if the irregularity $h^1(T, \mathcal{O}_T) = 0$ and if a positive multiple IK_T of the canonoial Weil divisor K_T is linearly equivalent to zero. Without loss of generality, we always assume from now on that a log Enriques surface has no Du Val singular points (see the comments after [Z1, Proposition 1.3]).

The smallest integer $I > 0$ satisfying $IK_T \sim 0$ is called the (global) *index* of T. It can be proved that $I \leq 66$ (cf. [Z1]). Recently, R. Blache [Bl] has shown that $I \leq 21$. He also studied the "generalized" log Enriques surfaces where log canonical singular points are allowed.

Rational log Enriques surfaces T can be regarded as degenerations of K3 or Enriques surfaces, which in turn played important roles in Enriques-Kodaira's classification theory for surfaces. In [A], A. Alexeev [A] has proved the boundedness of families of these T. In 3-dimensional case, the base surfaces W of elliptically fibred Calabi-Yau threefolds $\Phi_{|D|} : X \to W$ with $D.c_2(X) = 0$ are rational log Enriques surfaces (cf. [O1–O4]).

Let T be a log Enriques surface of index I. The Galois $\mathbf{Z}/I\mathbf{Z}$-cover

$$\pi : Y := \operatorname{Spec}_{\mathcal{O}_T} \bigoplus_{i=0}^{I-1} \mathcal{O}_T(-iK_T) \to T$$

is called the (global) *canonical covering*. Clearly, Y is either an abelian surface or a K3 surface with at worst Du Val singular points. We note also that π is unramified over the smooth part $T - \operatorname{Sing} T$.

We say that T is of *Type* A_m or D_n if Y has a singular point of Dynkin type A_m or D_n; T is of *actual Type* $(\oplus A_m) \oplus (\oplus D_n) \oplus (\oplus E_k)$ if Sing Y is of type $(\oplus A_m) \oplus (\oplus D_n) \oplus (\oplus E_k)$.

Around 1989, M. Reid and I. Naruki asked the second author about the uniqueness of rational log Enriques surface to Type D_{19}. The determinations of all isomorphism classes of rational log Enriques surfaces T of Type A_{19}, D_{19}, A_{18} and D_{18} have been done in [OZ1, 2] (see also [R1]). As a corrolary, the minimal

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resolutions X_d of the canonical covers of such T are isomorphic to the unique K3 surface of Picard number 20 and discriminant d for $d = 3$ or 4. So there are only two such X_d.

Here we consider the cases A_{17} and D_{17}. We will get some new K3 surface other than X_d above (cf. Main Theorem 3). Our main results are as follows:

Theorem 1. (1) *There is no rational log Enriques surface of Type* D_{17}.

(2) *Each rational log Enriques surface of Type* A_{17} *has index* 2, 3, 4 *or* 5.

Remark 2. The isomorphism classes of rational log Enriques surfaces of Type A_{17} and index 2, 3 or 4 are determined in [Z3, Z4].

Main Theorem 3. (1) *There are, up to isomorphisms, exactly two rational log Enriques surfaces of index* 5 *and Type* A_{17}. *These two are given as* $T(9)$, $T(14)$ *in Example* 2.1, *and both of them are of actual Type* A_{17}.

(2) *Let* $Y(i) \to T(i)$ *be the canonical Galois* $\mathbf{Z}/5\mathbf{Z}$*-cover*, $g(i) : X(i) \to Y(i)$ *the minimal resolution and* $\Delta(i) := g(i)^{-1}(\mathrm{Sing}\, Y(i))$ *the exceptional divisor, which is of Dynkin type* A_{17}. *Write* $\mathrm{Gal}(Y(i)/T(i)) = \langle \sigma(i) \rangle$.

Then the pairs $(X(i), \langle \sigma(i) \rangle)$ *are equivariantly isomorphic to each other and the fixed locus* (*point wise*) $X(i)^{\sigma(i)}$ *is a disjoint union of* 3 *smooth rational curves, which are contained in* $\Delta(i)$, *and* 13 *points. Moreover*, $\mathrm{rank}\, \mathrm{Pic}\, X(i) = 18$ *and* $|\det(\mathrm{Pic}\, X_i)| = 5$.

The pair $(X(i), \langle \sigma(i) \rangle)$ above is characterised in the following result, which is sort of the generalisation of Shioda-Inose's pairs in [OZ1].

Main Theorem 4. *There is, up to isomorphisms, only one pair* $(X, \langle \sigma \rangle)$ *of K3 surface* X *and an order* 5 *subgroup* $\langle \sigma \rangle$ *of* $\mathrm{Aut}(X)$ *satisfying*:

σ^* *acts non-trivially on non-zero holomorphic* 2*-forms, the fixed locus* X^σ *contains no curves of genus* ≥ 2, *but contains at least* 3 *rational curves.*

Moreover, $(X, \langle \sigma \rangle)$ *is equivariantly isomorphic to* $(X(i), \langle \sigma(i) \rangle)$ *in Main Theorem* 3.

Remark 5. In [OZ4, Z5], we have proved similar results on K3 automorphisms of quite arbitrary order. In particular, we proved that for each of $p = 13, 17$ and 19, there is, up to equivariant isomorphisms, only one pair $(X, \langle \sigma \rangle)$ of K3 surface X and an order p subgroup $\langle \sigma \rangle$ of $\mathrm{Aut}(X)$ (with no any other conditions on X).

Main Theorems 3 and 4 imply that on the surface X (with the automorphism σ) in Main Theorem 4, there are 2 divisors $\Delta(i)$ ($i = 1, 2$) of the same Dynkin type A_{17} such that the triplets $(X, \langle \sigma \rangle, \Delta(i))$ are not equivariantly isomorphic to each other. By virtue of this phenomenon, we pose the following:

Question 6. Is it true that there exists only one K3 surface X with $\mathrm{rank}\, \mathrm{Pic}\, X = 18$ and $|\det(\mathrm{Pic}\, X)| = 5$, which can be contracted to a normal K3 surface Y with a type A_{17} Du Val singular point?

Remark 7. (1) In Theorem 3.1 of §3, we shall determine the isomorphism class of Pic X, as an abstract lattice for the surface X in Question 6 (see the proof of Theorem 3.1). It turns out that there are two ways of contractions $h_i : X \to Y_i$ of type A_{17} divisors Δ_i such that Pic $Y_i = \mathbf{Z}H_i$ and $H_1^2 = 10$, $H_2^2 = 90$.

(2) The phenomenon of coexistence of these two h_i or Δ_i occurs because there are two different embeddings of type A_{17} lattice into Pic X: one is primitive and the other is not (see the proof of Theorem 3.1).

The organisation of the paper is as follows. In §1, we consider automorphisms σ of order 5 on K3 surfaces, and describe in detail the action of σ around points lying on linear chains of smooth rational curves. A precise relation between the numbers of σ-fixed isolated points and curves is obtained in Lemma 1.4 by applying the fixed point theorem for holomorphic bundle, which was proved by Atiyah, Segal and Singer in [AS1, 2].

In §2, we construct precisely two rational log Enriques surfaces $T(9)$, $T(14)$ of index 5 and actual Type A_{17}, starting from a nodal cubic curve and adopting the so called Campedelli's approach in the terminology of [R1].

In the proof of Theorem 3.1, we determine Pic X for X in Question 6 and also construct 5 Jacobian elliptic fibrations on X (non-isomorphic to each other). To construct a Jacobian elliptic fibration, we define a divisor η_1 in Pic X who behaves, from the viewpoint of intersection with other divisors, just like an elliptic fiber, and prove the nefness of η_1, which is one of the hardest part and possibly a "new technique" applicable to quite a lot of general situations. Another technique in proving the equivalence of the existence of any 2 of the 5 elliptic fibrations is to fully apply T. Shioda's theory on Mordell Weil lattices [Sh].

§4 is devoted to the proofs of the theorems.

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1. Preliminaries

In this section, we shall fix the following notation:

T is a rational log Enriques surface of index I and $\pi : Y \to T$ the (global) canonical covering. $g : X \to Y$ is a minimal resolution and $\Gamma := g^{-1}(\operatorname{Sing} Y)$, the exeptional locus.

Note that π is a Galois covering such that $\operatorname{Gal}(Y/T) = \mathbf{Z}/I\mathbf{Z}$ and $Y/(\mathbf{Z}/I\mathbf{Z}) = T$. Clearly, there is a natural action of $\mathbf{Z}/I\mathbf{Z}$ on X such that the minimal resolution $g : X \to Y$ is $(\mathbf{Z}/I\mathbf{Z})$-equivariant. We need the following lemmas for the later use.

Lemma 1.1. *Let T be a rational log Enriques surface of index I with Y the canonical cover. Then $\sigma^*\omega = \zeta_I\omega$ for exactly one generator σ of $\mathbf{Z}/I\mathbf{Z}$, where $\zeta_I = \exp(2\pi\sqrt{-1}/I)$ and ω is a non-zero holomorphic 2-form on Y or on X.*

Proof. The result follows from the definition of I.

Lemma 1.2. *With the notations and assumptions in Lemma 1.1, we have:*

(1) *The g-exceptional divisor Γ is σ-stable.*

(2) *Every singular point on Y has a non-trivial stabilizer subgroup of $\langle\sigma\rangle = \mathbf{Z}/I\mathbf{Z}$. In particular, every connected component of Γ is σ-stable provided that I is prime.*

(3) *Every σ^i-fixed curve on X where $\sigma^i \neq id$, is contained in Γ and hence a rational curve.*

Proof. (1) is true because the singular locus Sing Y is σ-stable.

(2) follows from our additional assumption that $T = Y/\sigma$ has no Du Val singular points. (3) is true because $\pi : Y \to T$ is unramified outside the finite set Sing T.

Lemma 1.3. *With the assumption and notation in Lemma 1.1, assume further that $I = pq$ for positive integers p, q. Then $Y_1 := Y/\langle\sigma^q\rangle$ is a rational log Enriques surface of index p with the quotient morphism $Y \to Y_1$ as the canonical cover.*

Proof. Ths follows from the fact that the (global) canonical index is equal to the l.c.m. of local canonical indices.

Lemma 1.4. *Let X be a K3 surface with an order five automorphism σ such that $\sigma^*\omega = \zeta_5\omega$ (see Lemma 1.1 for notation). Let N_i $(i = 0, 1, 2, \ldots)$ be the number of σ-fixed curves of genus i, let $N := N_0 - \sum_{i\geq 2}(i-1)N_i$, and let M_i $(i = 1, 2)$ be the number of σ-fixed points at which σ can be diagonalized as $\sigma^* = (\zeta^{-i}, \zeta^{i+1})$.*

Then the 1-dimensional part of X^σ is a nonsingular divisor. We have $M_1 = 3 + 2N$, $M_2 = 1 + N$.

Proof. Since $\sigma^*\omega = \zeta\omega$, one has the diagonalization $\sigma^* = \mathrm{diag}(\zeta^{-i}, \zeta^{i+1})$ for $i = 0, 1$ or 2, around a σ-fixed point P with suitable local coordinates (x, y). If $i = 1, 2$, P is isolated in X^σ; if $i = 0$ then X^σ is equal to $\{y = 0\}$ and hence smooth.

We now calculate the holomorphic Lefschetz number $L(\sigma)$ in two ways as in [AS1, 2, pages 542 and 567]:

$$L(\sigma) = \sum_{i=0}^{2}(-1)^i \mathrm{Tr}(\sigma^* \mid H^i(X, \mathcal{O}_X)),$$

$$L(\sigma) = \sum_i a(P_i) + \sum_j b(C_j).$$

Here

$$a(P_i) = 1/\det(1 - \sigma^* \mid T_{P_i}) = 1/(1 - \zeta^{-k})(1 - \zeta^{k+1}),$$
$$b(C_j) = (1 - g(C_j))/(1 - \zeta^{-4}) - (\zeta^{-4} C_j^2)/(1 - \zeta^{-4})^2,$$

where P_i is an isolated σ-fixed point with $\sigma^* \mid P_i = (\zeta^{-k}, \zeta^{k+1})$, T_{P_i} is the tangent space to X at P_i, $g(C_j)$ the genus of C_j and ζ^4 the eigenvalue of the action σ_* on the normal bundle of C_j.

The first formula yields $L(\sigma) = 1 + \zeta^{-1}$ by the Serre duality $H^0(X, \mathcal{O}(K_X))^\vee \cong H^2(X, \mathcal{O}_X)$. Plugging this into the second formula for $L(\sigma)$, we get:

$$1 + \zeta^{-1} = M_1/(1 - \zeta^{-1})(1 - \zeta^2) + M_2/(1 - \zeta^{-2})^2 + N(1 + \zeta)/(1 - \zeta)^2.$$

Multiplying this equality by denominators we obtain the following one after simplification:

$$-\zeta - \zeta^2 + 2\zeta^{-1} = M_1\zeta^{-1} + M_2(1 + \zeta^{-2}) + N(\zeta + \zeta^2 - \zeta^{-1}).$$

Using the relation $\sum_{i=0}^4 \zeta^i = 0$ we can transform the above equality into the following:

$$(-M_1 + M_2 + N + 2) + (-M_1 + 2N + 3)\zeta + (-M_1 + 2N + 3)\zeta^2$$
$$+ (-M_1 + M_2 + N + 2)\zeta^3 = 0.$$

Since 1, ζ, ζ^2, ζ^3 are linearly independent over $\mathbf{Q}$, the coefficients in the above equality all vanish, and hence Lemma 1.4 follows.

Lemma 1.5. *Let X, σ, N be as in Lemma* 1.4. *Then we have*:

(1) *There are integers (s, t) with $s \geq 0$, $t \geq 1$ and $s + t \leq 5$ such that $N = 4 - (s + t)$, $\rho(X) = 22 - 4t$ and σ^* has the following diagonalizations, where T_X is the transcendental lattice of X [BPV, p. 238]:*

$$\sigma^* \mid (\operatorname{Pic} X \otimes \mathbf{C}) = \operatorname{diag}[I_{22-4(s+t)}, \operatorname{diag}[\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4]^{\oplus s}],$$
$$\sigma^* \mid (T_X \otimes \mathbf{C}) = \operatorname{diag}[\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4]^{\oplus t}.$$

(2) $N = 3$ *if and only if* $\rho(X) = 18$ *and* $\sigma^* \mid \operatorname{Pic} X = \mathrm{id}$.

(3) *Suppose* $N = 3$. *Then* $|\det(\operatorname{Pic} X)| = \operatorname{discr}(X) = 5$.

Proof. (1) We only need to show $N = 4 - (s + t)$ and for the rest, we refer to [N1, Theorem 3.1] and the fact that $B_2(X) = 22$. Consider the topological Euler number:

$$\chi_{top}(X^\sigma) = \sum_{i=0}^{4} (-1)^i \operatorname{Tr}(\sigma^* \mid H^i(X, \mathbf{C})).$$

On the one hand, $\chi_{top}(X^\sigma) = M_1 + M_2 + 2N_0 + \sum_{i \geq 2} N_i(2 - 2i) = M_1 + M_2 + 2N = 4 + 5N$ (cf. Lemma 1.4). On the other hand, $\operatorname{Tr}(\sigma^* \mid (\operatorname{Pic} X) \otimes \mathbf{C}) = (22 - 4s - 4t) - s$, and $\operatorname{Tr}(\sigma^* | T_X \otimes \mathbf{C}) = -t$. Thus $4 + 5N = 2 + (22 - 5s - 4t) - t$, and $N = 4 - (s + t)$.

(2) follows from $N = 4 - (s+t)$, $t \geq 1$ and $\rho(X) = 22 - 4t$.

(3) Suppose $N = 3$. Then $s = 0$, $t = 1$. Note that T_X is a $\mathbf{Z}[\langle\sigma^*\rangle]$-module. The diagonalization of $\sigma^* \mid (T_X \otimes \mathbf{C})$ and the fact that $\Phi_4(X) = \sum_{i=0}^4 X^i$ is the minimal polynomial of ζ_5 over $\mathbf{Q}$, imply:

Claim 1. *$g \in \mathbf{Z}[\langle\sigma^*\rangle]$ annihilates $t \in T_X - \{0\}$ if and only if $g = a\Phi(\sigma^*)$ for some $a \in \mathbf{Z}$. Hence T_X is a free $\mathbf{Z}[\langle\bar{\sigma}^*\rangle]$-module, where $\bar{\sigma}^* = \sigma^* + \langle\Phi_4(\sigma^*)\rangle$.*

By Claim 1, $T_X \cong (\mathbf{Z}[\langle\bar{\sigma}^*\rangle])^{\oplus r}$ for some $r \geq 1$ because $\mathbf{Z}[\langle\bar{\sigma}^*\rangle]$ is a P.I.D. Since $4 = \operatorname{rank} T_X = 4r$, one has $r = 1$. So there is a $\mathbf{Z}[\langle\bar{\sigma}^*\rangle]$-module isomorphism: $\tau : \mathbf{Z}[\langle\bar{\sigma}^*\rangle] \to T_X$. Hence we have:

Claim 2. *$e_i = \tau(\bar{\sigma}^{i*})$ $(i = 1,2,3,4)$ form a $\mathbf{Z}$-basis of T_X so that $\sigma^* e_i = e_{i+1}$ $(i = 1,2,3)$ and $\sigma^* e_4 = -(e_1 + e_2 + e_3 + e_4)$.*

Since $\sigma^* \mid \operatorname{Pic} X = \mathrm{id}$, the natural isomorphism $T_X^\vee / T_X \cong H^2(X, \mathbf{Z}) / (\operatorname{Pic} X \oplus T_X) \cong (\operatorname{Pic} X)^\vee / \operatorname{Pic} X$ [BPV, Lemma 2.5, p. 13] implies that $\sigma^* \mid (T_X^\vee / T_X) = \mathrm{id}$. Now for any $x + T_X \in T_X^\vee / T_X$, one has $x \equiv \sigma^{i*} x \,(\operatorname{mod} T_X)$ for all i. Hence $5x \equiv \Phi_4(\sigma^*)x \equiv 0 \,(\operatorname{mod} T_X)$. So, $T_X^\vee / T_X \cong (\mathbf{Z}/5\mathbf{Z})^{\oplus r}$ for some $1 \leq r \leq 4$ by noting that $\operatorname{rk} T_X = 4$ and by [Ko, Theorem in §0] for $\langle\sigma\rangle \leq H_X$ now (cf. (2)).

We assert that $T_X^\vee / T_X = \langle \sum_{i=1}^4 ie_i/5 \rangle \cong \mathbf{Z}/5\mathbf{Z}$. Indeed, for any $x \in T_X^\vee - T_X$, one can write $x = \sum_{i=1}^4 a_i e_i / 5$, where $a_i \in \mathbf{Z}$. Since $\sigma^* x - x \equiv 0 \,(\operatorname{mod} T_X)$, we see that

$$a_1 + a_4, \quad a_1 - a_2 - a_4, \quad a_2 - a_3 - a_4, \quad a_3 - 2a_4$$

are all 0 (mod 5). Hence $a_i \equiv ia_1$ (mod 5) for all $i = 1,2,3,4$. Thus $x \equiv a_1 \sum_{i=1}^4 ie_i/5 \,(\operatorname{mod} T_X)$. Since x is not in T_X, g.c.d.$(5, a_1) = 1$ and hence $sx \equiv \sum_{i=1}^4 ie_i/5 \,(\operatorname{mod} T_X)$ for some $s \in \mathbf{Z}$. This proves the assertion.

Now (3) follows from this assertion and the fact that $\operatorname{discr}(X) = |\det(T_X)| = |T_X^\vee / T_X|$. This completes the proof of Lemma 1.5.

Lemma 1.6 (5-Go Lemma). *Let X, σ be as in Lemma 1.4. Assume that $\sum_{i=1}^n C_i$ is a linear chain of σ-stable smooth rational curves C_i with $C_i . C_{i+1} = 1$. Set $P_i := C_i \cap C_{i+1}$.*

If $n = 3$, there is a σ-fixed curve D with $D.(C_1 + C_3) = 1$. If $n = 5$, exactly one of C_i is σ-fixed, say C_r, and the quadruplet $\sigma^* | P_1$, $\sigma^* | P_2$, $\sigma^* \mid P_3$, $\sigma^* \mid P_4$ of diagonalized local σ^*-actions, is equal to the unique portion of the following recursive sequence such that $\sigma^* \mid P_r = (1, \zeta)$:

$$(\zeta, 1), (1, \zeta), (\zeta^{-1}, \zeta^2), (\zeta^{-2}, \zeta^{-2}), (\zeta^2, \zeta^{-1}),$$

$$(\zeta, 1), (1, \zeta), (\zeta^{-1}, \zeta^2), (\zeta^{-2}, \zeta^{-2}), (\zeta^2, \zeta^{-1}), \ldots.$$

Proof. We use the observation at the first paragraph of the proof of Lemma 1.4 and the fact that if $\sigma^* \mid P_{i-1} = (\zeta_5^{1-s}, \zeta_5^s)$ so that ζ_5^s is the eigenvalue of σ w.r.t. the tangent to C_i at P_{i-1} then $\sigma^* \mid P_i = (\zeta_5^{-s}, \zeta_5^{s+1})$.

Lemma 1.7. *Let X, σ be as in Lemma* 1.4. *Assume that $\Phi : X \to \mathbf{P}^1$ is an elliptic fibration and η is a singular fiber consisting of σ-stable curves. Then η fits one of the following 5 cases*:

(1) $\eta = \sum_{i=1}^{5n} C_i$, *where* $C_i.C_{i+1} = C_{5n}.C_1 = 1$, *is of Kodaira type* I_{5n} *for some* $1 \leq n \leq 3$. *Moreover*, $C_1, C_6, \ldots, C_{5n-4}$ *are only σ-fixed curves in η, after relabelling.*

(2) $\eta = C_1 + C_2 + 2(C_3 + C_4 + \cdots + C_{5n+3}) + C_{5n+4} + C_{5n+5}$, *where* $C_1.C_3 = C_{5n+3}.C_{5n+5} = C_i.C_{i+1} = 1$ $(i = 2, 3, \ldots, 5n+3)$, *is of Kodaira type* I_{5n}^* *for some* $n = 0, 1, 2$. *Moreover*, $C_3, C_8, \ldots, C_{5n+3}$ *are only σ-fixed curves in η.*

(3) *η is of Kodaira type IV^*, III^* (resp. II^*). The branch component R (resp. the branch component R and the tip component furthest away from R) is σ-fixed.*

(4) $\eta = C_1 + C_2 + C_3$ *is of Kodaira type IV. Each C_j is σ-stable but not σ-fixed. σ can be diagonalized as* $\sigma^* \mid P_0 = (\zeta^{-2}, \zeta^{-2})$ *at the common point* P_0 : $C_1 \cap C_2 \cap C_3$, *and as* $\sigma^* \mid P_j = (1, \zeta)$ *at the second σ- fixed point P_j on C_j.*

(5) $\eta = C_1 + C_2$ *is of Kodaira type III. Each C_j is σ-stable but not σ-fixed. σ can be diagonalized as* $\sigma^* \mid P_0 = (\zeta^{-1}, \zeta^2)$ *at the common point* $P_0 := C_1 \cap C_2$, *and as* $\sigma^* \mid P_j = (\zeta^{-2}, \zeta^{-2})$ *at the second σ-fixed point P_j on C_j.*

Proof. This follows from the analysis of σ^*-action at points in X^σ as in Lemma 1.6.

§2. Examples of index 5 and Type A_{17}

In the present section, we shall construct two isomorphism classes $T(9), T(14)$ of rational log Enriques surfaces of index 5 and actual Type A_{17} (cf. Main Theorem 3).

Example 2.1. Let Σ_4' be a nodal cubic curve in $\mathbf{P}^2$ with P_1 as a inflexion point and P_2 as its node. Denote by Π_6' (resp. Π_7') the tangent (resp. one of two tangents) to Σ_4' at P_1 (resp. P_2). We prove the following lemma. This lemma and the precise construction of $T(i)$ below will also be used in proving Main Theorem 3(1).

Lemma 2.2. *After a change of coordinates, the data above can be specified as follows*: Σ_4' *is given by* $Y^2Z = X^2(X+Z)$, $P_1 = [0:1:0]$, $P_2 = [0:0:1]$, $\Pi_6' = \{Z = 0\}$, *and* $\Pi_7' = \{Y - X = 0\}$.

Proof. First, we may assume that $P_1 = [0:1:0]$ and $\Pi_6' = \{Z = 0\}$ after changing coordinates. Now Σ_4' is given by $Y^2Z = X^3 + aX^2Z + bXZ^2 + cZ^3$ [R2, Ex. 2.10, p. 41]. May assume that the node $P_2 = [0:0:1]$. Hence $b = c = 0$ and $a \neq 0$. Now one of the projective transformations $(X, Y, Z) = (X', \pm\sqrt{a}Y', Z'/a)$ will change the data to those in Lemma 2.2.

We now take the set of data $\Sigma_4' : Y^2Z = X^2(X+Z)$, etc. as in Lemma 2.2. Let $\nu : V \to \mathbf{P}^2$ be the unique blowing-up of P_1, P_2 and 7 their infinitely near points, such that $\nu^{-1}(\Sigma_4' + \Pi_6' + \Pi_7')$ is given in Figure 1, where $\Sigma_4 = \nu'(\Sigma_4')$, etc.,

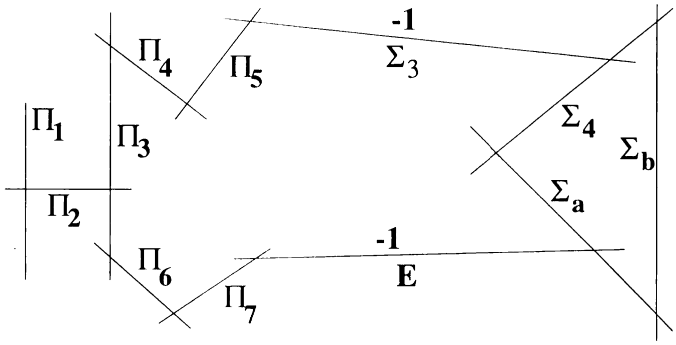

Fig. 1

where $E^2 = \Sigma_3^2 = -1$ and all other curves have intersection -2. The relation $\Sigma_4' \sim 2\Pi_6' + \Pi_7'$ induces:

(2.1.1) $\xi_1 := 3\Pi_3 + 2(\Pi_2 + \Pi_4 + \Pi_6) + \Pi_1 + \Pi_5 + \Pi_7 \sim \xi_2 := \Sigma_4 + \Sigma_a + \Sigma_b.$

So ξ_i are fibers of an elliptic fibration $\varphi : V \to \mathbf{P}^1$ and E, Σ_3 are cross-sections of φ with $E.\Sigma_a = E.\Pi_7 = 1$. By the way, the only remaining third singular fiber ξ_3 of φ is of Kodaira type I_1.

Consider the relation

(2.1.2) $$\mathcal{O}_V(\xi_1 + 4\xi_2) \cong \mathcal{O}_V(\xi_1)^{\otimes 5}.$$

Consider the Galois $\mathbf{Z}/5\mathbf{Z}$-covering:

$$\pi : W = \mathrm{Spec}_{\mathcal{O}_V} \bigoplus_{i=0}^4 \mathcal{O}(-i\xi_1) \to V.$$

This W has 3 Du Val points of type $\langle 5, 4\rangle$ (resp. or $\langle 5, 1\rangle$, or $\langle 5, 2\rangle$) over the 3 intersection points in ξ_2 (resp. the 3 points $\Pi_3 \cap \Pi_i$ for $i = 2, 4, 6$; or the 3 points $\Pi_i \cap \Pi_{i+1}$ for $i = 1, 4, 6$). Resolving these singularities and blowing down uniquely and smoothly curves lying over ξ_1, we get a K3 surface X with an elliptic fibration $\psi : X \to \mathbf{P}^1$ induced from φ, so that the fibers η_1, η_2 lying over ξ_1, ξ_2 is of Kodaira type IV, I_{15}, respectively.

Clearly, we may take a generator $\sigma \in \mathrm{Gal}(W/V) \cong \mathbf{Z}/5\mathbf{Z}$ such that $\sigma^*\omega = \zeta_5\omega$ where ω is a non-zero holomorphic 2-form on X and $\zeta_5 = \exp(2\pi\sqrt{-1}/5)$. By the way, if one takes one fiber η_3 of ψ lying over ξ_3, then 5 fibers $\sigma^i\eta_3$ $(i = 0, 1, \dots, 4)$ of Kodaira type I_1 are only fibers lying over ξ_3.

Denote by F, Γ_i the strict transforms on X of E, Σ_i for $i = 3, 4, a, b$. The graph of $F + \Gamma_3 + \eta_1 + \eta_2$ is given in Figure 2. To be precise, one can write uniquely $\eta_1 = F_1 + \Gamma_1 + \Gamma_2$ so that F_1, Γ_1, Γ_2 lie over the points $\Pi_3 \cap \Pi_i$ for $i = 6, 2, 4$, respectively.

Clearly, each curve in $F + \Gamma_3 + \eta_1 + \eta_2$ is σ-stable (in fact, $\sigma^* \mid \mathrm{Pic}\, X = \mathrm{id}$, see Lemma 4.1) and the fixed locus

$$X^\sigma = \mathrm{Supp}(\Gamma_4 + \Gamma_a + \Gamma_b) \coprod \{P_i\}_{i=1}^{13},$$

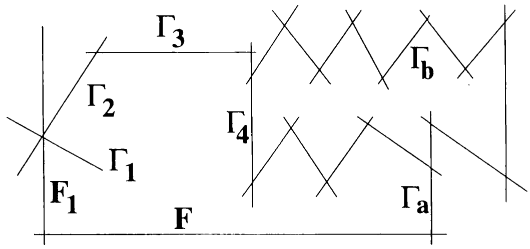

Fig. 2

where the first 12 P_i's are intersection points (not on Γ_k for all $k = 4, a, b$) in $F + \Gamma_3 + \eta_1 + \eta_2$ and P_{13} is a point lying on Γ_1.

Now $\Gamma_1 + \Gamma_2 + \Gamma_3 + \eta_2$ contains exactly two divisors $\Gamma(a)$, $\Gamma(b)$ of Dynkin type A_{17}:

$$\Gamma_1 - \Gamma_2 - \Gamma_3 - \cdots - \Gamma_{16} - \Gamma_{17}.$$

One is when $(a, b) = (9, 14)$ and the other when $(a, b) = (14, 9)$. Let $g : X \to Y(i)$ be the contraction of $\Gamma(i)$ to a point Q_1. Then the induced σ-action on $Y(i)$ has Q_1 and the image of $F \cap F_1$ as only fixed points. Clearly, $T(i) = Y(i)/\sigma$ is a rational log Enriques surface of index 5 and actual Type A_{17}.

Remark 2.3. In [Z2, Example 6.12], we constructed a rational log Enriques surface T of index 5 and Type A_{17}. Instead of T we used (V', D') there. To be precise, D' is a union of the following two linear chains on the smooth rational surface V' and $V' \to T$ is the contraction of D'

$$(-2)—(-2)—(-2)—(-3)—(-2)—(-3)—(-2)—(-2)—(-2),\ (-2)—(-3).$$

Since the (-1)-curve F_2' in [Z1, Example 6.12 and Figure (8)] links the only (-2)-curve in the second connected component of D' to one of two (-3)-curves in the first, the strict transform F on X of F_2' is a smooth rational curve with $F.\Gamma = F.\Gamma_{14} = 1$ in the notations of the proof of Main Theorem 3, after relabelling; hence this $T \cong T(14)$.

§ 3. A sublattice of type A_{17}

In this section, we shall prove:

Theorem 3.1. *Let X be a K3 surface of Picard number* 18 *and* $|\det(\mathrm{Pic}\, X)| = 5$. *Assume that there is a linear chain Γ of* 17 *smooth rational curves Γ_i's on X with $\Gamma_i.\Gamma_{i+1} = 1$. Then we have:*

(1) *There is an elliptic fibration $\psi : X \to \mathbf{P}^1$ such that ψ has fibers η_1 and η_2 of Kodaira types* (I_3 *or* IV) *and* I_{15}, *with Γ_3 as a cross-section* (*after relabelling Γ_i as Γ_{18-i} if necessary*), *and $\eta_1 = F_1 + \Gamma_1 + \Gamma_2$, $\eta_2 = F_2 + \sum_{i=4}^{17} \Gamma_i$ where F_i's are smooth rational curves with $F_1.\Gamma_i = F_2.\Gamma_j = 1$ $(i = 1, 2; j = 4, 17)$.*

(2) *There is a unique cross-section F of ψ such that $F.F_1 = F.(\Gamma_9 + \Gamma_{14}) = 1$.*

(3) *Let $X \to Y$ be the contraction of Γ, and let H denote the ample generator of* Pic Y *and also its pull-back on X.*

If $F.\Gamma_9 = 1$, then $H^2 = 10$, and Γ (the one generated by Γ_i's) is an index 3 sublattice of its primitive-closure $\tilde{\Gamma}$ in Pic X, *so that $\tilde{\Gamma} \oplus \mathbf{Z}H$ is a sublattice of index 2 in* Pic X.

If $F.\Gamma_{14} = 1$, then $H^2 = 90$ and Γ is a primitive sublattice of Pic X *such that $\Gamma \oplus \mathbf{Z}H$ is a sublattice of index 18 in* Pic X.

Remark 3.2. If $F.\Gamma_9 = 1$, we relabel in the following way: $\Gamma_i' := \Gamma_i$ $(i = 1,2,3,4)$, $\Gamma_5' := F_2$, $\Gamma_j' = \Gamma_{23-j}$ $(j = 6,7,\dots,17)$. Then $F.\Gamma_{14}' = 1$. In other words, by replacing Γ by a new Γ' of Dynkin type A_{17}, we can always assume that $F.\Gamma_{14}' = 1$ (or $F.\Gamma_9' = 1$ by a similar argument).

The proof of Theorem 3.1 consists of Lemmas 3.3–3.5 below. Let X, $\Gamma = \sum_{i=1}^{17} \Gamma_i$, H be as in Theorem 3.1. In the sequel, we shall use the same Γ to denote the sublattice in Pic X generated by Γ_i's. Note that $\Gamma^\perp \subseteq \operatorname{Pic} X$ is generated by the nef and big divisor H.

Lemma 3.3. *Assume that Γ is a primitive sublattice in* Pic X. *Then we have:*

(1) *Suppose that $h \in \operatorname{Pic} X$ satisfies $\operatorname{Pic} X = \Gamma + \mathbf{Z}h$ (the existence of such h is from the primitivity of Γ in* Pic X). *Then $\pm h \equiv h_+ := \frac{1}{18}(H + 7\sum_{i=1}^{17} i\Gamma_i) \pmod{\Gamma}$, after relabelling Γ_i as Γ_{18-i} if necessary. Moreover, $H^2 = 90$ and $|\operatorname{Pic} X : \Gamma \oplus \mathbf{Z}H| = 18$.*

(2) *$\{\Gamma_1, \Gamma_2, \dots, \Gamma_{17}, h_+\}$ form a $\mathbf{Z}$-basis of* Pic X, *and the intersection form of this basis is given by: $h_+^2 = -46$, $h_+.\Gamma_{17} = -7$, $h_+.\Gamma_i = 0$ $(i = 1,2,\dots,16)$.*

(3) *There are smooth rational curves F, F_1, F_2 such that $F \sim 2h_+ - \sum_{i=1}^{17} i\Gamma_i + (\Gamma_{15} + 2\Gamma_{16} + 3\Gamma_{17})$, $F_1 \sim 3h_+ - \sum_{i=1}^{17}(3+i)\Gamma_i + \Gamma_1$, $F_2 \sim 3h_+ - \sum_{i=1}^{17}(4+i)\Gamma_i + (3\Gamma_1 + 2\Gamma_2 + \Gamma_3)$.*

(4) *Theorem 3.1 is true with $F.\Gamma_{14} = 1$, by letting F, F_1, F_2 be as in (3) and $\eta_1 := F_1 + \Gamma_1 + \Gamma_2$, $\eta_2 := F_2 + \sum_{i=4}^{17} \Gamma_i$, $\psi := \Phi_{|\eta_1|}$.*

Proof. (1) Let $h \in \operatorname{Pic} X$ so that $\operatorname{Pic} X = \Gamma + \mathbf{Z}h$. Claim (1.1) below can be similarly proved as in [OZ1].

CLAIM 1. *Set $n = |\operatorname{Pic} X : \Gamma \oplus \mathbf{Z}H|$. Then we have:*

(1.1) *After replacing h by $-h$ if necessary, $nh \equiv H \pmod{\Gamma}$ and hence $h = \frac{1}{n}(H + \sum_{i=1}^{17} a_i\Gamma_i)$ for some integers a_i.*

(1.2) *n divides $|\det(\Gamma)| = 18$. Moreover, $5n^2 = 18H^2$. Hence $n = 6, 18$.*

Note that $\frac{1}{n}(a_1, a_2, \dots, a_{17})$ is the unique solution of the linear system:

$$\left(h - \sum_{i=1}^{17} x_i\Gamma_i\right)\Gamma_j = 0 \quad (j = 1,2,\dots,17).$$

Since $\det(\Gamma_i.\Gamma_j) = -18$, $18a_i/n \in \mathbf{Z}$. Hence $18H/n = 18h - \sum_{i=1}^{17} \frac{18a_i}{n}\Gamma_i = sH$ for some integer s. So $18/n = s$ and $n \mid 18$. The second assertion of Claim (1.2) follows from the observation that $|\det(\mathrm{Pic}\, X)|n^2 = |\det(\Gamma \oplus \mathbf{Z}H)|$.

Note that $(\sum_{i=1}^{17} a_i\Gamma_i).\Gamma_j = n(h.\Gamma_j) \equiv 0 \pmod n$ for all j. Hence

$$-2a_1 + a_2, \quad a_{i-1} - 2a_i + a_{i+1}\ (i = 2, 3, \ldots, 16), \quad a_{16} - 2a_{17}$$

are all 0 $(\mathrm{mod}\, n)$. So $a_i \equiv ia_1 \pmod n$ for all $1 \le i \le 17$. Thus $\left(h + \frac{1}{n}\sum_{i=1}^{17}(ia_1 - a_i)\Gamma_i\right)^2$ is an integer, which is equal to

$$\frac{1}{n^2}\left(H + a_1\sum_{i=1}^{17} i\Gamma_i\right)^2 = \frac{1}{n^2}(H^2 - 18 \times 17a_1^2)$$

$$= \frac{5}{18} - \frac{18 \times 17a_1^2}{n^2}.$$

For the latter to be an integer, $n = 18$ and $a_1 = \pm 7$ (mod 18) (cf. Claim 1.2)).

The above argument also shows that $h \equiv h_\pm := \frac{1}{18}(H \pm 7\sum_{i=1}^{17} i\Gamma_i) \pmod \Gamma$. Since $h_- \equiv \frac{1}{18}(H + 7\sum_{i=1}^{17}\Gamma_{18-i}) \pmod \Gamma$, the assertion (1) is proved.

(2) follows from the definition of h_+ and a direct calculation.

(3) Use the same F, F_1, F_2 to denote $2h_+ - \sum_{i=1}^{17} i\Gamma_i + (\Gamma_{15} + 2\Gamma_{16} + 3\Gamma_{17})$, $3h_+ - \sum_{i=1}^{17}(3+i)\Gamma_i + \Gamma_1$, $3h_+ - \sum_{i=1}^{17}(4+i)\Gamma_i + (3\Gamma_1 + 2\Gamma_2 + \Gamma_3)$, respectively. We shall show that each of $|F|$, $|F_1|$ and $|F_2|$ contains a smooth rational curve as a member.

A direct calculation shows Claims (2.1), (2.2) and (2.3) below.

CLAIM 2. (2.1) $H.h_+ = 5$, $F^2 = F_1^2 = F_2^2 = -2$, $H.F = 10$, $H.F_1 = 15 = H.F_2$.

(2.2) *The intersection number (0 or 1) between any two distinct divisors of* F, F_1, F_2, Γ_i $(i = 1, 2, \ldots, 17)$ *are as discribed in Figure 2 with* $(a, b) = (14, 9)$, *if we regard these divisors as irreducible curves, e.g.* $F_2.\Gamma_4 = F_2.\Gamma_{17} = F.F_1 = F.\Gamma_{14} = 1$, $F.F_2 = F.\Gamma_i = 0$ $(1 \le i \le 17, i \ne 14)$.

(2.3) $F_1 + \Gamma_1 + \Gamma_2 \sim F_2 + \sum_{i=4}^{17}\Gamma_i$.

(2.4) $|F| \ne \emptyset$, $|F_i| \ne \emptyset$ $(i = 1, 2)$. *Hence we assume* $F \ge 0$, $F_i \ge 0$.

$F^2 = -2$ and the Riemann-Roch theorem imply that $|F| \ne \emptyset$, or $|-F| \ne \emptyset$. Since $F.H > 0$, where H is nef and big, we have $|F| \ne \emptyset$. Similarly, we can finish the proof of Claim (2.4).

CLAIM 3. *Set* $G := F_1 + \Gamma_1 + \Gamma_2$.

(3.1) $G.\Gamma_3 = 1$, $G.\Gamma_i = 0$ $(1 \le i \le 17, i \ne 3)$, $G.F = 1$, $G.F_i = 0$ $(i = 1, 2)$, $G^2 = 0$.

(3.2) *G is a numerically effective divisor.*

Claim (3.1) can be verified easily using Claim 2. Suppose the contrary that Claim (3.2) is false. Then there is a smooth rational curve E_1 ($\neq \Gamma_i$ for any i) such that $G.E_1 \leq -1$ by noting that $|G| \neq \emptyset$ (Claim (2.4)).

By the proof of Theorem 1(3) in [S, p. 573], there is an effective divisor N, a union of E_1 and other smooth rational curves, such that $P := G - N$ is a numerically (non-trivial and) effective divisor with $P^2 = 0$; to be precise, P is the image of G by a composite of reflections of $\operatorname{Pic} X$. By [S, Theorem 1, p. 559], $P \sim m\eta$, where $m \in \mathbf{Z}_{>0}$ and η is an elliptic curve.

Write $N = nh_+ + \sum_{i=1}^{17} n_i\Gamma_i$, $P = ph_+ + \sum_{i=1}^{17} p_i\Gamma_i$, where n, n_i, p, $p_i \in \mathbf{Z}$. Clearly, N, P are positive multiples of H, modulo Γ; in particular, $p \geq 1$ and $n \geq 1$ because $E_1 \leq N$. On the other hand $(n+p)h_+ \equiv G \equiv 3h_+ \pmod{\Gamma}$. Hence $n + p = 3$. Thus $(n, p) = (1, 2)$, $(2, 1)$.

Set $c_i := P.\Gamma_i \in \mathbf{Z}_{\geq 0}$. We have

$$c_1 = P.\Gamma_1 = -2p_1 + p_2,$$

$$c_i = P.\Gamma_i = p_{i-1} - 2p_i + p_{i+1}, \quad (i = 2, 3, \ldots, 16),$$

$$c_{17} = P.\Gamma_{17} = -7p + p_{16} - 2p_{17}.$$

Solving this linear system, we obtain:

$$p_1 = -\frac{1}{18}\left(7p + \sum_{j=1}^{17}(18-j)c_j\right), \quad p_i = ip_1 + \sum_{j=1}^{i-1}(i-j)c_j, \quad (2 \leq i \leq 17),$$

$$7ip + 18p_i = -\sum_{j=1}^{i-1}(18-i)jc_j - \sum_{j=i}^{17} i(18-j)c_j.$$

We have also $P.h_+ = P.(H + 7\sum_{i=1}^{17} i\Gamma_i)/18 = (5p + 7\sum_{i=1}^{17} ic_i)/18$. Thus we can calculate:

$$(*) \quad 5p^2 = 5p^2 - 18P^2 = 5p^2 - 18P.\left(ph_+ + \sum_{i=1}^{17} p_i\Gamma_i\right)$$

$$= -\sum_{i=1}^{17}(7ip + 18p_i)c_i = \sum_{i=2}^{17}\sum_{j=1}^{i-1}(18-i)jc_ic_j + \sum_{i=1}^{17}\sum_{j=i}^{17} i(18-j)c_ic_j$$

$$= \sum_{j=1}^{16}\sum_{i=j+1}^{17}(18-i)jc_ic_j + \sum_{j=1}^{17}\sum_{i=1}^{j} i(18-j)c_ic_j.$$

If $c_j \geq 1$, then $20 \geq 5p^2 \geq (18-j)c_jjc_j \geq j(18-j)$, and hence $j = 1$, 17. Thus $c_k = 0$ $(2 \leq k \leq 16)$, and $20 \geq 5p^2 \geq \sum_{j=1,17} j(18-j)c_j^2$. Hence either $c_1 = 1$ and $c_i = 0$ for all $2 \leq i \leq 17$, or $c_{17} = 1$ and $c_i = 0$ for all $1 \leq i \leq 16$. But then the equality $(*)$ implies that $5p^2 = 17$, a contradiction. Hence Claim (3.2) is true.

The above argument also shows that $G \sim m\eta$ for some $m \in \mathbf{Z}_{>0}$ and an elliptic curve η. Since $G.\Gamma_3 = 1$, $m \doteq 1$ and Γ_3 is a cross-section of the elliptic fibration $\psi := \Phi_{|\eta|}$. Now $\eta_1 := G = F_1 + \Gamma_1 + \Gamma_2$ and $\eta_2 := F_2 + \sum_{i=4}^{17} \Gamma_i$ are singular fibers of ψ.

First $\eta_1 \neq \eta_2$, for otherwise $\eta_1 = \eta_2$ contains 16 curves Γ_i $(i \neq 3)$ and at least two more curves. This leads to $18 = \rho(X) \geq 2 + (\#\eta_1 - 1) \geq 19$, a contradiction [Sh, Cor. 5.3]. The same argument shows that $\#\eta_1 = 3$, $\#\eta_2 = 15$, each singular fiber η_i $(i \geq 3)$ is of Kodaira type I_1 or II, and the Mordell-Weil group of ψ is torsion. Thus η_2 is of Kodaira type I_{15} and η_1 is of type I_3 or IV. Hence F_i is irreducible and is the unique member in $|F_i|$, which is a smooth rational curve.

To finish (3), it still needs to show that $|F|$ contains an irreducible member. Here we may assume $F \geq 0$. Since $F.G = 1$ (Claim 3), $F = F' + C$ where F' is a cross-section of ψ, and C is contained in fibers.

As in the proof of Claim (3.2), $F = F' + C$ does not contain either of F_i because $F \equiv 2h_+ \pmod{\Gamma}$ while $F_i \equiv 3h_+ \pmod{\Gamma}$. Now $0 = F.F_2$ (Claim 2) implies that $F = F' + C$ does not contain Γ_4 or Γ_{17}. Inductively, $F.\Gamma_i = F.\Gamma_j = F.\Gamma_k = 0$ $(i = 4, 5, \ldots, 13; j = 17, 16, 15; k = 4, 3, 2)$ in Claim 2, implies that F does not contain Γ_{i+1} or Γ_{j-1} or Γ_{k-1}. Hence F does not contain any of $\Gamma_i (1 \leq i \leq 17)$. So C is a union of fibers. Since $-2 = F^2 = (F' + C)^2 = -2 + C^2 + 2CF' \geq -2$, $C = 0$ and $F = F'$ is an (irreducible) cross-section with $F.\Gamma_{14} = F.F_1 = 1$ (Claim 2). This prove (3). In fact, by the arguments so far (cf. Lemmas 3.4 and 3.5), Theorem 3.1 for the present case is also proved.

Lemma 3.4. *Assume that Γ is a not a primitive sublattice in* Pic X. *Let $\tilde{\Gamma}$ be the primitive closure of Γ in* Pic X. *Write $\tilde{\Gamma}^{\perp} = \Gamma^{\perp} = \mathbf{Z}H$ with the nef and big H. Then we have*:

(1) $|\tilde{\Gamma} : \Gamma| = 3$, *and* $|\det(\tilde{\Gamma})| = 2$. *Moreover,* $H^2 = 10$ *and* $|\mathrm{Pic}\, X : (\tilde{\Gamma} \oplus \mathbf{Z}H)| = 2$.

(2) *Suppose that $\delta \in \tilde{\Gamma}$ satisfies $\tilde{\Gamma} = \Gamma + \mathbf{Z}\delta$. Then $\pm\delta \equiv \delta_+ := \frac{1}{3}\sum_{i=1}^{17} i\Gamma_i$ $\pmod{\Gamma}$. Hence δ_+, Γ_i $(i = 2, 3, \ldots, 17)$ form a $\mathbf{Z}$-basis of $\tilde{\Gamma}$.*

(3) *Suppose that $h \in \mathrm{Pic}\, X$ satisfies $\mathrm{Pic}\, X = \tilde{\Gamma} + \mathbf{Z}h$. Then $h \equiv h_+ := \frac{1}{2}(H + \delta_+) \pmod{\tilde{\Gamma}}$.*

(4) *$\{\delta_+, \Gamma_2, \ldots, \Gamma_{17}, h_+\}$ form a $\mathbf{Z}$-basis of* Pic X, *and the intersection matrix of this basis is given by: $\delta_+^2 = -34$, $\delta_+.\Gamma_i = 0$ $(2 \leq i \leq 16)$, $\delta_+.\Gamma_{17} = -6$, $\delta_+.h_+ = -17$, $h_+^2 = -6$, $h_+.\Gamma_i = 0$ $(2 \leq i \leq 16)$, $h_+.\Gamma_{17} = -3$.*

(5) *There are smooth rational curves F, F_1, F_2 such that $F \sim -2\delta_+ + \sum_{i=10}^{17}(i-9)\Gamma_i + h_+$, $F_1 \sim -6\delta_+ + \sum_{i=2}^{17}(2i-3)\Gamma_i + h_+$, and $F_2 \sim -3\delta_+ + \sum_{i=5}^{17}(i-4)\Gamma_i + h_+$.*

(6) *Theorem 3.1 is true with $F.\Gamma_9 = 1$, by letting F, F_1, F_2 be as in (5) and $\eta_1 := F_1 + \Gamma_1 + \Gamma_2$, $\eta_2 := F_2 + \sum_{i=4}^{17} \Gamma_i$, $\psi := \Phi_{|\eta_1|}$.*

Proof. The first part of (1) follows from the fact that $18 = |\det(\Gamma)| = |\det(\tilde{\Gamma})|\,|\tilde{\Gamma} : \Gamma|^2$.

(2) Let $\delta \in \tilde{\Gamma}$ so that $\tilde{\Gamma} = \Gamma + \mathbf{Z}\delta$. By (1), $\delta = \frac{1}{3}\sum_{i=1}^{17} a_i\Gamma_i$ for some integers a_i. Note that

$$3\delta.\Gamma_1 = -2a_1 + a_2, \quad 3\delta.\Gamma_i = a_{i-1} - 2a_i + a_{i+1}$$

are all 0 (mod 3). Hence $a_i \equiv ia_1$ (mod 3). Thus $\delta \equiv \frac{a_1}{3}\sum_{i=1}^{17} i\Gamma_i$ (mod Γ). Since $\delta \notin \Gamma$, $a_1 \equiv \pm 1$ (mod 3). Now (2) follows.

(3) Set $n := |\mathrm{Pic}\, X : (\tilde{\Gamma} \oplus \mathbf{Z}H)|$. As in Lemma 3.3, we can prove that $h = \frac{1}{n}(H + a_1\delta_+ + \sum_{i=2}^{17} a_i\Gamma_i)$ for some integers a_i; moreover n divides $|\det(\tilde{\Gamma})| = 2$. The latter, together with $2H^2 = |\det(\tilde{\Gamma} \oplus \mathbf{Z}H)| = n^2|\det(\mathrm{Pic}\, X)| = 5n^2$, implies that $n = 2$ and $H^2 = 10$. This proves the second part of (1).

We shall use the calculation that $\delta_+^2 = -34$, $\delta_+.\Gamma_{17} = -6$, $\delta_+.\Gamma_i = 0$ $(1 \le i \le 16)$. Note that

$$2h.\Gamma_1 = a_2, \quad 2h.\Gamma_2 = -2a_2 + a_3, \quad 2h.\Gamma_i = a_{i-1} - 2a_i + a_{i+1} \ (3 \le i \le 16)$$

are all 0 (mod 2). Hence $a_i \equiv 0$ (mod 2) for all $2 \le i \le 17$. So $h \equiv \frac{1}{2}(H + a_1\delta_+)$ (mod Γ).

To finish (3), we have only to show that $a_1 \equiv 1$ (mod 2). In fact, if a_1 is even, then $h \equiv H/2$ (mod $\tilde{\Gamma}$) and hence $H/2 \in \Gamma^{\perp} \subseteq \mathrm{Pic}\, X$, a contradiction to the fact that H is a generator of $\Gamma^{\perp}$. This also proves (3).

(4) is from a direct calculation.

(5) As in Lemma 3.3, we can prove Claims (1.1)–(1.5) below.

Claim 1. (1.1) *$H.F = H.F_i = 5$ $(i = 1, 2)$, $\delta_+.F = 3$, $\delta_+.F_1 = 1$, $\delta_+.F_2 = 7$, $h_+.F = 4$, $h_+.F_1 = 3$, $h_+.F_2 = 6$; $F^2 = F_i^2 = -2$ $(i = 1, 2)$.*

(1.2) *The intersection number (0 or 1) between any two distinct divisors of F, F_1, F_2, Γ_i $(i = 1, 2, \ldots, 17)$ are as discribed in Figure 2 with $(a, b) = (9, 14)$, if we regard these divisors as irreducible curves, e.g. $F_2.\Gamma_4 = F_2.\Gamma_{17} = F.F_1 = F.\Gamma_9 = 1$, $F.F_2 = F.\Gamma_i = 0$ $(1 \le i \le 17, i \ne 9)$.*

(1.3) $F_1 + \Gamma_1 + \Gamma_2 \sim F_2 + \sum_{i=4}^{17} \Gamma_i$.

(1.4) $|F| \ne \phi$, $|F_i| \ne \phi$ $(i = 1, 2)$.

(1.5) *For $G := F_1 + \Gamma_1 + \Gamma_2$, one has $G.\Gamma_3 = 1$, $G.\Gamma_i = 0$ $(1 \le i \le 17$, $i \ne 3)$, $G.F = 1$, $G.F_i = 0$ $(i = 1, 2)$, $G^2 = 0$.*

(1.6) *G is a numerically effective divisor.*

(1.7) *It is impossible that $F \ge F_i$ for $i = 1$ or 2.*

Suppose the contrary that Claim (1.6) is false. Then, as in Lemma 3.3, $G = P + N$ so that $P \equiv ph_+$ (mod $\tilde{\Gamma}$), $N \equiv nh_+$ (mod $\tilde{\Gamma}$) for some positive integers p, n. Hence $(p + n)h_+ \equiv G \equiv F_1 \equiv h_+$ (mod $\tilde{\Gamma}$), and $p + n = 1$, a contradiction. So Claim (1.6) is true.

For $i = 1$ (resp. 2), $3(F - F_i) = \sum_{j=1}^{17} b_j\Gamma_j$ where $b_j \in \mathbf{Z}$ and $b_8 = -7$ (resp. -4). Hence $|F - F_i| = \phi$ for the Kodaira dimension $\kappa(X, \Gamma) = 0$. This proves Claim(1.7).

Now (5) and (6) can be proved similarly as in Lemma 3.3. (cf. Lemmas 3.3 and 3.5).

Lemma 3.5. *With the assumptions and notations in Theorem 3.1, there is a unique smooth rational curve F such that $F.\Gamma = F.(\Gamma_9 + \Gamma_{14}) = 1$.*

Proof. The existence of such F is proved in Lemmas 3.3 and 3.4. Suppose the contrary that there are two different cross-sections F', F'' each of them having intersection 1 with Γ and also with $\Gamma_9 + \Gamma_{14}$. There are 3 possible cases: $F'.\Gamma_9 = F''.\Gamma_9 = 1$, or $F'.\Gamma_{14} = F''.\Gamma_{14} = 1$, or $F'.\Gamma_9 = F''.\Gamma_{14} = 1$. But then if letting $\Gamma_{18} := F' - F''$, we have respectively $\det(\Gamma_i.\Gamma_j) = 36(2 + F'.F'') > 0$, $36(2 + F'.F'') > 0$, and $7 + 36(F'.F'') > 0$, a contradiction to the fact that $\operatorname{Pic} X$ has signature (1, 17). This proves Lemma 3.5.

Proposition 3.6. *Let X be a K3 surface of Picard number* 18 *and* $|\det(\operatorname{Pic} X)| = 5$. *Then the existence of a Jacobian elliptic fibration ψ on X with a cross-section P_0 having two fibers $\{\eta_1, \eta_2\}$ of one of 5 Kodaira types (i) $\{II^*, III^*\}$, (ii) $\{I_{10}, III^*\}$, (iii) $\{I_5^*, IV^*\}$, (iv) $\{I_{10}^*, I_2$ or $III\}$, (v) $\{I_{15}, I_3$ or $IV\}$, implies the existence of 4 new Jacobian elliptic fibrations having fibers $\{\eta_1', \eta_2'\}$ of the remaining 4 types; in other words, the existence of one type will imply the existence of all 5 types.*

Moreover, in each of 5 cases, any singular fiber ($\neq \eta_1, \eta_2$) has Kodaira type I_1 or II.

Proof. We shall proceed in the way "(v) ⇒ (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v). We shall fully apply [Sh]. Let $E = E(\psi)$ denote the Mordell-Weil lattice spanned by all cross-sections of ψ, $E^0 = E(\psi)^0$ the torsion-free and index-finite sublattice $\{P \in E \mid P$ and P_0 meet the same irreducible component in each fiber$\}$, T the sublattice of $NS(X)$ generated by the zero section P_0 and all irreducible components in all fibers of ψ.

Claim 1. (1.1) *In Cases (i)–(v), we have respectively,* $\operatorname{rk}(E) = 1$, $E \cong \mathbf{Z}/2\mathbf{Z}$, $\operatorname{rk}(E) = 1$, $\operatorname{rk}(E) = 1$ *and* $E \cong \mathbf{Z}/3\mathbf{Z}$.

(1.2) *The last assertion of Proposition 3.6 is true.*

In Cases (ii) and (v), the calculation $\operatorname{rk}(E) = \rho(X) - \operatorname{rk}(T) = 18 - \operatorname{rk}(T) \le 16 - (\#\eta_1 - 1) - (\#\eta_2 - 1) = 0$ [Sh, Cor. 5.3], implies Claim (1.2) and $\operatorname{rk}(E) = 0$. By [Sh, Th 8.7 and Def 7.3], $|E|^2 = \det(T)/\det(NS(X)) = \det(T)/5 = 2^2, 3^2$ respectively. Hence Claim (1.1) is true.

In Cases (i), (iii), (iv), we have $\operatorname{rk}(E) \le 1$ as above; and if Claim (1.2) is false, i.e., if $\operatorname{rk}(E) = 0$ then ψ has a fiber η_3 with 2 components and every fiber ($\neq \eta_i$ for $i = 1, 2, 3$) is irreducible, which leads to that $|E|^2 = \det(T)/5$, a contradiction to an easy calculation that $\det(T)$ is not divisible by 5 [Sh, Def 7.3]. This proves Claim 1.

"(v) ⇒ (i)". Assume that ψ, F, η_1, η_2 fit Case(v). Take a torsion element $P_1 \in E - \{P_0\}$. Then, by [Sh, Th 8.6, Table(8.16) and the proof of Th 8.4], the height pairing

$$0 = \langle P_1, P_1 \rangle = 2\chi(\mathcal{O}_X) + 2(P_1.P_0) - \sum_v contr_v(P_1).$$

Thus, $P_1.P_0 = 0$, P_0 and P_1 meet different irreducible components in η_2, and η_1 contains a linear chain of 4 curves linking the irreducible components of η_1 meeting P_0 and P_1. It is easy to see that there is an elliptic fibration ψ' so that $P_0 + P_1 + \eta_1 + \eta_2$ contains a cross-section of ψ' and two fibers of ψ' fitting Case (i).

"*(i)* ⇒ *(ii)*". Assume that ψ, F, η_1, η_2 fit Case (i). By [Sh, Table (8.16)], for any $P_1 \in E - \{P_0\}$, one has $\langle P_1, P_1 \rangle = 4 + 2(P_1.P_0) - (0 \text{ or } 3/2) > 0$, whence P_1 is not torsion [Sh, proof of Th 8.4]. So E is torsion free of rank 1. Thus we can write $E = \mathbf{Z}P_1$, $E_0 = n\mathbf{Z}P_1$. By [Sh, Th 8.7 and (8.17)], $n^2\langle P_1, P_1 \rangle = \det(E^0) = \det(NS(X))|E : E_0|^2/\det(T) = 5n^2/2$. Thus $5/2 = \langle P_1, P_1 \rangle = 4 + 2(P_1.P_0) - 3/2$ (hence P_1 and P_0 meet different tip components of η_2), and $P_1.P_0 = 0$. Then there is an elliptic fibration ψ' so that $P_0 + P_1 + \eta_1 + \eta_2$, together with an auxiliary smooth rational curve, contains a cross-section of ψ' and two fibers of ψ' fitting Case(ii).

"*(ii)* ⇒ *(iii)*". Assume that ψ, F, η_1, η_2 fit Case (ii). Take a torsion element $P_1 \in E - \{P_0\}$. Then $0 = \langle P_1, P_1 \rangle = 2\chi(\mathcal{O}_X) + 2(P_1.P_0) - \sum_v contr_v(P_1) = 4 + 2(P_1.P_0) - i(10 - i)/10 - (0 \text{ or } 3/2)$ for some $0 \le i \le 9$. Hence $(P_1.P_0) = 0$, $i = 5$ and we choose 3/2 instead of 0, whence P_0 and P_1 meet different tip components of η_2, and η_1 contains a linear chain of 4 curves linking the irreducible components of η_1 meeting P_0 and P_1. Now there is an elliptic fibration ψ' so that $P_0 + P_1 + \eta_1 + \eta_2$ contains a cross-section of ψ' and two fibers of ψ' fitting Case (iii).

"*(iii)* ⇒ *(iv)*". Assume that ψ, F, η_1, η_2 fit Case (iii). Clearly, there is an elliptic fibration ψ' so that $P_0 + \eta_1 + \eta_2$ contains a cross-section of ψ', a fiber η_1' of Kodaira type I_{10}^* and a curve F_2 disjoint from η_1'. Let η_2' be the fiber of ψ' containing F_2. If $\#(\eta_2') \ge 3$, then, as in Claim 1, $(\#\eta_2') = 3$, $\mathrm{rk}(E) = 0$ and $|E|^2 = \det(T)/5 = 4 \times 3/5$, a contradiction. Thus $\#(\eta_2') = 2$ and ψ' fits Case (iv).

"*(iv)* ⇒ *(v)*". Assume that ψ, F, η_1, η_2 fit Case (iv). Let $n := |E : E^0|$.

CLAIM 2. $E_{tor} \cong \mathbf{Z}/2\mathbf{Z}$.

Suppose the contrary that E is torsion free. Then we can write $E = \mathbf{Z}P_1$, $E^0 = n\mathbf{Z}P_1$. By [Sh, Th 8.7], $n^2\langle P_1, P_1 \rangle = \det(E^0) = 5n^2/(4 \times 2)$. This implies that $5/8 = \langle P_1, P_1 \rangle = 2\chi(\mathcal{O}_X) + 2(P_1.P_0) - \sum_v contr_v(P_1) = 4 + 2(P_1.P_0) - (0 \text{ or } 1 \text{ or } 7/2) - (0 \text{ or } 1/2)$, which is impossible (by multiplying by 8).

Suppose the contrary that $P_1 \ne P_2$ are two torsion elements in E. Then $0 = \langle P_i, P_i \rangle = 4 + 2(P_i.P_0) - (0 \text{ or } 1 \text{ or } 7/2) - (0 \text{ or } 1/2)$, whence $P_i.P_0 = 0$, we choose 7/2 and 1/2, and P_i and P_0 meet different components of η_2 and P_i and P_0 meet tip components of η_1 sprouting from different branchs. On the other hand, $0 = \langle P_1, P_2 \rangle = \chi(\mathcal{O}_X) + (P_1.P_0) + (P_2.P_0) - (P_1.P_2) - \sum_v contr_v(P_1, P_2) = 2 - (P_1.P_2) - (7/2 \text{ or } 3) - 1/2 < 0$, a contradiciton. This proves Claim 2.

By Claim 2, we can write $E = \mathbf{Z}P_3 \oplus \mathbf{Z}P_1$, $E^0 = \frac{n}{2}\mathbf{Z}P_3$, where P_1 is the unique torsion element (of order 2). Now $5n^2/8 = \det(E^0) = \frac{n^2}{4}\langle P_3, P_3 \rangle$ implies that $5/2 = \langle P_3, P_3 \rangle = 4 + 2(P_3.P_0) - (0 \text{ or } 1 \text{ or } 7/2) - (0 \text{ or } 1/2)$. Hence either

Case (iv-a) $P_3.P_0 = 0$, P_3 and P_0 meet different components of η_2 and P_3 and P_0 meet two neighbouring tip components of η_1, or Case (iv-b) $P_3.P_0 = 1$, P_3 and P_0 meet the same component in η_2 and P_3 and P_0 meet two tip components of η_1 sprouting from different branchs.

On the other hand, $0 = \langle P_1, P_3 \rangle = \chi(\mathcal{O}_X) + (P_1.P_0) + (P_3.P_0) - (P_1.P_3) - \sum_v contr_v(P_1, P_3) = 2 + (P_3.P_0) - (P_1.P_3) - (1/2 + 1/2$ in Case (iv-a); $7/2 + 0$ or $3 + 0$ in Case (iv-b)). Thus, in Case (iv-a), $P_1.P_3 = 1$; in Case (iv-b), $P_1.P_3 = 0$ and P_1 and P_3 meet two neighbouring tip components of η_1. Then there is an elliptic fibration ψ' so that $P_0 + P_1 + P_3 + \eta_1 + \eta_2$, together with an auxiliary smooth rational curve, contains a cross-section of ψ' and two fibers of ψ' fitting Case (v). This proves Proposition 3.6.

Corollary 3.7. *Let X be as in Theorem* 3.1. *Then for each* $1 \le i \le 5$, *there is a Jacobian elliptic fibration having fibers* $\{\eta_1, \eta_2\}$ *fitting Case* (*i*) *in Proposition* 3.6.

§4. Proofs of Theorems

We first prove Theorem 1. Theorem 1(1) is proved in [Z4]. Now we prove Theorem 1(2). Let T be a rational log Enriques surface of Type A_{17} and index I. By [Z3, Z4], $I = 2, 3, 4, 5$ or 10. Suppose the contrary that $I = 10$. We shall use the notations in Lemma 1.1. Now Y/σ^2 (resp. Y/σ^5) is a rational log Enriques surface of Type A_{17} and index 5 (resp. 2), and hence $\rho(X) = 18$ (resp. $\rho(X) \ge 19$) by Lemma 4.1 below and [Z4, Lemma 3.1 and Corollary 3.4]. We reach a contradiction. So Theorem 1(2) is true.

Next we prove Main Theorem 3(1). Let T be a rational log Enriques surface of index 5 and Type A_{17}. We employ the notations at the beginning of §1 and in Lemma 1.1:

$$\pi : Y \to T, \quad g : X \to Y, \quad \Gamma = g^{-1}(\mathrm{Sing}\, Y), \quad \langle \sigma \rangle = \mathrm{Gal}(Y/T), \quad \sigma^* \omega = \zeta_5 \omega.$$

We denote by $\Gamma(1) = \sum_{i=1}^{17} \Gamma_i$ where $\Gamma_i.\Gamma_{i+1} = 1$, the unique connected component of Γ of Dynkin type A_{17}.

Lemma 4.1. *Let T be a rational log Enriques surface of index 5 and Type* A_{17}. *Then we have*:

(1) *The Picard number* $\rho(X) = 18$ *and* $\sigma^* \,|\, \mathrm{Pic}\, X = \mathrm{id}$.

(2) *T is of actual Type* A_{17}, *i.e.*, $\Gamma = \Gamma(1)$. *The fixed locus* X^σ *is equal to*

$$\mathrm{Supp}(\Gamma_4 + \Gamma_9 + \Gamma_{14}) \coprod$$

$$\{p_1, p_{1,2}, p_{2,3}, p_{5,6}, p_{6,7}, p_{7,8}, p_{10,11}, p_{11,12}, p_{12,13}, p_{15,16}, p_{16,17}, p_{17}, q\},$$

where $p_{i,i+1} = \Gamma_i \cap \Gamma_{i+1}$, $p_j \in \Gamma_j$, *and q is a point not on* Γ.

Moreover, σ^* *can be expressed as* $(\zeta_5^{-2}, \zeta_5^{-2})$ (*resp.* $(\zeta_5^2, \zeta_5^{-1})$) *around the 4 points* $p_{1,2}, p_{6,7}, p_{11,12}, p_{16,17}$ (*resp. the 9 other isolated points in* X^σ), *with suitable coordinates.*

Proof. By Lemmas 1.1 and 1.2, the hypotheses in Lemma 1.5 are satisfied, and we shall use the notations there. So $4t = \operatorname{rank} T_X = 22 - \rho(X) \le 4$ because X contains $\Gamma(1)$ which is of Dynkin type A_{17}. Thus $t = 1$ and $\rho(X) = 18$.

On the other hand, each component Γ_i of $\Gamma(1)$ is σ-stable because $\operatorname{ord}(\sigma) = 5$ while the graph-automorphism group of $\Gamma(1)$ has order 2 (cf. Lemma 1.2). So $(22 - 4s - 4t) \ge 17$ and hence $s = 0$ (cf. Lemma 1.5). This proves (1).

$1 + \#(\Gamma) \le \rho(X) = 18$ implies that $\Gamma = \Gamma(1)$. The rest of (2) follows from Lemmas 1.4–1.6.

We now continue the proof of Main Theorem 3(1). In view of Lemmas 4.1 and 1.5, we can apply Theorem 3.1. We shall use the notations ψ, η_i, F, F_i $(i = 1, 2)$, etc. there. Clearly, the isolated σ-fixed point q not on Γ, equals $F \cap F_1$, and hence $X^\sigma \subseteq \operatorname{Supp}(\eta_1 + \eta_2)$.

Since $\sigma^* \mid \operatorname{Pic} X = \mathrm{id}$, σ permutes fibers of ψ. Since the cross-section F contains only two σ-fixed points $F \cap \eta_i$ $(i = 1, 2)$, η_i $(i = 1, 2)$ are only σ-stable fibers of ψ. Now $24 = \chi(X) = \sum_{i \ge 1} \chi(\eta_i)$ where η_i runs over the set of all singular fibers, implies that η_1, η_2 are of Kodaira type IV, I_{15} (Theorem 3.1), and that if we let η_3 be any singular fiber other than η_1, η_2 then η_3 is of Kodaira type I_1 and only

$$\eta_1, \eta_2, \sigma^{j*}\eta_3 \ (0 \le j \le 4)$$

are singular fibers of ψ.

Resolving 13 quotient singularities of X/σ (under 13 isolatetd σ-fixed points) and blowing down uniquely and smoothly some curves under η_i $(i = 1, 2)$ we get a rational surface S so that ψ induces a relatively minimal elliptic fibration $\varphi : S \to \mathbf{P}^1$ whose only singular fibers ζ_i $(i = 1, 2, 3)$ (under η_i) are of Kodaira type IV^*, I_3, I_1.

Let E, Σ_i $(i = 3, 4, 9, 14)$ be the image on S of F, Γ_i. Then $E + \Gamma_3 + \xi_1 + \xi_2$ is given in Figure 1 where $(a, b) = (9, 14)$ or $(14, 9)$ if $F.\Gamma_9 = 1$ or $F.\Gamma_{14} = 1$ accordingly (cf. Theorem 3.1). Now Lemma 2.2 and the uniqueness of the blowing-down $\nu : S \to \mathbf{P}^2$ there show that the rational log Enriques surface $T = Y/\sigma$ is isomorphic to $T(9)$ or $T(14)$ in Example 2.1 accordingly. This proves Main Theorem 3(1).

Theorem 3(2) follows the proof of Theorem 3(1), Lemma 4.1 and the construction of $T(i)$ in Example 2.1.

Finally, we prove Theorem 8 below which will imply Main Theorem 4.

Theorem 8. *There is, upto isomorphisms, only one pair* (X, σ) *of K3 surface* X *and an order* 5 *subgroup* $\langle \sigma \rangle$ *of* $Aut(X)$ *satisfying*:

$\sigma^*\omega = \zeta_5$ *for a non-zero holomorphic 2-form* ω *where* $\zeta_5 = \exp(2\pi\sqrt{-1}/5)$, *and the number* $N = N_0 - \sum_{i \ge 1}(i - 1)N_i$ *defined in Lemma 1.5 satisfies* $N \ge 3$.

Proof. By Lemma 1.5, $N = 3$, $\rho(X) = 18$, $\sigma^* \mid \operatorname{Pic} X = \mathrm{id}$, $|(\operatorname{Pic} X)^\vee / (\operatorname{Pic} X)| = |\det(\operatorname{Pic} X)| = 5$. By [N2, Cor.1.13.5], $\operatorname{Pic} X = U \oplus T$ and hence there is an elliptic fibration $\psi : X \to \mathbf{P}^1$ with a cross-section F. Note that σ stabilizes F and permutes fibers of ψ because $\sigma^* \mid \mathrm{Pic}\, X = \mathrm{id}$.

Since no elliptic curve has an order 5 automorphism with a fixed point, a general fiber η of ψ is not σ-stable for otherwise $\sigma \in \mathrm{Aut}(\eta)$ fixes $F \cap \eta$. Thus the cross-section F $(\cong \mathbf{P}^1)$ is not σ-fixed and hence has exactly two σ-fixed points which lie on fibers η_1, η_2 say. Therefore, we have:

CLAIM 1. *Only η_1, η_2 are σ-stable fibers of ψ. Hence $X^\sigma \subseteq \mathrm{Supp}(\eta_1 + \eta_2)$. In particular, $N_i = 0$ for all $i \geq 2$ and $N_0 = N = 3$.*

Claim 1, Lemma 1.7, $N_0 = 3$ and $\sum_\alpha \chi(\eta_\alpha) = \chi(X) = 24$, where η_α runs over the set of all singular fibers of ψ, imply:

CLAIM 2. *Only η_1, η_2, $\sigma^{i*}\eta_3$ $(0 \leq i \leq 4)$ are singular fibers of ψ, where η_3 is of Kodaira type I_1 and $\{\eta_1, \eta_2\}$ has one of the following Kodaira types:*

(8-1) $\{II^*, III^*\}$, (8-2) $\{I_{10}, III^*\}$, (8-3) $\{I_5^*, IV^*\}$, (8-4) $\{I_{10}^*, III\}$, (8-5) $\{I_{15}, IV\}$.

In view of Proposition 3.6, we may assume that ψ fits Case (8-5). Then $F + \eta_1 + \eta_2$ contains a linear chain of 17 (orderly) smooth rational curves $\Gamma = \sum_{i=1}^{17} \Gamma_i$. By Lemmas 1.4–1.6, X^σ is a disjoint union of 3 curves Γ_4, Γ_9, Γ_{14} and 13 isolated points (12 of them are on Γ). Let $X \to Y$ be the contraction of Γ. Then Y/σ is clearly a rational log Enriques surface of index 5 and Type A_{17}. Thus, $(X, \langle\sigma\rangle)$ is equivariantly isomorphic to $(X(9), \langle\sigma(9)\rangle)$ in Theorem 3(2). This proves Theorem 8.

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