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Homological codimension of modular rings of invariants and the Koszul complex

By

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Abstract

Let $\rho: G \hookrightarrow \mathrm{GL}(n, F)$ be a representation of a finite group over the field F of characteristic p, and $h_1, \cdots, h_m \in F[V]^G$ invariant polynomials that form a regular sequence in $F[V]$. In this note we introduce a tool to study the problem of whether they form a regular sequence in $F[V]^G$. Examples show they need not. We define the cohomology of G with coefficients in the Koszul complex

$$(\mathscr{K}, \partial) = (F[V] \otimes E(s^{-1}h_1, \cdots, s^{-1}h_n),\ \partial(s^{-1}h_i) = h_i : i = 1, \cdots, n),$$

which we denote by $H^*(G; (\mathscr{K}, \partial))$, and use it to study the homological codimension of rings of invariants of permutation representations of the cyclic group of order p, for $p \neq 0$, and to answer the above question in this case.

0. Introduction.

Let G be a finite group and $\rho: G \hookrightarrow \mathrm{GL}(n, F)$ a representation of G over the field F. Suppose that $h_1, \cdots, h_k \in F[V]^G$ are invariant polynomials (we assume familiarity with the basic ideas, definitions, and notations of invariant theory of finite groups as found for example in [22]) that form a regular sequence in $F[V]$. We pose the question: do they form a regular sequence in $F[V]^G$? In general the answer will be no. For example if F is a Galois field of characteristic p and $\boldsymbol{d}_{n,0}, \cdots, \boldsymbol{d}_{n,n-1} \in F[V]^G$ the Dickson polynomials (see [22] chapter 8), then $\boldsymbol{d}_{n,0}, \cdots, \boldsymbol{d}_{n,n-1}$ are certainly a regular sequence in $F[V]$, but, are a regular sequence in $F[V]^G$ if and only if $F[V]^G$ is Cohen-Macaulay, and this certainly need not be the case (see e.g. [22] chapter 6 and [23] §4).

This study began in an attempt to verify the ***depth conjecture*** of Landweber and Stong in some concrete examples by using the methods (not the results) of [8], even though the depth conjecture has been proved by other methods by Borguiba and Zarati [3] (see also [23] §6). These computations appear in §3 and §4.

In contrast to [8], where the ground field is algebraically closed, we take advantage of the fact that, over a finite field F there is a universal ring of invariants for representations of degree n, namely the Dickson algebra $\boldsymbol{D}^*(n)$. Since $\boldsymbol{D}^*(n) \subseteq F[V]^G$ is always a finite extension, the homological codimension of $F[V]^G$

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as a ring is the same as the homological codimension of $F[V]^G$ as a $D^*(n)$-module. The Dickson algebra $D^*(n)$ is the polynomial algebra $F[d_{n,n-1}, \cdots, d_{n,0}]$ and hence has finite global dimension, so a famous equality of Auslander and Buchsbaum [1] allows us to convert the computation of the homological codimension of $F[V]^G$ over $D^*(n)$ into an equivalent computation of the homological (i.e. projective) dimension of $F[V]^G$ over $D^*(n)$. For this we introduce a spectral sequence, which, loosely speaking, is a Koszul-Serre dual to the one used by Ellingsrud and Skjelbred in [8]. We hope in this way to make this circle of ideas available to a larger audience then seems to have been attracted by [8] alone.

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§1. A Motivational Example

Let $\rho: G \hookrightarrow \mathrm{GL}(n, F)$ be a representation of a finite group G over the field F. If $|G| \in F^\times$ then the Reynolds operator

$$\pi^G = \frac{1}{|G|} \mathrm{Tr}^G : F[V] \to F[V]^G$$

defines a splitting of the inclusion $F[V]^G \hookrightarrow F[V]$. If $h_1, \cdots, h_k \in F[V]^G$ form a regular sequence in $F[V]$, then they are algebraically independent, and $F[V]$ is a free $F[h_1, \cdots, h_k]$-module. Since $F[V]^G \subseteq F[V]$ is an $F[V]^G$-direct summand π^G is an $F[V]^G$-linear map (see [22] §2.4), it is also an $F[h_1, \cdots, h_k]$-direct summand, and hence projective as an $F[h_1, \cdots, h_k]$-module. In this graded connected context, projective and free are the same thing, so $F[V]^G$ is a free $F[h_1, \cdots, h_k]$-module, and therefore (see e.g. [22] 6.2) $h_1, \cdots, h_k$ is also a regular sequence in $F[V]^G$. So the question posed at the outset is only of interest in the modular case (for finite groups, i.e., where the characteristic of the ground field divides the group order).

The following example, due to M.D.Neusel, brings out the subtle nature of the problem.

Example 1 (M.D. Neusel). Consider the representation $\sigma_3 : Z/2 \hookrightarrow \mathrm{GL}(6, F)$, obtained by letting $Z/2$ permute the two *vector* variables x_1, x_2, x_3 with y_1, y_2, y_3 where F is a field of characteristic 2. The Poincaré series of $F[x_1, x_2, x_3, y_1, y_2, y_3]^{Z/2}$ can be computed by using either Molien's theorem to compute the Poincaré series over C and the last conclusion of [22] theorem 4.2.4 or [22] 4.2.8. In either case we obtain:

$$P(\boldsymbol{F}[x_1, x_2, x_3, y_1, y_2, y_3]^{\mathbf{Z}/2}, t) = \frac{1}{2}\left[\frac{(1+t)^3+(1-t)^3}{(1-t)^3(1-t^2)^3}\right] = \frac{1+3t^2}{(1-t)^3(1-t^2)^3}$$
$$= 1 + 3t + 12t^2 + 28t^3 + \cdots.$$

Therefore the space of invariant linear forms has dimension 3, the space of invariant quadratic forms dimension 12, the space of invariant cubic forms dimension 28, etc. In degrees 1 and 2 it is relatively easy to find bases for the invariant forms:

$$l_i = x_i + y_i \qquad i = 1, 2, 3$$

is a basis for the invariant linear forms, and the 6 products

$$l_i l_j \qquad 1 \leq i \leq j \leq 3$$

together with the 6 quadratic polynomials

$$q_i = x_i y_i \qquad i = 1, 2, 3$$
$$Q_3 = x_1 x_2 + y_1 y_2$$
$$Q_2 = x_1 x_3 + y_1 y_3$$
$$Q_4 = x_2 x_3 + y_2 y_3$$

form a basis for the space of invariant quadratic forms. The relation

$$l_1 l_2 l_3 = Q_1 l_1 + Q_2 l_2 + Q_3 l_3 + 2(x_1 x_2 x_3 + y_1 y_2 y_3)$$

in $\boldsymbol{K}[x_1, x_2, x_3, y_1, y_2, y_3]$, for any field $\boldsymbol{K}$, is due to M.D. Neusel, and shows that in $\boldsymbol{F}[x_1, x_2, x_3, y_1, y_2, y_3]^{\mathbf{Z}/2}$ the linear form l_3 is a zero divisor modulo the ideal generated by l_1 and l_2 (recall $\boldsymbol{F}$ has characteristic 2). However the forms l_1, l_2, l_3 are clearly a regular sequence in $\boldsymbol{F}[x_1, x_2, x_3, y_1, y_2, y_3]$.

The situation is even more complicated: in section 3 we will see that the homological codimension of $\boldsymbol{F}[x_1, x_2, x_3, y_1, y_2, y_3]^{\mathbf{Z}/2}$ is 5, and therefore by the theorem of Bourguiba and Zarati [3] the five Dickson polynomials of least degrees $\boldsymbol{d}_{6,5}, \boldsymbol{d}_{6,4}, \boldsymbol{d}_{6,3}, \boldsymbol{d}_{6,2}, \boldsymbol{d}_{6,1}$ form a regular sequence in $\boldsymbol{F}[x_1, x_2, x_3, y_1, y_2, y_3]^{\mathbf{Z}/2}$ as well as in $\boldsymbol{F}[x_1, x_2, x_3, y_1, y_2, y_3]$. By contrast the five Dickson polynomials of highest degrees $\boldsymbol{d}_{6,4}, \boldsymbol{d}_{6,3}, \boldsymbol{d}_{6,2}, \boldsymbol{d}_{6,1}, \boldsymbol{d}_{6,0}$ are a regular sequence in $\boldsymbol{F}[x_1, x_2, x_3, y_1, y_2, y_3]$ but not in $\boldsymbol{F}[x_1, x_2, x_3, y_1, y_2, y_3]^{\mathbf{Z}/2}$. We will return to this circle of problems in §5 after we have developed adequate tools in the next sections.

To delve into the problems posed by this example we recall, [22] §6.2, that $h_1, \cdots, h_k \in \boldsymbol{F}[V]^G$ is a regular sequence if and only if the Koszul complex

$$\mathscr{L} = \boldsymbol{F}[V]^G \otimes E(s^{-1}h_1, \cdots, s^{-1}h_k)$$
$$\partial(s^{-1}h_i) = h_i \text{ for } i = 1, \cdots, k$$

is acyclic. For the sake of clarity here is the definition of the Koszul complex and notation that we are using (see [22] §6.2, [21] part II §1).

Definition. Let A be a graded connected commutative algebra over a field $\boldsymbol{F}$ and $a_1, \cdots, a_n \in A$. The **Koszul complex** of A with respect to $a_1, \cdots, a_n \in A$ is the differential graded commutative algebra

$$\mathscr{K} = \mathscr{K}(a_1, \cdots, a_n) = A \otimes E(s^{-1}a_1, \cdots, s^{-1}a_n)$$

where $E(s^{-1}a_1, \cdots, s^{-1}a_n)$ denotes a graded[1]-exterior algebra with $\deg(s^{-1}a_i) = -1 + \deg(a_i)$, and the differential ∂ is defined by requiring

$$\partial|_A = 0$$

$$\partial(s^{-1}a_i) = a_i \quad \text{for } i = 1, \cdots, n$$

$$\partial(x \cdot y) = \partial(x)y + (-1)^{\deg(x)} x\partial(y) \qquad \forall x, y \in \mathscr{K}.$$

Introduce the Koszul complex

$$\mathscr{K} = \boldsymbol{F}[V] \otimes E(s^{-1}h_1, \cdots, s^{-1}h_k)$$

$$\partial(s^{-1}h_i) = h_i \text{ for } i = 1, \cdots, k.$$

The group G acts on $\mathscr{K}$ via the representation ρ on $\boldsymbol{F}[V]$ and trivially on $E(s^{-1}h_1, \cdots, s^{-1}h_k)$. Moreover $\mathscr{L} = \mathscr{K}^G$. By hypothesis $h_1, \cdots, h_k \in \boldsymbol{F}[V]$ is a regular sequence, so $(\mathscr{K}, \partial)$ is acyclic. Hence the question at hand becomes: is $(\mathscr{K}^G, \partial)$ also acyclic? More generally we would want to relate $H^*(\mathscr{K}, \partial)^G$ and $H^*(\mathscr{K}^G, \partial)$. One way to do so is to note that $M^G = H^0(G; M)$ for any G-module M. Since the functor $H^0(G; -)$ is not exact in general it is not suprising that the higher derived functors (see e.g. [10]) of $H^0(G; -)$ enter into the discussion.

§2. Koszul Cohomology of a Group

As a tool to deal with the problems encountered in the introduction and §1 we set up a spectral sequence[2] relating the cohomology of a G-cocomplex to the cohomology of the fixed cocomplex. In order to keep the discussion as simple as possible[3] we suppose that $\rho: G \subsetneq \mathrm{GL}(n, \boldsymbol{F})$ is a representation of a finite group G over the field $\boldsymbol{F}$ and that $h_1, \cdots, h_k \in \boldsymbol{F}[V]^G$ are invariant polynomials which form a regular sequence in $\boldsymbol{F}[V]$. Let $(\mathscr{K}, \partial)$ denote the Koszul complex

$$\mathscr{K} = \boldsymbol{F}[V] \otimes E(s^{-1}h_1, \cdots, s^{-1}h_k)$$

[1] This is the totalization of the Koszul complex bigraded as in [21].
[2] In fact this spectral sequence is a form of Koszul-Serre dual, or local cohomology dual, to the spectral sequence employed by Ellingsrud and Skjelbred in [8].
[3] It will be immediately clear to the experts that a much broader discussion is possible, however, my aim here is not generality, but to provide a tool to make concrete computations in invariant theory.

$$\partial(s^{-1}h_i)=h_i \text{ for } i=1,\cdots,k$$

with the extended action of G as in the previous section. We denote by $F(G)$ the group algebra of G over F and let $\mathscr{P}(G)\to F$ denote a projective resolution of F regarded as an $F(G)$-module via the augmentation homomorphism $\varepsilon: F(G)\to F$, such as for example the bar construction $\mathscr{B}(G)$ of G over F (see e.g. [4] or [6]). Introduce the double complex[4]

$$\mathscr{C}=\mathrm{Hom}_{F(G)}(\mathscr{P}(G),\mathscr{K}).$$

A bit of care is needed with the gradings to turn this into an acceptable double complex, i.e., one satisfying the standard grading conventions (see e.g. [15]). The differential d^* coming from the projective resolution $\mathscr{P}(G)$ appears in the contravariant variable of $\mathrm{Hom}_{F(G)}(-,-)$ and hence the differential d^* induced by d on $\mathscr{C}$ ***increases*** the grading in $\mathscr{C}$ coming from $\mathscr{P}(G)$, i.e., $d^*:\mathscr{C}^{s,t}\to\mathscr{C}^{s+1,t}$. Therefore we must grade the Koszul complex $\mathscr{K}$ compatibly, i.e., so that ∂ (which appears in the covariant variable of $\mathrm{Hom}_{F(G)}(-,-)$) also raises the resolution degree (this time in $\mathscr{K}$) by 1, and the induced differential on $\mathscr{C}$ satisfies $\partial:\mathscr{C}^{s,t}\to\mathscr{C}^{s,t+1}$. This is the standard grading of the Koszul complex à la Eilenberg-Moore [21]. In other words, we bigrade $\mathscr{K}$ by giving the elements of $F[V]$ resolution degree 0 (and internal degree $\deg(f)$) and the elements $s^{-1}h_i$ the resolution degree -1 (and internal degree $\deg(h_i)$) for $i=1,\cdots,k$. Doing so $(\mathscr{C},d^*,\partial)$ becomes a double complex (of graded modules! the grading $\mathscr{C}^{s,t}$ on coming from the grading of $F[V]$).

This describes the grading convention we will employ throughout this manuscript.

Since $(\mathscr{C},d^*,\partial)$ is a double complex we can totalize it, i.e., we can form the associated graded complex $\{\mathrm{Tot}(\mathscr{C})^m=\bigoplus_{s+t=m}\mathscr{C}^{s,t}\mid m\in \mathbf{Z}\}$ with differential $d^*+\partial$. We denote the cohomology of this complex by $H^*(G;(\mathscr{K},\partial))$ and refer to it laxly as the **cohomology of G with coefficients in the Koszul complex** $(\mathscr{K},\partial)$.

Associated to this double complex are two spectral sequences, [15] chapter XI, which are independent of the choice of the resolution $\mathscr{P}(G)$. Using the spectral sequence where we first apply the Koszul differential and then the differential from the projective resolution $\mathscr{P}(G)$ we obtain:

Proposition 2.1. *With the preceding hypotheses and notations the augmentation map of complexes*

$$\eta:(\mathscr{K},\partial)\to\left(\frac{F[V]}{(h_1,\cdots,h_k)},0\right)$$

4 See also [4] chapter VII section 5 for a similar construction in connection with the cohomology of semidirect products.

induces an isomorphism

$$H^*(G;(\mathscr{K},\partial)) \to H^*\left(G;\frac{F[V]}{(h_1,\cdots,h_k)}\right).$$

Proof. The map η is an equivalence of complexes, and hence so is

$$\mathrm{Hom}_{F(G)}(\mathscr{B}(G),\mathscr{K}) \to \mathrm{Hom}_{F(G)}\left(\mathscr{B}(G),\frac{F[V]}{(h_1,\cdots,h_k)}\right)$$

from which the desired conclusion follows.

The spectral sequence where we first apply the projective resolution differential (e.g. bar construction differential) and then the Koszul complex differential will become our main technical tool. If we denote this spectral sequence by $\{E_r, d_r\}$ then as a consequence of 2.1 ($\Rightarrow$ denotes **converges to**)

$$E_r \Rightarrow H^*\left(G;\frac{F[V]}{(h_1,\cdots,h_k)}\right).$$

Since G acts trivially on $E(s^{-1}h_1,\cdots,s^{-1}h_k)$ the term E_1 of the spectral sequence takes the form

$$E_1 = H^*(G;F[V])\otimes E(s^{-1}h_1,\cdots,s^{-1}h_k).$$

Therefore the term E_2 may be identified with the cohomology of yet another Koszul complex, namely for the elements

$$h_1,\cdots,h_k \in H^0(G;F[V]) = F[V]^G \subseteq H^*(G;F[V]).$$

Therefore, recalling that $h_1,\cdots,h_k \in F[V]^G$ are algebraically independent, we see

$$E_2^{s,t} = \mathrm{Tor}^s_{F[h_1,\cdots,h_k]}(H^t(G;F[V]),F)$$

where $H^*(G;F[V])$ is regarded as an $F[h_1,\cdots,h_k]$-module via the inclusion

$$F[h_1,\cdots,h_k] \subseteq F[V]^G = H^0(G;F[V]) \subseteq H^*(G;F[V])$$

and the product in group cohomology. To summarize, we have shown:

Proposition 2.2. *With the preceding hypotheses and notations there is a convergent second quadrant spectral sequence* $\{E_r, d_r\}$ *with*

$$E_r \Rightarrow H^*\left(G;\frac{F[V]}{(h_1,\cdots,h_k)}\right)$$

$$E_2^{s,t} = \mathrm{Tor}^s_{F[h_1,\cdots,h_k]}(H^t(G;F[V]),F).$$

Proof. We need only remark that convergence is a consequence of the fact that $E_2^{s,t}=0$ if $s<-k$.

This spectral sequence is a precursor of Grothendieck's local cohomology spectral sequence [11].

As a simple application of this spectral sequence we reprove a result of Landweber and Stong [14], that serves as a model for further applications.

Proposition 2.3. *Suppose that $\rho: G \hookrightarrow \mathrm{GL}(n, F)$, $n \geq 2$, is a representation of a finite group G over the field F. If $h_1, h_2 \in F[V]^G$ are a regular sequence in $F[V]$, then they are a regular sequence in $F[V]^G$ also.*

Proof. The polynomial algebra $F[h_1, h_2]$ has global dimension 2, so the functors $\mathrm{Tor}^s_{F[h_1,h_2]}(-,-)$ are identically zero for $s<-2$. The following diagram representing E_2

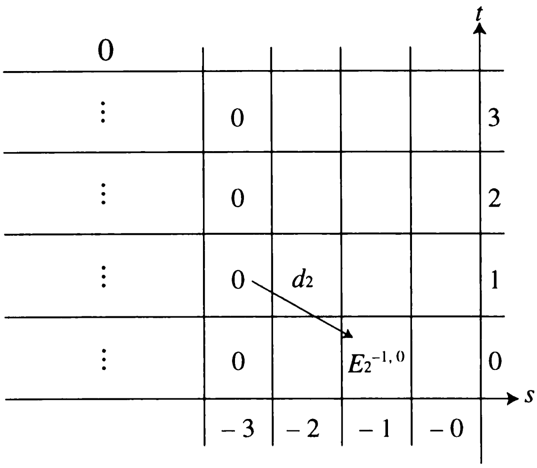

Figure 2.1. E_2 in the dimension 2 case

shows that there is no way that a nonzero differential can either arrive at or leave from $E_2^{-1,0}$. The elements of $E_2^{-1,0}$ have negative total degree. $H^*(G; \frac{F[V]}{(h_1,h_2)})$ is zero in negative degrees, and hence $E_\infty^{s,t}$ is also zero in negative degrees. Therefore

$$0=E_2^{-1,0}=\mathrm{Tor}^{-1}_{F[h_1,h_2]}(H^0(G;F[V]),F)=\mathrm{Tor}^{-1}_{F[h_1,h_2]}(F[V]^G,F)$$

and the result follows.

§3. Vector invariants of $Z/2$

In this section we will apply the spectral sequence of 2.2 to rings of invariants in characteristic 2. To see why this case is particularly amenable we begin with a number of general remarks. If $\rho: G \hookrightarrow \mathrm{GL}(n, F)$ is a representation of a finite group G over the field F of characteristic p and $P=\mathrm{Syl}_p(G) \leq G$ is a p-Sylow subgroup of

G then one has by [23] 4.6:

$$\text{hom}-\text{codim}(F[V]^G) \geq \text{hom}-\text{codim}(F[V]^P).$$

So to establish lower bounde for the homological codimension we may restrict ourselves to the case of p-groups, and henceforth we assume $G=P$ to be a finite p-group. We also assume that F is a Galois field with $q=p^\nu$ elements. The Dickson algebra (see e.g. [22] §8.1) $D^*(n)$ is a subalgebra of $F[V]^P$ and the extensions $D^*(n) \subseteq F[V]^P \subseteq F[V]$ are finite. Therefore (see the do it yourself kit [5] exercise 1.2.26)

$$\text{hom}-\text{codim}_{D^*(n)}(F[V]^P) = \text{hom}-\text{codim}(F[V]^P).$$

The Auslander-Buchsbaum equality [5] 1.3.3

$$n = \text{hom}-\text{codim}_{D^*(n)}(F[V]^P) = \text{hom}-\text{codim}_{D^*(n)}(F[V]^P)$$

in turn allows us to convert the original homological codimension computation to one of the homological dimension of $F[V]^P$ as $D^*(n)$-module. It is here that the spectral sequence

$$E_r \Rightarrow H^*(P; F[V]_{\mathrm{GL}(n,F)})$$
$$E_2^{s,t} = \mathrm{Tor}^s_{D^*(n)}(H^t(P; F[V]), F)$$

can be of use. (We have written $F[V]_{\mathrm{GL}(n,F)}$ for $F[V] \otimes_{D^*(n)} F$ for the ring of coinvariants of the group $\mathrm{GL}(n,F)$ which coincides with standard notation.)

Notice that $H^*(P; F[V]_{\mathrm{GL}(n,F)})$ is zero for $*<0$ and therefore there are no elements of negative total degree in E_∞. In other words, as the following diagram shows

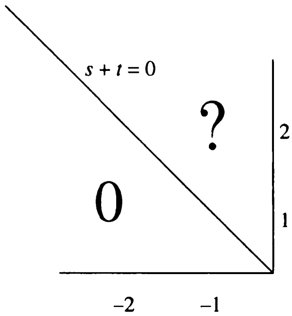

Figure 3.1. The vanishing area $s+t<0$ for E_∞

the terms $E_\infty^{s,t}$ for $s+t<0$ are all zero. This is a key fact that will be exploited in computations. The terms on the border of the vanishing area, i.e., where $s+t=0$ (referred to as the **vanishing line**) are connected with the **stable invariants** introduced in [12] and studied in [16].

To work with this spectral sequence we need to obtain information about $H^*(P; F[V])$ and how the Dickson algebra acts on it. It would seem that since we do not know $F[V]^P = H^0(P; F[V])$ that this would be a hopeless undertaking. However, in any case, it is natural to start with the smallest exmple $P = Z/p$, the cyclic group of order p. If $\gamma \in Z/p$ is a generator then the periodic complex [6]

$$0 \overset{\varepsilon}{\leftarrow} F(Z/p) \overset{\partial}{\leftarrow} F(Z/p) \overset{\mathrm{Tr}^{Z/p}}{\leftarrow} F(Z/p) \overset{\partial}{\leftarrow} F(Z/p) \overset{\mathrm{Tr}^{Z/p}}{\leftarrow} \cdots$$

where:

(i) $F(Z/p)$ is the group algebra of Z/p over F,
(ii) ε is the augmentation homomorphism,
(iii) ∂ denotes multiplication by the element $1-\gamma \in F(Z/p)$, and
(iv) $\mathrm{Tr}^{Z/p} = \partial^{p-1}$ denotes multiplication by $1+\gamma+\cdots+\gamma^{p-1} \in F(Z/p)$,

is a free resolution of F as $F(Z/p)$-module. Therefore applying the functor $\mathrm{Hom}_{F(Z/p)}(-, F[V])$ to this complex we see that $H^*(Z/p; F[V])$ is the cohomology of the cocomplex

$$0 \rightarrow F[V] \overset{\partial}{\rightarrow} F[V] \overset{\mathrm{Tr}^{Z/p}}{\rightarrow} F[V] \overset{\partial}{\rightarrow} F[V] \overset{\mathrm{Tr}^{Z/p}}{\rightarrow} \cdots$$

and hence

$$(*) \qquad H^i(Z/p; F[V]) = \begin{cases} 0 & \text{for } i<0 \\ F[V]^{Z/p} & \text{for } i=0 \\ F[V]^{Z/p}/\mathrm{Im}(\mathrm{Tr}^{Z/p}) & \text{for } i \text{ even and } i>0 \\ \ker(\mathrm{Tr}^{Z/p})/\mathrm{Im}(\partial) & \text{for } i \text{ odd and } i>0 \end{cases}$$

(N.b. we have used that $F[V]^{Z/p} = \ker(\partial)$ in the preceding formula for the even cohomology groups.) In particular when $p=2$ we have

$$(*_2) \qquad H^i(Z/2; F[V]) = \begin{cases} 0 & \text{for } i<0 \\ F[V]^{Z/2} & \text{for } i=0. \\ F[V]^{Z/2}/\mathrm{Im}(\mathrm{Tr}^{Z/2}) & \text{for } i>0. \end{cases}$$

So for $p=2$, and need only determine the quotient of $F[V]^{Z/2}$ by the image of the transfer to compute $H^i(Z/2; F[V])$ for $i>0$.

By using the Jordan form (or otherwise) one can see that a representation $Z/2$ of 2 over a field of characteristic 2 is always a permutation representation. A permutation representation of $Z/2$ decomposes into a sum of trivial and regular representations. The trivial part will cause us no problems and from [24], as a consequence of the remarkable lemma 3.1 of [17], we have for sums of the regular representation:

Theorem 3.1. *Let F be a field of characteristic 2 and $\tau_m : Z/2 \hookrightarrow \mathrm{GL}(2m, F)$ the representation of $Z/2$ implemented by the permutation matrix*

$$\begin{bmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{bmatrix} \in \mathrm{GL}(2m, \boldsymbol{F}).$$

Let $x_1, \cdots, x_m, y_1, \cdots, y_m$ be the standard basis for the dual vector space V^* of $V = \boldsymbol{F}^{2m}$. Then

(i) $\mathrm{Im}(\mathrm{Tr}^{\boldsymbol{Z}/2})$ *is a prime ideal of height m in $\boldsymbol{F}[V]^{\boldsymbol{Z}/p}$, and*
(ii) $\boldsymbol{F}[V]^{\boldsymbol{Z}/2}/\mathrm{Im}(\mathrm{Tr}^{\boldsymbol{Z}/2}) \cong \boldsymbol{F}[q_1, \cdots, q_m]$ *is a polynomial algebra on the quadratic forms $q_i = x_i y_i$, $i = 1, \cdots, m$.*

With the aid of this result we can compute $\mathrm{Tor}_{\boldsymbol{D}^*(2m)}(\boldsymbol{F}[V]^{\boldsymbol{Z}/2}/\mathrm{Im}(\mathrm{Tr}^{\boldsymbol{Z}/2}), \boldsymbol{F})$ for the representation τ_m as follows: the ideal $\mathrm{Im}(\mathrm{Tr}^{\boldsymbol{Z}/2})$ in $\boldsymbol{F}[x_1, \cdots, x_m, y_1, \cdots, y_m]^{\boldsymbol{Z}/2}$ is the vector space spanned by the binomials

$$x^A y^B + x^B y^A$$

where $A = (a_1, \cdots, a_m)$ and $B = (b_1, \cdots, b_m)$ are distinct index sequences of nonnegative integers[5], i.e. $A \neq B$. The subalgebra $\boldsymbol{F}[q_1, \cdots, q_m] \subset \boldsymbol{F}[x_1, \cdots, x_m, y_1, \cdots, y_m]^{\boldsymbol{Z}/2}$ is the linear span of the monomials $x^C y^C$ where $C = (c_1, \cdots, c_n)$ is an index sequence of nonnegative integers, and therefore $\boldsymbol{F}[q_1, \cdots, q_m]$ intersects the ideal $\mathrm{Im}(\mathrm{Tr}^{\boldsymbol{Z}/2})$ in the zero ideal (0). The prime ideal $(l_1, \cdots, l_m) \subset \boldsymbol{F}[x_1, \cdots, x_m, y_1, \cdots, y_m]$, where $l_j = x_j + y_j$ for $j = 1, \cdots, m$, lies over $\mathrm{Im}(\mathrm{Tr}^{\boldsymbol{Z}/2})$ and the map

$$\pi : \boldsymbol{F}[x_1, \cdots, x_m, y_1, \cdots, y_m]^{\boldsymbol{Z}/2}/\mathrm{Im}(\mathrm{Tr}^{\boldsymbol{Z}/2}) \to \boldsymbol{F}[x_1, \cdots, x_m, y_1, \cdots, y_m]/(l_1, \cdots, l_m)$$

is a monomorphism. We may identify $\boldsymbol{F}[x_1, \cdots, x_m, y_1, \cdots, y_m]/(l_1, \cdots, l_m)$ with $\boldsymbol{F}[u_1, \cdots, u_m]$ in such a way that π maps q_i to u_i^2 for $i = 1, \cdots, m$. The Dickson algebra $\boldsymbol{D}^*(2m) \subset \boldsymbol{F}[x_1, \cdots, x_m, y_1, \cdots, y_m]^{\boldsymbol{Z}/2}$ is a subalgebra of the tensor product $\underbrace{\boldsymbol{D}^*(2) \otimes \cdots \otimes \boldsymbol{D}^*(2)}_{m}$, which is in turn a subalgebra of $\boldsymbol{F}[x_1, \cdots, x_m, y_1, \cdots, y_m]^{\boldsymbol{Z}/2}$. The map π sends the Dickson polynomials $\boldsymbol{d}_{2,0}, \boldsymbol{d}_{2,1}$ in the i-th factor to 0 and u_i^2 respectively, for $i = 1, \cdots, m$. Therefore the map induced by π

$$\underbrace{\boldsymbol{D}^*(2) \otimes \cdots \otimes \boldsymbol{D}^*(2)}_{m} \to \boldsymbol{F}[x_1, \cdots, x_m, y_1, \cdots, y_m]^{\boldsymbol{Z}/2}/\mathrm{Im}(\mathrm{Tr}^{\boldsymbol{Z}/2}) = \boldsymbol{F}[u_1, \cdots, u_m]$$

maps onto the subalgebra $\boldsymbol{F}[q_1, \cdots, q_m]$ and has kernel the Borel ideal[6] generated by the m elements $1 \otimes \cdots \otimes \boldsymbol{d}_{2,0} \otimes \cdots \otimes 1$, where $\boldsymbol{d}_{2,0}$ occurs once in each factor. A

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[5] For an index sequence $E = (e_1, \cdots, e_m)$ of nonnegative in integers we write $x^E = x_1^{e_1} \cdots x_m^{e_m}$.
[6] A **Borel ideal** is an ideal generated by a regular sequence.

standard change of rings argument [21] therefore yields:

Proposition 3.2. *With the preceding hypotheses and notations we have*

$$\mathrm{Tor}^{*}_{\underset{m}{\otimes} D^*(2)}(F[x_1,\cdots,x_m,y_1,\cdots,y_m]^{Z/2}/\mathrm{Im}(\mathrm{Tr}^{Z/2}),F)=E(v_1,\cdots,v_m)$$

where v_i has homological degree-1 (and internal degree 3) for $i=1,\cdots,m$. In particular

$$\mathrm{Tor}^{s}_{\underset{m}{\otimes} D^*(2)}(F[x_1,\cdots,x_m,y_1,\cdots,y_m]^{Z/2}/\mathrm{Im}(\mathrm{Tr}^{Z/2}),F)=0$$

for $s<-m$.

Corollary 3.3. *With the preceding hypotheses and notations we have for all $t>0$*

$$\mathrm{Tor}^{*}_{D^*(2m)}(H^t(Z/2;F[x_1,\cdots,x_m,y_1,\cdots,y_m]),F)=[\underset{m}{\otimes} D^*(2)//D^*(2m)]\otimes E(v_1,\cdots,v_m)$$

where v_i has homological degree-1 (and internal degree 3) for $i=1,\cdots,m$. In particular

$$\mathrm{Tor}^{s}_{D^*(2m)}(H^t(Z/2;F[x_1,\cdots,x_m,y_1,\cdots,y_m]),F)=0$$

for $t>0$ and $s<-m$.

Proof. $\underset{m}{\otimes} D^*(2)$ is a free $D^*(2m)$ module, and the result follows from a standard change of rings argument and the formula $(*_2)$ for $H^t(Z/2;F[x_1,\cdots,x_m,y_1,\cdots,y_m])$ when $t>0$.

Theorem 3.4 (Ellingsrud and Skjelbred [8]). *Let $\rho:Z/2 \subsetneq \mathrm{GL}(n,F)$ be a representation of the cyclic group $Z/2$ of order 2 over a field F of characteristic 2. Then* hom-codim$(F[V]^{Z/2})=\min\{n,2+\dim_F(V^{Z/2})\}$.

Proof. Let $V=F^n$. Using the Jordan form (or otherwise) one sees that it is possible to choose a basis $x_1,\cdots,x_m,y_1,\cdots,y_m,z_1,\cdots,z_k$ for V^* such that $Z/2$ acts on V^* by interchanging the vector variable $(x_1,\cdots,x_m)$ with the vector variable $(y_1,\cdots,y_m)$ and fixing $z_1,\cdots,z_k$. Therefore

$$F[V]^{Z/2}=F[x_1,\cdots,x_m,y_1,\cdots,y_m]^{Z/2}\otimes F[z_1,\cdots,z_k]$$

and hom-codim$(F[V]^{Z/2})=$hom-codim$(F[x_1,\cdots,x_m,y_1,\cdots,y_m]^{Z/2})+k$ and we may suppose $k=0$ and $n=2m$. The case $m=1$ being elementary, we also may suppose $m>1$.

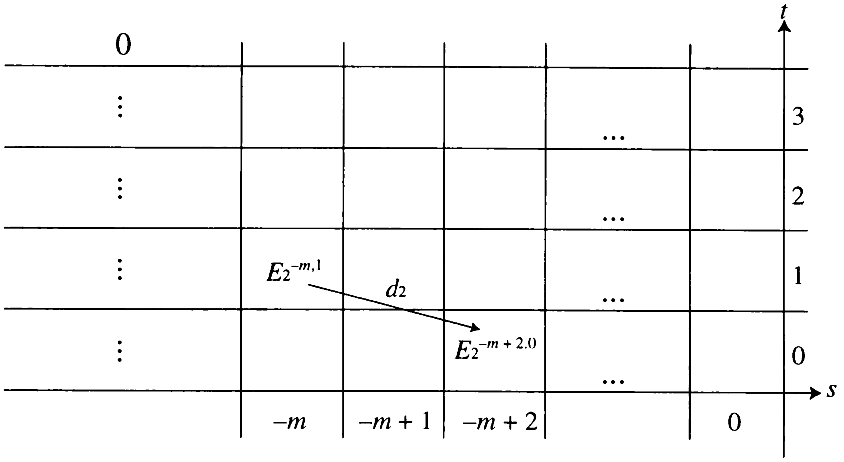
Consider the spectral sequence of 2.2 in this case, for which we have

$$E_r \Rightarrow H^*(\mathbf{Z}/2; \mathbf{F}[V]_{\mathrm{GL}(2m,\mathbf{F})})$$
$$E_2^{s,t} = \mathrm{Tor}_{\mathbf{D}^*(2m)}(H^t(\mathbf{Z}/2; \mathbf{F}[V]), \mathbf{F}).$$

From 3.3 it follows that

$$E_2^{s,t} = 0 \quad \text{for } s < -m \text{ and } t > 0$$
$$E_2^{-m,1} \neq 0.$$

This leads to the following picture for E_2:



It follows that none of the terms $E_2^{-2m,0}, \cdots, E_2^{-m+1,0}$ can be the target of a nonzero differential. Since all differentials starting on the s-axis are zero, if any of these terms were nonzero, it would represent nonzero elements of negative degree in $H^*(\mathbf{Z}/2; \mathbf{F}[V]_{\mathrm{GL}(2m,\mathbf{F})})$ which cannot exit. Hence we conclude $E_2^{-2m,0} = \cdots = E_2^{-m+1,0} = 0$. The term $E_2^{-m,1}$ is nonzero and the indicated differential is the only possible nonzero differential either originating, or terminating, at $E_2^{-m,1}$. Since $E_\infty^{-m,1} = 0$ it follows that $d_2 : E_2^{-m,1} \to E_2^{-m+2,0}$ must be an isomorphism. In particular

$$0 \neq E_2^{-m+2,0} = \mathrm{Tor}_{\mathbf{D}^*(2m)}^{-m+2}(\mathbf{F}[V]^{\mathbf{Z}/2}), \mathbf{F})$$

so $\text{hom-dim}_{\mathbf{D}^*(2m)}(\mathbf{F}[V]^{\mathbf{Z}/2}) = m-2$ from which it follows that $\text{hom-codim}(\mathbf{F}[V]^{\mathbf{Z}/2}) = m+2$ as required.

§4. Permutation Representations of $\mathbf{Z}/p$ in characteristic p

Let p be an odd prime, $\mathbf{Z}/p$ the cyclic group of order p and X a finite $\mathbf{Z}/p$-set, i.e., a permutation representation of $\mathbf{Z}/p$. Since $p \in \mathbf{N}$ is a prime the only possible orbits of $\mathbf{Z}/p$ on X are fixed points and free orbits[7]: the action is **semifree**. Let B denote

[7] A **free orbit** is an orbit with p elements cyclically permuted by $\mathbf{Z}/p$.

the $\boldsymbol{Z}/p$-set with underlying set the elements of $\boldsymbol{Z}/p$ and $\boldsymbol{Z}/p$-action by left multiplication: B is a generic free orbit, and[8]

$$X = \underbrace{B \sqcup \cdots \sqcup B}_{m} \sqcup X^{\boldsymbol{Z}/p}$$

as $\boldsymbol{Z}/p$-set for a unique integer $m \in \boldsymbol{N}_0$.

If $\boldsymbol{F}$ is a field we denote by $V_X = \operatorname{Fun}(X, \boldsymbol{F})$ the $\boldsymbol{F}$-vector space of all functions from X to $\boldsymbol{F}$, which is the corresponding $\boldsymbol{F}$-linear representation. In this way X becomes identified as $\boldsymbol{Z}/p$-set with a basis for V_X^*, and the algebra $\boldsymbol{F}[V_X]$ with the algebra of polynomials $\boldsymbol{F}[X]$ in the variables X. Note that the representation V_X is defined over the prime subfield of $\boldsymbol{F}$ and so there is no loss of generality in the following discussion in supposing that $\boldsymbol{F} = \boldsymbol{F}_p$.

We are going to study the homological codimension of the ring of invariants $\boldsymbol{F}[X]^{\boldsymbol{Z}/p}$, where $\boldsymbol{F}$ is a field of characteristic p, and show

$$\text{hom-codim}(\boldsymbol{F}[X]^{\boldsymbol{Z}/p}) = 2 + \dim_{\boldsymbol{F}}(V_X^{\boldsymbol{Z}/p}) = 2 + m + |X^{\boldsymbol{Z}/p}|,$$

confirming a formula of [8]. Our strategy is the same as in §3: we use the Auslander-Buchsbaum equality to convert a homological codimension computation into the computation of the homological dimension of $\boldsymbol{F}[X]^{\boldsymbol{Z}/p}$ as a module over the Dickson algebra $\boldsymbol{D}(|X|)$, which we do with the aid of the spectral sequence 2.2

$$E_r \Rightarrow H^*(\boldsymbol{Z}/p\,; \boldsymbol{F}[X]_{\mathrm{GL}(|X|, \boldsymbol{F})})$$
$$E_2^{s,t} = \operatorname{Tor}^s_{\boldsymbol{D}^*(|X|)}(H^t(\boldsymbol{Z}/p\,; \boldsymbol{F}[X]), \boldsymbol{F}).$$

It is not hard to see that

$$\boldsymbol{F}[X]^{\boldsymbol{Z}/p} = \boldsymbol{F}[X^{\boldsymbol{Z}/p}] \otimes \boldsymbol{F}[X \backslash X^{\boldsymbol{Z}/p}]^{\boldsymbol{Z}/p}$$

where $X \backslash X^{\boldsymbol{Z}/p} = \underbrace{B \sqcup \cdots \sqcup B}_{m}$ denotes the **free part** of the $\boldsymbol{Z}/p$-set X. Hence from the viewpoint of invariant theory there is no loss of generality in assuming that $X^{\boldsymbol{Z}/p} = \varnothing$, i.e., the action of $\boldsymbol{Z}/p$ on X is **free**. Another way to say this is that V_X is the m-fold direct sum of the **regular representation** of $\boldsymbol{Z}/p$. So, without loss of generality, we assume until further notice that X is a free $\boldsymbol{Z}/p$-set, and we set $|X| = mp$.

To make use of 2.2 we need to compute $H^i(\boldsymbol{Z}/p; \boldsymbol{F}[X])$ as a $\boldsymbol{D}^*(mp)$-module. This we do with the standard resolution of Cartan-Eilenberg as described in §3. Recall (formula (∗) from §3)

$$(*) \qquad H^i(\boldsymbol{Z}/p\,; \boldsymbol{F}[X]) = \begin{cases} 0 & \text{for } i < 0 \\ \boldsymbol{F}[X]^{\boldsymbol{Z}/p} & \text{for } i = 0 \\ \boldsymbol{F}[X]^{\boldsymbol{Z}/p}/\operatorname{Im}(\operatorname{Tr}^{\boldsymbol{Z}/p}) & \text{for } i \text{ even and } i > 0 \\ \ker(\operatorname{Tr}^{\boldsymbol{Z}/p})/\operatorname{Im}(\partial) & \text{for } i \text{ odd and } i > 0. \end{cases}$$

[8] $\sqcup$ means **disjoint union**.

As a consequence of [17] lemma 3.1 we have for $j>0$ from [24] theorem 2:

$$H^{2j}(\boldsymbol{Z}/p\,;\boldsymbol{F}[X])=\boldsymbol{F}[X]^{\boldsymbol{Z}/p}/\mathrm{Im}(\mathrm{Tr}^{\boldsymbol{Z}/p})=\boldsymbol{F}[c_p(X_1),\cdots,c_p(X_m)]$$

where,

— $X_1,\cdots,X_m$ denote the orbits of $\boldsymbol{Z}/p$ on X, so each X_i is isomorphic to B as $\boldsymbol{Z}/p$-set,
— $X=X_1\sqcup\cdots\sqcup X_m$, and
— $c_p(X_i)$ denotes the p-th Chern class of the orbit X_i for $i=1,\cdots,m$.

The action of the Dickson algebra $\boldsymbol{D}^*(mp)$ on $H^{2j}(\boldsymbol{Z}/p\,;\boldsymbol{F}[X])$ is via the composition

$$\pi:\boldsymbol{D}^*(mp)\subset\boldsymbol{F}[X]^{\boldsymbol{Z}/p}\to\boldsymbol{F}[X]^{\boldsymbol{Z}/p}/\mathrm{Im}(\mathrm{Tr}^{\boldsymbol{Z}/p})\to\boldsymbol{F}[c_p(X_1),\cdots,c_p(X_m)].$$

There are algebra inclusions

$$\boldsymbol{D}^*(mp)\subset\underbrace{\boldsymbol{D}^*(p)\otimes\cdots\otimes\boldsymbol{D}^*(p)}_{m}\subset\boldsymbol{F}[X]^{\boldsymbol{Z}/p}$$

induced by the inclusions $\boldsymbol{Z}/p<\times_m\mathrm{GL}(p,\boldsymbol{F})<\mathrm{GL}(mp,\boldsymbol{F})$ and the map π satisfies

$$\pi(d_{p,j})=\begin{cases}0 & \text{for } j=0,\cdots,p-2\\ c_p(X_i)^{p^{i-1}(p-1)} & \text{for } j=p-1\end{cases}$$

when restricted to the i-th factor $\boldsymbol{D}^*(p)$ of the tensor product. Hence the kernel of the map $\pi|_{\otimes_m \boldsymbol{D}^*(p)}$ is a Borel ideal and $\boldsymbol{F}[c_p(X_1),\cdots,c_p(X_m)]$ is a free module over the image. Using the Koszul complex and a standard change of rings argument [21] we find that $\text{hom-dim}_{\otimes_m \boldsymbol{D}^*(p)}\boldsymbol{F}[X]^{\boldsymbol{Z}/p}=m(p-1)$. Finally the algebra $\otimes_m\boldsymbol{D}^*(p)$ is a free $\boldsymbol{D}^*(mp)$-module, so we conclude:

Proposition 4.1. *With the preceding hypotheses and notations we have* $\text{hom-dim}_{\boldsymbol{D}^*(mp)}(H^{2j}(\boldsymbol{Z}/p\,;\boldsymbol{F}[X])=m(p-1)$ *for all* $j>0$.

Our next task is to describe the cohomology modules $H^i(\boldsymbol{Z}/p\,;\boldsymbol{F}[X])$ for i odd and positive as modules over the Dickson algebra. To this end we require a lemma.

Lemma 4.2. *With the preceding hypotheses and notations the maps* ∂, $\mathrm{Tr}^{\boldsymbol{Z}/p}:\boldsymbol{F}[X]\to\boldsymbol{F}[X]$ *satisfy*

$$\mathrm{Im}(\mathrm{Tr}^{\boldsymbol{Z}/p})=\mathrm{Im}(\partial)\cap\ker(\partial)$$

$$\ker(\mathrm{Tr}^{\boldsymbol{Z}/p})=\mathrm{Im}(\partial)+\ker(\partial).$$

Proof. Note that the action of $\boldsymbol{Z}/p$ on the homogeneous component $\boldsymbol{F}[X]_d$

of $\boldsymbol{F}[X]$ of degree d is the linear action associated to the permutation representation of $\boldsymbol{Z}/p$ on the set $\boldsymbol{SP}^d(X)$, the d-fold **symmetric product** of the $\boldsymbol{Z}/p$-set X, which[9] is identified with the monomial basis for $\boldsymbol{F}[X]_d$ as $\boldsymbol{F}$-vector space. Since p is a prime the action of $\boldsymbol{Z}/p$ on $\boldsymbol{SP}^d(X)$ is semifree. Let $\boldsymbol{SP}^d(X)^{\text{free}}$ denote the ***set*** of free orbits of the action of $\boldsymbol{Z}/p$ on $\boldsymbol{SP}^d(X)$, i.e., the elemints of $\boldsymbol{SP}^d(X)$ are free $\boldsymbol{Z}/p$-orbits in $\boldsymbol{SP}^d(X)$ and

$$\boldsymbol{SP}^d(X)\backslash \boldsymbol{SP}^d(X)^{\boldsymbol{Z}/p} = \bigcup_{U\in \boldsymbol{SP}^d(X)^{\text{free}}} U.$$

For any $f\in \boldsymbol{F}[X]_d$ we may write

$$f = \sum_{U\in \boldsymbol{SP}^d(X)^{\text{free}}} \left(\sum_{u\in U} \alpha_u(U)\cdot u\right) + \sum_{\omega\in \boldsymbol{SP}^d(X)^{\boldsymbol{Z}/p}} \beta_\omega \cdot \omega$$

where $\alpha_u(U)$ and β_ω belong to $\boldsymbol{F}$ for all $U\in \boldsymbol{SP}^d(X)^{\text{free}}$ and $u\in U$ as well as $\omega\in \boldsymbol{SP}^d(X)^{\boldsymbol{Z}/p}$. This expression for f is unique.

For $U\in \boldsymbol{SP}^d(X)^{\text{free}}$ a free orbit we may write $U=\{u_1,\cdots,u_p\}$ with the action of $\boldsymbol{Z}/p$ given by cyclic permutation of $u_1,\cdots,u_p$, i.e.,

$$\gamma(u_i) = \begin{cases} u_{i+1} & \text{for } i=1,\cdots,p-1 \\ u_1 & \text{for } i=p \end{cases}$$

where $\gamma\in \boldsymbol{Z}/p$ is a generator (independent of U). Therefore, for $f\in \boldsymbol{F}[V]_d$ we find

$$\gamma(f) = \sum_{U\in \boldsymbol{SP}^d(X)^{\text{free}}} (\alpha_{u_1}(U)\cdot u_2 + \cdots + \alpha_{u_{p-1}}(U)\cdot u_p + \alpha_{u_p}(U)\cdot u_1) + \sum_{\omega\in \boldsymbol{SP}^d(X)^{\boldsymbol{Z}/p}} \beta_\omega\cdot\omega$$

so we obtain

$$\partial(f) = \sum_{U\in \boldsymbol{SP}^d(X)^{\text{free}}} [(\alpha_{u_1}(U) - \alpha_{u_p}(U))\cdot u_1 + \cdots + (\alpha_{u_p}(U) - \alpha_{u_{p-1}}(U))\cdot u_p]$$

and

$$\mathrm{Tr}^{\boldsymbol{Z}/p}(f) = \sum_{U\in \boldsymbol{SP}^d(X)^{\text{free}}} \alpha(U)\cdot(u_1 + \cdots + u_p)$$

where

$$\alpha(U) = \alpha_{u_1}(U) + \cdots + \alpha_{u_p}(U).$$

From these equations it is an easy matter to deduce the following criteria:

(i) $f\in\ker(\partial)$ if and only if $\alpha_{u_1}(U) = \cdots = \alpha_{u_p}(U)$ for all $U\in \boldsymbol{SP}^d(X)^{\text{free}}$,

(ii) $h\in\mathrm{Im}(\partial)$ if and only if $\alpha(U)=0$ for all $U\in \boldsymbol{SP}^d(X)^{\text{free}}$ and $\beta_\omega=0$ for all

[9] For a discussion of the invariant theory of permutation representations and symmetric products see [22] §4.1 and §4.2.

$\omega \in SP^d(X)^{Z/p}$,

(iii) $f \in \ker(\mathrm{Tr}^{Z/p})$ if and only if $\alpha(U)=0$ for all $U \in SP^d(X)^{\mathrm{free}}$,

(iv) $f \in \mathrm{Im}(\mathrm{Tr}^{Z/p})$ if and only if $\alpha_{u_1}(U) = \cdots = \alpha_{u_p}(U)$ for all $U \in SP^d(X)^{\mathrm{free}}$ and $\beta_\omega = 0$ for all $\omega \in SP^d(X)^{Z/p}$.

Only (ii) requires any comment at all: here one just needs to notice that the system of linear equations

$$\begin{bmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 1 & -1 \\ -1 & 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_q \end{bmatrix} = \begin{bmatrix} a \\ a \\ \vdots \\ \vdots \\ a \end{bmatrix}$$

can always be solved for $b_1, \cdots, b_p \in F$ because

$$\begin{bmatrix} a \\ a \\ \vdots \\ \vdots \\ a \end{bmatrix} \in \mathrm{Span}_F \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Since $\alpha_{u_1}(U) = \cdots = \alpha_{u_p}(U)$ implies $\alpha(U)=0$ it follows that condition (iii) is equivalent to either condition (i), or (ii), or both and hence $\ker(\mathrm{Tr}^{Z/p}) = \ker(\partial) + \mathrm{Im}(\partial)$. Likewise, condition (iv) is equivalent to conditions (i) and (ii) taken together, and hence $\mathrm{Im}(\mathrm{Tr}^{Z/p}) = \ker(\partial) \cap \mathrm{Im}(\partial)$ as required.

Proposition 4.3. *With the preceding hypotheses and notations we have*

$$H^{2j+1}(Z/p; F[X]) = H^{2(j+1)}(Z/p; F[X])$$

for all $j \geq 0$.

Proposition 4.4. *With the precceding hypothese and notations we have* $\text{hom-dim}_{D^*(mp)}(H^{2j+1}(Z/p; F[X]) = m(p-1)$ *for all* $j > 0$.

Theorem 4.5 (Ellingsrud and Skjelbred [8]). *Let p be an odd prime, Z/p the cyclic group of order p and X a finite Z/p-set. If F is a field of characteristic p, then*

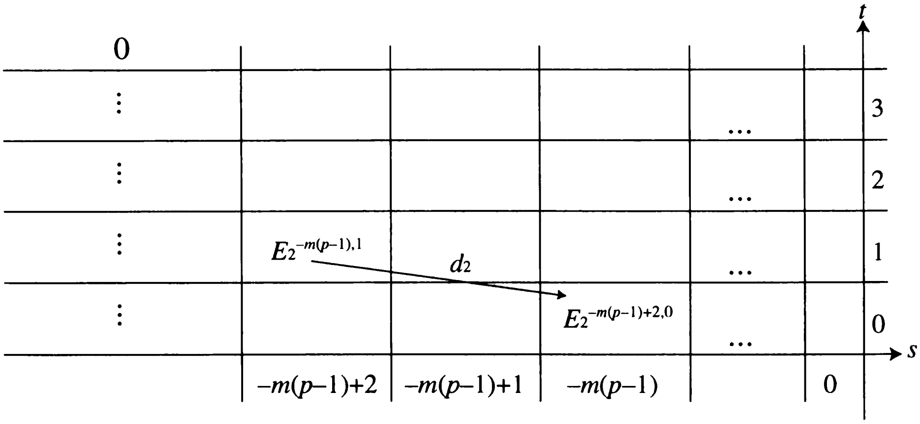
$$\text{hom-codim}(F[X]^{Z/p}) = 2 + \dim_F(V_X^{Z/p}) = 2 + m + |X^{Z/p}|$$

where $m = \dfrac{|X| \backslash |X^{Z/p}|}{p}$ *is the number of free orbits of Z/p on X.*

Proof. Without loss of generality we may suppose that $X^{\boldsymbol{Z}/p}=\varnothing$, $m>0$ and $\boldsymbol{F}$ is finite. We consider the spectral sequence 2.2

$$E_r \Rightarrow H^*(\boldsymbol{Z}/p;\boldsymbol{F}[X]_{\mathrm{GL}(mp,\boldsymbol{F})})$$
$$E_2^{s,t}=\mathrm{Tor}^s_{\boldsymbol{D}^*(mp)}(H^t(\boldsymbol{Z}/p;\boldsymbol{F}[X]),\boldsymbol{F}).$$

By propositions 4.1 and 4.4 we have the precise vanishing line for $E_2^{s,t}$, $t>0$, is $s=-(m(p-1))+1$. This leads to the following diagram for E_2:



which together with the fact that $E_\infty^{s,t}=0$ for $s+t<0$ implies that

$$0=E_2^{s,0}=\mathrm{Tor}^s_{\boldsymbol{D}^*(mp)}(\boldsymbol{F}[X]^{\boldsymbol{Z}/p},\boldsymbol{F}) \quad \text{for } s<-m(p-1)+1$$

and that the indicated differential is an isomorphism. Since

$$E_2^{-m(p-1),1}=\mathrm{Tor}^{-m(p-1)}_{\boldsymbol{D}^*(mp)}(H^1(\boldsymbol{Z}/p;\boldsymbol{F}[X]),\boldsymbol{F})\neq 0$$

we conclude that

$$\mathrm{Tor}^{-m(p-1)+2}_{\boldsymbol{D}^*(mp)}(\boldsymbol{F}[X]^{\boldsymbol{Z}/p},\boldsymbol{F})\neq 0$$

and hence $\text{hom-dim}_{\boldsymbol{D}^*(mp)}(\boldsymbol{F}[X]^{\boldsymbol{Z}/p})=m(p-1)-2$ and the result follows from the Auslander-Buchsbaum equality.

Corollary 4.6 (Fossum and Griffiths [9]). *Let p be a prime, $\boldsymbol{Z}/p \subsetneq \mathrm{GL}(p,\boldsymbol{F})$ the regular representation of the cyclic group $\boldsymbol{Z}/p$ of order p over a field $\boldsymbol{F}$ of characteristic p. Then $\boldsymbol{F}[V]^{\boldsymbol{Z}/p}$ is Cohen-Macaulay if and only if $p=2$ or 3.*

Proof. For $p=2$ the result is clear. For odd p, if the ring of invariants is Cohen-Macualy, substitute in the formula of 4.5 to obtain $p=\text{hom-codim}(\boldsymbol{F}[V]^{\boldsymbol{Z}/p})$ $=2+\dim_{\boldsymbol{F}}(V^{\boldsymbol{Z}/p})=2+1=3.$

§5. Regular Sequences in invariant rings of permutation representations

The computations of sections §3 and 4 combined with the tools of §2 may be used to reexamine the motivational problem posed in §1 to good advantage in the case of permutation representations. Let us begin by examining the case of a representation $\rho: \mathbf{Z}/2 \subsetneq \mathrm{GL}(n, \mathbf{F})$. Without loss of generality we may suppose that the representation permutes a basis $\mathscr{B}$ of the dual vector space V^* and to simplify the notations we furthermore suppose that $\mathscr{B}^{\mathbf{Z}/2} = \emptyset$. So $n = 2m$ and ρ is implemented by the permutation matrix

$$\begin{bmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{bmatrix} \in \mathrm{GL}(2m, \mathbf{F}).$$

Let $x_1, \cdots, x_m, y_1, \cdots, y_m$ be the standard basis for the dual vector space V^* of $V = \mathbf{F}^{2m}$ and set $l_i = x_i + y_i$, $q_i = x_i y_i$, $i = 1, \cdots, m$. These polynomials are invariant and $q_1, \cdots, q_m$ is a regular sequence in $\mathbf{F}[V]$. Note carefully, that we have choosen to work with the obvious quadratic invariants instead of the linear ones as in example 1 of §1. This is of course influenced by the cohomological computations of §3 and proposition 2.2. Choose any two polymials $f, h \in \mathbf{F}[V]^{\mathbf{Z}/2}$ such that $q_1, \cdots, q_m, f, h$ is a regular sequence in $\mathbf{F}[V]$. For example we could choose f and h to be distinct linear forms l_i, l_j.

Proposition 5.1. *With the preceding notations we have that* $q_1, \cdots, q_m$, f, $h \in \mathbf{F}[V]^{\mathbf{Z}/2}$ *is a regular sequence.*

Proof. We consider the spectral sequence of proposition 2.2 for the elements $q_1, \cdots, q_m, f, h$. From the discussion of $H^i(\mathbf{Z}/2; \mathbf{F}[V])$ in §3 we have

$$E_2^{-s,t} = \mathrm{Tr}^{-s}_{\mathbf{F}[q_1, \cdots, q_m, f, h]}(\mathbf{F}[V]^{\mathbf{Z}/2}/\mathrm{Im}(\mathrm{Tr}^{\mathbf{Z}/2}), \mathbf{F}) \quad \text{for } t > 0.$$

By [24] theorem 1

$$\mathbf{F}[V]^{\mathbf{Z}/2}/\mathrm{Im}(\mathrm{Tr}^{\mathbf{Z}/2}) \cong \mathbf{F}[\bar{q}_1, \cdots, \bar{q}_m]$$

where the $^-$ indicates residue classes, so a simple change of rings spectral sequence argument (see e.g. [21]) shows

$$\mathrm{Tor}^{-s}_{\mathbf{F}[q_1, \cdots, q_m, f, h]}(\mathbf{F}[V]^{\mathbf{Z}/2}/\mathrm{Im}(\mathrm{Tr}^{\mathbf{Z}/2}), \mathbf{F}) \cong \mathrm{Tor}^{-s}_{\mathbf{F}[f, h]}(\mathbf{F}, \mathbf{F}) \cong E(s^{-1}f, s^{-1}h).$$

Hence we conclude that $E_2^{-s,t} = 0$ for $t > 0$ and $s > 2$. The proof may be completed as in proposition 2.3 (refer to figure 2.1).

On the other hand, suppose we choose three polynomials $f_1, f_2, f_3 \in F[V]^{Z/2}$ such that $q_1, \cdots, q_m, f_1, f_2, f_3$ is a regular sequence in $F[V]$. Again, we could choose these amongst the linear forms $l_1, \cdots, l_m$, but because of §1 we know this choice could not lead to a regular sequence in $F[V]^{Z/2}$, but there are lots of other possible choices for f_1, f_2, f_3. If we examine the spectral sequence of proposition 2.2 in this case we find

$$\begin{aligned} E_2^{-s,t} &= \mathrm{Tor}^{-s}_{F[q_1,\cdots,q_m,f_1,f_2,f_3]}(F[V]^{Z/2}/\mathrm{Im}(\mathrm{Tr}^{Z/2}), F) \\ &= \mathrm{Tor}^{-s}_{F[q_1,\cdots,q_m,f_1,f_2,f_3]}(F[V]^{Z/2}/\mathrm{Im}(\mathrm{Tr}^{Z/2}), F) \cong \mathrm{Tor}^{-s}_{F[f_1,f_2,f_3]}(F, F) \\ &= E(s^{-1}f_1, s^{-1}f_2 s^{-1}f_3) \qquad \text{for } t>0. \end{aligned}$$

In particular $E_2^{-3,1} \neq 0$ and has negative total degree. Since $E_2^{-s,*}=0$ for $s<3$ there are no nonzero differentials terminating at $E_2^{-3,1}$ and the only possible nonzero differential originating there is $d_2: E_2^{-3,1} \to E_2^{-1,0}$. Therefore

$$E_2^{-1,0} = \mathrm{Tor}^{-1}_{F[q_1,\cdots,q_m,f_1,f_2,f_3]}(F[V]^{Z/2}, F) \neq 0$$

so $F[V]^{Z/2}$ is not a free $F[q_1, \cdots, q_m, f_1, f_2, f_3]$-module and $q_1, \cdots, q_m, f_1, f_2, f_3 \in F[V]^{Z/2}$ is not a regular sequence.

In fact it is not necessary to start with the quadratic invariants $q_1, \cdots, q_m$. We could start with any m polynomials in $F[V]^{Z/2}$ whose residue classes in $F[V]^{Z/2}/\mathrm{Im}(\mathrm{Tr}^{Z/2})$ form a regular sequence and the same argument would apply. Putting all these facts together, and adding the argument of M.D. Neusel used in example 1 of §1 we arrive at the following definitive result for regular sequences of maximal length.

Corollary 5.2. *Let $\rho: Z/2 \hookrightarrow \mathrm{GL}(n, F)$ be a permutation representation of $Z/2$ over the field F of characteristic 2. Choose a basis for F^n so ρ is implemented by the permutation matrix*

$$\begin{bmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & 0 & & & \\ & \ddots & & & & \\ 0 & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & & \\ 0 & & & 1 & & \\ & & & & \ddots & 0 \\ & & & & & 1 \end{bmatrix} \in \mathrm{GL}(2m+k, F).$$

Let $x_1, \cdots, x_m, y_1, \cdots, y_m, u_1, \cdots, u_k$ be the standard basis for the dual vector space V^* of $V = F^{2m+k}$ and set $q_i = x_i y_i$, $i = 1, \cdots, m$. If $h_1, \cdots, h_{m+k+2} \in F[V]^{Z/2}$ form a regular sequence in $F[V]$ *then they are a regular sequence in* $F[V]^{Z/2}$ *if and only if some* $m+k$ *of them form a regular sequence in* $F[V]^{Z/2}/\mathrm{Im}(\mathrm{Tr}^{Z/2})=F[\bar{q}_1,\cdots,\bar{q}_m,\bar{u}_1,\cdots,\bar{u}_k]$.

Since at most two invariant polynomials in $F[V]^{Z/2}$ of a regular sequence in can lie in the ideal $\mathrm{Im}(\mathrm{Tr}^{Z/2})\subset F[V]^{Z/2}$ when F has characteristic 2 there is a slight bonus to our computations. Recall that the **grade** of an ideal[10] is the length of the longest regular sequence that it contains. Corollary 5.2 implies:

Corollary 5.3. *Let* X *be a finite* $Z/2$ *set and* $\rho: Z/2\to \mathrm{GL}(n,F)$ *the corresponding permutation representation over the a field* F *of characteristic* 2. *Write* $n=2m+k$ *where* $k=|X^{Z/2}|$ *and is the number of free orbts of* $Z/2$ *on* X. *Assume* ρ *is faithfull so* $m>0$. *Then the ideal* $\mathrm{Im}(\mathrm{Tr}^{Z/2})$ *has grade* $\min(2,m)$.

The case of Z/p,p an odd prime, proceeds in a completely analogous vein. From proposition 2.2 and theorem 2 of [24] we obtain:

Proposition 5.4. *Let* p *be a prime and* X *a finite* Z/p *set. Decompose* X *into orbits*

$$X=\underbrace{X_1\sqcup\cdots\sqcup X_m}_{\longleftarrow\ m\ \longrightarrow}\sqcup X^{Z/p}$$

where each of the orbits $X_1,\cdots,X_m$ *is a free orbit. Let* $N_i=c_p(X_i)$ *be the p-th Chern class of the orbits* X_i *for* $i=1,\cdots,m$ *and* $u_1,\cdots,u_k$ *the distinct elements of* $X^{Z/p}$. *The forms* $N_1,\cdots,N_m,u_1,\cdots,u_k$ *form a regular sequence in* $F[X]$. *Extend these to a regular sequence in* $F[X]$ *of length* $m+k+2$ *by choosing two appropriate invariant forms* $f,h\in F[X]^{Z/p}$; *for example two linear forms in* $\mathrm{Span}_F(X_1\ \cdots\ X_m)$. *Then* $N_1,\cdots,N_m,u_1,\cdots,u_k,f,h$ *is a regular sequence in* $F[X]^{Z/p}$. *It is not possibel to extend* $N_1,\cdots,N_m,u_1,\cdots,u_k$ *to a regular sequence in* $F[X]^{Z/p}$ *of length* $m+k+3$.

Corollary 5.5. *With the notations of* 5.4 *we have that a sequence* $h_1,\cdots,h_{m+k+2}\in F[X]^{Z/p}$ *that is a regular sequence in* $F[X]$ *is also a regular sequence in* $F[X]^{Z/p}$ *if and only if some* $m+k$ *of them form a regular sequence in* $F[X]^{Z/p}/\mathrm{Im}(\mathrm{Tr}^{Z/p})=F[N_1,\cdots,N_m,u_1,\cdots,u_k]$.

Corollary 5.6. *Let* X *be finite* Z/p *set and* $\rho: Z/p\to\mathrm{GL}(n,F)$ *the corresponding permutaion representation over a field of* F *characteristic* p. *Write* $n=pm+k$ *where* $k=|X^{Z/p}|$ *and* m *is the number of free orbts of* Z/p *on* X. *Assume* ρ *is faithfull so* $m>0$. *Then the ideal* $\mathrm{Im}(\mathrm{Tr}^{Z/p})$ *has grade* $\min(2,m)$.

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[10] See for example [5]. This concept has also been called the **girth** in some of the literature. See for example [7].

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Referenes

[1] M. Auslander and D.A. Buchsbaum, Codimension and Multiplicity, Ann. of Math., **68** (1958), 625–657.

[2] M.-J. Bertin, Anneaux dinvariants d'anneaux de polynômes en caractéristique p, C. R. Acad. Sci. Paris, **264** (Série A) (1967), 653–656.

[3] D. Bourguiba and S. Zarati, Depth and Steenrod Operations, Inventiones Math., **128** (1997), 589–602.

[4] K.S. Brown, Cohomology of Groups, Springer-Verlag, Heidelberg, Berlin, New York, 1982.

[5] W. Bruns and J.Herzog, Cohen-Macaulay Rings, Cambridge Studies in Advanced Math 39, Cambridge Univ. Press, Cambridge, 1993.

[6] H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, 1956.

[7] P.E. Conner and L. Smith, On the Complex Bordism of finite Complexes, I.H.E.S. Journal de Math., **37** (1969), 117–221.

[8] G. Ellingsrud and T. Skjelbred, Profondeur d'anneaux d'invariants en caractéristique p, Comp. Math., **41** (1980), 233–244.

[9] R.M. Fossum and P.A. Griffith, Complete Local Factorial Rings which are not Cohen-Macaulay in Characterisic p, Ann. scient. Éc. Norm. Sup., 4[e] série, **8** (1975), 189–200.

[10] A. Grothendieck, Sur quelques Points d'Algèbre Homologique, Tohoku Math J., **9** (1957), 119–221.

[11] A. Grothendieck, Seminar de Geometric Algebrique, SGA 2, IHES Bois de Marie, 1964.

[12] V.G. Kac and D.H. Peterson, Gerenalized Invariants of Groups Generated by Reflections, in: Progress in Mathematics 60, Geometry of Today, Roma 1984, Birkhauser Verlag, Boston, 1985.

[13] N. Killius, Some Modular Invariant Theory of Finite Groups with particular Emphasis on the Cyclic Group, Diplomarbeit, Univ. Göttingen, 1996.

[14] P.S. Landweber and R.E. Stong, The Depth of Rings of Invariants over Finite Fields, Proc. New York Number Theory Seminar, 1984, Lecture Notes in Math. 1240, Springer-Verlag, New York 1987.

[15] S. Mac Lane, Homology, Springer-Verlag, Heidelberg, Berlin, 1974.

[16] F. Neumann, M.D. Neusel and L. Smith, Rings of Generalized and Stable Invariants of Pseudoreflections and Pseudoreflection Groups, J. of Algebra, **182** (1996), 85–122.

[17] M.D. Neusel, Mme. Bertin's $Z/4$ Revisited, Preprint Number 1, Institut für Algebra und Geometrie, Universität Magdeburg, 1997.

[18] M.D. Neusel, Mme. Bertin's $Z/4$ Revisited, (revised version), to appear.

[19] D.R. Richman, On Vector Invariants over Finite Fields, Adv. in Math., **81** (1990), 30–65.

[20] D.R. Richman, Invariants of Finite Groups over Fields of Characteristic p, Adv. in Math., **124** (1996), 25–48.

[21] L. Smith, Homological Algebra and the Eilenberg-Moore Spectral Sequence, Trans. of the Amer. Math. Soc., **129** (1967), 58–93.

[22] L. Smith, Polynomial Invariants of Finite Groups, (second corrected printing), A.K. Peters Ltd., Wellesley, MA 1997.

[23] L. Smith, Polynomial Invariants of Finite Groups, A Survey of Recent Developments, Bull. of the Amer. Math. Soc., **34** (1997), 211–250.

[24] L. Smith, Modular Vector Invariants of Cyclic Permutation Representations, to apear in Can. Math. Bull.