J. Math. Kyoto Univ. (JMKYAZ)
38-4 (1998), 749–768

On sufficiently connected manifolds which are homotopy equivalent

By

Norio KOBAYAKAWA

§0. Introduction

Let M be a simply connected closed smooth m-manifold satisfying the following hypotheses :

(H1) $H_i(M)=0$ except for $i=0$, p, q, $m=p+q$ $(0<p<q)$,

(H2) The tangent bundle of M is trivial on its p-skeleton.

Here the second hypothesis is satisfied if $p\equiv 3,5,6,7$ (mod 8) or if M is a π-manifold. Such manifolds as M are called (p,q)-*primary* in [11]. A p-sphere bundle over the q-sphere and a connected sum of such bundles are (p,q)-primary. There also exist (p,q)-primary manifolds which are essentially different from such connected sums (cf.[5], [6]). In the classification of manifolds, primary manifolds play an important and fundamental role. We are concerned with the problem whether two (p,q)-primary manifolds which are (tangentially) homotopy equivalent are homeomorphic or not in the metastable range $2p>q>1$.

In this paper, we study the two cases $(p,q)=(n-1,n+1)$ $(n\geq 5)$ and $(p,q)=(n-2,n+1)$ $(n\geq 6)$. We show that in these cases, such manifolds as M which are (tangentailly) homotopy equivalent are homeomorphic and diffeomorphic modulo homotopy spheres in almost all cases. Furthermore, we show similar results for $(n-2)$-connected $2n$-manifolds $(n\geq 5)$ with torsion free homology groups. For $(n-1)$-connected $2n$-manifolds $(n>2)$ (that is, if $p=q>2$), we know such a property as above without assuming (H2) by [31] and [19]. For $(p,q)=(n,n+1)$ $(n\geq 2)$, such a property holds also if $H_p(M)$ is torsion free (cf [34]). For $(p,q)=(n-4,n+1)$ $(n\geq 10)$ or $(n-5,n+1)$ $(n\geq 12)$, such a property holds also in almost all cases (cf. [8]). We note that if $n-5\leq p\leq n-3$, $q=n+1$, there exist certain manifolds satisfying (H1), (H2) which are homemorphic but different from each other more than homotopy spheres as to differentiable structures (cf. [8]). The hypothesis (H1) implies that the homology groups are torsion free if $p<q-1$. For torsion cases, there exist certain manifolds with torsion homology groups which satisfy the conditions similar to (H1),(H2) and are tangentially homotopy equivalent but are not homeomorphic (cf. Example 7.9 of [20]).

Henceforth manifolds are connected, closed, smooth, and oriented, and homotopy

---

Communicated by G. Nishida, January 19, 1998
Revised May 1, 1988

equivalences and diffemorphisms are orientation preserving. The proofs of the theorems in this section are given in the following sections.

We have the following theorems. Here two m-manifolds M, M' are called *tangentially homotopy equivalent* if there exists a homotopy equivalence $f: M \to M'$ such that the tangent bundle τM is stably equivalent to $f^*(\tau M')$. We say that M, M' are *diffeomorphic mod* Θ_m if M is diffeomorphic to $M' \# \Sigma$ for some homotopy sphere Σ of Θ_m. If M, M' are diffeomorphic mod Θ_m, then M, M' are homeomorphic.

Theorem 1. *Let M, M' be simply connected $2n$-manifolds satisfying the hypotheses* ($H1$), ($H2$) *for* $(p,q)=(n-1,n+1)$ $(n \geq 5)$.

(i) *Let* $n \equiv 3,7$ (mod 8). *If M, M' are tangentially homotopy equivalent, then M, M' are diffeomorphic* mod Θ_{2n}.

(i) *Let* $n=8$ *or* $n \equiv 2,4,5,6$ (mod 8). *If M, M' are homotopy equivalent, then M, M' are diffeomorphic* mod Θ_{2n}.

(iii) *Let* $n \equiv 0,1$ (mod 8). *If M, M' are π-manifolds which are homotopy equivalent, then M, M' are diffeomorphic* mod Θ_{2n}.

Theorem 2. *Let M, M' be simply connected* $(2n-1)$*-manifolds satisfying the hypotheses* (H1), (H2) *for* $(p,q)=(n-2,n+1)$ $(n \geq 6)$.

(i) *Let* $n \equiv 3,7$ (mod 8). *If M, M' are tangentially homotopy equivalent, then M, M' are diffeomorphic* mod Θ_{2n-1}.

(ii) *Let* $n \equiv 0,2,4,5,6$ (mod 8). *If M, M' are homotopy equivalent, then M, M' are diffeomorphic* mod Θ_{2n-1}.

(iii) *Let* $n \equiv 1$ (mod 8). *If M, M' are π-manifolds which are homotopy equivalent, then M, M' are diffeomorphic* mod Θ_{2n-1}.

In particular, using Theorem 3 of [5] and the splitting theorem of [3], we have the following.

Theorem 3. *Let M, M' be* $(n-2)$*-connected* $2n$*-manifolds* $(n \geq 5)$ *which have torsion free* $(n-1)$*-th homology groups and have tangent bundles which are trivial on the* $(n-1)$*-skeletons.*

(i) *Let* $n=8$ *or* $n \equiv 3,4,7$ (mod 8). *If M, M' are tangentially homotopy equivalent, then M, M' are diffeomorphic* mod Θ_{2n}.

(ii) *Let* $n \equiv 2,5,6$ (mod 8). *If M, M' are homotopy equivalent, then M, M' are diffeomorphic* mod Θ_{2n}.

(iii) *Let* $n \equiv 0,1$ (mod 8). *If M, M' are π-manifolds which are homotopy equivalent, then M, M' are diffeomorphic* mod Θ_{2n}.

In the above theorem, we note that the tangent bundles of M, M' are trivial on the $(n-1)$-skeletons if $n \equiv 0,4,6,7$ (mod 8) or if M, M' are π-manifolds.

It is still not known whether every (iii) of the above three theorems can be valid or not without the assumption that M, M' are π-manifolds.

We have the following which is clear for the cases (ii), (iii).

Corollary 4. *Let M, M' be manifolds in Theorem* 1, 2, *or* 3. *In each case of* (i), (ii), (iii) *of the theorems, the following three are equivalent, where m is the dimension of M, M', and M, M' are π-manifolds in case of* (iii).

(a) *M, M' are homotopy equivalent, tangentially in case of* (i).

(b) *M, M' are homeomorphic.*

(c) *M, M' are diffeomorphic Θ_m.*

These results are obtained using the homotopy classification theorems of [11]. The invariant defined there can be replaced by more computable one (Lemma 1.2). In order to compute the invariants, certain Whitehead products are studied and some new facts which supplement the results of [24] are given (Lemma 4.1 and Lemma 4.2). The results of computations (Theorem 1.7 and Theorem 1.8) will be used also in the subsequent paper to classify the above manifolds completely up to homotopy equivalence.

I would like to express my thanks to professor H. Ishimoto who suggested these problems and gave considerable help and advice to me.

§1. Homotopy classification theorems

In this section, we study the homotopy classification theorem of simply connected m-manifolds satisfying the hypotheses (H1), (H2) in the introduction.

We remember the following diagram which is commutative up to sign (cf. [13, I]):

(D1)
$$\begin{array}{ccccc} & \overset{\partial}{\nearrow} & \pi_{q-1}(SO_p) & \xrightarrow{S} & \pi_{q-1}(SO_{p+1}) \\ \pi_q(S^p) & & \downarrow J & & \downarrow J \\ & \underset{P}{\searrow} & \pi_{p+q-1}(S^p) & \xrightarrow{E} & \pi_{p+q}(S^{p+1}), \end{array}$$

where $P=[\ ,\iota_p]$, the Whitehead product with the orientation generator ι_p of $\pi_p(S^p)$, and the lower sequence is exact if $2p>q-1$. Let $\lambda: S(\pi_{q-1}(SO_p))\to\pi_{p+q-1}(S^p)/\mathrm{Im}\,P$ be the homomorphism defined well by $\lambda(S\xi)=\{J\xi\}$. Let θ be an element of $\pi_{q-1}(S^p)$ $(p<q-1)$. Then the inclusion map $i: S^p\to S^p\cup_\theta D^q$ induces the homomorphisms $i_*:\pi_{p+q-1}(S^p)\to\pi_{p+q-1}(S^p\cup_\theta D^q)$ and $\bar{i}_*:\pi_{p+q-1}(S^p)/\mathrm{Im}\,P\to\pi_{p+q-1}(S^p\cup_\theta D^q)/i_*(\mathrm{Im}\,P)$. We define the homomorphism $\bar{\lambda}: S(\pi_{q-1}(SO_p))\to\pi_{p+q-1}(S^p\cup_\theta D^q)/i_*(\mathrm{Im}\,P)$ by $\bar{\lambda}=\bar{i}_*\circ\lambda$.

In [11], the invariant $\bar{\lambda}$ is used to obtain the homotopy classification theorem of manifolds is question. For an element $\theta\in\pi_{q-1}(S^p)$ $(p<q-1)$, the kernel of $\bar{\lambda}$ is closely related to the subgroup $G(\theta)$ of $\pi_{q-1}(SO_{p+1})$ defined by $G(\theta)=S(J^{-1}(\mathrm{Im}\,\theta_*))$, where the homomorphisms are as follows (cf. [13, II]):

$$\pi_{p+q-1}(S^{q-1})\xrightarrow{\theta_*}\pi_{p+q-1}(S^p)\xleftarrow{J}\pi_{q-1}(SO_p)\xrightarrow{S}\pi_{q-1}(SO_{p+1}).$$

In fact, if there exists a p-sphere bundle over the q-sphere with θ as the attaching map of the q-cell, then $\operatorname{Im}\theta_* = \operatorname{Ker} i_*$ (cf. (3.2) of [13, II] and Lemma 4.4 of [26]), and so we have $\operatorname{Ker}\bar{\lambda} = G(\theta)$. But, such a bundle does not always exist. We have the following lemma in a somewhat weak form.

Lemma 1.1. *If* $p<q-1$, $2p>q-1$, *and* $p, q>2$, *then*

$$\operatorname{Im}\theta_* + \operatorname{Im}P = \operatorname{Ker} i_* + \operatorname{Im}P$$

for $i_*: \pi_{p+q-1}(S^p) \to \pi_{p+q-1}(S^p \cup_\theta D^q)$ *and* $P: \pi_q(S^p) \to \pi_{p+q-1}(S^p)$.

Proof. Let $\bar{\theta}$ be the orientation generator of $\pi_q(S^p \cup_\theta D^q, S^p) \cong H_q(S^p \cup_\theta D^q, S^p) \cong \boldsymbol{Z}$. Then we have the follwing diagram which is commutative up to sign:

$$\begin{array}{ccccccc}
 & \bar{\chi}\swarrow & \pi_{p+q}(D^q, S^{q-1}) & \underset{\cong}{\overset{\partial}{\to}} & \pi_{p+q-1}(S^{q-1}) & & \\
\pi_{p+q}(S^q) & & \downarrow \bar{\theta}_* & & \downarrow \theta_* & & \\
 & \chi\nwarrow & \pi_{p+q}(S^p \cup_\theta D^q, S^p) & \overset{\partial}{\to} & \pi_{p+q-1}(S^p) & \overset{i_*}{\to} & \pi_{p+q-1}(S^p \cup_\theta D^q),
\end{array}$$

where χ, $\bar{\chi}$ are induced by shrinking S^p, S^{q-1}, to a point respectively.

By the Blakers-Massey Theorem, $\bar{\chi}$ is surjective if $p=q-2$ and isomorphic if $p<q-2$. So we have

$$\pi_{p+q}(S^p \cup_\theta D^q, S^p) = \operatorname{Im}\bar{\theta}_* + \operatorname{Ker}\chi$$

for $p<q-1$, which is a direct sum if $p<q-2$. On the other hand, by the homotopy exact sequence of Theorem (2.1) of [12], we know that $\operatorname{Ker}\chi = \operatorname{Im}Q$ for the homomorphism $Q: \pi_{p+1}(S^p) \to \pi_{p+q}(S^p \cup_\theta D^q, S^p)$ defined by $Q(\gamma) = [\bar{\theta}, \gamma]$, $\gamma \in \pi_{p+1}(S^p)$, the relative Whitehead product with $\bar{\theta}$. Hende we have

$$\pi_{p+1}(S^p \cup_\theta D^q, S^p) = \operatorname{Im}\bar{\theta}_* + \operatorname{Im}Q$$

for $p<q-1$, which is a direct sum if $p<q-2$.

Thus, from the diagram, $\operatorname{Ker} i_* = \operatorname{Im}\partial = \operatorname{Im}\theta_* + \partial(\operatorname{Im}Q)$. Since $\partial[\bar{\theta}, \gamma] = [\partial\bar{\theta}, \gamma] = [\theta, \gamma]$ for $[\bar{\theta}, \gamma] \in \operatorname{Im}Q$, $\gamma \in \pi_{p+1}(S^p)$, and $E[\theta, \gamma] = 0$ for the suspension homomorphism E, we know that $E(\partial(\operatorname{Im}Q)) = 0$ and so $\partial(\operatorname{Im}Q) \subset \operatorname{Im}P$ from the lower exact sequence of the diagram (D1). So, we have $\operatorname{Ker} i_* \subset \operatorname{Im}\theta_* + \operatorname{Im}P$, and therefore

$$\operatorname{Ker} i_* + \operatorname{Im}P \subset \operatorname{Im}\theta_* + \operatorname{Im}P.$$

Conversely, it is clear that $\operatorname{Im}\theta_* \subset \operatorname{Ker} i_*$ from the diagram, and so

$$\operatorname{Ker} i_* + \operatorname{Im}P \supset \operatorname{Im}\theta_* + \operatorname{Im}P.$$

This completes the proof.

Lemma 1.2. *If* $p<q-1$, $2p>q-1$, *and* $p, q>2$, *then*

$$\operatorname{Ker}\bar{\lambda}=G(\theta)=J^{-1}(\operatorname{Im}(E\theta)_*)\cap\operatorname{Im}S$$

for $J:\pi_{q-1}(SO_{p+1})\to\pi_{p+q}(S^{p+1})$.

Proof. Since $\bar{\lambda}=\bar{\imath}_*\circ\lambda$ and $\operatorname{Ker}\bar{\imath}_*=(\operatorname{Ker}i_*+\operatorname{Im}P)/\operatorname{Im}P=(\operatorname{Im}\theta_*+\operatorname{Im}P)/\operatorname{Im}P$ from Lemma 1.1, an element $\nu\in\pi_{q-1}(SO^{p+1})$ belongs to $\operatorname{Ker}\bar{\lambda}$ if and only if there exists an element $\xi\in\pi_{q-1}(SO_p)$ such that $S\xi=\nu$ and $J\xi\in\operatorname{Im}\theta_*+\operatorname{Im}P$. So, ν belongs to $\operatorname{Ker}\bar{\lambda}$ if and only if ν belongs to $S(J^{-1}(\operatorname{Im}\theta_*+\operatorname{Im}P))$, that is, $\operatorname{Ker}\bar{\lambda}=S(J^{-1}(\operatorname{Im}\theta_*+\operatorname{Im}P))$. Since $P=-J\circ\partial$ at the diagram (D1), it is easily seen that $J^{-1}(\operatorname{Im}\theta_*+\operatorname{Im}P=J^{-1}(\operatorname{Im}\theta_*)+J^{-1}(\operatorname{Im}P)$. So, we have

$$\operatorname{Ker}\bar{\lambda}=SJ^{-1}(\operatorname{Im}\theta_*)+SJ^{-1}(\operatorname{Im}P).$$

Then, since $J^{-1}(\operatorname{Im}P)=J^{-1}(\operatorname{Im}(-J\circ\partial))=J^{-1}(J(\operatorname{Im}\partial))=\operatorname{Im}\partial+J^{-1}(0)$, we have $SJ^{-1}(\operatorname{Im}P)=S(\operatorname{Im}\partial)+S(J^{-1}(0))=S(J^{-1}(0))\subset SJ^{-1}(\operatorname{Im}\theta_*)$. Thus we have $\operatorname{Ker}\bar{\lambda}=SJ^{-1}(\operatorname{Im}\theta_*)=G(\theta)$.

The latter half of the assertion is known straightforwardly from the diagram (D1) combined with the commutative diagram

(D2)
$$\begin{array}{ccc}\pi_{p+q-1}(S^p) & \xrightarrow{E} & \pi_{p+q}(S^{p+1})\\ \theta_*\uparrow & & \uparrow(E\theta)_*\\ \pi_{p+q-1}(S^{q-1}) & \xrightarrow{E} & \pi_{p+q}(S^q)\end{array}$$

where we note that $E:\pi_{p+q-1}(S^{q-1})\to\pi_{p+q}(S^q)$ is surjective since $p<q-1$.

If $\theta=0$ in particular, then i_*, $\bar{\imath}_*$ are injective, and so we have $\operatorname{Ker}\lambda=\operatorname{Ker}\bar{\lambda}$. Therefore, we have

Corollary 1.3. *Let* $2p>q-1$ *and* $p,q>2$. *Then*

$$\operatorname{Ker}\lambda=G(0)=J^{-1}(0)\cap\operatorname{Im}S$$

for $J:\pi_{q-1}(SO_{p+1})\to\pi_{p+q}(SO^{p+1})$.

In fact, we have $J(\pi_{q-1}(SO_p))/\operatorname{Im}P\cong\operatorname{Im}(E\circ J)=J(\operatorname{Im}S)$ from the diagram (D1), and so λ is equivalent to $-J|\operatorname{Im}S$. Therefore $\operatorname{Ker}\lambda=J^{-1}(0)\cap\operatorname{Im}S$, and $G(0)=J^{-1}(0)\cap\operatorname{Im}S$ is known straightforwardly from the diagram (D1).

Let $\mathscr{H}(p+q+1,k,q)$ be the set of handlebodies obtained by gluing q-handles, k in number, disjointly to a $(p+q+1)$-disk. Then a simply connected m-manifold M satisfying (H1), (H2) is represented $\operatorname{mod}\Theta_m$ $(m=p+q)$ as the boundary of a handlebody W of $\mathscr{H}(p+q+1,k,q)$ with $k=\operatorname{rank}H_p(M)$, where we assume that $p<q-1$ and $2p>q>1$. In fact, killing the generators of $H_p(M)$ by surgery, M is modified to a homotopy sphere Σ, and so $M\#(-\Sigma)$ to the standard sphere. Therefore, constructing conversely, $M\#(-\Sigma)$ is the boundary of a handlebody $\mathscr{H}(p+q+1,k,q)$.

A handlebody W of $\mathscr{H}(p+q+1,k,q)$ $(2p>q>1)$ is determined up to diffeomorphism by the invariant system $(H;\phi,\alpha)$ defined by Wall[32]. Here $H=H_p(W)$ is a free abelian group of rank k, $\phi:H\times H\to\pi_q(S^{p+1})$ is a certain bilinear form symmetric up to sign, and $\alpha:H\to\pi_{q-1}(SO_{p+1})$ is a map assigning each $x\in H\cong\pi_q(W)$ the characteristic element of the normal bundle of the imbedded q-sphere which represents x. α is a quadratic form with associated bilinear form $\partial\circ\phi$, $\partial:\pi_q(S^{p+1})\to\pi_{q-1}(SO_{p+1})$ (cf. [32]).

Handlebody Classification Theorem (Wall[32]). *Let W, W' be handlebodies of $\mathscr{H}(p+q+1,k,q)$ $(2p>q>1)$ with the invariant systems $(H;\phi,\alpha)$, $(H';\phi',\alpha')$ respectively. Then W, W' are diffeomorphic if and only if there exists an isomorphism $h:H\to H'$ such that $\phi=\phi'\circ(h\times h)$ and $\alpha=\alpha'\circ h$.*

We note that ϕ is a homotopy invariant of the boundary of W by Propositon 1 of [7, II].

Henceforce we assume that $(p,q)=(n-1,n+1)$ $(n\geq 4)$ or $(n-2,n+1)$ $(n\geq 6)$ and put $\theta=\eta_{n-1}$ or η_{n-2}^2 according as (p,q) is the former or the latter, where η_{n-1}, η_{n-2}^2 are the essential elements of $\pi_{q-1}(S^p)\cong \boldsymbol{Z}/2\boldsymbol{Z}$ respectively. Let $\rho:\pi_{q-1}(SO_{p+1})\to\pi_{q-1}(SO_{p+1})/G(\theta)$, $\rho_0:\pi_{q-1}(SO_{p+1})\to\pi_{q-1}(SO_{p+1})/G(0)$ be the canonical maps. Let W, W' be handlebodies of $\mathscr{H}(p+q+1,k,q)$ with the invariant systems $(H;\phi,\alpha)$, $(H';\phi',\alpha')$ respectively. In [7, I,II,III], [9], and [11], the homotopy classification theorems for the boundaries of handlebodies were mentioned in terms of invariant systems and the homomorphisms ρ_0, ρ, λ, and $\bar{\lambda}$. Since $\operatorname{Ker}\bar{\lambda}$, $\operatorname{Ker}\lambda$ may be replaced respectively by $G(\theta)$, $G(0)$ by Lemma 1.2 and Corollary 1.3, we can restate and unify the homotopy classification theorems as follows:

Theorem 1.4. *If ϕ, ϕ' are non-singular, then ∂W, $\partial W'$ are homotopy equivalent if and only if there exists an isomorphism $h:H\to H'$ such that $\phi=\phi'\circ(h\times h)$ and $\rho\circ\alpha=\rho\circ(\alpha'\circ h)$.*

Theorem 1.5. *Let* rank ϕ, rank $\phi'<k$. Then *$\partial W,\partial W'$ are homotopy equivalent if and only if there exist an isomorphism $h:H\to H'$ satisfying $\phi=\phi'\circ(h\times h)$ and a direct sum decomposition $H=H_0\oplus H_1$ orthogonal w.r.t. ϕ such that*

(i) *$\phi|H_0\times H_0=0$ and $\phi|H_1\times H_1$ is non-singular,*

(ii) *$\rho_0\circ\alpha=\rho_0\circ(\alpha'\circ h)$ on H_0 and $\rho\circ\alpha=\rho\circ(\alpha'\circ h)$ on H_1.*

Remark. Since $G(0)\subset G(\theta)$, (ii) induces that $\rho\circ\alpha=\rho\circ(\alpha'\circ h)$ on the whole of H. The canonical map ρ_0 is equivalent to λ on $S\pi_{q-1}(SO_p)\subset\pi_{q-1}(SO_{p+1})$, since $G(0)=S(\operatorname{Ker}J)$ and we have $S\pi_{q-1}(SO_p)/G(0)\cong\pi_{q-1}(SO_p)/\operatorname{Im}\partial+\operatorname{Ker}J\cong J\pi_{q-1}(SO_p)/\operatorname{Im}P$. For $J:\pi_{q-1}(SO_{p+1})\to\pi_{p+q}(S^{p+1})$, the maps ρ_0, λ are equvalent to $-J$ on $\operatorname{Im}S\subset\pi_{q-1}(SO_{p+1})$.

From Handlebody Classification Theorem, we have the following.

Corollary 1.6. *Assume that $G(\theta)=0$. If $\partial W, \partial W'$ are homotopy equivalent, then W, W' are diffeomorphic and so $\partial W, \partial W'$ are diffeomorphic.*

Thus the calculation of $G(\theta)$ is an essential problem. We have the following results which will be proved in §4, where we abbreviate $\boldsymbol{Z}/k\boldsymbol{Z}$ as $\boldsymbol{Z}_k$ and $m(2t)$ is the denominator of $B_t/4t$, B_t is the t-th Bernoulli number, and we put $l(2t)=m(2t)/2$.

Theorem 1.7. *Let $(p,q)=(n-1,n+1)$ and put $\theta=\eta_{n-1}$ $(n\geq 4)$. Then $G(\eta_{n-1})$ is given by the following table*:

n	4	5	6	7	8
$\pi_n(SO_n)$	$\boldsymbol{Z}_2+\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$	0	$\boldsymbol{Z}$	$\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2$
$G(\eta_{n-1})$	$\boldsymbol{Z}_2+0$	0	0	$120\boldsymbol{Z}$	0

$n\geq 9$	$4t-1$		$4t$		$4t+1$		$4t+2$
	t: odd	t: even	t: odd	t: even	t: odd	t: even	
$\pi_n(SO_n)$	$\boldsymbol{Z}$		$\boldsymbol{Z}_2+\boldsymbol{Z}_2$	$\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$	$\boldsymbol{Z}_2+\boldsymbol{Z}_2$	$\boldsymbol{Z}_2$
$G(\eta_{n-1})$	$l(2t)\boldsymbol{Z}$	$m(2t)\boldsymbol{Z}$	0	$0+0+\boldsymbol{Z}_2$	0	$0+\boldsymbol{Z}_2$	0

Theorem 1.8. *Let $(p,q)=(n-1,n+1)$ and put $\theta=\eta^2_{n-2}$ $(n\geq 6)$. Then $G(\eta^2_{n-2})$ is given by the following table*:

n	6	7	8	9
$\pi_n(SO_{n-1})$	0	$\boldsymbol{Z}$	$\boldsymbol{Z}_2+\boldsymbol{Z}_2$	$\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2$
$G(\eta^2_{n-1})$	0	$60\boldsymbol{Z}$	0	0

$n\geq 10$	$4t-1$		$4t$		$4t+1$		$4t+2$
	t: odd	t: even	t: odd	t: even	t: odd	t: even	
$\pi_n(SO_{n-1})$	$\boldsymbol{Z}$		$\boldsymbol{Z}_2$	$\boldsymbol{Z}_2+\boldsymbol{Z}_2$	$\boldsymbol{Z}_2+\boldsymbol{Z}_2$	$\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2$	$\boldsymbol{Z}_8$
$G(\eta^2_{n-2})$	$l(2t)\boldsymbol{Z}$	$m(2t)\boldsymbol{Z}$	0	0	0	$0+0+\boldsymbol{Z}_2$	0

§2. Proofs of Theorem 1 and Theorem 2

In this section, Theorem 1 and Theorem 2 are proved at the same time. Let W be a handlebody of $\mathscr{H}(p+q+1,k,q)$ $(2p>q>1)$ with the invariant system $(H;\phi,\alpha)$. If $q=4t$ $(t>0)$, we have the following commutative diagram easily from Lemma 1.1 of [16] and Theorem 35.12 of [28] (cf. also Lemma 9.2 of [5] and p.731 of [6]):

$$\text{(D3)}\qquad \begin{array}{ccc} H_q(W) & \xrightarrow{\alpha} & \pi_{q-1}(SO_{p+1}) \\ {\scriptstyle \langle p_t(W),\ \rangle}\downarrow & & \downarrow {\scriptstyle i^S_*} \\ Z & \xleftarrow{c\times} & Z \cong \pi_{q-1}(SO), \end{array}$$

where $i^S : SO_{p+1} \to SO$ is the inclusion map, $p_t(W)$ is the t-th Pontrjagin class, and

$$c = \pm \begin{cases} 2(2t-1)! & \text{if } t \text{ is odd,} \\ (2t-1)! & \text{if } t \text{ is even.} \end{cases}$$

Let W' be also a handlebody of $\mathscr{H}(p+q+1,k,q)$ with the invariant system $(H';\phi',\alpha')$. It may be assumed that $M=\partial W$, $M'=\partial W'$, mod Θ_m $(m=p+q)$, and $k=\operatorname{rank} H_p(M)=\operatorname{rank} H_p(M')$.

Proof of (i). Let $q=n+1=4t$ $(t>1)$. Since i^S_* is injective in this case (cf.[34]), the invariant α is determined by $p_t(W)$. From the assumption, we have a tangential homotopy equivalence $f: M \to M'$, and by Theorem 2 of [27], we may assume that f if a tangential homotopy equivalence from ∂W to $\partial W'$. Let i, i' be the inclusion maps of $\partial W, \partial W'$ into W, W' respectively and let $h: H=H_q(W) \to H'=H_q(W')$ be the isomorphism defined by $h=i'_* \circ f_* \circ i_*^{-1}$. Then, $p_t(W)=(i^{*-1} \circ f^* \circ i'^*)p_t(W')$, and from the above diagram, it is easily seen that $\alpha=\alpha' \circ h$. On the other hand, $\phi=\phi' \circ (h \times h)$ and by Proposition 1 of [7,II]. Hence $(H;\phi,\alpha)$ is isomorphic to $(H';\phi',\alpha')$ and so W is diffeomorphic to W' by Handlebody Classification Theorem. Thus we have $M=M' \bmod \Theta_m$.

Remark 2.1. In the above proof, we may assume only that $p_t(M)=f^*p_t(M')$ for the given homotopy equvalence f.

Proof of (ii). In these cases of n, we have $G(\eta_{n-1})=0$ by Theorem 1.7 and $G(\eta_{n-2}^2)=0$ by Theorem 1.8. Therefore $M=M' \bmod \Theta_m$ by Corollary 1.6.

To prove (iii), we need the following.

Lemma 2.2. *Let M be a simply connected m-dimensional π-manifold satisfying the hypothesis* (H1) *of* §0. *Let $k=\operatorname{rank} H_p(M)$ and suppose that $H_p(M)$ is torsion free if $p=q-1$. Then, $M(\bmod \Theta_m)$ bounds a parallelizable handlebody W of $\mathscr{H}(p+q+1,k,q)$ $(p+q=m)$.*

Proof. By killing the generators of $H_p(M)$ $(\cong \pi_p(M))$ by surgery, M is modified to a homotopy sphere Σ. Furthermore, by Lemma 6.2 of [18], we can perform it by framed surgery (cf. §6 of [18]). That is, for a given trivialization $\mathfrak{g}$ of the stable tangent bundle of M, there exists a parallelizable $(m+1)$-manifold N with a trivialization $\mathfrak{G}$ of τN such that $\partial N = M \cup (-\Sigma)$ and $\mathfrak{g} = \mathfrak{G} | M$.

Then, giving a trivialization $\mathfrak{e}$ to the stable tangent bundle of $-\Sigma$, we can make the connected sum $M \# (-\Sigma)$ with the trivialization $\tilde{\mathfrak{g}} = \mathfrak{g} \# \mathfrak{e}$. Hence, considering the above construction again, there exists a parallelizable $(m+1)$-manifold $\tilde{N}$ bounded by $M \# (-\Sigma)$ and $(-\Sigma) \# \Sigma = S^m$. Let $W = \tilde{N} \cup D^{m+1}$, which is considered as a handlebody of $\mathscr{H}(p+q+1, k, q)$. W has a trivialization of τW on its m-skeleton since

$$H^i(W, \tilde{N}; \pi_{i-1}(SO_{m+1})) \cong H^i(D^{m+1}, S^m; \pi_{i-1}(SO_{m+1})) = 0$$

for $i \leq m$. The obstruction to build up a trivialization of τW on the whole of W using it lies in $H^{m+1}(W; \pi_m(SO_{m+1})) = 0$. Thus W is parallelizable.

Proof of (iii). We may assume that $M = \partial W$, $M' = \partial W'$, $\bmod \Theta_m$ and W, W' are parallelizable handlebodies of $\mathscr{H}(p+q+1, k, q)$ by Lemma 2.2, where $k = \operatorname{rank} H_p(M) = \operatorname{rank} H_p(M')$. Since W, W' are parallelizable, we know that $i^S_* \alpha$, $i^S_* \alpha'$ are trivial. In the present case, i^S_* maps the last direct summand of $\pi_{q-1}(SO_{p+1})$ in the list of Theorem 1.7 and Theorem 1.8 isomorphically onto $\pi_{q-1}(SO) \cong \boldsymbol{Z}_2$ (cf. Table 3 of [34]). Therefore α, α' always take the values with trivial last components. We note that $G(\eta_{n-1})$, $G(\eta^2_{n-2})$ are just the last direct summands of $\pi_{q-1}(SO_{p+1})$ in each case of n of (iii) by Theorem 1.7 and Theorem 1.8.

Let M, M' be homotopy equivalent. Then there exists an isomorphism $h: H \to H'$ such that $\phi = \phi' \circ (h \times h)$ and $\rho \circ \alpha = \rho \circ (\alpha' \circ h)$ on H by Theorem 1.4, Theorem 1.5, and the remark. Since ρ is injective on the images of α, α' from the above, we have $\alpha = \alpha' \circ h$ on H, and so W is diffeomorphic to W' by Handlebody Classification Theorem. Thus M is diffeomorphic to $M' \bmod \Theta_m$.

This completes the proof of Theorem 1 and Theorem 2.

§3. Proofs of Theorem 3 and Corollary 4

The following is essentially known by Theorem 5 of [31] partly and by [19] as an application of the theory of surgery. However, we show it again since our proof is elemental and complementary to [31].

Theorem 3.1. *Let M, M' be $(n-1)$-connected $2n$-manifolds $(n \geq 3)$. Let M, M' be homotopy equivalent, and suppose that the homotopy equivalence $f: M \to M'$ satisfies $p_t(M) = f^* p_t(M')$ if $n = 4t$ $(t > 0)$. Then M, M' are diffeomorphic* $\bmod \Theta_{2n}$.

Proof. Let $M = W \cup D^{2n}$, $M' = W' \cup D^{2n}$. We consider W, W' as handlebodies of $\mathscr{H}(2n, k, n)$ $(k = \operatorname{rank} H_n(M))$ with the invariant systems $(H; \phi, \alpha)$, $(H'; \phi', \alpha')$ respectively. Let $\tilde{\phi}: H_n(M) \times H_n(M) \to \boldsymbol{Z}$ be the intersection form and let $\tilde{\alpha}: H_n(M) \to \pi_{n-1}(SO_n)$ be the map assigning each $x \in H_n(M)$ the characteristic element of the normal bundle of the imbedded n-sphere which represents x (cf. Theorem 1 of [32]). Then, from the definitions, we have $\phi = \tilde{\phi} \circ (i_* \times i_*)$, $\alpha = \tilde{\alpha} \circ i_*$, where $i: W \subset M$ is the inclusion map and $i_*: H = H_n(W) \to H_n(M)$ is an isomorphism. We define $\tilde{\phi}'$,

$\tilde{\alpha}'$, and i' similarly for M' and have $\phi' = \tilde{\phi}' \circ (i'_* \times i'_*)$, $\alpha' = \tilde{\alpha}' \circ i'_*$.

Let $f: M \to M'$ be a homotopy equivalence. Then we have $\tilde{\phi} = \tilde{\phi}' \circ (f_* \times f_*)$, $J \circ \tilde{\alpha} = J \circ (\tilde{\alpha}' \circ f_*)$ by Lemma 8 of [31] (cf. also Proposition 5.1 of [7,1]), where $J: \pi_{n-1}(SO_n) \to \pi_{2n-1}(S^n)$ is the J-homomorphism. Since i_*, i'_* are isomorphisms, we can define an isomorphism $h: H \to H'$ by $h = (i'_*)^{-1} \circ f_* \circ i_*$. Then, easily we have $\phi = \phi' \circ (h \times h)$ and $J \circ \alpha = J \circ (\alpha' \circ h)$. So, if we show that $\alpha = \alpha' \circ h$, then the invariant systems of W, W' are isomorphic. Hence W, W' are diffeomorphic by Handlebody Classification Theorem, are therefore M, M' are diffeomorphic mod Θ_{2n}.

Let $n \not\equiv 0$ (mod 4). Then we have $\alpha = \alpha' \circ h$ immediately since $J: \pi_{n-1}(SO_n) \to \pi_{2n-1}(S^n)$ is injective (cf. Proposition 2.1 of [34]). Let $n = 4t$ $(t > 0)$ and suppose that $p_t(M) = f^* p_t(M')$. The isomorphism $\bar{h} = i^* \circ f^* \circ (i'^*)^{-1}: H^n(W') \to H^n(W)$ satisfies $\langle \bar{h}(\varphi), x \rangle = \langle \varphi, h(x) \rangle$ for $x \in H = H_n(W)$, $\varphi \in H^n(W') = \mathrm{Hom}(H_n(W'), \boldsymbol{Z})$. So we have $p_t(W) = i^* p_t(M) = i^* \circ f^* p_t(M') = i^* \circ f^* \circ (i'^*)^{-1} p_t(W') = \bar{h} p_t(W')$ and therefore $\langle p_t(W), x \rangle = \langle p_t(W'), h(x) \rangle$ for $x \in H$. Thus, from the diagram (D3), we have $i^S_* \circ \alpha = i^S_* \circ (\alpha' \circ h)$ for $i^S_*: \pi_{n-1}(SO_n) \cong \boldsymbol{Z} + \boldsymbol{Z} \to \pi_{n-1}(SO) \cong \boldsymbol{Z}$ $(n = 4t)$.

Let $n = 4t$ and $t \geq 3$. It is known that $\pi_{n-1}(SO_n) = \mathrm{Ker}\, i^S_* \oplus S\pi_{n-1}(SO_{n-1}) \cong \boldsymbol{Z} + \boldsymbol{Z}$ for $S: \pi_{n-1}(SO_{n-1}) \to \pi_{n-1}(SO_n)$ induced from the inclusion $SO_{n-1} \subset SO_n$. Put $\alpha: H \to \pi_{n-1}(SO_n)$ as $\alpha = (\alpha_1, \alpha_2)$ and similarly α' as $\alpha' = (\alpha'_1, \alpha'_2)$. Then, since i^S_* is isomorphic on the second direct summand, we have $\alpha_2 = \alpha'_2 \circ h$ from the above. On the other hand, for the homomorphism $\pi_* = \pi_{n-1}(SO_n) \to \pi_{n-1}(SO^{n-1}) = \boldsymbol{Z}$ induced from the projection, we have

$$\begin{aligned} \pi_* \alpha_1(x) &= \pi_* \alpha(x) = x \cdot x = \phi(x, x) = \phi'(h(x), h(x)) \\ &= h(x) \cdot h(x) = \pi_* \alpha'(h(x)) = \pi_* \alpha'_1(h(x)) \ (x \in H). \end{aligned}$$

So $\alpha_1 = \alpha'_1 \circ h$ since π_* is injective on $\mathrm{Ker}\, i^S_*$ if n is even. Thus we have $\alpha = \alpha' \circ h$.

Let $n = 4$ or 8. Then still $\pi_{n-1}(SO_n) \cong \boldsymbol{Z} + \boldsymbol{Z}$ by a well-known basis $\{u_1, u_2\}$ such that $\mathrm{Ker}\, i^S_*$ is generated by $u_1 - 2u_2$, u_2 comes from the generator of $\pi_{n-1}(SO_{n-1}) \cong \boldsymbol{Z}$ by the homomorphism S, and $i^S_* u_2$ generates $\pi_{n-1}(SO) \cong \boldsymbol{Z}$. Therefore the basis $\{u_1 - 2u_2, u_2\}$ gives again the above direct sum decomposition of $\pi_{n-1}(SO_n)$. Hence the argument is quite similar.

This completes the proof.

Proof of Theorem 3. Let M, M' be $(n-2)$-connected $2n$-manifolds $(n \geq 5)$ with trosion free $(n-1)$-th homology groups and tangent bundles which are trivial on the $(n-1)$-skeletons. By Theorem 3 of [5], there exists a decomposition $M' = M'_1 \# M'_2$ such that M'_1 is $(n-2)$-connected, $H_n(M'_1) = 0$, and M'_2 is $(n-1)$-connected, $H_n(M'_2) \cong H_n(M')$. Let $f: M \to M'$ be a homotopy equivalence and let $N'_i = M'_i - \mathrm{Int}\, D^{2n}_i$ $(i = 1, 2)$ so that $M' = N'_1 \cup N'_2$, $N'_1 \cap N'_2 = \partial N'_1 = \partial N'_2 = S^{2n-1}$. Then, by Theorem 1.1 of [3], there exist $2n$-submanifolds of M such that $M = N_1 \cup N_2$, $\tilde{S} = N_1 \cap N_2 = \partial N_1 = \partial N_2$ and f is homotopic to a map $g: (M, N_1, N_2, \tilde{S}) \to (M', N'_1, N'_2, S^{2n-1})$ whose restrictions to respective manifolds are homotopy equivalences.

Here $\tilde{S}$ is a homotopy $(2n-1)$-sphere but we can show that $\tilde{S}$ is diffeomorphic to the standard $(2n-1)$-sphere. In fact, since N_1 is homotopy equivalent to N_1', $H_i(N_1)$ is non-trivial only for $i=0$, $n-1$, and $n+1$, and those homology groups are torsion free. So, since $N_1(\subset M)$ is parallelizable on its $(n-1)$-skeleton, we can kill the generators of $H_{n-1}(N_1)$ by surgery so that $\tilde{S}=\partial N_1$ is the boundary of a contractible manifold. Therefore $\tilde{S}$ is diffeomorphic to the standard $(2n-1)$-sphere, and we have $M=M_1\# M_2$ for $M_i=N_i\cup D^{2n}$, $i=1,2$.

Since $g(\partial N_i)=\partial N_i'$ $(i=1,2)$ we can extend $g|N_i$ to a homotopy equivalence $\tilde{g}_i: M_i\to M_i'$ $(i=1,2)$. If f is tangential, then g is also a tangential homotopy equivalence, and so $p_t(N_1)=(g|N_1)^*p_t(N_1')$ for $n=4t-1$ and $p_t(N_2)=(g|N_2)^*p_t(N_2')$ for $n=4t$. Hence we have $p_t(M_1)=(\tilde{g}_1)^*p_t(M_1')$ if $n=4t-1$ and $p_t(M_2)=(\tilde{g}_2)^*p_t(M_2')$ if $n=4t$. Therefore, by Theorem 1, Remark 2.1, and Theorem 3.1, we obtain the results according to the corresponding cases.

This completes the proof.

Remark 3.2. In the above proof, we may assume only that $p_t(M)=f^*p_t(M')$ for $n=4t-1$, $4t$ $(t>1)$ for the given homotopy equivalence f.

Proof of Corollary 4. It is clear that (a), (b), and (c) are equivalent for the cases of (ii), (iii) of Theorems 1, 2, and 3. For the case (i), it is sufficient to show that (b) induces (a). Let $f: M\to M'$ be an orientation preserving homomorphism. Since $H_*(M)$, $H^*(M)$ are torsion free and so $H^*(M;\boldsymbol{Q}/\boldsymbol{Z})$ is of torsion, the homomorphisms $H^i(M;\boldsymbol{Z})\to H^i(M;\boldsymbol{Q})$, $i=n,n+1$, are injective. This situation is similar for M'. Hence, from the topological invariance of rational Pontrjagin classes ([25]), we have $p_t(M)=f^*p_t(M')$ for $n=4t-1, 4t$. Therefore, from respective (i) of the theorems and Remarks 2.1, 3.2, M is diffemorphic to $M' \bmod \Theta_m$. Thus we have (c) and so (a) by Theorem 2 of [27].

§4. Calculations of $G(\eta_{n-1})$ and $G(\eta_{n-1}^2)$

In this section, we prove Theorem 1.7 and Theorem 1.8. Those are given by calculating $G(\theta)$ for $(p,q)=(n-1,n+1)$, $\theta=\eta_{n-1}$ $(n\geq 4)$ and for $(p,q)=(n-1,n+2)$, $\theta=\eta_{n-2}^2$ $(n\geq 6)$.

From the diagrams (D1) and (D2), we have the following diagram which is commutative up to sign and we frequently refer to it in our calculations:

$$
\text{(D4)}\qquad
\begin{array}{ccccccccc}
\pi_n(SO_{n-2}) & \xrightarrow{S^{(4)}} & \pi_n(SO_{n-1}) & \xrightarrow{S^{(3)}} & \pi_n(SO_n) & \xrightarrow{S^{(2)}} & \pi_n(SO_{n+1}) & \xrightarrow{S^{(1)}} & \pi_n(SO) \\
\downarrow{\scriptstyle J^{(4)}} & & \downarrow{\scriptstyle J^{(3)}} & & \downarrow{\scriptstyle J^{(2)}} & & \downarrow{\scriptstyle J^{(1)}} & & \downarrow{\scriptstyle J} \\
\pi_{2n-2}(S^{n-2}) & \xrightarrow{E^{(4)}} & \pi_{2n-1}(S^{n-1}) & \xrightarrow{E^{(3)}} & \pi_{2n}(S^n) & \xrightarrow{E^{(2)}} & \pi_{2n+1}(S^{n+1}) & \xrightarrow{E^{(1)}} & \Pi_n \\
\uparrow{\scriptstyle (\eta_{n-2})_*} & & \uparrow{\scriptstyle (\eta_{n-1})_*} & & \uparrow{\scriptstyle (\eta_n)_*} & & \uparrow{\scriptstyle (\eta_{n+1})_*} & & \uparrow{\scriptstyle \eta_*} \\
\pi_{2n-2}(S^{n-1}) & \xrightarrow{E^{(4)}} & \pi_{2n-1}(S^n) & \xrightarrow{E^{(3)}} & \pi_{2n}(S^{n+1}) & \underset{\cong}{\xrightarrow{E^{(2)}}} & \pi_{2n+1}(S^{n+2}) & \underset{\cong}{\xrightarrow{E^{(1)}}} & \Pi_{n-1} \\
\uparrow{\scriptstyle (\eta_{n-1})_*} & & \uparrow{\scriptstyle (\eta_n)_*} & & \uparrow{\scriptstyle (\eta_{n+1})_*} & & \uparrow{\scriptstyle (\eta_{n+2})_*} & & \uparrow{\scriptstyle \eta_*} \\
\pi_{2n-2}(S^n) & \underset{\cong}{\xrightarrow{E^{(4)}}} & \pi_{2n-1}(S^{n+1}) & \underset{\cong}{\xrightarrow{E^{(3)}}} & \pi_{2n}(S^{n+2}) & \underset{\cong}{\xrightarrow{E^{(2)}}} & \pi_{2n+1}(S^{n+3}) & \underset{\cong}{\xrightarrow{E^{(1)}}} & \Pi_{n-2}
\end{array}
$$

Here $S^{(i)}$, $E^{(i)}$, and $J^{(i)}$ denote respectively the homomorphism induced from the inclusion map, the suspension homomorphism, and the J-homomorphism for each i. The diagram can be extended to the left if necessary and $E^{(i)}: \pi_{2n-3}(S^{n-1}) \to \pi_{2n-2}(S^n)$ is surjective.

The homotopy groups of rotation groups are known by [17] and [2]. There exists a splitting exact sequence

$$0 \to \pi_{n+1}(V_{m+n-r,m}) \xrightarrow{\partial} \pi_n(SO_{n-r}) \xrightarrow{i^S_*} \pi_n(SO) \to 0$$

for $n > 8$, $r < 4$ except the cases $n = 14$ and $n = 8s + 7$, $r = 3$, where m is sufficiently large.

If $n \equiv 2, 4, 5, 6 \pmod 8$, then $\pi_n(SO) = 0$ and we determine the generators of $\pi_n(SO_{n-r})$ by those of $\pi_{n+1}(V_{m+n-r,m})$ carried by the isomorphism ∂. If $n \equiv 0, 1$ (mod 8), then there exists the largest number $r_0 > r$ such that $i^S_*: \pi_n(SO_{n-r}) \to \pi_n(SO)$ is isomorphic. So we adopt the element of $\pi_n(SO_{n-r_0})$ corresponding to the generator of $\pi_n(SO)$ as its generator. Then, the generators of $\pi_{n+1}(V_{m+n-r,m})$, $\pi_n(SO_{n-r_0})$ carried by ∂ and $S: \pi_n(SO_{n-r_0}) \to \pi_n(SO_{n-r})$ are designated as the generators of $\pi_n(SO_{n-r})$. For $n = 8$, confer p.782 of [7, III]. Those generators are temporarily denoted by u, v, w, etc. with numbers suffixed if necessary. For $n \equiv 0, 1$ (mod 8), we assign the final number to the generator which corresponds to the generator of $\pi_n(SO)$ by i^S_*.

For the cases $n \equiv 3, 7$ (mod 8), we can also set the generators of $\pi_n(SO_{n-r})$ in a canonical way (cf. pp.8, 9 of [8]), and the notations are similar. For $n = 14$, we know $\pi_{14}(SO_{10}) = \pi_{14}(SO_{11}) = Z_8$, $\pi_{14}(SO_{12}) = Z_4 + Z_{24}$ by [15].

The following lemmas supplement the results of Nomura[24] (cf. also Lemmas 2.3, 3.3 of [7, III]).

Lemma 4.1. *If* $n \equiv 0, 1, 2$ (mod 4) *and* $n \geq 5$, $n \neq 6$, *then* $[\eta_n, \iota_n]$ does not belong to $\mathrm{Im}(\eta_n)_*$ *for* $(\eta_n)_*: \pi_{2n}(S^{n+1}) \to \pi_{2n}(S^n)$.

Proof. For $n \equiv 0, 1$ (mod 4), the result is due to [24]. Let $n \equiv 2$ (mod 4), $n > 6$. Then we have

$$\begin{array}{ccccc} \pi_n(SO_{n-1}) & \overset{S^{(3)}}{\to} & \pi_n(SO_n) & \overset{S^{(2)}}{\to} & \pi_n(SO_{n+1}) \\ \| & & \| & & \| \\ Z_8 w & & Z_4 v & & Z_2 u \end{array}$$

and $S^{(3)}w=v$, $S^{(2)}v=u$ (cf. [34]). Suppose that $[\eta_n, \iota_n]=(\eta_n)_*\beta$ for some $\beta\in\pi_{2n}(S^{n+1})$. Since $S^{(2)}(2v)=0$, by the diagrams (D1), (D4), we have $[\eta_n, \iota_n]=J^{(2)}(2v)=J^{(2)}S^{(3)}(2w) =E^{(3)}J^{(3)}(2w)$. Let $\beta=E^{(3)}\gamma$ for some $\gamma\in\pi_{2n-1}(S^n)$. Then $[\eta_n, \iota_n]=(\eta_n)_*\beta=(\eta_n)_*E^{(3)}\gamma =E^{(3)}(\eta_{n-1})_*\gamma$ and so $E^{(3)}J^{(3)}(2w)=E^{(3)}(\eta_{n-1})_*\gamma$. So we have $J^{(3)}(2w)-(\eta_{n-1})_*\gamma \in \operatorname{Ker} E^{(3)}=\langle[\eta^2_{n-1}, \iota_{n-1}]\rangle$. Let $J^{(3)}(2w)-(\eta_{n-1})_*\gamma=c[\eta^2_{n-1}, \iota_{n-1}]$, $c=0$ or 1. Since $\pi_{2n-1}(S^n)=Z[\iota_n, \iota_n]\oplus E\pi_{2n-2}(S^{n-1})$ for even n ($n\neq 2,4,8$), the element γ is given by $\gamma=d[\iota_n, \iota_n]+E\delta$ for some integer d and $\delta\in\pi_{2n-2}(S^{n-1})$. Therefore $(\eta_{n-1})_*\gamma =d[\eta_{n-1}, \eta_{n-1}]+\eta_{n-1}\circ E\delta$. Thus we have

$$J^{(3)}(2w)=c[\eta^2_{n-1}, \iota_{n-1}]+d[\eta_{n-1}, \eta_{n-1}]+\eta_{n-1}\circ E\delta$$

Therefore $J^{(3)}(4w)=2J^{(3)}(2w)=0$. Since $J^{(3)}$ is injective by Propositon 2.1 of [34], this is a contradiction.

Lemma 4.2. *If* $n\equiv 1,2 \pmod 4$ *and* $n\geq 5$, $n\neq 6$, *then* $[\eta^2_{n-1}, \iota_{n-1}]$ *does not belong to* $\operatorname{Im}(\eta^2_{n-1})_*$ *for* $(\eta^2_{n-1})_*:\pi_{2n-1}(S^{n+1})\to\pi_{2n-1}(S^{n-1})$.

Proof. For the case $n\equiv 1 \pmod 4$, the result is due to [24]. Let $n=8s+2$ ($s>0$). Then we have

$$\begin{array}{ccccc} \pi_n(SO_{n-2}) & \overset{S^{(4)}}{\to} & \pi_n(SO_{n-1}) & \overset{S^{(3)}}{\to} & \pi_n(SO_n). \\ \| & & \| & & \| \\ Z_8 x_1+Z_{24}x_2 & & Z_8 w & & Z_4 v \end{array}$$

Here $S^{(4)}(x_1)=w$, $S^{(4)}(x_2)=0$ (cf. p.6 of [8]), and $S^{(3)}(w)=v$ (cf. [34]). So $\partial:\pi_{n+1}(S^{n-2})\cong Z_{24}\to\pi_n(SO_{n-2})$ is injective and we can take x_2 as $x_2=\partial v_{n-2}$, where v_{n-2} is the generator of $\pi_{n+1}(S^{n-2})$.

We heve the following diagram commutative up to sign:

$$\text{(D5)}\qquad \begin{array}{ccccc} & & \pi_n(SO_{n-2}) & & \\ & \overset{\partial}{\nearrow} & \downarrow J^{(4)} & \overset{\pi_*}{\searrow} & \\ Z_{24}\cong\pi_{n+1}(S^{n-2}) & & & & \pi_n(Sn^{-3})\cong Z_{24}\ (n>7), \\ & \underset{P}{\searrow} & & \underset{H}{\nearrow} & \\ & & \pi_{2n-2}(S^{n-2}) & & \end{array}$$

where π_* is induced from the projection and H is the Hopf homomorphism. We note that π_* is sujective for $n=8s+2$ ($s>0$) since in the exact sequence

$$\pi_n(SO_{n-2}) \xrightarrow{\pi_*} \pi_n(SO_{n-3}) \xrightarrow{\partial} \pi_{n-1}(SO_{n-3}) \xrightarrow{S} \pi_{n-1}(SO_{n-2}),$$

$S:\pi_{8s+1}(SO_{8s-1})=Z_2+Z_2 \to \pi_{8s+1}(SO_{8s})=Z_2+Z_2+Z_2$ is injective by Diagram 2 (p.786) of [7,III]. By (5.32) of [33] and from the diagram (D5), we have $\pi_*(x_2)=\pi_*(\partial\nu_{n-2})=HJ^{(4)}(\partial\nu_{n-2})=H[\nu_{n-2},\iota_{n-2}]=\pm 2\nu_{n-3}$. Since π_* is surjective, this implies that $\pi_*(x_1)=\pm k\nu_{n-3}$ for some odd integer k, $1\leq k\leq 11$. Hence the order of $\pi_*(x_1)$ is just 8.

Suppose that $[\eta^2_{n-1},\iota_{n-1}]=(\eta^2_{n-1})_*\beta$ for some β $\pi_{2n-1}(S^{n+1})$ and let $\beta=E^2\gamma$, $\gamma\in\pi_{2n-3}(S^{n-1})$. Then by the diagrams (D1), (D4) we have $J^{(3)}(4w)=[\eta^2_{n-1},\iota_{n-1}]=(\eta^2_{n-1})_*E^2\gamma=E^{(4)}(\eta^2_{n-2})_*E\gamma$ and $J^{(3)}(4w)=J^{(3)}S^{(4)}(4x_1)=E^{(4)}J^{(4)}(4x_1)$. Therefore we have $J^{(4)}(4x_1)-(\eta^2_{n-1})_*E\gamma\in\operatorname{Ker}E^{(4)} = \langle[\nu_{n-2},\iota_{n-2}]\rangle = \langle J^{(4)}(x_2)\rangle$. Here $J^{(4)}(x_2) = -[\nu_{n-2},\iota_{n-2}]$ has the order 24 by 2.14 of [29]. ($J^{(4)}$ is also injective by Proposition 3.2 of [8]). Then we have $J^{(4)}(4x_1)-(\eta^2_{n-1})_*E\gamma=cJ^{(4)}x_2$ for some integer c, which must be 0 or 12 (mod 24) since the left-hand side is of order 2. Thus, applying the Hopf homomorphism to it from the diagram (D5), we have

$$HJ^{(4)}(4x_1)=4HJ^{(4)}(x_1)=4\pi_*(x_1)\neq 0$$

since $\pi_*(x_1)$ has the order 8, and

$$H(\eta^2_{n-2})_*E\gamma=HE^{(5)}(\eta^2_{n-3})_*\gamma=0,$$
$$H(cJ^{(4)}x_2)=c\pi_*(x_2)=\pm 2c\nu_{n-3}=0.$$

This is a contradiction.

Let $n=8s+6$ ($s>0$). If $s=1$, then $\pi_{2n-1}(S^{n+1})=\Pi_{12}=0$ and so $\operatorname{Im}(\eta^2_{n-1})_*=0$ for $(\eta^2_{n-1})_*:\pi_{2n-1}(S^{n+1})\to\pi_{2n-1}(S^{n-1})$. Since $[\eta^2_{n-1},\iota_{n-1}]\neq 0$ by Lemma 5.1 of [4], this means that $[\eta^2_{n-1},\iota_{n-1}]$ does not belong to $\operatorname{Im}(\eta^2_{n-1})_*$ for $n=14$.

Let $s>1$. Then we have

$$\begin{array}{ccccc} \pi_n(SO_{n-2}) & \xrightarrow{S^{(4)}} & \pi_n(SO_{n-1}) & \xrightarrow{S^{(3)}} & \pi_n(SO_n). \\ \| & & \| & & \| \\ Z_4x_1+Z_{16}x_2+Z_3x_3 & & Z_8w & & Z_4v \end{array}$$

Here $S^{(3)}(w)=v$ again and $S^{(4)}$ is surjective since $\pi_*:\pi_n(SO_{n-1})\to\pi_n(S^{n-2})$ is trivial for $n=8s+6$ by Lemma 2.2 of [6]. Now we may assume that $S^{(4)}(x_2)=w$ as follows: Let $S^{(4)}(x_1)=m_1w$, $S^{(4)}(x_2)=m_2w$, and let $S^{(4)}(l_1x_1+l_2x_2)=w$ for some integers m_1, m_2, and l_1, l_2. Then $w=(l_1m_1+l_2m_2)w$ and so $l_1m_1+l_2m_2\equiv 1$ (mod 8). Since x_1 is of order 4, we have $S^{(4)}(4x_1)=4m_1w=0$, and m_1 must be even. Hence l_2, m_2 must be odd. Since $(m_2,8)=1$, there exists an odd integer m'_2 such that $m'_2m_2\equiv 1$ (mod 8). Let $x'_2=m'_2x_2$. Then $S^{(4)}(x'_2)=m'_2S^{(4)}(x_2)=m'_2m_2w=w$ and x'_2 is another generator for x_2 since $(m'_2,16)=1$. Thus we may assume that $S^{(4)}(x_2)=w$ (This holds also for $s=1$, $\pi_{14}(SO_{12})=Z_4x_1+Z_8x_2+Z_3x_3$).

We have the following exact sequence for $n=8s+6$ ($s>1$):

$$\begin{array}{ccccccc} \pi_n(SO_{n-2}) & \xrightarrow{\pi_*} & \pi_n(S^{n-3}) & \xrightarrow{\partial} & \pi_{n-1}(SO_{n-3}) & \xrightarrow{S} & \pi_{n-1}(SO_{n-2}). \\ \| & & \| & & \| & & \| \\ Z_4x_1+Z_{16}x_2+Z_3x_3 & & Z_8\tilde{v}_{n-3}+Z_3\alpha_{n-3} & & Z_2+Z_2 & & Z_2+Z_2 \end{array}$$

Since $\operatorname{Im}\partial=\operatorname{Ker}S=Z_2+0$ (cf. p.6 of [8]), we know that $\operatorname{Im}\pi_*=\operatorname{Ker}\partial=\langle 2\tilde{v}_{n-3}\rangle+\langle\alpha_{n-3}\rangle\cong Z_4+Z_3$.

Now suppose that $[\eta_{n-1}^2, \iota_{n-1}]=(\eta_{n-1}^2)_*\beta$ for some $\beta\in\pi_{2n-1}(S^{n+1})$, $n=8s+6$ $(s>1)$, and let $\beta=E^2\gamma$, $\gamma\in\pi_{2n-3}(S^{n-1})$. Then we have $[\eta_{n-1}^2, \iota_{n-1}]=J^{(3)}(4w)=4J^{(3)}S^{(4)}(x_2)=4E^{(4)}J^{(4)}(x_2)$ and $(\eta_{n-1}^2)_*\beta=(\eta_{n-1}^2)_*E^2\gamma=E^{(4)}(\eta_{n-2}^2)_*(\mathrm{E}\gamma)$. Hence we have

$$4J^{(4)}(x_2)-(\eta_{n-1}^2)_*E\gamma\in\operatorname{Ker}E^{(4)}=\langle\tilde{v}_{n-2}, \iota_{n-2}]+[\alpha_{n-2}, \iota_{n-2}]\rangle.$$

Here $[\tilde{v}_{n-2}, \iota_{n-2}]$, $[\alpha_{n-2}, \iota_{n-2}]$ have the orders 8, 3 respectively (cf. [21] and p.8 of [8]). Since $J^{(4)}$ is injective by Proposition 3.2 of [8], the left-hand side of the above is of order 4, and so

$$4J^{(4)}(x_2)-(\eta_{n-2}^2)_*E\gamma=\pm 2[\tilde{v}_{n-2}, \iota_{n-2}].$$

Thus applying the Hopf homomorphism to it and by the diagram (D5), we have the following:

$$H(4J^{(4)}(x_2))=4HJ^{(4)}(4x_2)=4\pi_*(x_2)=0$$

since $\pi_*(x_2)$ belongs to $\langle 2\tilde{v}_{n-3}\rangle\cong Z_4$ as shown in the above.

$$H(\eta_{n-2}^2)_*E\gamma=HE^{(5)}(\eta_{n-3}^2)_*\gamma=0,$$

and by (5.32) of [33],

$$H(2[\tilde{v}_{n-2}, \iota_{n-2}])=\pm 4\tilde{v}_{n-3}.$$

Since $4\tilde{v}_{n-3}$ is of order 2, this is a contradiction.

This completes the proof.

Proposition 4.3. *Let* $n=4t$ $(t:odd>0)$.
(i) $G(\eta_{n-1})=0$ *if* $t>1$ *and* $G(\eta_3)=S\pi_4(SO_3)=Z_2v_1$, *where* $\pi_n(SO_n)=Z_2v_1+Z_2v_2$.
(ii) $G(\eta_{n-1}^2)=0$.

Proof. (i) For $n=4t$ $(t:odd>0)$, we have

$$\begin{array}{ccccc} \pi_n(SO_{n-1}) & \xrightarrow{S^{(3)}} & \pi_n(SO_n) & \xrightarrow{S^{(2)}} & \pi_n(SO_{n+1}), \\ \| & & \| & & \| \\ Z_2w & & Z_2v_1+Z_2v_2 & & Z_2u \end{array}$$

where $S^{(3)}w=v_1$ and $S^{(2)}v_1=0$, $S^{(2)}v_2=u$ (cf. [34]). So $v_1=\partial\eta_n$ for $\partial:\pi_{n+1}(S^n)\to\pi_n(SO_n)$. Since $J^{(2)}v_1=J^{(2)}\partial\eta_n=[\eta_n, \iota_n]$, which does not belong to $\mathrm{Im}(\eta_n)_*$ by Lemma 4.1 for $n>4$, we have $J^{(2)-1}(\mathrm{Im}(\eta_n)_*)\cap\mathrm{Im}\,S^{(3)}=0$. Hence $G(\eta_{n-1})=0$ if $n>4$ by Lemma 1.2. Let $n=4$. Then the homomorphism $(\eta_3)_*:\pi_7(S^4)\cong\boldsymbol{Z}+\boldsymbol{Z}_2\to\pi_7(S^3)\cong\boldsymbol{Z}_2$ maps ν_4 generating the free part of $\pi_7(S^4)$ to $\nu'\circ\eta_6$ which generates $\pi_7(S^3)$ (cf. (5.9) of [30]). Since $J^{(3)}$ is isomorphic (cf. Proposition 2.1 of [34]), we have $G(\eta_3)=S^{(3)}J^{(3)-1}(\mathrm{Im}(\eta_3)_*)=S^{(3)}\pi_4(SO_3)=\boldsymbol{Z}_2v_1$.

(ii) Since $S^{(4)}:\pi_n(SO_{n-2})\to\pi_n(SO_{n-1})$ is trivial for $n=4t$ (t: $odd>0$) (cf. [8] pp.6–7), we have the conclusion from the definition of $G(\eta^2_{n-2})$.

Proposition 4.4. *Let* $n=4t$ (t: *even*>0).

(i) $G(\eta_{n-1})=\boldsymbol{Z}_2v_3$ *if* $t>2$ *and* $G(\eta_7)=0$, *where* $\pi_n(SO_n)=\boldsymbol{Z}_2v_1+\boldsymbol{Z}_2v_2+\boldsymbol{Z}_2v_3$.

(ii) $G(\eta^2_{n-2})=0$.

Proof. These are already known by Propositions 4.2, 4.3, and 5.2 of [7, III].

Proposition 4.5. *Let* $n=4t+1$ (t: *odd*).

(i) $G(\eta_{n-1})=0$ *for* $t>0$.

(ii) $G(\eta^2_{n-2})=0$ *for* $t>1$.

Proof. We have

$$\begin{array}{ccccc}\pi_n(SO_{n-2}) & \overset{S^{(4)}}{\rightarrow} & \pi_n(SO_{n-1}) & \overset{S^{(3)}}{\rightarrow} & \pi_n(SO_n), \\ \| & & \| & & \| \\ \boldsymbol{Z}_2x_1+\boldsymbol{Z}_2x_2 & & \boldsymbol{Z}_2w_1+\boldsymbol{Z}_2w_2 & & \boldsymbol{Z}_2v\end{array}$$

where $S^{(4)}x_1=0$, $S^{(4)}x_2=w_1$, and $S^{(3)}w_1=0$, $S^{(3)}w_2=v$ (cf. [8] p.6 and [34]). Since $\pi_n(SO_{n+1})\cong\boldsymbol{Z}$, we have $S^{(2)}v=0$ and so $v=\partial\eta_n$ for $\partial:\pi_{n+1}(S^n)\to\pi_n(SO_n)$. Hence $J^{(2)}v=J^{(2)}\partial\eta_n=[\eta_n, \iota_n]$, which never belong to $\mathrm{Im}(\eta_n)_*$ by Lemma 4.1. Thus we have $G(\eta_{n-1})=0$ by Lemma 1.2.

Similarly we know that $w_1=\partial\eta^2_{n-1}$ for $\partial:\pi_{n+1}(S^{n-1})\to\pi_n(SO_{n-1})$ and that $J^{(3)}w_1=[\eta^2_{n-1}, \iota_{n-1}]$ which does not belong to $\mathrm{Im}(\eta^2_{n-1})_*$ by Lemma 4.2. Hence we have $G(\eta^2_{n-2})=0$ similarly.

Proposition 4.6. *Let* $n=4t+1$ (t: *even*>0).

(i) $G(\eta_{n-1})=\boldsymbol{Z}_2v_2$, *where* $\pi_n(SO_n)=\boldsymbol{Z}_2v_1+\boldsymbol{Z}_2v_2$.

(ii) $G(\eta^2_{n-2})=\boldsymbol{Z}_2w_3$ *if* $t>2$ *and* $G(\eta^2_7)=0$, *where* $\pi_n(SO_{n-1})=\boldsymbol{Z}_2w_1+\boldsymbol{Z}_2w_2+\boldsymbol{Z}_2w_3$.

Proof. (i) We have

$$\begin{array}{ccccccc} \pi_n(SO_{n-1}) & \xrightarrow{S^{(3)}} & \pi_n(SO_n) & \xrightarrow{S^{(2)}} & \pi_n(SO_{n+1}) & \xrightarrow{S^{(1)}} & \pi_n(SO), \\ \| & & \| & & \| & & \| \\ Z_2w_1+Z_2w_2+Z_2w_3 & & Z_2\bar{v}_1+Z_2v_2 & & Zu_1+Z_2u_2 & & Z_2\tau \end{array}$$

where $S^{(3)}w_1=0$, $S^{(3)}w_2=v_1$, $S^{(3)}w_3=v_2$; $S^{(2)}v_1=0$, $S^{(2)}v_2=u_2$; $S^{(1)}u_1=0$, $S^{(1)}u_2=\tau$ (vf. p.786, Diagram 2 of [7,III]). Since $S^{(2)}v_1=0$, we have $v_1=\partial\eta_n$ for $\partial:\pi_{n+1}(S^n)\to\pi_n(SO_n)$. So $J^{(2)}v_1=J^{(2)}\partial\eta_n=[\eta_n,\iota_n]$, which does not belong to $\mathrm{Im}(\eta_n)_*$ by Lemma 4.1.

On the other hand, it is known that $J\tau$ belongs to $\mathrm{Im}(\eta^2)_*$ if $t>2$ by [21]. Therefore, as is shown in the proof of Assertion 2 of Proposition 5.3 of [7, III], we know that $J^{(2)}v_2$ belongs to $\mathrm{Im}(\eta_n^2)_*$ and so belongs to $\mathrm{Im}(\eta_n)_*$ for $t>2$. If $t=2$ $(n=9)$, $J\tau=v^3$ by [14] and $v^3=\eta\circ\bar{v}$ for a basis element $\bar{v}$ of $\Pi_8\cong Z_2+Z_2$. So $J\tau\in\mathrm{Im}\,\eta_*$ for $t=2$. Then, replacing η_k^2 by η_k, an argument similar to the proof of the same Assertion 2 induces that $J^{(2)}v_2$ belongs to $\mathrm{Im}(\eta_n)_*$, where we note that $E^{(4)}:\pi_{2n-2}(S^{n-1})\to\pi_{2n-1}(S^n)$ is surjective since n is odd.

Thus we have $J^{(2)}v_2\in\mathrm{Im}(\eta_n)_*$ for even $t>0$ and therefore we know that $G(\eta_{n-1})=Z_2v_2$ by Lemma 1.2.

(ii) is already known by Proposition 5.3 of [7,III].

Proposition 4.7. *Let* $n=4t+2$ $(t>0)$. *Then* $G(\eta_{n-1})=0$ *and* $G(\eta_{n-2}^2)=0$.

Proof. Since $\pi_n(SO_5)=\pi_6(SO_6)=0$, the assertion is clear for $t=1$. Let $t>1$. Then we have

$$\begin{array}{ccccc} \pi_n(SO_{n-1}) & \xrightarrow{S^{(3)}} & \pi_n(SO_n) & \xrightarrow{S^{(2)}} & \pi_n(SO_{n+1}). \\ \| & & \| & & \| \\ Z_8w & & Z_4v & & Z_2u \end{array}$$

where $S^{(3)}w=v$, $S^{(2)}v=u$ and $J^{(2)}:\pi_n(SO_n)\to\pi_{2n}(S^n)$, $J^{(3)}:\pi_n(SO_{n-1})\to\pi_{2n-1}(S^{n-1})$ are injective (cf. [34]). Since the elements of $\mathrm{Im}(\eta_n)_*$, $(\eta_n)_*:\pi_{2n}(S^{n+1})\to\pi_{2n}(S^n)$, have order 2, only the element $J^{(2)}(2v)$ may possibly by in $\mathrm{Im}(\eta_n)_*$. However, we have $2v=\partial\eta_n$ for $\partial:\pi_{n+1}(S^n)\to\pi_n(SO_n)$ since $S^{(2)}(2v_1)=0$, and therefore $J^{(2)}(2v)=J^{(2)}\partial\eta_n=[\eta_n,\iota_n]$, which does not belong to $\mathrm{Im}(\eta_n)_*$ by Lemma 4.1. Hence we have $\mathrm{Im}\,J^{(2)}\cap\mathrm{Im}\,(\eta_n)_*=0$. Thus $G(\eta_{n-1})=0$ by Lemma 1.2. A similar argument holds also for $G(\eta_{n-1}^2)$ by Lemma 4.2.

The situation of the case $n=4t-1$ $(t>1)$ is a little different from those of the other cases. Let $m(2t)$ be the denominator of $B_t/4t$, where B_t is the t-th Bernouli number, and let $l(2t)=m(2t)/2$. We have the following.

Proposition 4.8. *Let* $n=4t-1$ $(t>1)$.

(i) *If* t *is odd, then* $G(\eta_{n-1})=l(2t)Z$, $G(\eta_{n-2}^2)=l(2t)Z$.

(ii) *If t is even, then $G(\eta_{n-1})=m(2t)\boldsymbol{Z}$ $(t>2)$, $G(\eta_6)=120\boldsymbol{Z}$, $G(\eta_{n-2}^2)=m(2t)\boldsymbol{Z}$ $(t>2)$, and $G(\eta_5^2)=60\boldsymbol{Z}$.*
Here, $\pi_n(SO_n)=\boldsymbol{Z}$ and $\pi_n(SO_{n-1})=\boldsymbol{Z}$ for $t>1$.

Proof. Let $t>2$. We have

$$\begin{array}{ccccccc} \pi_n(SO_{n-1}) & \overset{S^{(3)}}{\to} & \pi_n(SO_n) & \overset{S^{(2)}}{\to} & \pi_n(SO_{n+1}) & \overset{S^{(1)}}{\to} & \pi_n(SO), \\ \| & & \| & & \| & & \| \\ \boldsymbol{Z}w & & \boldsymbol{Z}v & & \boldsymbol{Z}u_1+\boldsymbol{Z}u_2 & & \boldsymbol{Z}\tau \end{array}$$

where $S^{(3)}w=v$, $S^{(2)}v=u_2$, $S^{(1)}u_1=0$, $S^{(1)}u_2=\tau$ (cf.[34]). Since $S^{(2)}$, $S^{(3)}$ are injective, $E^{(3)}:\pi_{2n-1}(S^{n-1})\to\pi_{2n}(S^n)$ and $E^{(2)}:\pi_{2n}(S^n)\to\pi_{2n+1}(S^{n+1})$ are also injective by the diagram (D1). Furthermore, since $n+1$ is even and $n>7$, we have $\pi_{2n+1}(S^{n+1}) =[\iota_{n+1},\iota_{n+1}]\oplus E^{(2)}\pi_{2n}(S^n)$. So $E^{(1)}:\pi_{2n+1}(S^{n+1})\to\pi_{2n+2}(S^{n+2})=\Pi_n$, which is surjective, is isomorphic on the image $E^{(2)}\pi_{2n}(S^n)$ since $\operatorname{Ker}E^{(1)}=\langle[\iota_{n+1},\iota_{n+1}]\rangle$. Therefore $\pi_{2n}(S^n)$, $\pi_{2n-1}(S^{n-1})$ are mapped injectively (ismorphically in fact) to Π_n by suspensions. Thus, from the diagram (D4), we have the following diagram which is commutative up to sign:

$$\begin{array}{ccccc} \pi_n(SO_{n-1}) & \underset{\cong}{\overset{S^{(3)}}{\longrightarrow}} & \pi_n(SO_n) & \underset{\cong}{\overset{i_*^S}{\longrightarrow}} & \pi_n(SO)=\boldsymbol{Z} \\ \downarrow{\scriptstyle J^{(3)}} & & \downarrow{\scriptstyle J^{(2)}} & & \downarrow{\scriptstyle J} \\ \pi_{2n-1}(S^{n-1}) & \overset{E^{(3)}}{\longrightarrow} & \pi_{2n}(S^n) & \overset{E^{(2)}}{\longrightarrow} & \Pi_n \\ \uparrow & & \uparrow{\scriptstyle (\eta_n)_*} & & \uparrow{\scriptstyle \eta_*} \\ {\scriptstyle (\eta_{n-1}^2)_*} & & \pi_{2n}(S^{n+1}) & \underset{\cong}{\longrightarrow} & \Pi_{n-1} \\ \big| & & & & \uparrow{\scriptstyle \eta_*} \\ \pi_{2n-1}(S^{n+1}) & & \underset{\cong}{\longrightarrow} & & \Pi_{2n-2} \end{array}$$

So we have $\operatorname{Im}J^{(3)}\cong\operatorname{Im}J^{(2)}\cong\operatorname{Im}J$, $\operatorname{Im}(\eta_n)_*\cong\operatorname{Im}\eta_*$, and $\operatorname{Im}(\eta_{n-1}^2)_*\cong\operatorname{Im}\eta_*^2$, and hence

$$\text{(1)}\qquad \begin{cases} \operatorname{Im}J^{(2)}\cap\operatorname{Im}(\eta_n)_*\cong\operatorname{Im}J\cap\operatorname{Im}\eta_*, \\ \operatorname{Im}J^{(3)}\cap\operatorname{Im}(\eta_{n-1}^2)_*\cong\operatorname{Im}J\cap\operatorname{Im}\eta_*^2. \end{cases}$$

$\operatorname{Im}J$ is isomorphic to the cyclic group $\boldsymbol{Z}_{m(2t)}$ by Theorem 1.5 of [1], and the elements of $\operatorname{Im}\eta_*$, $\operatorname{Im}\eta_*^2$ are of order 2. Hence $\operatorname{Im}J\cap\operatorname{Im}\eta_*$ and $\operatorname{Im}J\cap\operatorname{Im}\eta_*^2$ are isomorphic to 0 or $\boldsymbol{Z}_2$. Let $t=2s+1$ $(s>0)$. Then, by Proposition 4.2 of [21], there exists an element $\xi_s\in\operatorname{Im}J\subset\Pi_{8s+3}$ with the order 8 such that $4\xi_s=\eta^2\mu_s$ for a certain element $\mu_s\in\Pi_{8s+1}$. This means that $\operatorname{Im}J\cap\operatorname{Im}\eta_*=\operatorname{Im}J\cap\operatorname{Im}\eta_*^2\cong\boldsymbol{Z}_2$. Hence, we have (i) by (1) and Lemma 1.2, where we note that $S^{(4)}:\pi_n(SO_{n-2})$

$\to \pi_n(SO_{n-1})$ is surjective (cf. [8], p.8).

Let $t=2s$ $(s>1)$. We use Adams' invariants (cf. [1]). Since $e_C(\eta\circ\theta)=d_C(E\theta)e_C(\eta)$ for any $\theta\in\Pi_{n-1}$ $(n=8s-1,\ s>1)$ and $d_C=0$ by Proposition 3.2 (c) and Proposition 7.1 of [1] respectively, we have $e_C(\eta\circ\theta)=0$ for any $\theta\in\Pi_{n-1}$. Here, $e_C=e'_R$ by Proposition 7.14 of [1]. So we know that $\operatorname{Im}\eta_*\subset\operatorname{Ker}e'_R$. On the other hand, Theorem 1.6 of [1], [22], and the remark (p.284) of [23] show that

$$\Pi_{8s-1}=\operatorname{Im}J\oplus\operatorname{Ker}e'_R.$$

Hence we have

$$\operatorname{Im}J\cap\operatorname{Im}\eta_*=\operatorname{Im}J\cap\operatorname{Im}\eta_*^2=0 \tag{2}$$

since $\operatorname{Im}\eta_*^2\subset\operatorname{Im}\eta_*\subset\operatorname{Ker}e'_R$. Thus we have (ii) from (1), (2) and by Lemma 1.2. The results for $t=2$ $(n=7)$ are already known by Propositions 4.1, 5.1 of [7, III]. This completes the proof.

Department of Mathematics
Faculty of Science
Kanazawa University

References

[1] J. F. Adams, On the groups $J(X)$-IV, Topology, **5** (1966), 21–71.
[2] M. G. Barratt and M. E. Mahowald, The metastable homotopy of $O(n)$, Bull. Amer. Math. Soc., **70** (1964), 758–760.
[3] W. Browder, Embedding 1-connected manifolds, Bull. Amer. Math. Soc., **72** (1966), 225–231.
[4] W. C. Hsiang, J. Levine and R. H. Szczarba, On the normal bundle of a homotopy sphere embedded in euclidean space, Topology, **3** (1965), 173–181.
[5] H. Ishimoto, On the classification of $(n-2)$-connected $2n$-manifolds with torsion free homology groups, Publ. RIMS, Kyoto Univ., **9** (1973), 211–260.
[6] H. Ishimoto, On the classification of some $(n-3)$-connected $(2n-1)$-manifolds, Publ. RIMS, Kyoto Univ., **11** (1976), 723–747.
[7] H. Ishimoto, Homotopy classification of connected sums of sphere bundles over spheres I, Nagoya Math. J., **83** (1981), 15–36; II, Publ. RIMS, Kyoto Univ., **18** (1982), 307–324; III, ibid., **19** (1983), 773–811.
[8] H. Ishimoto and K. Miyoshi, On certain manifolds which are tangentially homotopy equivalent, Sci. Rep. Kanazawa Univ., **30** (1985), 1–13.
[9] H. Ishimoto, On an extension of the James-Whitehead theorem about sphere bundles over spheres, Proc. Japan Acad., **64A** (1988), 356–359.
[10] H. Ishimoto, On cellular decompositions of the boundaries of handlebodies and Wall's invariants, Sci.Rep.Kanazawa Univ., **35** (1990), 1–23.
[11] H. Ishimoto, On a globalization of the James-Whitehead theorem about sphere bundles over spheres, Quart. J. Math. Oxford, **46** (1995), 453–469.
[12] I. M. James, On the homotopy groups of certain pairs and triads, Quart. J. Math. Oxford, **5** (1954), 260–270.
[13] I. M. James and J. H. C. Whitehead, The homotopy theory of sphere bundles over spheres (I), Proc. London Math. Soc., **4** (1954), 196–218; (II), ibid., **5** (1955), 148–166.
[14] H. Kachi, On the homotopy groups of rotation groups and the J-homomorphism, Dissertation (in Japanese).
[15] H. Kachi, On the homotopy groups of rotation groups R_n, J. Fac. Sci., Shinshu Univ., **3** (1968), 13–33.

[16] M. A. Kervaire, A note on obstructions and characteristic classes, Amer. J. Math., **81** (1959), 773–784.
[17] M. A. Kervaire, Some nonstable homotopy groups of Lie groups, Illinois J. Math., **4** (1960), 161–169.
[18] M. A. Kervaire and J. W. Milnor, Groups of homotopy spheres: I, Ann. of Math., **77** (1963), 504–537.
[19] R. K. Lashof, Some theorems of Browder and Novikov on homotopy equivalent manifolds with an application, Notes prepared by Rudolfo de Sapio, Chicago, 1963.
[20] I. Madsen, L. R. Taylor and B. Williams, Tangential homotopy equivalences, Comment. Math. Helv., **55** (1980), 445–484.
[21] M. Mahowald, The metastable homotopy of S, Mem. Amer. Math. Soc. **72**, Amer. Math. Soc., 1967.
[22] M. Mahowald, The order of the image of the *J*-homomorphism, Bull. Amer. Math. Soc., **76** (1970), 1310–1313.
[23] J. W. Milnor and J. D. Stasheff, Characteristic classes, Ann. of Math. Studies **76**, Princeton Univ. Press, 1974.
[24] Y. Nomura, Note on some Whitehead products, Proc.Japan Acad., **50** (1974), 48–52.
[25] S. P. Novikov, Topological invariance of rational Pontrjagin classes, Dokl. Akad. Nauk SSSR, **163** (1965), 298–300; Soviet Math., **6** (1965), 921–923.
[26] S. Sasao, Homotopy types of spherical fibre spaces over spheres, Pacific J. Math., **52** (1974), 207–219.
[27] K. Shiraiwa, A note on tangent bundles, Nagoya Math. J., **29** (1967), 259–267.
[28] N. Steenrod, The Topology of fibre bundles, Princeton Math. Series **14**, Princeton Univ. Press, 1974.
[29] S. Thomeier, Einige Ergebnisse über Homotopiegruppen von Sphären, Math. Ann., **164** (1966), 225–250.
[30] H. Toda, Composition methods in homotopy groups of spheres, Ann. of Math. Studies **49**, Princeton Univ. Press, 1962.
[31] C. T. C. Wall, Classification of $(n-1)$-connected $2n$-manifolds, Ann. of Math., **75** (1962), 163–189.
[32] C. T. C. Wall, Classification problems in differential topology-I, Classification of handlebodies, Topology, **2** (1963), 253–261.
[33] G. W. Whitehead, A generalization of the Hopf invariant, Ann. of Math., **51** (1950), 192–237.
[34] J. Yoshida and H. Ishimoto, Classification of certain manifolds with sufficient connectedness, Sci. Rep. Kanazawa Univ., **24** (1979), 61–71.