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1-cocycles on the group of diffeomorphisms

By

Hiroaki SHIMOMURA

§1. Introduction

Let M be a d-dimensional paracompact C^∞-manifold and $\mathrm{Diff}(M)$ be the group of all C^∞-diffeomorphisms on M. Among the subgroups of $\mathrm{Diff}(M)$, we take here the group $\mathrm{Diff}_0(M)$ which consists of all $g \in \mathrm{Diff}(M)$ with compact supports, that is the set $\{P \in M |\ g(P) \neq P\}$ is relatively compact. Up to the present time, unitary representations $(U, \mathscr{H})$ of $\mathrm{Diff}_0(M)$ or of its subgroups ($\mathscr{H}$ is the representation Hilbert space of U) are constructed and considered by many authors, for example [4], [5], [6], [7], [8], [9], [10], [12] and [19]. The first purpose of this paper is a trial to construct some differentiable method to analyze these representations $(U, \mathscr{H})$ of $\mathrm{Diff}_0(M)$ or of its subgroups. Roughly speaking, we wish to consider a differential representation of a given one. So the first step we should do is to define a suitable Lie algebra $\mathscr{G}_0$ of $\mathrm{Diff}_0(M)$, regarding it as an infinite dimensional Lie group. For the case of compact manifold, it is well known for a pretty long time ago that $\mathrm{Diff}(M) = \mathrm{Diff}_0(M)$ is an infinite dimensional Lie group whose modelled space is a Fréchet space called strong inductive limit of Hilbert spaces by a few authors. (cf [13]). So after them, we are naturally derived that we should take the set $\Gamma_0(M)$ of all C^∞-vector fields X with compact supports as the Lie algebra $\mathscr{G}_0$, and it is appropriate to take the map $\mathrm{Exp}(X)$ as the exponential map exp from $\Gamma_0(M)$ to $\mathrm{Diff}_0(M)$, where $\{\mathrm{Exp}(tX)\}_{t \in \boldsymbol{R}}$ is the 1-parameter transformation group generated by $X \in \Gamma_0(M)$. Thus formally we have self adjoint operators $dU(X)$ on $\mathscr{H}$ by Stone's result,

$$U(\mathrm{Exp}(tX)) = \exp(\sqrt{-1}\, t\, dU(X)),$$

and simaltaneouly there arise many problems for such $dU(X)$ and for Exp maps. Among them the following questions are fundamental.

(1) Does $\sqrt{-1}dU$ become a linear representation under suitable restrictions of the domain of each $dU(X)$?

(2) Is the common domain of $\{dU(X)\}_{X \in \Gamma_0(M)}$ rich such one like Gårding space?

(3) Is the subgroup generated by $\mathrm{Exp}(X)$, $X \in \Gamma_0(M)$ dense in $\mathrm{Diff}_0(M)$?

It is easily expected that the linearity of $\sqrt{-1}dU$ mostly depends on the usual

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formula which is easily derived from Campbell-Hausdorff formula of which will be made sure in the next section. In conclusion the question (1) is affirmative at least for the finite dimensional representations. Now the theory of product integral is so useful for (3). It turns that the above subgroup is dense in the connected component $\mathrm{Diff}_0^*(X)$ of id, where id is the identity map. This will be carried out also in the next section. As a direct cosequence of these results we will show in Section 3 that there is no continuous finite dimensional representations of $\mathrm{Diff}_0^*(X)$ except for a trivial one. Lastly as for the question (2), it will be expected that a lot of discussions are required for satisfactory solutions. However since we will only consider here the finite dimensional representations of subgroups of $\mathrm{Diff}_0^*(X)$, so it has no problem at the present time. We will not concern with this problem in this paper.

The second and main purpose of this paper is a characterization of continuous 1-cocycles θ defined on $M\times\mathrm{Diff}_0(M)$ using the above differential methods. A $\boldsymbol{T}^1$-valued function θ is said to be a continuous 1-cocycle if

(1) for any fixed $P\in M$, $\theta(P,g)$ is a continuous function of g with respect to the inductive limit topology τ (later τ on $\mathrm{Diff}_0(M)$ will be explained exactly), and

(2) for any $P\in M$ and for any $g_1, g_2\in\mathrm{Diff}_0(M)$ we have $\theta(P,g_1)\theta(g_1^{-1}(P),g_2)=\theta(P,g_1g_2)$. These 1-cocycles, especially the following typical ones, appear in the various kinds of representations of $\mathrm{Diff}_0(M)$. (cf. [5])

(a) $c(g^{-1}(P))/c(P)$, which is called 1-coboundary type, where c is an arbitrary $\boldsymbol{T}^1$-valued continuous function on M.

(b) $\left(\dfrac{d\mu_g}{d\mu}(P)\right)^{\sqrt{-1}s}$, which is called Jacobian type, where μ is any σ-finite smooth measure on M which is locally equivalent to the Lebesgue measure on $\boldsymbol{R}^d$, μ_g is the image measure of μ under the map g and s is a real parameter.

(c) $\eta(g)$, which is called character type, where η is any continuous unitary character on $\mathrm{Diff}_0(M)$.

We call standard type continuous 1-cocycles which consist of 1-coboundary term, Jacobian term and characer term. Section 3 is devoted to the study of 1-cocycles and in it it is shown that if the manifold M is simply connected, then any continuous 1-cocycle is of standard type, which is expected by T. Hirai in the local case $M=\boldsymbol{R}^d$. However the simply connectedness condition is not the necessary one, because the same results hold on compact connected Lie groups. On the other hand there exist also continuous 1-cocycles of non standard type on the manifold $M=\boldsymbol{R}\times\boldsymbol{T}^1$. We will give a concrete example and decide the general form of 1-cocycles on the cylinder in the second part of this section. The last section is devoted to the study of natural representations $(U_\theta(g), \mathrm{L}^2_\mu(M))$,

$$U_\theta(g): f(P)\in\mathrm{L}^2_\mu(M)\mapsto\theta(P,g)\sqrt{\frac{d\mu_g}{d\mu}(P)}f(g^{-1}(P))\in\mathrm{L}^2_\mu(M).$$

It will be shown there that these representations are all irreduceible and they are equivalent if and only if the corresponding 1-cocycles are 1-cohomologus. These results are simple consequences of the local form of such 1-cocycles.

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§2. Diffeomorphism Group as an Infinite Dimensional Lie Grop

2.1. Topology on $\mathrm{Diff}_0(M)$. Let K be a compact subset of M and put

$$\mathrm{Diff}(K):=\{g\in \mathrm{Diff}_0(M)\,|\,\mathrm{supp}\,g\subseteq K\}.$$

We shall introduce to the set Diff(K) a topology τ_K of uniform convergence on K with every derivative of higher order. It is clear that τ_K is a group topology. Now Diff(K) is naturally imbedded into an infinite dimensional Lie group by the following procedure. Let us take a compact submanifold L with boundary containing K. L is obtained by first covering K with finitely many open sets which are diffeomorphic to disks of $\boldsymbol{R}^d$ and by next smoothing their boundaries. Next we patch L and the copy L' together along the boundary of L and form the double N of L, $N:=L\cup L'$ (the double of L). Then N is a compact manifold without boundary and Diff(K) is regarded as a subgroup of Diff(N). Now it is already known that Diff(N) is an infinite dimensional Lie group whose modelled space is a strong inductive limit of Hilbert spaces (cf. [13]), especially it is a Fréchet space. Consequently Diff(K) is locally connected. That is there exists a fundamental system of arcwise connected open neibourhoods $\mathscr{U}$ at id. So the connected component Diff*(K) is also arcwise connected. Now it is clear that $\mathrm{Diff}_0(M) = \cup_K \mathrm{Diff}(K)$, where K runs through all compact sets. So it is natural to consider the inductive limit topology τ of τ_K on $\mathrm{Diff}_0(M)$. However the great care must be taken for τ, because τ is not a group topology, unless M is compact. (cf. [17] and [18]) The right and left translations and the inverse operation is continuous. However the map $(g,h)\mapsto gh$ is not continuous. Since Diff*(K) is an open subgroup of Diff(K) which increases for K, so $\mathrm{Diff}_0^*(M):=\cup_K \mathrm{Diff}^*(K)$ is a connected normal open and thus closed subgroup. It follows that $\mathrm{Diff}_0^*(M)$ is the connected component of id in $\mathrm{Diff}_0(M)$. Note that for any $g\in \mathrm{Diff}_0^*(M)$, there exists a continuous path, $t\in[0,1]\mapsto g_t\in \mathrm{Diff}_0^*(M)$ such that $g_0=\mathrm{id}$, $g_1=g$ and $\mathrm{supp}\,g_t$ is contained in a fixed compact set K.

2.2. Primitive Campbell-Hausdorff formula on the grop of diffeomorphism. The following theorem is an extension of the usual formula which is easily derived from Campbell-Hausdorff formula on (compact) Lie groups. Actually we can assure it by operating all transformations in (1) and (2) in Theorem 2.1. corresponding to right invariant vector fields to the unit element of the compact Lie group.

Theorem 2.1. *Let $X, Y \in \Gamma_0(M)$ and $\{\mathrm{Exp}(tX)\}_{t\in\mathbf{R}}$, $\{\mathrm{Exp}(tY)\}_{t\in\mathbf{R}}$ be 1-parameter subgroups of diffeomorphisms generated by X, Y, respectively. Then as n tends to $+\infty$,*

(1) $\left\{\mathrm{Exp}\left(\frac{tX}{n}\right)\circ\mathrm{Exp}\left(\frac{tY}{n}\right)\right\}^n$ *converges to* $\mathrm{Exp}(t(X+Y))$, *and*

(2) $\left\{\mathrm{Exp}\left(-\frac{tX}{\sqrt{n}}\right)\circ\mathrm{Exp}\left(-\frac{tY}{\sqrt{n}}\right)\circ\mathrm{Exp}\left(\frac{tX}{\sqrt{n}}\right)\circ\mathrm{Exp}\left(\frac{tY}{\sqrt{n}}\right)\right\}^n$ *converges to* $\mathrm{Exp}(-t^2[X,Y])$ *in τ_K uniformly on every compact interval of t, respectively, where K is any compact set containing* $\mathrm{supp}\,X$ *and* $\mathrm{supp}\,Y$.

Proof. Using the notation in 2.1, we imbed Diff(K) into Diff(N) which is a strong inductive limit of Hilbert Lie group (SILH-group). Put

$$h(s,t):=\mathrm{Exp}(sX)\circ\mathrm{Exp}(sY).$$

Then h is a t-independent map which is so called C^1-hair. For the definition of C^1-hair we quote the following reference [14] on regular Fréchet groups. Since SILH-groups are regular Fréchet groups, we are able to apply fundamental theorem 4.1. in [14] to $G=\mathrm{Diff}(N)$ and h. Then it follows that $\left\{\mathrm{Exp}\left(\frac{tX}{n}\right)\circ\mathrm{Exp}\left(\frac{tY}{n}\right)\right\}^n$ converges to some g_t in Diff(N) uniformly on every compact interval ot t and that g_t satisfies the equation,

$$\frac{dg_t}{dt}=dR_{g_t}u, \qquad g_0=\mathrm{id}, \tag{2.1}$$

where u is an element in the Lie algebra Diff(N) such that

$$u=\left.\frac{d}{ds}\right|_{s=0}\mathrm{Exp}(sX)\circ\mathrm{Exp}(sY).$$

Now take a local coordinate system $x_1,\cdots,x_d$ at $g_t(x_0)$, x_0 is any fixed point, and put $g_i(t):=x_i(g_t(x_0))$. Then we have

$$\begin{aligned}\frac{dg_i}{dt}(t)&=\left.\frac{d}{ds}\right|_{s=0}x_i(\mathrm{Exp}(sX)\circ\mathrm{Exp}(sY)\circ g_t(x_0))\\&=(X(x_i)+Y(x_i))(g_t(x_0)).\end{aligned}$$

So we have $g_t=\mathrm{Exp}(t(X+Y))$.

(2) is derived in a similar way. This time we put

$$h(s,t):=\mathrm{Exp}(-\sqrt{s}X)\circ\mathrm{Exp}(-\sqrt{s}Y)\circ\mathrm{Exp}(\sqrt{s}X)\circ\mathrm{Exp}(\sqrt{s}Y).$$

Then the limit point g_t satisfies the equation,

$$\frac{dg_i}{dt}(t) = \frac{d}{ds}\Big|_{s=0} x_i(\mathrm{Exp}(-\sqrt{s}X) \circ \mathrm{Exp}(-\sqrt{s}Y) \circ \mathrm{Exp}(\sqrt{s}X) \circ \mathrm{Exp}(\sqrt{s}Y) \circ g_t(x_0))$$
$$= -[X, Y](x_i)(g_t(x_0)). \quad \text{Q.E.D.}$$

Now we proceed to the question (3) stated in the Introduction.

Theorem 2.2. *Put* $\Gamma(K) := \{X \in \Gamma_0(M) \mid \operatorname{supp} X \subseteq K\}$. *Then the subgroup* G_K *generated by* $\mathrm{Exp}(X)$, $X \in \Gamma(K)$ *is dense in* $\mathrm{Diff}^*(K)$.

Proof. Let $\{g_t\}_{0 \le t \le 1} \subset \mathrm{Diff}^*(K)$ be a continuous path connecting id and $g \in \mathrm{Diff}^*(K)$. Taking a suitable partition of $[0, 1]$, $0 = t_0 < t_1 < \cdots < t_n = 1$, we may assume that $g_{t_i}^{-1} g_{t_{i+1}} \in \mathscr{U}$ $(i = 0, \cdots, n-1)$, where $\mathscr{U}$ is a neighbourhood of id which is diffeomorphic by a map ξ to an open convex set containing the origin of the modelled ILH-space $\Gamma(N)$. (N is the compact manifold containing K.) The explicit form of ξ, according to H. Omori in [13], is written as

$$\xi(u)(x) = \exp_x u(x), \tag{2.2}$$

where the last means the minimal geodisc for a Riemann structure on N starting at x along the direction $u(x)$. Thus there exists for each $i, u_i \in \Gamma(N)$ such that $\xi(u_i) = g_{t_i}^{-1} g_{t_{i+1}}$ and $\xi(u_i)(P) = P$ for all $P \in K^c$. So we get $u_i(P) = 0$ for all $P \in K^c$ and u_i are actually in $\Gamma(K)$. Now put $\gamma_i(t) := \xi(tu_i)$. Then $\gamma_i(t)$ is a C^∞-curve on $\mathrm{Diff}(N)$ and $v_i(t) := dR_{\gamma_i(t)}^{-1} \dot{\gamma}_i$ is also in $\Gamma(K)$. Because take any C^∞-function on M and take any $P \in K^c$. Since $\gamma_i(t)P = P$ for all $0 \le t \le 1$, we have for any C^∞-function on M,

$$\frac{d}{ds}\Big|_{s=0} f(\exp_P(sv_i(t))) = \frac{d}{dt} f(\gamma_i(t)P) = 0.$$

It follows that $(v_i(t)f)(P) = 0$ for all $P \in K^c$.

By the way $\gamma_i(t)$ is just equal to the product integral $\Pi_0^t(1 + v_i(s))ds$, where the product integral is defined as the limit of

$$\mathrm{Exp}((a_n - a_{n-1})v_i(a_{n-1})) \circ \mathrm{Exp}((a_{n-1} - a_{n-2})v_i(a_{n-2})) \circ \cdots \circ \mathrm{Exp}((a_1 - a_0)v_i(a_0)),$$

when the size $\mathrm{Max}_{1 \le i \le n}|a_i - a_{i-1}|$ of the partition, $0 = a_0 < a_1 < \cdots < a_n = t$ tends to 0. For details see p63–p66 in [13]. It follows that each $\gamma_i(t)$ and $\gamma(t) := \gamma_0(t) \cdots \gamma_{n-1}(t)$, especially $g = \gamma(1)$ is approximated by the elements of G_K. Q.E.D.

Here for the later discussions we shall list another version of Theorem 2.2. Let A be any fixed point of M and consider all diffemorphisms g with compact supports which leave A invariant. Now put

$\mathrm{Diff}_{0,A}(M) := \{g \in \mathrm{Diff}_0(M) \mid g(A) = A\}$, $\mathrm{Diff}_A(K) := \{g \in \mathrm{Diff}_{0,A}(M) \mid \operatorname{supp} g \subseteq K\}$, and $\mathrm{Diff}_A^*(K) := \{g \in \mathrm{Diff}_A(K) \mid$ there exists a continuous path $\{g_t\}_{0 \le t \le 1} \subset \mathrm{Diff}_A(K)$ such that $g_0 = \mathrm{id}$ and $g_1 = g\}$.

Theorem 2.3. *Put* $\Gamma_A(K):=\{X\in\Gamma(K)\,|\,X(A)=0\}$. *Then the subgroup* $G_{A,K}$ *generated by* $\mathrm{Exp}(X)$, $X\in\Gamma_A(K)$ *is dense in* $\mathrm{Diff}_A^*(K)$.

Proof is derived in a quite similar way with the above one. So we omit it.

The following theorem as an affirmative answere of the problem (3) is a direct consequence of Theorem 2.2.

Theorem 2.4. *The group* $\tilde{G}$ *generated by* $\mathrm{Exp}(X)$, $X\in\Gamma_0(M)$ *is dense in* $\mathrm{Diff}_0^*(M)$.

§3. 1-Cocycles on the Group of Diffeomorphisms

3.1. Finite dimensional representation of $\mathrm{Diff}_0(M)$.

Theorem 3.1. *There is no continuous representations of* $\mathrm{Diff}_0^*(M)$ *to* $GL(n, \boldsymbol{C})$ *except for trivial one.*

Proof. Let U be a continuous representation of $\mathrm{Diff}_0^*(M)$ to $GL(n,\boldsymbol{C})$. Take X from $\Gamma_0(M)$ and form a continuous 1-parameter subgroup $\{\mathrm{Exp}(tX)\}_{t\in\boldsymbol{R}}$. Then there exists some and unique $dU(X)\in M(n,\boldsymbol{C})$ such that

$$U(\mathrm{Exp}(tX))=\exp(t\,dU(X)). \tag{3.1}$$

By Theorem 2.1 dU is linear on $\Gamma_0(M)$ and

$$dU([X,Y])=[dU(X),dU(Y)], \tag{3.2}$$

for all $X, Y\in\Gamma_0(M)$. Since

$$dU(X)=\lim_{t\to 0}\frac{U(\mathrm{Exp}(tX))-E_n}{t}, \tag{3.3}$$

where E_n is the unit matrix, dU is continuous on $\Gamma_0(M)$ equipped with the inductive limit topology of $\Gamma(K)$'s, on which we give the usual C^∞-topology. For, $dU|\Gamma(K)$ is a limit of continuous functions on $\Gamma(K)$ by (3.3) and thus a set of all discontinuous points of $dU|\Gamma(K)$ is second category. On the other hand $\Gamma(K)$ is a Fréchet space, so there exists at least one continuous point of $dU|\Gamma(K)$ by Baire's theorem. As $dU|\Gamma(K)$ is linear, it is continuous on all the points of $\Gamma(K)$. Hence dU is continuous with the inductive limit topology. We wish to show that $dU=0$ and for it it is enough to admit the following lemma, because using a partition of unity, X is decomposed to finitely many C^∞-vector fileds whose supports are contained in some cubic neighbourhoods.

Lemma 3.1. *For a positive number* α, *put* $U_\alpha:=\{x\in\boldsymbol{R}^d\,|\,-\alpha<x_i<\alpha\ (i=1,\cdots,d)\}$, *and consider a Lie algebra* $\mathcal{G}_\alpha$ *consisting of* $\boldsymbol{R}^d$*-valued* C^∞*-functions* $F(x)=(F_i(x))_{1\le i\le d}$

on $\boldsymbol{R}^d$ *such that* $\operatorname{supp} F \subset U_\alpha$ *with the Lie bracket,*

$$[F, G] := \sum_{i=1}^{d} \{F_i(x)\frac{\partial G}{\partial x_i}(x) - G_i(x)\frac{\partial F}{\partial x_i}(x)\}.$$

Then there is no continuous linear representations dU *from* $\mathscr{G}_\alpha$ *to* $B(H)$ *except for trivial one, where the toplogy of* $\mathscr{G}_\alpha$ *is the usual one imposed on the space of test functions on* U_α *and* $B(H)$*, the space of all bounded operators on a complex finite or infinite dimensional Hilbert space* H*, is equipped with the weak operator topolology.*

Proof. Let for $1 \le k \le d$, $\hat{1}_k$ be the function defined by its kth component is equal to 1 and the other component is equal to 0. Further put

$$dU_k(\rho) := dU(\rho \hat{1}_k), \quad [\rho, \sigma]_k := \rho\frac{\partial \sigma}{\partial x_k} - \sigma\frac{\partial \rho}{\partial x_k} \quad (\rho, \sigma \in C_0^\infty(U_\alpha)),$$

where the set $C_0^\infty(U_\alpha)$ consists of all C^∞-functions whose supports are compact subsets of U_α. Then

$$dU_k([\rho, \sigma]_k) = dU_k(\sigma)dU_k(\rho) - dU_k(\rho)dU_k(\sigma). \tag{3.4}$$

Now we claim that

(*) for any $x_0 \in U_\alpha$ there exists a function $\rho \equiv \rho_{x_0} \in C_0^\infty(U_\alpha)$ such that

$$\rho(x_0) \neq 0, \quad \text{and} \quad dU_k(\rho) = 0.$$

For the proof we carry out it at $x_0 = 0$ for simplicity. Consider a function $\rho \in C_0^\infty(U_\alpha)$ with $\operatorname{supp} \rho \subset \overline{U_{\frac{\alpha}{4}}}$ and take a δ from $[\frac{\alpha}{2}, \frac{5\alpha}{8}]$. Put

$$\rho_t(x) := \rho(x_1, \cdots, x_{k-1}, x_k + t, x_{k+1}, \cdots x_d) \quad (|t| \le \delta).$$

Then since

$$\frac{\rho_{t+h} - \rho_t}{h} \to \frac{\partial \rho_t}{\partial x_k} \quad (h \to 0)$$

in the space of test functions, we get by the assumption on continuity

$$\langle dU_k\left(\frac{\partial \rho_t}{\partial x_k}\right)e_1, e_2\rangle_H = \frac{d}{dt}\langle dU_k(\rho_t)e_1, e_2\rangle_H \tag{3.5}$$

for all $e_1, e_2 \in H$. ($\langle \cdot, \cdot \rangle_H$ is the scalar product on H.) Take $\sigma_0 \in C_0^\infty(U_\alpha)$ such that $\sigma_0 \equiv 1$ on $\overline{U_{\frac{7\alpha}{8}}}$. Then by the definiton of $[\cdot, \cdot]_k$ and by (3.4) we have

$$\operatorname{ad}(dU_k(\sigma_0))(dU_k(\rho_t)) := [dU_k(\rho_t), dU_k(\sigma_0)] = dU_k\left(\frac{\partial \rho_t}{\partial x_k}\right). \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\frac{d}{dt}\langle dU_k(\rho_t)e_1, e_2\rangle_H = \langle \mathrm{ad}(dU_k(\sigma_0))(dU_k(\rho_t))e_1, e_2\rangle_H, \tag{3.7}$$

and repeating this procedure over n times,

$$\frac{d^n}{dt^n}\langle dU_k(\rho_t)e_1, e_2\rangle_H = \langle \mathrm{ad}(dU_k(\sigma_0))^n(dU_k(\rho_t))e_1, e_2\rangle_H. \tag{3.8}$$

Here using again the assumption on continuity and the resonance theorem, we get

$$M := \sup_{|h|\leq\delta} \|dU_k(\rho_t)\|_{op} < \infty. \tag{3.9}$$

Hence,

$$\sup_{|h|\leq\delta} \left|\frac{d^n}{dt^n}\langle dU_k(\rho_t)e_1, e_2\rangle_H\right| \leq 2^n M \|dU_k(\sigma_0)\|_{op}^n \|e_1\|_H \|e_2\|_H. \tag{3.10}$$

Therefore $\langle dU_k(\rho_t)e_1, e_2\rangle_H$ is an analytic function on $|t|<\delta$ for any fixed $e_1, e_2 \in H$, and the same holds for $\langle dU_k([\rho, \rho_t]_k)e_1, e_2\rangle_H$. Since $\mathrm{supp}\,\rho \cap \mathrm{supp}\,\rho_t = \emptyset$ for $\frac{\alpha}{2} \leq t \leq \delta$,

$$dU_k([\rho, \rho_t]_k) = 0$$

for all $|t|\leq\delta$. Especially taking a function of the form,

$$\rho(x) = \phi(x_k)\psi(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_d),$$

$\phi(0) = \psi(0) = 1$, $\phi'(0) = 0$ and $\phi'(t) > 0$ for sufficiently small t, we get a desired function as the one in (*), as

$$dU_k([\rho, \rho_t]_k) = 0, \qquad [\rho, \rho_t]_k(0) = \phi'(t) > 0.$$

Now we will finish the proof. From (*) we have

$$dU_k(\sigma) = 0$$

for all $\sigma \in C_0^\infty(U_\alpha)$ with $\mathrm{supp}\,\sigma \subset \{x | \rho_{x_0}(x) \neq 0\}$. Because we have $\sigma = [\rho_{x_0}, s]_k$ for a s defined by

$$s(x) := \rho_{x_0} \int_{-\alpha}^{x_k} \frac{\sigma}{\rho_{x_0}^2}(x_1, \cdots, x_{k-1}, t, x_{k+1}, \cdots x_d)dt.$$

Therefore using a partition of unity we get $dU_k(\sigma) = 0$ for all $\sigma \in C_0^\infty(U_\alpha)$ and the conclusion follows. Q.E.D.

From Theorem 3.1 we see that finite dimensional representations of $\mathrm{Diff}_0(M)$ actually come from the discrete group $\mathrm{Diff}_0(M)/\mathrm{Diff}_0^*(M)$. For example in the case that $M=\boldsymbol{T}^1$, $\mathrm{Diff}_0(M)$ consists of two components. One is the $\mathrm{Diff}_0^*(\boldsymbol{T}^1)$ and the other is the component of the reflection R. So a function χ defined by $\chi=1$ on $\mathrm{Diff}_0^*(\boldsymbol{T}^1)$ and $\chi=-1$ on $R\,\mathrm{Diff}_0^*(\boldsymbol{T}^1)$ is the unique non trivial unitary character on $\mathrm{Diff}_0(\boldsymbol{T}^1)$.

3.2. 1-cocycles of standard type. Let θ be a continuous 1-cocycle stated in the Introduction. The purpose of this subsection is to show that if the manifold M is simply connected, then any continuous 1-cocycle θ consists of standard type. We begin with a study of unitary characters associated with 1-cocycles. Take any point $A\in M$ and fix it. Then the function

$$\chi: h\in \mathrm{Diff}_{0,A}(M)\mapsto \theta(A,h)\in \boldsymbol{T}^1$$

is a unitary character. First we shall investigate such characters on $\mathrm{Diff}_{0,A}^*(M)$ $:=\cup_K \mathrm{Diff}_A^*(K)$.

Theorem 3.2. *Let χ be any unitary character on $\mathrm{Diff}_{0,A}^*(M)$. Then there exists some real number s such that*

$$\chi(\phi)=(J\phi(A))^{\sqrt{-1}s}$$

for all $\phi\in\mathrm{Diff}_{0,A}^*(M)$, where $J\phi(A)$ is the Jacobian of ϕ at A ($J\phi(A)$ does not depend on a particular choice of the local coordinate systems at A).

Proof. (I) Put $\Gamma_{0,A}(M):=\{X\in\Gamma_0(M)\mid X(A)=0\}$. Then for any $X\in\Gamma_{0,A}(M)$ there exists a real constant $\lambda(X)$ such that

$$\chi(\mathrm{Exp}(tX))=\exp(\sqrt{-1}t\lambda(X)) \tag{3.11}$$

for all $t\in\boldsymbol{R}$, where λ is a linear functional on $\Gamma_{0,A}(M)$ which satisfies

$$\lambda([X,Y])=0 \tag{3.12}$$

for all $X,Y\in\Gamma_{0,A}(M)$ by Theorem 2.1. As before we analyze λ locally. That is, first we cover $\{A\}\cup\operatorname{supp}X$ by finitely many open cubic neighbourhoods U_n $(n=0,\cdots,N)$ such that A belongs to only one of the sets U_n, say U_0. Then using a partition of unity, X is represented as a sum of $X_n\in\Gamma_{0,A}(M)$ whose support is contained in U_n $(n=0,\cdots,N)$. It is not hard to see that by virtue of Lemma 3.1 $\lambda(X_n)=0$ for all $n\neq 0$. For $n=0$ the following lemma is fundamental.

(II)

Lemma 3.2. *Using the same notation as in Lemma 3.1, we consider a Lie algebra $\mathscr{G}_\alpha^0:=\{F\in\mathscr{G}_\alpha\mid F(0)=0\}$. Then for each linear functional λ defined on $\mathscr{G}_\alpha^0$ with* the property, $\lambda([F,G])=0$ *for all* $F,G\in\mathscr{G}_\alpha^0$, *there exists a real constant s such that*

$$\lambda(F)=s\sum_{i=1}^{d}\frac{\partial F_i}{\partial x_i}(0).$$

Proof. 1-STEP. Noting that $x_jF\in\mathscr{G}_\alpha^0$ for all $F\in\mathscr{G}_\alpha$, let us prove first that

$$\lambda(x_j^2F)=0 \tag{3.13}$$

for each $1\leq j\leq d$. Take any $\boldsymbol{R}$-valued C^∞-function ρ with $\operatorname{supp}\rho\subset U_\alpha$, and put

$$\lambda_j(\rho)=\lambda(x_j^2\rho\hat{1}_j).$$

Since the jth component of $[x_j\rho\hat{1}_j, x_j\sigma\hat{1}_j]$ is equal to $x_j^2\left(\rho\frac{\partial\sigma}{\partial x_j}-\sigma\frac{\partial\rho}{\partial x_j}\right)$ and the other component is equal to 0, we have by the assumption

$$\lambda_j\left(\rho\frac{\partial\sigma}{\partial x_j}-\sigma\frac{\partial\rho}{\partial x_j}\right)=0. \tag{3.14}$$

Hence we get from the last part of the proof of Lemma 3.1

$$\lambda_j(\rho)=0.$$

Next for $k\neq j$ we put

$$\lambda_k(\rho)=\lambda(x_j^2\rho\hat{1}_k).$$

Then noting that the kth component of $[x_j\rho\hat{1}_k, x_j\sigma\hat{1}_k]$ is equal to $x_j^2\left(\rho\frac{\partial\sigma}{\partial x_k}-\sigma\frac{\partial\rho}{\partial x_k}\right)$ and the other component is equal to 0, we have

$$\lambda_k\left(\rho\frac{\partial\sigma}{\partial x_k}-\sigma\frac{\partial\rho}{\partial x_k}\right)=0,\quad \text{and therefore } \lambda_k(\rho)=0.$$

Therefore (3.13) follows directly.

2-STEP. Next we shall prove that

$$\lambda(x_jx_kF)=0 \tag{3.15}$$

for all $F\in\mathscr{G}_\alpha$ and for all $j\neq k$. Now for each $l\neq j,k$, put

$$\lambda_l(\rho)=\lambda((x_j+x_k)^2\rho\hat{1}_l).$$

Then the lth component of $[(x_j+x_k)\rho\hat{1}_l, (x_j+x_k)\sigma\hat{1}_l]$ is equal to $(x_j+x_k)^2\left(\rho\frac{\partial\sigma}{\partial x_l}-\sigma\frac{\partial\rho}{\partial x_l}\right)$, and the other component is equal to 0, so we get

$$\lambda_l\left(\rho\frac{\partial\sigma}{\partial x_l}-\sigma\frac{\partial\rho}{\partial x_l}\right)=0, \quad \text{and thererore } \lambda_l(\rho)=0.$$

Further if l is equal to one of j or k, say j, then the jth component of $[(x_j+x_k)\rho\hat{1}_j,$ $(x_j+x_k)\sigma\hat{1}_j]$ is equal to $(x_j+x_k)^2\left(\rho\frac{\partial\sigma}{\partial x_j}-\sigma\frac{\partial\rho}{\partial x_j}\right)$ and the other component is equal to 0. So we have $\lambda((x_j+x_k)^2\rho\hat{1}_j)=0$ as before. It follows that $\lambda((x_j+x_k)^2F)=0$ and (3.15) follows from (3.13).

3-STEP. Take any $F\in\mathscr{G}_\alpha^0$ and expand it by Taylor's formula,

$$F(x)=\sum_{i=1}^{d}\frac{\partial F}{\partial x_i}(0)x_i+\sum_{j,k=1}^{d}x_jx_k\int_0^1(1-t)\frac{\partial^2F}{\partial x_j\partial x_k}(tx)dt. \tag{3.16}$$

Further take an $\boldsymbol{R}$-valued C^∞-function φ which is equal to 1 on a neighbourhood of $\operatorname{supp}F\cup\{0\}$ and have a support contained in U_α. Then we get from 1-step and 2-step,

$$\lambda(F)=\lambda(\varphi F)=\sum_{i,j=1}^{d}a_{i,j}\frac{\partial F_j}{\partial x_i}(0),$$

where the number $a_{i,j}:=\lambda(x_i\varphi\hat{1}_j)$ is actually the same one for every φ which is equal to 1 on a neighbourhood of 0 and $\operatorname{supp}\varphi\subset U_\alpha$, so it is a constant independent of $F\in\mathscr{G}_\alpha^0$. Consequently for matrices $A:=(a_{i,j})_{1\le i,j\le d}$ and $J_F(0):=\left(\frac{\partial F_i}{\partial x_j}(0)\right)_{1\le i,j\le d}$ we have

$$\lambda(F)=\operatorname{tr}(AJ_F(0)). \tag{3.17}$$

4-STEP. Lastly we shall prove that A is a scalar matrix. By the assumption and by the following equality,

$$J_{[F,G]}(0)=J_G(0)J_F(0)-J_F(0)J_G(0), \tag{3.18}$$

we get

$$\operatorname{tr}(AJ_G(0)J_F(0))=\operatorname{tr}(AJ_F(0)J_G(0)). \tag{3.19}$$

Since for any matrix $P\in\mathscr{M}(d)$, there exists an $F\in\mathscr{G}_\alpha^0$ such that $J_F(0)=P$,

$$\operatorname{tr}(APQ)=\operatorname{tr}(AQP) \tag{3.20}$$

for all $P,Q\in\mathscr{M}(d)$. Now let e_i $(1\le i\le d)$ be the canonical base of $\boldsymbol{R}^d$ whose ith component is equal to 1 and the other is equal to 0. First for each $i\neq j$ choose $P,Q\in\mathscr{M}(d)$ as follows,

$$Pe_i=e_i,\quad Pe_k=0\quad \text{for } k\neq i,\quad \text{and}\quad Qe_i=e_j,\quad Qe_k=0\quad \text{for } k\neq i.$$

Then we have $\mathrm{tr}(APQ)=0$ and $\mathrm{tr}(AQP)=a_{i,j}$, and therefore A is a diagonal matrix. Next for each $i\neq j$, we set

$$Pe_j=e_i, \quad Pe_k=0 \quad \text{for } k\neq j \quad \text{and} \quad Qe_i=e_j, \quad Qe_k=0 \quad \textit{for } k\neq i.$$

Then we have $\mathrm{tr}(APQ)=a_{i,i}$ and $\mathrm{tr}(AQP)=a_{j,j}$, and therefore $a_{i,i}$ $((1\leq i\leq d)$ are the same one, say s and we have $A=sE$.

(III) Put

$$\phi_t:=\mathrm{Exp}(tX), \quad \text{and } \phi:=\mathrm{Exp}(X) \quad (X\in\Gamma_{0,A}(M)).$$

We claim that

$$J\phi(A)=\exp\left(\sum_{i=1}^{d}\frac{\partial F_i}{\partial x_i}(0)\right), \tag{3.21}$$

where $x_1,\cdots,x_d$ is a local coordinate system at A such that $x_i(A)=0$ $(i=1,\cdots,d)$, and $X=\Sigma_{i=1}^{d}F_i(x)\dfrac{\partial}{\partial x_i}$.

For, put

$$\phi_i(t,x)=x_i\circ\phi_t(x).$$

Then

$$\frac{d}{dt}\frac{\partial\phi_i}{\partial x_j}(t,0)=\frac{\partial}{\partial x_j}\bigg|_A\frac{\partial}{\partial t}x_i\circ\phi_t(x)=\frac{\partial}{\partial x_j}\bigg|_A F_i(\phi_t(x))=\sum_{k=1}^{d}\frac{\partial F_i}{\partial x_k}(0)\frac{\partial\phi_k}{\partial x_j}(t,0).$$

Hence a matrix $A_t:=\left(\dfrac{\partial\phi_i}{\partial x_j}(t,0)\right)_{1\leq i,j\leq d}$ satisfies

$$\frac{dA_t}{dt}=J_F(0)A_t, \quad A_0=E, \tag{3.22}$$

so we have $A_t=\exp(J_F(0))$, especially

$$J\phi(A)=\det(A(1))=\exp(\mathrm{tr}(J_F(0))).$$

(IV) Returning to the notation in (I), for X which is written as $X=X_0+X_1+\cdots+X_N$ we have

$$\chi(\mathrm{Exp}(X))$$
$$=\exp(\sqrt{-1}\lambda(X))=\exp(\sqrt{-1}\lambda(X_0))=\exp(\sqrt{-1}s\sum_{i=1}^{d}\frac{\partial F_{i,0}}{\partial x_i}(0))=(J_{\mathrm{Exp}(X_0)}(A))^{\sqrt{-1}s},$$

where $F_{i,0}(x)$ is the ith component of X_0 with respect to the local coordinate system

$x_1, \cdots, x_d$ at A. Since A belongs to none of $\operatorname{supp} X_n$ $(1 \leq n \leq N)$, $\operatorname{Exp}(X)$ is equal to $\operatorname{Exp}(X_0)$ for a small neighbourhood of A. Therefore we get

$$\chi(\operatorname{Exp}(X)) = (J_{\operatorname{Exp}(X)}(A))^{\sqrt{-1}s} \tag{3.23}$$

for all $X \in \Gamma_{0,A}(M)$. It follows that $\chi(\phi) = (J_\phi(A))^{\sqrt{-1}s}$ for all ϕ in the group $G_{0,A}$ generated by $\operatorname{Exp} X$, $X \in \Gamma_{0,A}(M)$ and the assertion of the theorem follows from Theorem 2.3. Q.E.D.

According to T. Hirai here we shall rewrite the original θ as follows, using a section $s_p \in \operatorname{Diff}_0(M)$, $s_P(A) = P$ by the map, $g \in \operatorname{Diff}_0(M) \mapsto g(A) \in M$ and a function c defined by $c(P) := \theta(A, s_P^{-1})$,

$$\theta(P, g) = \frac{c(g^{-1}(P))}{c(P)} \chi(s_P^{-1} g s_{g^{-1}(P)}), \tag{3.24}$$

where χ is a unitary character on $\operatorname{Diff}_{0,A}(M)$ defined by $\chi(h) := \theta(A, h)$. Note that

$$|J_{s_P^{-1} g s_{g^{-1}(P)}}(A)|^{-1} = \left(\frac{d\mu_{s_{g^{-1}(P)}}}{d\mu}(g^{-1}(P))\right)\left(\frac{d\mu_{s_P}}{d\mu}(P)\right)^{-1}\left(\frac{d\mu_g}{d\mu}(P)\right) \tag{3.25}$$

holds for any σ-finite smooth measure μ on M which is locally equivalent to the Lebesgue measure on $\boldsymbol{R}^d$.

Now let us take locally finite open coverings $\mathscr{V}^i := (V_l^i)_{l \in \boldsymbol{N}}$ $(i = 1, 2, \cdots, 6)$ which satisfies

(1) $\overline{V_l^i}$ is compact for each $1 \leq i \leq 6$ and $l \in \boldsymbol{N}$,

(2) $\overline{V_l^{i+1}} \subset V_l^i$ for each $1 \leq i \leq 5$ and $l \in \boldsymbol{N}$,

(3) V_l^i is diffeomorphic to $\boldsymbol{R}^d$ for each $1 \leq i \leq 3$ and $l \in \boldsymbol{N}$ and

(4) $\mathscr{V}^2$ is a covering which have a property such that whenever $V_i^2 \cap V_j^2 \neq \emptyset$, then $V_i^2 \cap V_j^2$ is connected.

The existence of such coverings is derived from the theory of simple covering. (cf. [15]) From (2) and (3), V_i^2 is regarded as a relatively compact open set of $\boldsymbol{R}^d$. Consequently for any fixed $A_i \in V_i^2$, there exists a section $s_P^i \in \operatorname{Diff}_0^*(V_i^1)$, $s_P^i(A_i) = P$ such that $P \in V_i^2 \mapsto s_P^i \in \operatorname{Diff}_0^*(V_i^1)$ is continuous. For example exponential maps generated by vector fields which are equal to $\overrightarrow{A_iP}$ for all $P \in V_i^2$ and vanishes outside some fixed open disk containing $\overline{V_i^2}$ are desired ones.

Here we consider the following condition (*) for $P \in M$ and $g \in \operatorname{Diff}_0(M)$.

(*) For $P \in M$ and $g \in \operatorname{Diff}_0(M)$, there exist some $i \in \boldsymbol{N}$ and a continuous path $\{g_t\}_{0 \leq t \leq 1} \subset \operatorname{Diff}_0(M)$ connecting id and g such that $g_t^{-1}(P) \in V_i^2$ for all $0 \leq t \leq 1$.

If such condition is satisfied, then $s_P^{i^{-1}} g_t s_{g_t^{-1}(P)}^i$ moves continuously in $\operatorname{Diff}_{0,A_i}(M)$ starting from id and ending to $s_P^{i^{-1}} g s_{g^{-1}(P)}^i$ as t runs from 0 to 1, and it follows that $s_P^{i^{-1}} g s_{g^{-1}(P)}^i \in \operatorname{Diff}_{0,A_i}^*(M)$. Hence carrying out some calculations, we get from Theorem 3.2,

$$(3.26)\qquad \chi(s_P^{i^{-1}} g s^i_{g^{-1}(P)})=\left(\frac{d\mu_{s^i_{g^{-1}(P)}}}{d\mu}(g^{-1}(P))\right)^{\sqrt{-1}s_i}\left(\frac{d\mu_{s^i_P}}{d\mu}(P)\right)^{-\sqrt{-1}s_i}\left(\frac{d\mu_g}{d\mu}(P)\right)^{\sqrt{-1}s_i},$$

where s_i is some real constant. Thus defining a function c_i on V_i^2 as

$$c_i(P):=\left(\frac{d\mu_{s^i_P}}{d\mu}(P)\right)^{\sqrt{-1}s_i}\theta(A_i, s_P^{i^{-1}}),$$

we get from (3.24) and (3.26),

$$(3.27)\qquad \theta(P,g)=\frac{c_i(g^{-1}(P))}{c_i(P)}\left(\frac{d\mu_g}{d\mu}(P)\right)^{\sqrt{-1}s_i},$$

for all (P,g) satisfying $(*)$. We take and fix such s_i and c_i for each i. s_i does not depend on i, if M is connected. For, suppose first that V_i^2 and V_j^2 have a common point P. Take an open neighbourhood of $U(P)$ of P which is diffeomorphic to $\boldsymbol{R}^d$. Then taking a continuous path $\{g_t\}_{0\leq t\leq 1}\subset \mathrm{Diff}_0(U(P))$ such that $g_t^{-1}(P)=P$ and $\frac{d\mu_g}{d\mu}(P)=a$ for each $a>0$, we see that $s_i=s_j$ holds. The existence of such a path is assured by considering maps which act as similar transfomations near at P.

Next let i_0 and j_0 be arbitrary integers. Take $P\in V_{i_0}$ and $Q\in V_{j_0}$. Then there exists a finitely many $\{V_{k_l}^2\}_{1\leq l\leq N}$ such that $k_1=i_0$, $k_N=j_0$ and $V_{k_l}^2\cap V_{k_{l+1}}^2\neq 0$ for $l=0,\cdots,L-1$. Consequently $s_{i_0}=s_{k_1}$, $s_{k_1}=s_{k_2},\cdots,s_{k_{N-1}}=s_{j_0}$ hold by the above arguments, and thus $s_{i_0}=s_{j_0}$. Let us denote this common value by s. Then we get

$$(3.28)\qquad \theta(P,g)=\frac{c_i(g^{-1}(P))}{c_i(P)}\left(\frac{d\mu_g}{d\mu}(P)\right)^{\sqrt{-1}s},$$

if (P,g) satisfies $(*)$.

Now we claim that if $V_i^2\cap V_j^2\neq\emptyset$, then c_i coincides with c_j up to a constant factor on this intersection. In fact take any point $P_0\in V_i^2\cap V_j^2$ and fix it. Since $V_i^2\cap V_j^2$ is connected, for any another point $Q\in V_i^2\cap V_j^2$, there exists $g\in\mathrm{Diff}_0(M)$ and a continuous path $\{g_t\}_{0\leq t\leq 1}$ connecting id and g such that $g_t^{-1}(P_0)\in V_i^2\cap V_j^2$ and $g^{-1}(P_0)=Q$. Thus the conclusion follows from the equality,

$$\frac{c_i(g^{-1}(P_0))}{c_i(P_0)}\left(\frac{d\mu_g}{d\mu}(P_0)\right)^{\sqrt{-1}s}=\frac{c_j(g^{-1}(P_0))}{c_j(P_0)}\left(\frac{d\mu_g}{d\mu}(P_0)\right)^{\sqrt{-1}s}.$$

Let us put

$$K_n:=\mathrm{Cl}(V_1^3\cup\cdots\cup V_n^3)\quad\text{and}\quad I_n:=\{i\in N\mid K_n\cap V_i^3\neq\emptyset\}.$$

Set

$\mathscr{U}_n:=\{g\in\mathrm{Diff}(K_n)\mid{}^{\exists}\{g_t\}$: continuous path connecting id and g s.t., $g_t\in\mathrm{Diff}(K_n)$, $g_t^{-1}(\overline{V_i^6})\subset V_i^5$, $g_t(\overline{V_i^5})\subset V_i^4$, $g_t^{-1}(\overline{V_i^4})\subset V_i^3$ and $g_t(\overline{V_i^3})\subset V_i^2$ hold for all $i\in I_n$ and

$0\leq t\leq 1\}$. $\mathscr{U}_n$ contains an open neighbourhood of id, and we have

$$\mathscr{U}_n \subseteq \mathrm{Diff}^*(K_n)\cap\mathscr{U}_m$$

for $m\geq n$. Let $n(P)$ be the smallest integer n such that $P\in V_i^3\cup\cdots\cup V_n^3$ for each $P\in M$. Then a set $\{g(P)\,|\,g\in\mathscr{U}_{n(P)}\}$ is a neighbourhood of P which is seen by similar arguments with the construction s_P^i in the preceding arguments. Thus there exists a connected open neighbourhood O_P of P such that $O_P\subseteq\{g(P)\,|\,g\in\mathscr{U}_{n(P)}\}$. Without loss of generality we may assume that $O_P\subseteq V_i^6$ for some $i\in N$. Under the above preparations we are now able to prove the following theorem.

Theorem 3.3. *Let M be a simply connected paracompacr C^∞-manifold. (In particular it is connected.) Then for any continuous 1-cocycle θ, there exists a continuous function c defined on M, a real parameter s and a unitary character η such that*

$$\theta(P,g)=\frac{c(g^{-1}(P))}{c(P)}\left(\frac{d\mu_g}{d\mu}(P)\right)^{\sqrt{-1}s}\eta(g),$$

where μ is any but fixed σ-finite smooth measure on M which is locally equivalent to the Lebesgue measure on $\boldsymbol{R}^d$. Moreover s and η are uniquely determined by a given θ and c is determined up to a constant factor.

Proof. Proof is derived by the theorem of Principle of monodoromy which we shall list it below for reference. (See, [1])

Theorem 3.4. *Let M be a simply connected space. Assume that we have assigned to every $P\in M$ a non empty set E_P, to every point (P,Q) of a certain subset D of $M\times M$ a mapping $\varphi_{P,Q}$ of E_P into E_Q, in such a way that the following conditions are satisfied.*

(1) *The set D is a connected open set containing the diagonal in $M\times M$,*

(2) *each $\varphi_{P,Q}$ is a one-to-one mapping of E_P onto E_Q, $\varphi_{P,P}$ is the identity mapping, and*

(3) *if $\varphi_{P,Q}$, $\varphi_{Q,R}$, $\varphi_{P,R}$ are all defined, we have $\varphi_{P,R}=\varphi_{Q,R}\circ\varphi_{P,Q}$.*
Then there exists a mapping ψ which assigns to every $P\in M$ an element $\psi(P)\in E_P$ in such a way that $\psi(Q)=\varphi_{P,Q}(\psi(P))$ whenever $\varphi_{P,Q}$ is defined.

We continue the proof of Theorem 3.3.

(I) Definition of E_P and D: Set $E_P:=\boldsymbol{T}^1$ for each $P\in M$, and $D:=\cup_{P\in M}(O_P\times O_P)$, which is a connected open set containing the diagonal.

(II) Definition of $\varphi_{P,Q}$: For each $(P,Q)\in D$, there exists some $i\in N$ such that $(P,Q)\in V_i^6\times V_i^6$. So we put

$$\varphi_{P,Q}(z)=\frac{c_i(Q)}{c_i(P)}z.$$

This definition does not depend on a particular choice of V_i^6 by the above discussions. Especially, $\varphi_{P,Q}$ is a bijection. Moreover, if $(P,Q)\in O_X\times O_X$, $(Q,R)\in O_Y\times O_Y$, and $(P,R)\in O_Z\times O_Z$, then $P=g_1(X)$, $Q=g_2(X)$ for some $g_1, g_2\in\mathscr{U}_{n(X)}$, $Q=g_3(Y)$, $R=g_4(Y)$ for some $g_3, g_4\in\mathscr{U}_{n(Y)}$, and $O_X\subseteq {}^{\exists}V_i^6$, $O_Y\subseteq {}^{\exists}V_j^6$, $O_Z\subseteq {}^{\exists}V_k^6$ for some $i,j,k\in N$. Thus, we have $g_i\in\mathscr{U}_l$ $(i=1,2,3,4)$ for $l:=\mathrm{Max}(n(X),n(Y))$ and $K_l\cap V_i^6\neq\emptyset$, $K_l\cap V_j^6\neq\emptyset$. Take pathes $g_{i,t}$ $(i=1,2,3,4)$ which have properties described in the definiton of $\mathscr{U}_l$. We have

$$g_{4,t}\circ g_{3,t}^{-1}\circ g_{2,t}\circ g_{1,t}^{-1}(\overline{V}_i^6)\subset g_{4,t}\circ g_{3,t}^{-1}\circ g_{2,t}(V_i^5)\subset g_{4,t}\circ g_{3,t}^{-1}(V_i^4)\subset g_{4,t}(V_i^3)\subset V_i^2 .$$

Consequently the diffeomorphism $g:=g_1\circ g_2^{-1}\circ g_3\circ g_4^{-1}$ and the point P satsfies the condtion (*). It follows from (3.27) that

$$\theta(P,g)=\frac{c_i(R)}{c_i(P)}\left(\frac{d\mu_g}{d\mu}(P)\right)^{\sqrt{-1}s}=\frac{c_k(R)}{c_k(P)}\left(\frac{d\mu_g}{d\mu}(P)\right)^{\sqrt{-1}s},$$

where the last equality follows from $P,R\in O_Z\subseteq V_k^6$. Similarly we have,

$$\theta(P,g_1g_2^{-1})=\frac{c_i(Q)}{c_i(P)}\left(\frac{d\mu_{g_1g_2^{-1}}}{d\mu}(P)\right)^{\sqrt{-1}s}, \quad\text{and}\quad \theta(Q,g_3g_4^{-1})=\frac{c_j(R)}{c_j(Q)}\left(\frac{d\mu_{g_3g_4^{-1}}}{d\mu}(Q)\right)^{\sqrt{-1}s}.$$

So we get $\dfrac{c_k(R)}{c_k(P)}=\dfrac{c_j(R)}{c_j(Q)}\dfrac{c_i(Q)}{c_i(P)}$ and the equality $\varphi_{P,R}=\varphi_{Q,R}\circ\varphi_{P,Q}$ follows. It follows from the above arguments that there exists a T^1-valued function $c(P)$ such that

$$(P,Q)\in D\cap(V_i^6\times V_i^6)\quad\text{implies that}\quad\frac{c_i(Q)}{c_i(P)}=\frac{c(Q)}{c(P)}.$$

(III) Next we cover K_n by finitely many open sets O_{P_i} $(i=1,\cdots,l)$, where P_i belongs to $V_1^2\cup\cdots\cup V_n^2$. Choose an open covering $\{G_i\}_{1\leq i\leq l}$ of K_n such that $\overline{G_i}\subset O_{P_i}$ $(1\leq i\leq l)$ and $\overline{G_i}$ are all compact. Then there exists an neighbourhood $\mathscr{U}'_n$ of id in Diff(K_n) such that if $g\in\mathscr{U}'_n$, there exists a continuous path $\{g_t\}_{0\leq t\leq 1}$ connecting id and g such that $g_t^{-1}(\overline{G_i}\cap K_n)\subset O_{P_i}$ for each $1\leq i\leq l$ and for all $0\leq t\leq 1$. Thus for any fixed point $P\in K_n$ and for any $g\in\mathscr{U}'_n$, we have $g_t^{-1}(P)\in O_{P_i}\subseteq {}^{\exists}V_k^6$ $(0\leq t\leq 1)$, where $i\in N$ is a number such that $P\in G_i\cap K_n$. It follows from (3.27) that

$$\theta(P,g)=\frac{c_k(g^{-1}(P))}{c_k(P)}\left(\frac{d\mu_g}{d\mu}(P)\right)^{\sqrt{-1}s}=\frac{c(g^{-1}(P))}{c(P)}\left(\frac{d\mu_g}{d\mu}(P)\right)^{\sqrt{-1}s},$$

because $(P,g^{-1}(P))$ also belongs to $D\cap(V_k^6\times V_k^6)$. Now put for all $P\in M$ and for all $g \in \mathrm{Diff}_0(M)$,

$$\zeta(P,g) := \frac{c(g^{-1}(P))}{c(P)} \left(\frac{d\mu_g}{d\mu}(P) \right)^{\sqrt{-1}s}.$$

Then we have $\zeta(P,g) = \theta(P,g)$ for any $P \in K_n$ and for any $g \in \mathscr{U}'_n$. Thus it holds also for all $g \in \mathrm{Diff}^*(K_n)$, because $\mathscr{U}'_n$ generates the whole group $\mathrm{Diff}^*(K_n)$. Consequently we have $\theta = \zeta$ on $M \times \mathrm{Diff}_0^*(M)$ due to $K_n \uparrow M$.

(IV) Let us check that c is a continuous funciton. For it take any point $P_0 \in M$ and take a local continuous section s_P, $s_P(P_0) = P$, around P_0 which satisfies $s_{P_0} = \mathrm{id}$. Then the continuity of c at P_0 follows from the equality,

$$\frac{c(P)}{c(P_0)} = \theta(P_0, s_P^{-1}) = \left(\frac{d\mu_{s_P^{-1}}}{d\mu}(P_0) \right)^{-\sqrt{-1}s}.$$

(V) Next we shall prove that $\eta := \dfrac{\theta}{\zeta}$ is a unitary character, which is derived from the following theorem.

Theorem 3.5. *Assume that a paracompact C^∞-manifold M is connected. If a continuous* 1*-cocycle η is identically equal to* 1 *on* $\mathrm{Diff}_0^*(M)$, *then it is a unitary character.*

Proof. Take any $P \in M$, any $g_0 \in \mathrm{Diff}_0^*(M)$ and any $h \in \mathrm{Diff}_0(M)$. We have

$$\eta(P, hg_0) = \eta(P,h)\eta(h^{-1}(P), g_0) = \eta(P,h).$$

Since the group $\mathrm{Diff}_0^*(M)$ is normal, $\eta(P,h) = \eta(P, g_0 h)$ also holds and thus we have

$$\eta(P,h) = \eta(P, g_0^{-1}h) = \eta(P, g_0^{-1})\eta(g_0(P), h) = \eta(g_0(P), h).$$

Hence $\eta(P,h)$ is independent of P, because $\mathrm{Diff}_0^*(M)$ acts transitively on M, and η is a unitary character. Q.E.D.

(VI) Uniqueness. We restrict θ to $\mathrm{Diff}_0^*(M)$ in order to omit the character term. Then the uniqueness of s is derived by taking some transformations such one like similar transformations at P_0, where P_0 is any fixed point. So 1-coboundary term $\dfrac{c(g^{-1}(P))}{c(P)}$ remains under considerations, however c is determined up to a constant factor by virtue of the transitivity of $\mathrm{Diff}_0^*(M)$. Consequently the remainder term of unitary characters should coincide with each other. Q.E.D.

From the above proof we see that the assertion of Theorem 3.3 holds if we can take a global continuous section s_P on M. As a special case of it we have the following corollary.

Corollary 3.6. *If M is a compact connected Lie group, then the same result as in Theorem 3.3 holds for continuous 1-cocycles θ. Namely, every continuous 1-cocycle is of standard type.*

Remark 3.1. If the M is not connected, then the general form of continuous 1-cocycles θ consists of 1-coboundary type, of Jacobian type and of the following type of 1-cocycles ξ.

Namely, decompose $M=\cup_{i=1}^{N}M_i$ into connected components M_i $(i=1,\cdots,N)$. Then $\mathrm{Diff}_0(M)/\mathrm{Diff}_0^*(M)$ acts as $\sigma(\bar{g})$ on $\{1,\cdots,N\}$ such that $g(M_i)=M_{\sigma(\bar{g})i}$. Under this notation the above 1-cocycle ξ is characterized as

$$\xi(P,g)=\hat{\xi}(i,\sigma(\bar{g})),$$

where $\hat{\xi}$ is an arbitrary 1-cocycle for the action of σ on the discrete space $\{1,\cdots,N\}\times\mathrm{Diff}_0(M)/\mathrm{Diff}_0^*(M)$ and i is number such that $P\in M_i$.

It follows that the simply connectedness condition is not a necessary one for the arguments of the canonical form of these 1-cocycles. Moreover Theorem 3.3 is no longer true if we omit the simply connectedness condition. We will give a counter example for it in the next subsection.

3.3. 1-cocycles on the cylinder. In this subsection we consider continuous 1-cocycles θ on $M=\boldsymbol{R}\times\boldsymbol{T}^1$. The elements in M will be denoted by (u,z), or $(u,\exp(\sqrt{-1}\theta))$. Let $g\in\mathrm{Diff}_0^*(\boldsymbol{R}\times\boldsymbol{T}^1)$ and take a continuous path $\{g_t\}_{0\le t\le 1}$ connecting id and g. Then for each fixed $(u,z)\in\boldsymbol{R}\times\boldsymbol{T}^1$, the second component $Z(t,u,z)$ of $g_t^{-1}(u,z)$ has an continuous angular function $\theta(t,u,z)$.

Lemma 3.3. *Put $\varphi_g(u,z):=\theta(1,u,z)-\theta(0,u,z)$. Then $\varphi:=\varphi_g$ does not depend on a particular choice of $\{g_t\}_{0\le t\le 1}$.*

Proof. It is a direct consequence of the properties of the covering space $(\boldsymbol{R}^1\to\boldsymbol{T}^1)$ and $\mathrm{supp}\,g$. Q.E.D.

For any real number Ω we put

$$\zeta_\Omega((u,z),g):=\exp(\sqrt{-1}\Omega\varphi(u,z)). \tag{3.29}$$

Lemma 3.4. *ζ_Ω is a continuous 1-cocycle on $\mathrm{Diff}_0^*(\boldsymbol{R}\times\boldsymbol{T}^1)$.*

Proof. First of all we shall prove the cocycle equality. Let $g,h\in\mathrm{Diff}_0^*(\boldsymbol{R}\times\boldsymbol{T}^1)$ and $\{g_t\}_{0\le t\le 1}$, $\{h_t\}_{0\le t\le 1}$ be continuous paths connecting id and g, id and h, respectively. Then a path defined $f_t:=g_{2t}$ for $0\le t\le 1/2$ and $f_t:=gh_{2t-1}$ for $1/2\le t\le 1$

connectes with id and $f:=gh$. Let us put $g^{-1}(u,z)=(v,w)$ and take continuous angular functions $\theta_g(t,u,z)$ along $\{g_t\}_{0\le t\le 1}$, $\theta_h(t,v,w)$ along $\{h_t\}_{0\le t\le 1}$, respectively. Then we have

$$\theta_g(1,u,z)=\theta_h(0,v,w)+2k\pi$$

for a $k\in \boldsymbol{Z}$. Hence $\theta_f(t,u,z)$ defined as below is a continuous angular function along $\{f_t\}_{0\le t\le 1}$,
$\theta_f(t,u,z):=\theta_g(2t,u,z)-2k\pi$ for $0\le t\le 1/2$, and $\theta_f(t,u,z):=\theta_h(2t-1,v,w)$ for $1/2\le t\le 1$. It follows that

$$\begin{aligned}\varphi_f(u,z)&=\theta_h(1,v,w)-\theta_g(0,u,z)+2k\pi\\&=\varphi_g(u,z)+\varphi_h(v,w),\end{aligned}$$

and therefore we have

$$\begin{aligned}\zeta_\Omega((u,z),gh)&=\zeta_\Omega((u,z),g)\cdot\zeta_\Omega((v,w),h)\\&=\zeta_\Omega((u,z),g)\cdot\zeta_\Omega(g^{-1}(u,z),h).\end{aligned}$$

Next we check the continuity. For it, we have only to show that $\zeta_\Omega((u,z),g)$ is continuous at id as a funciton of $g\in \mathrm{Diff}^*(\boldsymbol{R}\times \boldsymbol{T}^1)$ for each fixed (u,z). Evidently, for any given $\epsilon>0$ there exists a neighbourhood $\mathscr{U}$ of id such that $g\in\mathscr{U}$ implies there exists a continuous path $\{g_t\}_{0\le t\le 1}$ connecting id and g such that $\|g_t^{-1}(u,z)-(u,z)\|<\epsilon$ for all $0\le t\le 1$. So we have $|\exp(\sqrt{-1}\theta(t,u,z))-\exp(\sqrt{-1}\theta)|<\epsilon$, where θ is an argument of z, and therefore $|\varphi(u,z)|<2\arcsin\frac{\epsilon}{2}$. Thus we have

$$|\zeta_\Omega((u,z),g)-1|=|\exp(\sqrt{-1}\varphi(u,z))-1|<2\arcsin\frac{\epsilon}{2}. \qquad \text{Q.E.D.}$$

Lemma 3.5. *ζ_Ω is not of standard type, unless $\Omega\in\boldsymbol{Z}$. While if $\Omega=n\in\boldsymbol{Z}$, then it is a 1-coboundary. Namaly, $\zeta_\Omega((u,z),g)=\left(\frac{P_2(g^{-1}(u,z))}{z}\right)^n$, where P_2 is a second projection from $\boldsymbol{R}\times\boldsymbol{T}^1$ to $\boldsymbol{T}^1$.*

Proof. For the first part we have only to show an example such that $g=\mathrm{id}$ holds on some neighbourhood $P_0=(u_0,z_0)$, while $\zeta_\Omega(P_0,g)\ne 1$. For it take an $\boldsymbol{R}$-valued C^∞-function ρ with compact support such that $\rho(u)=1$ on a neighbourhood of u_0 and define $g\in \mathrm{Diff}_0^*(\boldsymbol{R}\times\boldsymbol{T}^1)$ such that $g(u,z):=(u,\exp(2\pi\sqrt{-1}\rho(u))z)$. Then $g_t:=(u,\exp(2\pi\sqrt{-1}t\rho(u))z)$ defines a continuous path connecting id and g, so we have $\zeta_\Omega(P_0,g)=\exp(-2\pi\sqrt{-1}\Omega)$. While g is equal to id on some neighbourhood P_0. The second part is obvious. Q.E.D.

Next we shall extend the domain of ζ_Ω to the whole group $\mathrm{Diff}_0(\boldsymbol{R}\times\boldsymbol{T}^1)$. So let $g\in\mathrm{Diff}_0(\boldsymbol{R}\times\boldsymbol{T}^1)$ and take R such that $g(u,z)=(u,z)$ holds for all $|u|\geq R$. Then $P_2(g(u,z))$ describes a continuous curve on $\boldsymbol{T}^1$ as u runs from $-R$ to R. We will denote its continuous angular function by $\phi(u,z)$. Put

$$Q_g(z)\equiv Q(z):=\phi(R,z)-\phi(-R,z).$$

Lemma 3.6. *$Q(z)$ is a continuous function of $z\in\boldsymbol{T}^1$.*

Proof. Since g is uniformly continous on $\boldsymbol{R}\times\boldsymbol{T}^1$, for any given $\epsilon>0$ there esists $\delta>0$ such that

$$|\exp(\sqrt{-1}(\phi(u,z)-\phi(u,z')))-1|\leq\|g(u,z)-g(u,z')\|<\epsilon \tag{3.30}$$

for all z, z' such that $|z-z'|<\delta$. We may assume that

$$|\phi(-R,z)-\phi(-R,z')|<2\arcsin\frac{\delta}{2},$$

so we get from (3.30)

$$|\phi(R,z)-\phi(R,z')|<2\arcsin\frac{\epsilon}{2}\text{ and therefore }|Q(z)-Q(z')|<2\left(\arcsin\frac{\delta}{2}+\arcsin\frac{\epsilon}{2}\right).$$

Q.E.D.

By the above, $Q(z)$ takes a constant value, say $2\pi n$, $n\in\boldsymbol{Z}$, on $\boldsymbol{T}^1$ which will be denoted by $\mathrm{Rot}(g)=n$. Put for all $n\in\boldsymbol{Z}$

$$G_n:=\{g\in\mathrm{Diff}_0(\boldsymbol{R}\times\boldsymbol{T}^1)\,|\,\mathrm{Rot}(g)=n\}.$$

Lemma 3.7. *Each connected component of $\mathrm{Diff}_0(\boldsymbol{R}\times\boldsymbol{T}^1)$ is contained in some G_n.*

Proof. Let $\{g_t\}_{0\leq t\leq 1}$ be a continuous path connecting g and g'. Then there exists a partition of $[0,1]:0=t_0<t_1<\cdots<t_m=1$ such that

$$\sup\{\|g_{t_i}(u,z)-g_{t_{i-1}}(u,z)\|\ \mid\ (u,z)\in\boldsymbol{R}\times\boldsymbol{T}^1\}<\epsilon$$

for all $1\leq i\leq m$. Take a continuous angular function $\phi_i(u,z)$ of g_{t_i}. We have

$$|\exp(\sqrt{-1}(\phi_i(u,z)-\phi_{i-1}(u,z)))-1|\leq\|g_{t_i}(u,z)-g_{t_{i-1}}(u,z)\|<\epsilon.$$

Since we may assume that $\phi_i(-R,z)=\phi_{i-1}(-R,z)$,

$$|\phi_i(u,z)-\phi_{i-1}(u,z)|<2\arcsin\frac{\epsilon}{2}\text{ and therefore }|Q_{g_{t_i}}(z)-Q_{g_{t_{i-1}}}(z)|<2\arcsin\frac{\epsilon}{2}.$$

Thus for a sufficiently small ϵ, we have $\mathrm{Rot}(g_{t_i})=\mathrm{Rot}(g_{t_{i-1}})$ for all $1\leq i\leq m$. Q.E.D.

Conversely,

Lemma 3.8. *If* $\mathrm{Rot}(g)=\mathrm{Rot}(g')$, *then* g *and* g' *belongs to the same componet.*

Proof. For the proof we use the following well known fact privately communicated by H. Omori.

Fact: For any $g\in \mathrm{Diff}_0(\boldsymbol{R}\times \boldsymbol{T}^1)$ there exists some $g_0\in \mathrm{Diff}_0(\boldsymbol{R}\times \boldsymbol{T}^1)$ consisting of the form, $g_0(u,z)=(u,h(u,z))$ such that g and g_0 belongs to the same component.

A proof of this fact is an application of the uniqueness and the continuity of Riemann mapping theorem to a suitable domain of the unit disk derived from the diffeomorphism g.

Consequently we may assume that

$$g(u,z)=(u,h(u,z)) \quad \text{and} \quad g'(u,z)=(u,h'(u,z)),$$

so we have

$$h(u,z)=\exp(\sqrt{-1}\phi(u,z)) \quad \text{and} \quad h'(u,z)=\exp(\sqrt{-1}\phi'(u,z)).$$

As we may also assume that $\phi(-R,z)=\phi'(-R,z)$, we have by the assumption $\phi(R,z)=\phi'(R,z)$. Put

$$g_t(u,z):=(u,h_t(u,z)), \quad \text{where} \quad h_t(u,z):=\exp(\sqrt{-1}(t\phi'(u,z)+(1-t)\phi(u,z))).$$

Then we have $g_0=g$, $g_1=g'$ and $h_t(u,z)=z$ for all $|u|\geq R$. We claim that g_t is a diffeomorphism for each t. For it we choose ϕ and ϕ' which satisfy $\phi(-R,z)=\phi'(-R,z)=\theta$ for $z=\exp(\sqrt{-1}\theta)$ with $0\leq\theta<2\pi$. Then $\phi(u,\exp(\sqrt{-1}\theta))$ is a continuous function of θ for each fixed u, as is seen from (3.30). It follows that $\phi(u,\exp(\sqrt{-1}\theta))$ is a C^∞-function on $\boldsymbol{R}\times(0,2\pi)$, because it coincides with an argument of $h(u,\exp(\sqrt{-1}\theta))$ on a neighbourhood of each point in $\boldsymbol{R}\times(0,2\pi)$. Thus the same holds for $h_t(u,\exp(\sqrt{-1}\theta))$ for each $t\in[0,1]$. To see the differentiability of h_t at $(u,1)$, we replace ϕ and ϕ' with ϕ_1 and ϕ'_1 which comes from the following condition likewise ϕ and ϕ',

$$\phi_1(-R,z)=\phi'_1(-R,z)=\theta \quad \text{for} \quad z=\exp(\sqrt{-1}\theta) \quad \text{with} \quad -\pi\leq\theta<\pi.$$

Since we have

$$\phi_1(u,z)+2\pi=\phi(u,z) \quad \text{if } \theta<0 \text{ and } \phi_1(u,z)=\phi(u,z) \quad \text{if } \theta\geq 0$$

and the same holds for ϕ'_1 and ϕ', h_t is still invariant under the change ϕ, ϕ' to ϕ_1, ϕ'_1. So repeating the above arguments for ϕ_1 and ϕ'_1, we see that h_t is everywhere C^∞-differentiable for each t.

Next we check that g_t is a bijection. It is easy to see that $J_g(u, \exp(\sqrt{-1}\theta))$, Jacobian of g at $(u, \exp(\sqrt{-1}\theta))$ $(0<\theta<2\pi)$, satisfies

$$(3.31)\qquad J_g(u, \exp(\sqrt{-1}\theta)) = \frac{\partial\phi}{\partial\theta}(u, \exp(\sqrt{-1}\theta)), \text{ and } \frac{\partial\phi}{\partial\theta}(-R, \exp(\sqrt{-1}\theta)) = 1.$$

So we have $\frac{\partial\phi}{\partial\theta}(u, \exp(\sqrt{-1}\theta)) > 0$ for $u \in \boldsymbol{R}$ and $0<\theta<2\pi$. The same holds for ϕ'. It follows that g_t is an injection for each t and the Jacobian g_t does not vanish everywhere. Finally, the surjection of g implies that

$$(3.32)\qquad \lim_{\theta\to 2\pi-0}\{\phi(u, \exp(\sqrt{-1}\theta)) - \phi(u, 1)\} = 2\pi.$$

Since the same holds for ϕ', we have

(3.33)

$$\lim_{\theta\to 2\pi-0}\{t\phi'(u, \exp(\sqrt{-1}\theta)) + (1-t)\phi(u, \exp(\sqrt{-1}\theta)) - (t\phi'(u, 1) + (1-t)\phi(u, 1))\} = 2\pi.$$

This shows that g_t is a surjection and that g_t actually belongs to $\mathrm{Diff}_0(\boldsymbol{R}\times \boldsymbol{T}^1)$. The continuity of the map, $t \to g_t$, is easily checked. Q.E.D.

In conclusion we have the following results which seem to be well known, but we list them for our later discussions.

(1) $G_0 = \mathrm{Diff}_0^*(\boldsymbol{R}\times \boldsymbol{T}^1)$.

(2) $G_n = gG_0 = G_0 g$ for each $g \in G_n$.

Take an $\boldsymbol{R}$-valued C^∞-function $\rho(u)$ on $\boldsymbol{R}$ such that $\rho(u)=0$ on $(-\infty, 0]$ and $\rho(u)=1$ on $[1, \infty)$, and define g_ρ as $g_\rho(u, z) := (u, \exp(2\pi\sqrt{-1}\rho(u))z)$.

(3) $g_\rho^n \in G_n$, and $\mathrm{Diff}_0(\boldsymbol{R}\times \boldsymbol{T}^1) = \cup_{n=-\infty}^{\infty} g_\rho^n G_0$,

(4) $\mathrm{Diff}_0(\boldsymbol{R}\times \boldsymbol{T}^1)/\mathrm{Diff}_0^*(\boldsymbol{R}\times \boldsymbol{T}^1) \simeq \boldsymbol{Z}$.

This group isomorphism is given by the homomorphism, $g \in \mathrm{Diff}_0(\boldsymbol{R}\times \boldsymbol{T}^1) \mapsto \mathrm{Rot}(g) \in \boldsymbol{Z}$.

Next we wish to show that it is able to extend ζ_Ω to the whole group as a continuous 1-cocycle. For simplicity we shall write ζ instead of ζ_Ω.

Definition 3.1. *Let g_ρ be as above. For any $a \in \boldsymbol{T}^1$ and for any $n \in \boldsymbol{Z}$, put*

$$\zeta_a(P, g_\rho^n) := a^n \frac{\zeta(P_0, g_\rho^{-n} h g_\rho^n)}{\zeta(P_0, h)},$$

where $P_0 = (0, 1)$ and $h \in G_0$ is a map such that $h^{-1}(P_0) = P$.

First of all let us assure that this definition does not depend on a particular choice of h. So let h, k be in G_0 such that $h^{-1}(P_0)=k^{-1}(P_0)$. Put $h_t^{-1}(P_0)=(u_t, \exp(\sqrt{-1}\theta_t))$, $k_t^{-1}(P_0)=(v_t, \exp(\sqrt{-1}\varphi_t))$, where $\{h_t\}_{0\le t\le 1}$, $\{k_t\}_{0\le t\le 1}$ be paths connecting id and h, and id and k respectively. We have

$$g_\rho^{-n}h_t^{-1}g_\rho^n(P_0)=g_\rho^{-n}h_t^{-1}(P_0)=(u_t, \exp(\sqrt{-1}(\theta_t-2\pi n\rho(u_t)))).$$

Thus,

$$\frac{\zeta(P_0, g_\rho^{-n}hg_\rho^n)}{\zeta(P_0, h)}=\frac{\exp(\sqrt{-1}\Omega(-2\pi n\rho(u_1)+\theta_1-\theta_0))}{\exp(\sqrt{-1}\Omega(\theta_1-\theta_0))}=\exp(-2\pi\sqrt{-1}\Omega n\rho(u_1)).$$

Similarly we have

$$\frac{\zeta(P_0, g_\rho^{-n}kg_\rho^n)}{\zeta(P_0, k)}=\exp(-2\pi\sqrt{-1}\Omega n\rho(v_1)).$$

So the definition is well defined by virtue of $u_1=v_1$.

Lemma 3.9. *For any* $n, m\in \boldsymbol{Z}$,

$$\zeta_a(P, g_\rho^n)\zeta_a(g_\rho^{-n}(P), g_\rho^m)=\zeta_a(P, g_\rho^{n+m}).$$

Proof. Put

$$P:=h^{-1}(P_0),\ g_\rho^{-n}(P):=k^{-1}(P_0),\ h_t^{-1}(P_0):=(u_t, \exp(\sqrt{-1}\theta_t)),$$
$$k_t^{-1}(P_0):=(v_t, \exp(\sqrt{-1}\varphi_t)).$$

Then

$$\zeta_a(P, g_\rho^n)=a^n\exp(-2\pi\sqrt{-1}\Omega n\rho(u_1)), \zeta_a(g_\rho^{-n}(P), g_\rho^m)=a^m\exp(-2\pi\sqrt{-1}\Omega m\rho(v_1)),$$

and

$$\zeta_a(P, g_\rho^{n+m})=a^{n+m}\exp(-2\pi\sqrt{-1}\Omega(n+m)\rho(u_1)).$$

This completes the proof.

Lemma 3.10. *For any* $f\in G_0$ *and for any* $n\in \boldsymbol{Z}$,

$$\zeta(P, g_\rho^n f g_\rho^{-n})=\zeta_a(P, g_\rho^n)\zeta(g_\rho^{-n}(P), f)\zeta_a(f^{-1}g_\rho^{-n}(P), g_\rho^{-n}).$$

Proof. Put

$$P:=h^{-1}(P_0),\ g_\rho^{-n}(P):=k^{-1}(P_0),\ h_t^{-1}(P_0):=(u_t, \exp(\sqrt{-1}\theta_t)),$$
$$k_t^{-1}(P_0):=(v_t, \exp(\sqrt{-1}\varphi_t)).$$

Then we have

$$f^{-1}g_\rho^{-n}(P)=(kf)^{-1}(P_0)$$

and the right hand side of the equality in the lemma is equal to

$$\begin{aligned}&\frac{\zeta(P_0,g_\rho^{-n}hg_\rho^n)}{\zeta(P_0,h)}\cdot\zeta(g_\rho^{-n}(P),f)\cdot\frac{\zeta(P_0,g_\rho^nkg_\rho^{-n})}{\zeta(P,kf)}\\&=\frac{\zeta(P_0,g_\rho^{-n}hg_\rho^n)\cdot\zeta(g_\rho^{-n}(P),f)}{\zeta(P_0,h)}\cdot\frac{\zeta(P_0,g_\rho^nkfg_\rho^{-n})}{\zeta(P_0,k)\cdot\zeta(g_\rho^{-n}(P),f)}\\&=\frac{\zeta(P_0,g_\rho^{-n}hg_\rho^n)}{\zeta(P_0,h)}\cdot\frac{\zeta(P_0,g_\rho^nkg_\rho^{-n})}{\zeta(P_0,k)}\cdot\zeta(P,g_\rho^nfg_\rho^{-n})\\&=\exp(-2\pi\sqrt{-1}\Omega n\rho(u_1))\exp(2\pi\sqrt{-1}\Omega n\rho(v_1))\zeta(P,g_\rho^nfg_\rho^{-n})\\&=\zeta(P,g_\rho^nfg_\rho^{-n}),\end{aligned}$$

where the last equality follows from $u_1=v_1$. Q.E.D.

Definition 3.2. *For any $n\in\boldsymbol{Z}$ and for any $h\in G_0$, put*

$$\zeta_{\Omega,a}(P,g_\rho^nh):=\zeta_a(P,g_\rho^n)\zeta(g_\rho^{-n}(P),h).$$

Lemma 3.11. *$\zeta_{\Omega,a}$ is a continuous 1-cocycle on $M\times\mathrm{Diff}_0(M)$ and it is an extension of ζ_Ω.*

Proof. For the cocycle equality, we have only to show that

$$\zeta_{\Omega,a}(P,g_\rho^nh)\zeta_{\Omega,a}((g_\rho^nh)^{-1}(P),g_\rho^mk)=\zeta_{\Omega,a}(P,g_\rho^nhg_\rho^mk)$$

for all $n,m\in\boldsymbol{Z}$ and for all $h,k\in G_0$. The left hand side of the above equality is equal to

$$\zeta_a(P,g_\rho^n)\zeta(g_\rho^{-n}(P),h)\zeta_a(h^{-1}g_\rho^{-n}(P),g_\rho^m)\zeta(g_\rho^{-m}h^{-1}g_\rho^{-n}(P),k).$$

While the right hand side is equal to

$$\begin{aligned}\zeta_{\Omega,a}(P,g_\rho^{n+m}g_\rho^{-m}hg_\rho^mk)&=\zeta_a(P,g_\rho^{n+m})\zeta(g_\rho^{-n-m}(P),g_\rho^{-m}hg_\rho^mk)\\&=\zeta_a(P,g_\rho^{n+m})\zeta(g_\rho^{-n-m}(P),g_\rho^{-m}hg_\rho^m)\zeta(g_\rho^{-m}h^{-1}g_\rho^{-n}(P),k)\\&=\zeta_a(P,g_\rho^n)\zeta_a(g_\rho^{-n}(P),g_\rho^m)\zeta_a(g_\rho^{-n-m}(P),g_\rho^{-m})\cdot\\&\quad\zeta(g_\rho^{-n}(P),h)\zeta_a(h^{-1}g_\rho^{-n}(P),g_\rho^m)\zeta(g_\rho^{-m}h^{-1}g_\rho^{-n}(P),k)\\&=\zeta_a(P,g_\rho^n)\zeta(g_\rho^{-n}(P),h)\zeta_a(h^{-1}g_\rho^{-n}(P),g_\rho^m)\zeta(g_\rho^{-m}h^{-1}g_\rho^{-n}(P),k).\end{aligned}$$

So the both sides coincides with each other. The continuity of $\zeta_{\Omega,a}$ is clearly reduced to the continuity of ζ_Ω which is already proved. Q.E.D.

Lemma 3.12. *If* 1*-cocycle* θ *on* $\mathrm{Diff}_0(\boldsymbol{R}\times \boldsymbol{T}^1)$ *is an extension of* ζ_Ω, *then there exists some* $a\in \boldsymbol{T}^1$ *such that* $\theta=\zeta_{\Omega,a}$.

Proof. Put $a:=\theta(P_0,g_\rho)$. Then we have for any $h\in G_0$,

$$\begin{aligned}\theta(h^{-1}(P_0),g_\rho^n)&=\frac{\theta(P_0,g_\rho^{-n}hg_\rho^n)}{\theta(P_0,g_\rho^{-n})\theta(P_0,h)}\\&=a^n\frac{\zeta_\Omega(P_0,g_\rho^{-n}hg_\rho^n)}{\zeta_\Omega(P_0,h)}\\&=\zeta_{\Omega,a}(h^{-1}(P_0),g_\rho^n).\end{aligned}$$

Further,

$$\theta(P,g_\rho^n h)=\theta(P,g_\rho^n)\theta(g_\rho^{-n}(P),h)=\zeta_{\Omega,a}(P,g_\rho^n)\zeta_{\Omega,a}(g_\rho^{-n}(P),h)=\zeta_{\Omega,a}(P,g_\rho^n h).$$

So we get $\theta=\zeta_{\Omega,a}$. Q.E.D.

Remark 3.2. A function η_a defined by $\eta_a(g_\rho^n h):=a^n$ is a unitary character on $\mathrm{Diff}_0(\boldsymbol{R}\times \boldsymbol{T}^1)$ and we have $\zeta_{\Omega,a}=\zeta_{\Omega,1}\eta_a$. Thus the essential part of the extension $\zeta_{\Omega,a}$ is $\zeta_{\Omega,1}$ which will be denoted again by ζ_Ω.

Theorem 3.7. *The general form of continuous* 1*-cocycles* θ *on the manifold* $M=\boldsymbol{R}\times \boldsymbol{T}^1$ *is as follows*:

$$\theta(P,g)=\frac{c(g^{-1}(P))}{c(P)}\left(\frac{d\mu_g}{d\mu}(P)\right)^{\sqrt{-1}s}\zeta_\Omega(P,g)\eta(g),$$

where $s\in \boldsymbol{R}$ *and* $0\leq\Omega<1$. *Besides,* s, Ω *and* η *are uniquely determined and* c *is determined up to a constant factor for a given* θ.

Proof. Put

$$I_k:=\{t\in \boldsymbol{R}\,|-k<t<k\},\qquad U_1:=\{z\in \boldsymbol{T}^1\,|\,z\neq -1\},\qquad U_2:=\{z\in \boldsymbol{T}^1\,|\,z\neq 1\},$$
$$V_{1,\epsilon}:=\{z\in \boldsymbol{T}^1\,|\ |z+1|>\epsilon\},\qquad V_{2,\epsilon}:=\{z\in \boldsymbol{T}^1\,|\ |z-1|>\epsilon\}$$

for a given $\epsilon>0$, and put $K_k:=\overline{I_k}\times \boldsymbol{T}^1$. Then it is easily deduced from the discussions in 3.2 that there exists an arcwise connected neighbourhood $\mathscr{U}_{k,\epsilon}$ of id in $\mathrm{Diff}(K_k)$ such that

(1) $g^{-1}(\overline{I_k}\times\overline{V_{i,\epsilon}})\subset \overline{I_k}\times V_{i,\frac{\epsilon}{2}}$ for all $g\in\mathscr{U}_{k,\epsilon}$ and
(2) for any $P\in \overline{I_k}\times\overline{V_{i,\epsilon}}$ and for any $g\in\mathscr{U}_{k,\epsilon}$,

$$\theta(P,g)=\frac{c_i(g^{-1}(P))}{c_i(P)}\left(\frac{d\mu_g}{d\mu}(P)\right)^{\sqrt{-1}s},$$

where c_i is a function defined on $I_{k+1}\times V_{i,\frac{\epsilon}{2}}$. c_i ($i=1,2$) and s are actually also depend on (ϵ,k), $c_i=c_{i,\epsilon,k}$, $s=s_{\epsilon,k}$. However if we assign to the value of c_i ($i=1,2$) at $(0,\sqrt{-1})$ the same value, say 1, then since $\frac{c_{i,\epsilon,k}}{c_{i,\epsilon',k'}}$, ($\epsilon'<\epsilon$, $k'>k$) is locally constant, using the connectedness we see that $\{c_{i,\epsilon,k}\}_{\epsilon>0,k\in N}$ defines a function c_i on $\boldsymbol{R}\times U_i$. The independence s of (ϵ,k) is more clear. Further the above consideration also implies that

$$c_1(t,z)=c_2(t,z) \quad \text{if } \operatorname{Im} z>0, \quad \text{and} \quad c_1(t,z)=\exp(-2\pi\sqrt{-1}\Omega)c_2(t,z) \text{ if } \operatorname{Im} z<0,$$

where Ω ($0\leq\Omega<1$) is some constant derived from $\exp(2\pi\sqrt{-1}\Omega):=\frac{c_2(0,-\sqrt{-1})}{c_1(0,-\sqrt{-1})}$.

Here let us take functions $q_i(z)$ on U_i such that $q_i(z):=\exp(\sqrt{-1}\theta_i)$ ($i=1,2$), where θ_i is an argument of z which satisfies, $-\pi<\theta_1<\pi$ and $0<\theta_2<2\pi$. Then, $\frac{c_1(u,z)}{q_1(z)}=\frac{c_2(u,z)}{q_2(z)}$ for all $z\in U_1\cap U_2$, and thus a function

$$c(P):=\frac{c_1(P)}{q_1(P_2(P))} \text{ on } \boldsymbol{R}\times U_1 \text{ and } c(P):=\frac{c_2(P)}{q_2(P_2(P))} \text{ on } \boldsymbol{R}\times U_2,$$

is well defined on the whole set. Now we define a new 1-cocycle ζ by

$$\zeta(P,g):=\theta(P,g)\frac{c(P)}{c(g^{-1}(P))}\left(\frac{d\mu_g}{d\mu}(P)\right)^{-\sqrt{-1}s}.$$

Then we get for any $P\in\overline{I_k}\times\overline{V_{i,\epsilon}}$ and for any $g\in\mathscr{U}_{k,\epsilon}$

$$\zeta(P,g)=\frac{q_i(P_2(g^{-1}(P)))}{q_i(P_2(P))}.$$

Now set $P:=(u,\exp(\sqrt{-1}\theta))$ and $g_t^{-1}(P):=(u_t,\exp(\sqrt{-1}\theta_t))$ for $P\in\overline{I_k}\times\overline{V_{1,\epsilon}}$ and for a continuous path $\{g_t\}_{0\leq t\leq 1}\subset\mathscr{U}_{k,\epsilon}$ connecting id and $g\in\mathscr{U}_{k,\epsilon}$. We choose an angle θ such that $-\pi<\theta<\pi$, from which we get $-\pi<\theta_t<\pi$ for all $0\leq t\leq 1$. So $\varphi(u,\exp(\sqrt{-1}\theta))=\theta_1-\theta_0$, and it follows that

$$\zeta_\Omega(P,g)=\exp(\sqrt{-1}\Omega(\theta_1-\theta_0))=\frac{q_1(P_2(g^{-1}(P)))}{q_1(P_2(P))}=\zeta(P,g).$$

Similar arguments derive the same result for $P\in\overline{I_k}\times\overline{V_{2,\epsilon}}$ and for $g\in\mathscr{U}_{k,\epsilon}$. Consequently for any $P\in K_k$ and for any $g\in\mathscr{U}_{k,\epsilon}$ we have $\zeta_\Omega(P,g)=\zeta(P,g)$. As $\mathscr{U}_{k,\epsilon}$

generates the group $\mathrm{Diff}^*(K_k)$, we get $\zeta_\Omega=\zeta$ on $K_k\times\mathrm{Diff}^*(K_k)$, and thus

$$\theta(P,g)=\frac{c(g^{-1}(P))}{c(P)}\left(\frac{d\mu_g}{d\mu}(P)\right)^{\sqrt{-1}s}\zeta_\Omega(P,g)$$

on $M\times\mathrm{Diff}_0^*(M)$. The rest of the proof is immediate and the uniqueness follows from similar considerations with those for Theorem 3.3. Q.E.D.

§4. Natural Representations of the Group of Diffeomorphisms

4.1. Irreducibility. In this subsection we consider natural representations U_θ of $\mathrm{Diff}_0(M)$ on $\mathrm{L}^2_\mu(M)$ defined by,

$$U_\theta(g): f(P)\mapsto\theta(P,g)\sqrt{\frac{d\mu_g}{d\mu}(P)}f(g^{-1}(P)). \tag{4.1}$$

First we show that they are all irreducible, if M is connected.

Lemma 4.1. *Let P be any fixed point in M and g be in* $\mathrm{Diff}_0^*(M)$ *such that there exists a continuous path $\{g_t\}_{0\le t\le 1}$ connecting* id *and g such that $g_t(P)=P$ for all $0\le t\le 1$. Then for any continuous 1-cocycle θ, there exists $s\in\boldsymbol{R}$ such that*

$$\theta(P,g)=\left(\frac{d\mu_g}{d\mu}(P)\right)^{\sqrt{-1}s}.$$

Proof. This lemma is nothing but Theorem 3.2 in 3.2. Q.E.D.

Theorem 4.1. *If M is connected, then the representations* $(U_\theta\mid\mathrm{Diff}_0^*(M),\mathrm{L}^2_\mu(M))$ *of* $\mathrm{Diff}_0^*(M)$ *are irreducible for all continuous 1-cocycles θ.*

Proof. Let $\mathscr{H}$ $(\neq 0)$ be an invariant subspace of the representation. Take a non zero $f\in\mathscr{H}$ and an open set $U\subset M$ which is diffeomorphic to $\boldsymbol{R}^d$ such that $f_U\neq 0$, where f_U is a function defined by $f_U(x):=f(x)$ for $x\in U$ and $f_U(x):=0$, otherwise. Put $\mathrm{Diff}_0^*(U):=\{g\in\mathrm{Diff}_0(U)\mid {}^\exists\{g_t\}_{0\le t\le 1}\subset\mathrm{Diff}_0(U)$: continuous path connecting id and $g\}$. Then by virtue of Theorem 3.3. we have

$$\theta(P,g)=\frac{c(g^{-1}(P))}{c(P)}\left(\frac{d\mu_g}{d\mu}(P)\right)^{\sqrt{-1}s}$$

for all $P\in U$ and for all $g\in\mathrm{Diff}_0^*(U)$ with a suitable continuous function c on U. Thus, the restricted representation of $\mathrm{Diff}_0^*(U)$ on the space $\mathrm{L}^2_\mu(U)$ of all square summable functions vanishing outside of U is equivalent to the usual representation $(U_s,\mathrm{L}^2_\lambda(\boldsymbol{R}^d))$ of $\mathrm{Diff}_0^*(\boldsymbol{R}^d)$,

$$U_s(g): f(x)\in\mathrm{L}^2_\lambda(\boldsymbol{R}^d)\mapsto\left(\frac{d\lambda_g}{d\lambda}(x)\right)^{\frac{1}{2}+\sqrt{-1}s}f(g^{-1}(x))\in\mathrm{L}^2_\lambda(\boldsymbol{R}^d), \tag{4.2}$$

where λ is the Lebesgue measure on $\boldsymbol{R}^d$, and the later one is irreducible (cf. Lemma 4.2). In particular, there exists some $g_0 \in \mathrm{Diff}_0^*(U)$ such that $U_\theta(g_0)f_U \neq f_U$. Now for any $P \in U^c$, we have by Lemma 4.1, $\theta(P, g)=1$. Hence $h := U_\theta(g_0)f - f$ is a non zero element which belongs to $\mathrm{L}_\mu^2(U) \cap \mathscr{H}$. Again by the irreducibility, we see that $\mathrm{L}_\mu^2(U)$ is generated by $U_\theta(g)h$, $g \in \mathrm{Diff}_0^*(U)$, so we get $\mathrm{L}_\mu^2(U) \subset \mathscr{H}$. By the assumption, $\mathrm{Diff}_0^*(M)$ acts transitively on M. It follows that for any $P \in M$ there exists a neighbourhood U_P of P such that $\mathrm{L}_\mu^2(U_P) \subset \mathscr{H}$. As $\mathrm{L}_\mu^2(M)$ is generated these $\mathrm{L}_\mu^2(U_P)$'s, we get that $\mathscr{H} = \mathrm{L}_\mu^2(M)$. Q.E.D.

4.2. Equivalence. Next we consider the mutual equivalence of $(U_\theta, \mathrm{L}_\mu^2(M))$. For the bigger group $\mathrm{Diff}_0(M)$ the assertion which is more general than the following one is already known as Lemma A.1 in Appendix in [5]. However for the group $\mathrm{Diff}_0^*(M)$ here we list it as the next lemma and prove it for completeness and for our later use.

Lemma 4.2. *Let λ be the Lebesgue measure on $\boldsymbol{R}^d$ and consider for each $s \in \boldsymbol{R}$ a representation U_s of* $\mathrm{Diff}_0^*(\boldsymbol{R}^d)$ *defined by* (4.2). *Then if there exists a non trivial intertwing operator T from $(U_s, \mathrm{L}_\lambda^2(\boldsymbol{R}^d))$ to $(U_{s'}, \mathrm{L}_\lambda^2(\boldsymbol{R}^d))$, we have $s=s'$ and* $T = \alpha \mathrm{Id}$ *with some constant $\alpha \in \boldsymbol{C}$.*

Proof. Let $B \in \mathscr{M}(d)$ and take for each $n \in \boldsymbol{N}$ an $\boldsymbol{R}$-valued C^∞-function $\rho_n(x)$ with compact support such that $\rho_n = 1$ on $\{x \in \boldsymbol{R}^d \mid \|x\| \leq n\}$. Then for a 1-parameter transformation subgroup $\varphi_{t,n}(x) := \mathrm{Exp}(t\tilde{B}_n)(x)$ generated by a vector field $\tilde{B}_n$, $\tilde{B}_n(x) := \rho_n(x)Bx$, $U_s(\varphi_{1,n})$ converges strongly to $U_{s,A}$ on $\mathrm{L}_\lambda^2(\boldsymbol{R}^d)$ as $n \to \infty$, where $A := \exp B$ and $U_{s,A}$ is a unitary operator on $\mathrm{L}_\lambda^2(\boldsymbol{R}^d)$ such that

$$U_{s,A}(f)(x) := |\det A|^{-(\frac{1}{2}+\sqrt{-1}s)} f(A^{-1}x) \quad (f \in \mathrm{L}_\lambda^2(\boldsymbol{R}^d)). \tag{4.3}$$

By the same procedure we can find a sequence $\{\psi_n\}_{n \in \boldsymbol{N}} \subset \mathrm{Diff}_0^*(\boldsymbol{R}^d)$ such that $U(\psi_n)$ converges to T_a for each $a \in \boldsymbol{R}^d$, where

$$T_a(f)(x) := f(x-a) \quad (f \in \mathrm{L}_\lambda^2(\boldsymbol{R}^d)).$$

It follows that

$$T \circ U_{s,A} = U_{s',A} \circ T, \tag{4.4}$$

$$T \circ T_a = T_a \circ T \tag{4.5}$$

for $A \in GL_0(d) := \{A \in GL(d) \mid \det A > 0\}$ and $a \in \boldsymbol{R}^d$.

Here we change T to $S := \mathscr{F} T \mathscr{F}^{-1}$, using the Fourier transform

$$\mathscr{F} : f(x) \mapsto \int_{\boldsymbol{R}^d} \exp(2\pi\sqrt{-1}\langle \xi, x \rangle) f(x) \lambda(dx).$$

Then from (4.5) there exists some $c \in L^{\infty}_{\lambda}(\boldsymbol{R}^d)$ such that

$$S(f)(x) = c(x)f(x).$$

Further (4.4) implies that for all $A \in GL_0(d)$ we have

$$c(Ax) = c(x)(\det A)^{\sqrt{-1}q} \tag{4.6}$$

for λ-a.e.x, where $q := s - s'$. In particular taking A from $SO(d)$, we see that $c(x)$ is rotationally-invariant. Namely there exists a Borel function γ on $[0, \infty)$ such that

$$c(x) := \gamma(\|x\|) \tag{4.7}$$

for λ-a.e.x. Next we take A from similar transformations, $Ax = kx$. Then it follows from (4.6) that for all $k > 0$ we have

$$\gamma(k\|x\|) = \gamma(\|x\|)k^{\sqrt{-1}qd} \tag{4.8}$$

for λ-a.e.x. Hence by virtue of Fubini's theorem there exists some $x_0 \neq 0$ such that

$$\gamma(k\|x_0\|) = \gamma(\|x_0\|)k^{\sqrt{-1}qd} \tag{4.9}$$

for a.e.k and therefore

$$\gamma(k) = \alpha k^{\sqrt{-1}qd} \tag{4.10}$$

for a.e.k with a non zero constant $\alpha := \dfrac{\gamma(\|x_0\|)}{\|x_0\|^{\sqrt{-1}qd}}$. It follows from (4.6) and (4.10) that

$$\|Ax\|^{\sqrt{-1}qd} = \|x\|^{\sqrt{-1}qd}(\det A)^{\sqrt{-1}q}$$

for λ-a.e.x. So we should have $q = 0$ and $c(x)$ becomes a constant α. Q.E.D.

Theorem 4.2. *Let M be a paracompact C^{∞}-manifold and assume that M is connected. Then two representaions $(U_{\theta_1}, L^2_{\mu}(M))$ and $(U_{\theta_2}, L^2_{\mu}(M))$ of* $\mathrm{Diff}^*_0(M)$ *(of* $\mathrm{Diff}_0(M)$ *resp.) is equivalent if and only if θ_1 and θ_2 are 1-cohomologous in* $\mathrm{Diff}^*_0(M)$ *(in* $\mathrm{Diff}_0(M)$ *resp.). That is, there exists a $\boldsymbol{T}^1$-valued continuous function c on M such that $\theta_1(P, g) = \theta_2(P, g)\dfrac{c(g^{-1}(P))}{c(P)}$ for all $P \in M$ and for all $g \in$* $\mathrm{Diff}^*_0(M)$ *($g \in$* $\mathrm{Diff}_0(M)$ *resp.).*

Proof. The sufficiency is obvious. We prove the necessity. Let T be an intertwining unitary operator from $(U_{\theta_1}, L^2_{\mu}(M))$ to $(U_{\theta_2}, L^2_{\mu}(M))$. Take any open neighbourhood U for each $P \in M$ which is diffeomorphic to $\boldsymbol{R}^d$ and form a space $L^2_{\mu}(U)$ of square summable functions zero outside of U. Further take any non zero

$k_0 \in L^2_\mu(U)$ and take some $g_0 \in \mathrm{Diff}^*_0(U)$ such that $U_{\theta_1}(g_0)k_0 - k_0 \neq 0$. Then we have

$$T(U_{\theta_1}(g_0)k_0 - k_0) = U_{\theta_2}(g_0) \circ Tk_0 - Tk_0,$$

which belongs again to $L^2_\mu(U)$ due to Lemma 4.1. So we have shown that there exists a non zero $k \in L^2_\mu(U)$ such that $Tk \in L^2_\mu(U)$. It follows that $T(L^2_\mu(U)) = L^2_\mu(U)$, as $(U_{\theta_1} \mid \mathrm{Diff}^*_0(U), L^2_\mu(U))$ is irreducible. Now on the set $U \times \mathrm{Diff}^*_0(U)$, we have

$$\theta_i(P, g) = \frac{c_i(g^{-1}(P))}{c_i(P)} \left(\frac{d\mu_g}{d\mu}(P) \right)^{\sqrt{-1}s_i}$$

with a $\boldsymbol{T}^1$-valued continuous function c_i and a real constant s_i $(i = 1, 2)$. Therefore by virtue of Lemma 4.2, we have $s_1 = s_2$ and there exists some $\alpha_U \in \boldsymbol{T}^1$ such that

$$(4.11) \qquad (Tf)(P) = \alpha_U \frac{c_1(P)}{c_2(P)} f(P)$$

for all $f \in L^2_\mu(U)$. Put $c_U(P) := \alpha_U \dfrac{c_1(P)}{c_2(P)}$. Clearly, $c_U = c_V$ on $U \cap V$, unless this intersection is empty. Thus $\{c_U\}_U$ defines a continuous $\boldsymbol{T}^1$-valued function c on M such that

$$(4.12) \qquad T(f)(P) = c(P) f(P)$$

for all $f \in L^2_\mu(M)$. Consequently for all $g \in \mathrm{Diff}^*_0(M)$ and for all $f \in L^2_\mu(M)$ we have

$$(4.13) \qquad c(P)\theta_1(P, g)\sqrt{\frac{d\mu_g}{d\mu}}(P) f(g^{-1}(P)) = c(g^{-1}(P))\theta_2(P, g)\sqrt{\frac{d\mu_g}{d\mu}}(P) f(g^{-1}(P))$$

for μ-a.e.P. Therefore the desired result follows directly. The same proof works in the case of $\mathrm{Diff}_0(M)$. Q.E.D.

DEPARTMENT OF MATHEMATICS
FUKUI UNIVERSITY

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