

Note on a paper of N. Iwasaki

To the memory of Professor N. Iwasaki

By

Tatsuo NISHITANI

1. Introduction

Let $P(x, D)$ be a differential operator of order m defined in an open set Ω in \mathbf{R}^n with principal symbol $p(x, \xi)$ which can be factorized as

$$p(x, \xi) = \prod_{j=1}^m q_j(x, \xi), \quad q_j(x, \xi) = \xi_1 - \lambda_j(x, \xi')$$

where $q_j(x, \xi)$ are real valued pseudodifferential symbol of order 1 and $x = (x_1, x') = (x_1, x_2, \dots, x_n)$, $\xi = (\xi_1, \xi') = (\xi_1, \xi_2, \dots, \xi_n)$. We assume that the characteristics of q_j intersects normally and non-involutively each other, that is

$$\{q_i, q_j\} \neq 0 \quad \text{on } q_i = q_j = 0 \quad \text{for } i \neq j$$

where $\{q_i, q_j\}$ denotes the Poisson bracket of q_i and q_j .

According to Iwasaki [2], we define the signature of a triplet (q_i, q_j, q_k) at z^0 where $q_i(z^0) = q_j(z^0) = q_k(z^0) = 0$. Let us say that three real numbers a, b, c have the same sign if they are simultaneously positive or simultaneously negative. We say $\text{sgn}(q_i, q_j, q_k)(z^0) = +$ if $\{q_i, q_j\}(z^0)$, $\{q_j, q_k\}(z^0)$, $\{q_k, q_i\}(z^0)$ have the same sign and $\text{sgn}(q_i, q_j, q_k)(z^0) = -$ otherwise.

When $m = 3$, in [2], Iwasaki proved that in order that the Cauchy problem of $P(x, D)$ is well posed the lower order terms must verify additional conditions at z^0 more than Ivrii-Petkov condition if $\text{sgn}(q_1, q_2, q_3)(z^0) = +$. On the other hand, in [3] we proved that if the propagation cone at every triple characteristic is transversal to the doubly characteristic set and the lower order terms verify the Ivrii-Petkov condition then the Cauchy problem is well posed.

Here we recall that the localization p_{z^0} of p is the first non-trivial term in the Taylor expansion of p at z^0 which is a hyperbolic polynomial on $T_{z^0}(T^*\Omega)$ with respect to $\Theta = -H_{x_1} \in T_{z^0}(T^*\Omega)$ (see [1]) where H_f denotes the Hamilton vector field of $f \in C^\infty(T^*\Omega)$. The propagation cone at z^0 is the dual cone of the hyperbolic cone $\Gamma(p_{z^0}, \Theta)$ (for the definition, see [1]) with respect to the canonical symplectic structure on $T_{z^0}(T^*\Omega)$ induced by the 2-form $d\xi \wedge dx$.

In this note we show that, when $m = 3$ the condition $\text{sgn}(q_1, q_2, q_3)(z^0) = -$ is equivalent to that the propagation cone at z^0 is transversal to the doubly characteristic set (Corollary 2.4). Let $m > 3$. At z^0 where $q_j(z_0) = 0, 1 \leq j \leq m$ we can induce an order relation on the set $\{q_1, \dots, q_m\}$ provided that $\text{sgn}(q_i, q_j, q_k)(z^0) = -$ for every triplet (q_i, q_j, q_k) . Using this order relation we give another formulation of the condition that the propagation cone at z^0 intersects transversally to the doubly characteristic set (Theorem 2.3).

2. Result

Let $q_i(x, \xi)$ be classical pseudodifferential symbols of order 1 defined near z^0 where $q_j(z^0) = 0$ and assume that the differentials dq_j are linearly independent at z^0 . We also assume that $\{q_\mu, q_\nu\}(z^0) \neq 0$ for every pair $\mu, \nu, \mu \neq \nu$. Let

$$(2.1) \quad p(x, \xi) = \prod_{j=1}^m q_j(x, \xi)$$

and assume that $p(x, \xi)$ is (microlocally) hyperbolic with respect to Θ . Thus we may suppose that $q_i(x, \xi)$ are real-valued and $dq_i(\Theta) > 0$ at z^0 . Then it is clear that

$$\Gamma(p_{z^0}, \Theta) = \{X \in T_{z^0}(T^*\Omega) \mid dq_j(X) > 0, \forall j\}.$$

Denoting by $C(p_{z^0}, \Theta)$ the propagation cone at z^0 we easily see that

$$C(p_{z^0}, \Theta) = \left\{ X \in T_{z^0}(T^*\Omega) \mid X = \sum \alpha_j H_{q_j}(z^0), \alpha_j \geq 0 \right\}.$$

To simplify notations we write Γ_{z^0} and C_{z^0} for $\Gamma(p_{z^0}, \Theta)$ and $C(p_{z^0}, \Theta)$ respectively. Let $S_{ij} = \{(x, \xi) \mid q_i(x, \xi) = q_j(x, \xi) = 0\}$ for $i \neq j$ and define the map π_{ij} :

$$\pi_{ij} : T_{z^0}(T^*\Omega) \ni X \mapsto (dq_i(X), dq_j(X)) \in \mathbf{R}^2.$$

Then we have

Lemma 2.1. *We have*

$$(2.2) \quad C_{z^0} \cap T_{z^0} S_{ij} = \{0\}$$

if and only if $\pi_{ij}(C_{z^0})$ is a proper cone in \mathbf{R}^2 . Moreover this is equivalent to

$$-H_{c_i q_i + c_j q_j}(z^0) \in \Gamma_{z^0}$$

with some $c_i, c_j \in \mathbf{R}$.

Proof. We assume that $C_{z^0} \cap T_{z^0} S_{ij} = \{0\}$. We examine that $\pi_{ij}(C_{z^0})$ is a proper cone in \mathbf{R}^2 . Otherwise $\pi_{ij}(C_{z^0})$ would contain a line. Thus there exist $0 \neq X_k = \sum \alpha_{k\mu} \pi_{ij}(H_{q_\mu}), k = 1, 2$ such that $X_1 + X_2 = 0$. This implies that

$$Y = \sum (\alpha_{1\mu} + \alpha_{2\mu}) H_{q_\mu} \in T_{z^0} S_{ij}.$$

Since $\sum(\alpha_{1\mu} + \alpha_{2\mu}) > 0$ and hence $Y \neq 0$ this contradicts the assumption. Conversely assume that $\pi_{ij}(C_{z^0})$ is a proper cone in \mathbf{R}^2 . Let $X = \sum \alpha_\mu H_{q_\mu}(z^0) \in C_{z^0} \cap T_{z^0}S_{ij}$. From $dq_i(X) = dq_j(X) = 0$ it follows that

$$\sum \alpha_\mu \pi_{ij}(H_{q_\mu}) = 0.$$

Since $\pi_{ij}(H_{q_\mu}) \neq 0$ we have $\alpha_\mu = 0$ and hence $X = 0$. That is $C_{z^0} \cap T_{z^0}S_{ij} = \{0\}$.

We next assume that $Z = -H_{c_i q_i + c_j q_j}(z^0) \in \Gamma_{z^0}$ with some $c_i, c_j \in \mathbf{R}$ so that

$$dq_\mu(Z) = c_i dq_i(H_{q_\mu}) + c_j dq_j(H_{q_\mu}) > 0, \quad \forall \mu.$$

This shows that $\pi_{ij}(C_{z^0})$ is a proper cone in \mathbf{R}^2 . Conversely we assume that $\pi_{ij}(C_{z^0})$ is a proper cone. Then we can choose c_i, c_j such that

$$\langle (c_i, c_j), \pi_{ij}(H_{q_\mu}) \rangle = (c_i dq_i + c_j dq_j)(H_{q_\mu}) > 0, \quad \forall \mu.$$

This proves $-H_{c_i q_i + c_j q_j}(z^0) \in \Gamma_{z^0}$. Hence the assertion.

Let us set $a_{\mu\nu} = \{q_\mu, q_\nu\}(z^0)$ and we express the condition (2.2) in terms of $a_{\mu\nu}$. Note that $Z = a_{\alpha\beta}H_{q_\gamma} + a_{\beta\gamma}H_{q_\alpha} + a_{\gamma\alpha}H_{q_\beta} \in T_{z^0}S_{\alpha\beta\gamma}$ where

$$S_{\alpha\beta\gamma} = \{(x, \xi) \mid q_\alpha(x, \xi) = q_\beta(x, \xi) = q_\gamma(x, \xi) = 0\}.$$

If $a_{\alpha\beta}, a_{\beta\gamma}, a_{\gamma\alpha}$ have the same sign then it follows that $Z \in C_{z^0}$ which contradicts to $C_{z^0} \cap T_{z^0}S_{\alpha\beta\gamma} = \{0\}$. Thus if (2.2) holds for every pair i, j ($i \neq j$) then one has $\text{sgn}(q_\alpha, q_\beta, q_\gamma)(z^0) = -$ for every triplet (α, β, γ) (cf. Lemma 4.1 in [4]).

We say that $q_\alpha \gg q_\beta$ if $a_{\alpha\beta} > 0$ or $\alpha = \beta$. Assuming that $\text{sgn}(q_\alpha, q_\beta, q_\gamma)(z^0) = -$ for every triplet (α, β, γ) , the relation \gg becomes an order relation which is easily verified.

Lemma 2.2. Assume that $\text{sgn}(q_\alpha, q_\beta, q_\gamma)(z^0) = -$ for every triplet (α, β, γ) . Let $q_i \gg q_j$. Then $\pi_{ij}(C_{z^0})$ is a proper cone if and only if

$$\frac{a_{vi}}{a_{vj}} > \frac{a_{\mu i}}{a_{\mu j}}$$

for every q_ν, q_μ with $q_\nu \ll q_j \ll q_i \ll q_\mu$.

Proof. Recall that $\pi_{ij}(H_{q_\kappa}) = (a_{\kappa i}, a_{\kappa j})$. Note that $(a_{\kappa i}, a_{\kappa j})$ lies in the first, the third and the second quadrant if $q_i \ll q_\kappa, q_\kappa \ll q_j$ and $q_j \ll q_\kappa \ll q_i$ respectively. Let

$$\min_{q_\mu \cdot q_i \ll q_\mu} \frac{a_{\mu j}}{a_{\mu i}} = \frac{a_{\mu_0 j}}{a_{\mu_0 i}}.$$

Then it is easy to see that the condition

$$\langle (a_{\mu_0 j}, -a_{\mu_0 i}), (a_{vi}, a_{vj}) \rangle = a_{\mu_0 j} a_{vi} - a_{\mu_0 i} a_{vj} < 0$$

for every ν with $q_\nu \ll q_j$ is necessary and sufficient for $\pi_{ij}(C_{z^0})$ to be a proper cone. This proves the assertion.

Theorem 2.3. *Let $m > 3$. Then the following two conditions are equivalent.*

- (i) $C_{z^0} \cap T_{z^0} S_{ij} = \{0\}$ for every i, j ($i \neq j$).
- (ii) $\text{sgn}(q_i, q_j, q_k)(z^0) = -$ for every triplet (i, j, k) and for every quadruplet $(\alpha, \beta, \gamma, \delta)$ with $q_\alpha \ll q_\beta \ll q_\gamma \ll q_\delta$ we have

$$\frac{\{q_\alpha, q_\gamma\}(z^0)}{\{q_\alpha, q_\beta\}(z^0)} > \frac{\{q_\delta, q_\gamma\}(z^0)}{\{q_\delta, q_\beta\}(z^0)}.$$

Proof. It is clear from Lemma 2.2.

Corollary 2.4. *Let $m = 3$. Then the following two conditions are equivalent.*

- (i) $C_{z^0} \cap T_{z^0} S_{ij} = \{0\}$ for every i, j ($i \neq j$).
- (ii) $\text{sgn}(q_1, q_2, q_3)(z^0) = -$.

DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL OF SCIENCE
OSAKA UNIVERSITY

References

- [1] M. F. Atiyah, R. Bott and L. Gårding, Lacunas for hyperbolic differential operators with constant coefficients I, *Acta Math.*, **124** (1970), 109–189.
- [2] N. Iwasaki, Bicharacteristic curves and wellposedness for hyperbolic equations with non involutive multiple characteristics, *J. Math. Kyoto Univ.*, **34** (1994), 41–46.
- [3] K. Kajitani, S. Wakabayashi and T. Nishitani, The Cauchy problem for hyperbolic operators of strong type, *Duke Math. J.*, **75** (1994), 1–56.
- [4] T. Nishitani, Propagation of singularities for hyperbolic operators with transverse propagation cone, *Osaka J. Math.*, **27** (1990), 1–16.