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Limit theorem for symmetric statistics with respect to Weyl transformation: Disappearance of dependency

By

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1. Introduction

It is known that there are several kinds of deterministic sequences $\{x_n\}_{n=1}^{\infty}$ on $T^m=[0,1)^m$ having the following property : For any function $F: T^m \to R$ of finite variation, we have

$$\left|\int_{T^m} F(x)dx - \frac{1}{N}\sum_{n=1}^{N} F(x_n)\right| = O(N^{-1+\varepsilon}), \qquad N\to\infty. \quad (^\forall\varepsilon>0) \tag{1}$$

These sequences are called *low discrepancy sequences* ([2]). The convergence (1) can be used for numerical integrations in T^m, which is called the *quasi Monte Carlo method.* Since the usual Monte Carlo method (=random sampling method) converges at the rate of $O(N^{-1/2})$, this method is more effective for numerical integrations.

However, many authors have reported that the quasi Monte Carlo method does not converge so fast as it is expected, if the dimension is very high. In extreme cases, it is observed to converge at the rate of $O(N^{-1/2})$, namely, exactly as slow as the Monte Carlo method. This phenomenon is often called "*the curse of dimensionality*", and it has been explained by some intuitive arguments (e.g. [15]), but no rigorous discussion has ever been made to explain the observed convergence rate, for example, $O(N^{-1/2})$ in extreme cases. Of course, even the curse of dimensionality cannot contradict with the convergence rate (1), so that it must be an intermediate or transient state, which will eventually disappear and the rate $O(N^{-1+\varepsilon})$ will appear after that.

In this paper, we tried to explain "the curse of dimensionality" in extreme cases by a rigorous probabilistic discussion for the low-discrepancy sequences generated by the Weyl transformation (= irrational rotation). In doing this, we were inspired by the following claim of Sobol' et al. ([11, 12]):

CLAIM (Sobol' et al.). *In high dimensions, the quasi Monte Carlo method is no*

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more effective than the Monte Carlo method, that is, it seems to converge at the rate of exactly $O(N^{-1/2})$, if the integrands depend equally on each coordinate.

In order to extract an essence from the phenomenon caused by the very large dimensions, we investigated the limit behavior when m (=dimension) $\to \infty$. (One may think that $m \to \infty$ is not realistic, but to the contrary, it is getting more and more realistic. For example, in a simulation of quantum field theory, we sometimes have to implement more than 10^4-dimensional numerical integrations.)

The above claim being in our mind, we exclusively investigated the case when the integrands are symmetric in each coordinate. However, "symmetry" alone is not enough. Indeed, we have the following example:

Example 1. Define $F: \boldsymbol{T}^m \to \boldsymbol{R}$ by

$$F(x) := \sin\left(2\pi \sum_{i=1}^{m} x_i\right), \qquad x = (x_1, \cdots, x_m) \in \boldsymbol{T}^m.$$

Then F is a symmetric function. But applying the Weyl transformation with irrational numbers $(\alpha_1, \cdots, \alpha_m)$ to F is nothing but applying the Weyl transformation with an irrational number $\alpha_1 + \cdots + \alpha_m$ to a 1-dimensional function "$\sin 2\pi x$". Hence, it is very effective even if m is very large.

The class of integrands which we finally found appropriate for the purpose is that of *symmetric statistics* ([4]). Let $\sigma_n^m(x; h)$ be the symmetric statistic on $\boldsymbol{T}^m$ with a canonical kernel function $h \in L_2(\boldsymbol{T}^n)$ (see, Definition 6 below for details). Consider the sequence $\{m^{-n/2}\sigma_n^m(x + n\alpha^{(m)}; h)\}_{n=1}^{\infty}$, where $\alpha^{(m)}$ is the first m-coordinates of an irrational vector $\alpha \in \boldsymbol{T}^{\infty}$, as a stationary process on $(\boldsymbol{T}^m, dx^m)$. Then what we obtained in this paper is the following (Main Theorem and its Corollary): *Under a certain condition, the sequence of processes $\{m^{-n/2}\sigma_n^m(x + n\alpha^{(m)}; h)\}_{n=1}^{\infty}$, converges as $m \to \infty$ in law to the sequence of independent copies of the multiple Wiener integrals with the kernel function h.*

This result directly connects the purely deterministic sequences with the fully random sequences, and it may well give a probabilistic explanation to the claim of Sobol' et al. for the Weyl transformation.

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2. Observation of elementary case

In this section, we will observe an elementary case to see the heart of the matter, which shows how naturally the disappearance of dependency takes place.

2.1. Presentation of C.L.T. Let $\boldsymbol{T}^m = [0, 1)^m$ be the m-dimensional torus and let $\boldsymbol{P}^m$ be the Lebesgue probability measure on it. As usual, the addition in $\boldsymbol{T}^m$ is defined in each coordinate with modulo 1. For each $F: \boldsymbol{T}^m \to \boldsymbol{R}$ and each $\alpha \in \boldsymbol{T}^m$, we will regard the sequence of functions $\{F(\cdot + n\alpha)\}_{n=0}^{\infty}$ as a sequence of random variables defined on the probability space $(\boldsymbol{T}^m, \boldsymbol{P}^m)$.

To formulate the problem rigorously, we have to let $m \to \infty$ so that our basic probability space should be $(\boldsymbol{T}^\infty, \boldsymbol{P}^\infty)$, where $\boldsymbol{T}^\infty := [0,1)^\infty$ and $\boldsymbol{P}^\infty$ is the infinite direct product of the 1-dimensional Lebesgue probability measure.

Let $f: \boldsymbol{T}(:= \boldsymbol{T}^1) \to \boldsymbol{R}$ be a continuous function which is not a constant. Define a function $F^m: \boldsymbol{T}^\infty \to \boldsymbol{R}$, $m \in \boldsymbol{N}$, by

$$F^m(x) := \frac{1}{\sqrt{m V_f}} \sum_{i=1}^{m} (f(x_i) - M_f), \qquad x = (x_1, x_2, \cdots) \in \boldsymbol{T}^\infty, \tag{2}$$

where $M_f = \int_{\boldsymbol{T}} f(t)dt$ and $V_f = \int_{\boldsymbol{T}} (f(t) - M_f)^2 dt$. Note that the function F^m is a normalized symmetric statistic of order 1 (see Definition 6 below).

Definition 1. For each probability measure μ on $\boldsymbol{T}$, we put

$$\boldsymbol{T}_\mu^\infty := \left\{ \alpha = (\alpha_i)_{i=1}^\infty \in \boldsymbol{T}^\infty ;\ \frac{1}{m} \sum_{i=1}^{m} \delta_{\alpha_i}(dx) \text{ weakly converges to } \mu(dx) \text{ as } m \to \infty \right\}.$$

In particular, if μ is the Lebesgue measure, we denote it by $\boldsymbol{T}_{dx}^\infty$, that is,

$$\boldsymbol{T}_{dx}^\infty := \left\{ \alpha = (\alpha_i)_{i=1}^\infty \in \boldsymbol{T}^\infty ;\ \frac{1}{m} \sum_{i=1}^{m} \delta_{\alpha_i}(dx) \text{ weakly converges to } dx \text{ as } m \to \infty \right\}. \tag{3}$$

In the sequel, we use the following notation: By "$\overset{\text{f.d.}}{\Rightarrow}$", we mean the convergence of random variables in each finite dimensional distribution.

Then, our first theorem is just a central limit theorem (C.L.T.).

Theorem 1. *For each $\alpha = (\alpha_1, \alpha_2, \cdots) \in \boldsymbol{T}^\infty$, define a sequence of random variables $\{X_n^m(\cdot\,;\alpha)\}_{n=0}^\infty$ on the probability space $(\boldsymbol{T}^\infty, \boldsymbol{P}^\infty)$ by*

$$X_n^m(x;\alpha) := F^m(x + n\alpha), \qquad x = (x_1, x_2, \cdots) \in \boldsymbol{T}^\infty. \tag{4}$$

Then, if $\alpha \in \boldsymbol{T}_\mu^\infty$ where μ has a density with respect to the Lebesgue measure, the sequence of random variables $\{X_n^m(\cdot\,;\alpha)\}_{n=0}^\infty$ converges to a strongly mixing stationary Gaussian sequence. In particular, if $\alpha \in \boldsymbol{T}_{dx}^\infty$, we have

$$\{X_n^m(\cdot\,;\alpha)\}_{n=0}^\infty \underset{m\to\infty}{\overset{\text{f.d.}}{\Rightarrow}} N(0,1)\text{-}i.i.d.\ random\ sequence.$$

Here $N(0,1)$ is the Gaussian distribution with mean 0 and variance 1.

Before the proof, we will give some comments to the theorem.

We first note that we are particularly interested in the case that the function f in the theorem is smooth, such as $f(t)=\sin 2\pi t$. It is because we want to emphasize that even if the integrand is smooth, the generated sequence becomes very random if the dimension is so high. (It is known that if a 1-dimensional integrand $f:[0,1)\to \boldsymbol{R}$ is very irregular, the quasi Monte Carlo method generates very random sequences. This subject was discussed in several papers, such as [3, 5, 13].)

Let $\tilde{\boldsymbol{T}}_{\mu}^{\infty}$ be the set of all $\alpha=(\alpha_1,\alpha_2,\cdots)\in \boldsymbol{T}_{\mu}^{\infty}$ such that $\{1,\alpha_1,\cdots,\alpha_m\}$ are linearly independent over $\boldsymbol{Q}$ for each m, and that each α_i is algebraic over $\boldsymbol{Q}$. If $\alpha=(\alpha_1,\alpha_2,\cdots)\in \tilde{\boldsymbol{T}}_{\mu}^{\infty}$, the Weyl transformation with $(\alpha_1,\cdots,\alpha_m)$ generates low discrepancy sequences in all $\boldsymbol{T}^m$ ([10]). A typical example of elements of $\tilde{\boldsymbol{T}}_{dx}^{\infty}$ is the following:

$$\alpha=(\alpha_i)_{i=1}^{\infty}, \quad \text{with} \quad \alpha_i=\sqrt{p_i} \pmod 1, \tag{5}$$

where p_i is the i-th prime number (see [7, 8]). We guess that many people are using this typical α in the quasi Monte Carlo method by means of the Weyl transformation for high dimensional numerical integrations, because it was suggested by an influential paper [8].

Note that the Weyl transformation with irrationals $\alpha_1,\cdots,\alpha_m$ is *uniquely ergodic* ([14]), if $\{1,\alpha_1,\cdots,\alpha_m\}$ are linearly independent over $\boldsymbol{Q}$. Therefore if $\alpha\in\tilde{\boldsymbol{T}}_{\mu}^{\infty}$ where μ has a density, this implies together with Theorem 1 that the asymptotic relative frequency distribution of the deterministic samples $\{X_n^m(x;\alpha)\}_{n=0}^{\infty}$ in each dimension is very close to that of the limit strongly mixing stationary Gaussian sequence. Hence, it must be hard to distinguish the deterministic samples $\{X_n^m(x;\alpha)\}_{n=0}^{\infty}$ from the samples of the limit sequence by statistical tests.

2.2. Proof of C.L.T. For any $L\in\boldsymbol{N}$ and any $\beta_0,\cdots,\beta_{L-1}\in\boldsymbol{R}$, we consider the linear combination

$$\sum_{n=0}^{L-1}\beta_n X_n^m(\cdot\,;\alpha). \tag{6}$$

Note that for each $\alpha=(\alpha_1,\alpha_2,\cdots)\in\boldsymbol{T}^{\infty}$, (6) is a sum of independent random variables as follows:

$$\sum_{i=1}^{m} Z_{mi}(\cdot\,;\alpha), \tag{7}$$

where $\{Z_{mi}(\cdot\,;\alpha)\}_{i=1}^{\infty}$ are defined by

$$Z_{mi}(x;\alpha):=\frac{1}{\sqrt{mV_f}}\sum_{n=0}^{L-1}\beta_n(f(x_i+n\alpha_i)-M_f), \qquad x=(x_1,x_2,\cdots)\in\boldsymbol{T}^{\infty}. \tag{8}$$

By the definition, their expectations $\boldsymbol{E}^{\infty}[Z_{mi}(\cdot\,;\alpha)]=0$ and

$$|Z_{mi}(x;\alpha)|\leq\left(\sum_{n=0}^{L-1}|\beta_n|\right)\frac{2\|f\|_\infty}{\sqrt{mV_f}},\qquad 1\leq{}^\forall i\leq m,\ {}^\forall x\in \boldsymbol{T}^\infty,$$

from which a triangular array $\{Z_{mi}(\cdot;\alpha)\}_{1\leq i\leq m}$ satisfies the Lindeberg condition. To apply the Lindeberg-Feller theorem (see Theorem 27.2 of [1]), let us check the convergence of $\sum_{i=1}^m \boldsymbol{E}^\infty[Z_{mi}(\cdot;\alpha)^2]$ as $m\to\infty$. We first compute $\sum_{i=1}^m \boldsymbol{E}^\infty[Z_{mi}(\cdot;\alpha)^2]$ as

$$\begin{aligned}
&\sum_{i=1}^m \boldsymbol{E}^\infty[Z_{mi}(\cdot;\alpha)^2]\\
&=\sum_{i=1}^m\frac{1}{mV_f}\sum_{n,n'=0}^{L-1}\beta_n\beta_{n'}\boldsymbol{E}^\infty[(f(x_i+n\alpha_i)-M_f)(f(x_i+n'\alpha_i)-M_f)]\\
&=\sum_{i=1}^m\frac{1}{mV_f}\sum_{n,n'=0}^{L-1}\beta_n\beta_{n'}\int_{\boldsymbol{T}}(f(t+n\alpha_i)-M_f)(f(t+n'\alpha_i)-M_f)dt\\
&=\sum_{n,n'=0}^{L-1}\beta_n\beta_{n'}\frac{1}{V_f}\int_{\boldsymbol{T}}(f(t)-M_f)\frac{1}{m}\sum_{i=1}^m(f(t+(n'-n)\alpha_i)-M_f)dt,
\end{aligned}$$

where in the last line we have used the translation invariance of the Lebesgue measure. Since $\alpha=(\alpha_i)_{i=1}^\infty\in\boldsymbol{T}_\mu^\infty$,

$$\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^m(f(t+(n'-n)\alpha_i)-M_f)=\int_{\boldsymbol{T}}(f(t+(n'-n)s)-M_f)\mu(ds).$$

Substituting this into the expression above we have

$$\lim_{m\to\infty}\sum_{i=1}^m\boldsymbol{E}^\infty[Z_{mi}(\cdot;\alpha)^2]=\sum_{n,n'=0}^{L-1}\beta_n\beta_{n'}R^\infty(n'-n),$$

where

$$R^\infty(k)=\frac{1}{V_f}\int_{\boldsymbol{T}}(f(t)-M_f)dt\int_{\boldsymbol{T}}(f(t+ks)-M_f)\mu(ds),\qquad k\in\boldsymbol{Z}. \tag{9}$$

Consequently, by the Lindeberg-Feller theorem, the linear combination (6) converges in law to a Gaussian random variable with mean 0 and variance $\sum_{n,n'=0}^{L-1}\beta_n\beta_{n'}R^\infty(n'-n)$, and hence, the sequence $\{X_n^m(\cdot;\alpha)\}_{n=0}^\infty$ converges in law to a stationary Gaussian sequence with covariance function $R^\infty(\cdot)$.

If $\alpha\in\boldsymbol{T}_{dx}^\infty$, namely, μ is the Lebesgue measure, we see $R^\infty(k)\equiv 0$ for $k\neq 0$, which shows the limit is a Gaussian i.i.d. random sequence. For a general μ which has a density, it is easy to see by the Riemann-Lebesgue lemma that

$$\int_{\boldsymbol{T}}(f(t+ks)-M_f)\mu(ds)\to 0,\qquad \text{as}\quad |k|\to\infty,$$

which shows $R^\infty(k)\to 0$ as $|k|\to\infty$, that is, the limit sequence is strongly mixing. Now the proof is done.

Remark 1. Since we wish no more complicated calculations, we will not state the assertions in the forthcoming sections for general probability measure μ, but only for the Lebesgue measure. Namely, we will deal with only the cases where the dependency disappears.

3. General case

We will investigate the cases when the integrands are symmetric statistics of general orders. At this time, *multiple Wiener integrals* appear as their limits.

3.1. Preliminaries. We will here introduce necessary notions by following Dynkin-Mandelbaum [4].

Definition 2. For each $n \in N$, we define

$$\mathscr{L}_2^n := L_2(\boldsymbol{T}^n, \boldsymbol{P}^n) = L_2([0,1)^n; dx_1 \cdots dx_n).$$

If $n=1$, we will write $\mathscr{L}_2^1$ simply by $\mathscr{L}_2$.

Definition 3. (i) We define the *symmetrizer* $\mathscr{S}: \mathscr{L}_2^n \to \mathscr{L}_2^n$ by

$$(\mathscr{S}h)(x_1, \cdots, x_n) := \frac{1}{n!} \sum_{\sigma \in S_n} h(x_{\sigma(1)}, \cdots, x_{\sigma(n)}), \qquad h \in \mathscr{L}_2^n$$

where S_n is the symmetric group over the set $\{1, 2, \cdots, n\}$.

(ii) $\mathscr{S}\mathscr{L}_2^n := \{h \in \mathscr{L}_2^n; \mathscr{S}h = h\}$. If $n=1$, we have $\mathscr{S}\mathscr{L}_2^1 = \mathscr{L}_2$.

(iii) $$\mathscr{C}\mathscr{S}\mathscr{L}_2^n := \left\{h \in \mathscr{S}\mathscr{L}_2^n; \int_T h(x_1, \cdots, x_{n-1}, y)dy = 0, \text{ a.a.}(x_1, \cdots, x_{n-1})\right\}.$$

An element of $\mathscr{C}\mathscr{S}\mathscr{L}_2^n$ is called *canonical*. If $n=1$, we write

$$\mathscr{C}\mathscr{S}\mathscr{L}_2^1 = \mathscr{C}\mathscr{L}_2 = \left\{h \in \mathscr{L}_2; \int_T h(y)dy = 0\right\}.$$

Definition 4. We define

$$H := \left\{\{h_n\}_{n=1}^\infty; h_n \in \mathscr{C}\mathscr{S}\mathscr{L}_2^n; \sum_{n=1}^\infty \frac{1}{n!} \|h_n\|_{\mathscr{L}_2^n}^2 < \infty\right\}.$$

Then H is a Hilbert space with an inner product

$$(h^{(1)}, h^{(2)})_H := \sum_{n=1}^\infty \frac{1}{n!} (h_n^{(1)}, h_n^{(2)})_{\mathscr{L}_2^n}, \quad h^{(1)} = \{h_n^{(1)}\}_{n=1}^\infty, \ h^{(2)} = \{h_n^{(2)}\}_{n=1}^\infty \in H$$

Definition 5. (i) For $\psi_1, \cdots, \psi_n \in \mathscr{L}_2$ we put

$$\psi_1 \otimes \cdots \otimes \psi_n(x_1, \cdots, x_n) := \psi_1(x_1) \times \cdots \times \psi_n(x_n) \in \mathscr{L}_2^n.$$

In particular, we put $\psi^{\otimes n} := \underbrace{\psi \otimes \cdots \otimes \psi}_{n} \in \mathscr{S}\mathscr{L}_2^n$.

(ii) For $\phi \in \mathscr{C}\mathscr{L}_2$, we put $h^\phi := \{\phi^{\otimes n}\}_{n=1}^\infty \in H$.

If $\{\phi_k\}_{k=0}^\infty$ is a complete orthonormal system (abbreviated as CONS) in $\mathscr{L}_2$, then $\{\phi_{i_1} \otimes \cdots \otimes \phi_{i_n}\}_{i_1, \cdots, i_n = 0}^\infty$ is a CONS in $\mathscr{L}_2^n$.

Definition 6. (i) For each $h \in \mathscr{S}\mathscr{L}_2^n$, we define the *symmetric statistic* $\sigma_n^m(x; h)$, $x \in \boldsymbol{T}^\infty$, by

$$\sigma_n^m(x; h) := \begin{cases} \sum_{1 \le i_1 < \cdots < i_n \le m} h(x_{i_1}, \cdots, x_{i_n}), & n \le m \\ 0, & n > m. \end{cases}$$

(ii) For $h = \{h_n\}_{n=1}^\infty \in H$, we define

$$Y_m(x; h) := \sum_{n=1}^\infty m^{-n/2} \sigma_n^m(x; h_n).$$

The function $Y_m(\cdot; h)$ is an infinite-dimensional analogue of the symmetric statistics. The coefficients $m^{-n/2}$ are normalizing factors. It is easy to see that $Y_m(\cdot; h) \in L_2(\boldsymbol{T}^\infty, \boldsymbol{P}^\infty)$. Our main theorem below will be stated for each σ_n^m as well as Y_m. And then, independent multiple Wiener integrals will appear as the limits.

Definition 7. Let $\{(B^{(p)}(t))_{0 \le t \le 1}\}_{p \in \boldsymbol{N}}$, defined on a probability space $(\Omega, \mathscr{F}, P)$, be a sequence of *independent* 1-dimensional Brownian motions starting at the origin.

(i) For $\phi \in \mathscr{L}_2$, we define the *Wiener integral* by

$$I_1^{(p)}(\phi) := \int_0^1 \phi(s) dB^{(p)}(s).$$

(ii) For $h \in \mathscr{S}\mathscr{L}_2^n$, we define the *multiple Wiener integral* by

$$I_n^{(p)}(h) := \int_0^1 \cdots \int_0^1 h(s_1, \cdots, s_n) dB^{(p)}(s_1) \cdots dB^{(p)}(s_n).$$

Proposition 1 ([6]). (i) *For each* $\phi \in \mathscr{L}_2$, *we have*

$$I_1^{(p)}(\phi^{\otimes n}) = n! \int_0^1 \phi(s_1) dB^{(p)}(s_1) \int_0^{s_1} \phi(s_2) dB^{(p)}(s_2) \cdots$$
$$\cdots \int_0^{s_{n-2}} \phi(s_{n-1}) dB^{(p)}(s_{n-1}) \int_0^{s_{n-1}} \phi(s_n) dB^{(p)}(s_n)$$

$$=n!\|\phi\|^n_{\mathscr{L}_2}H_n\left(I_1^{(p)}\left(\frac{\phi}{\|\phi\|_{\mathscr{L}_2}}\right)\right),$$

where $\{H_n\}_{n=0}^{\infty}$ *are Hermite polynomials* : $H_n(\xi)=\frac{(-1)^n}{n!}e^{\xi^2/2}\frac{d^n}{d\xi^n}(e^{-\xi^2/2})$.

(ii) *For each* $h\in\mathscr{S}\mathscr{L}_2^n$ *and each* $k\in\mathscr{S}\mathscr{L}_2^m$, *we have*

$$E[I_n^{(p)}(h)]=0,$$

$$E[I_n^{(p)}(h)I_m^{(p)}(k)]=\begin{cases}0 & \text{if } n\neq m\\ n!(h,k)_{\mathscr{L}_2^n} & \text{if } n=m.\end{cases}$$

3.2. Presentation of Main Theorem. Now, it is possible to mention the main theorem of this paper.

Main Theorem. *For any* $h=\{h_n\}_{n=1}^{\infty}\in H$, *we consider* $\{Y_m(x+p\alpha;h)\}_{p\in\boldsymbol{N}}$, $\alpha\in\boldsymbol{T}^{\infty}$, *to be random variables defined on the probability space* $(\boldsymbol{T}^{\infty},\boldsymbol{P}^{\infty})$. *Then we have*

$$\{Y_m(x+p\alpha;h)\}_{p\in\boldsymbol{N}}\underset{m\to\infty}{\overset{\text{f.d.}}{\Rightarrow}}\left\{\sum_{n=1}^{\infty}\frac{1}{n!}I_n^{(p)}(h_n)\right\}_{p\in\boldsymbol{N}},$$

if and only if $\alpha\in\boldsymbol{T}_{dx}^{\infty}$.

Corollary. *For any sequence* $h=\{h_n\}_{n=1}^{\infty}$ *such that each* h_n *is in* $\mathscr{C}\mathscr{S}\mathscr{L}_2^n$, *the distribution of an array of random variables* $\{m^{-n/2}\sigma_n^m(x+p\alpha;h_n)\}_{p,n\in\boldsymbol{N}}$ *on the probability space* $(\boldsymbol{T}^{\infty},\boldsymbol{P}^{\infty})$ *converges to that of an array of i.i.d. random variables* $\{\frac{1}{n!}I_n^{(p)}(h_n)\}_{p,n\in\boldsymbol{N}}$, *if and only if* $\alpha\in\boldsymbol{T}_{dx}^{\infty}$.

Obviously, Theorem 1 (C.L.T.) in case $\alpha\in\boldsymbol{T}_{dx}^{\infty}$ is an easy consequence of this corollary.

3.3. Proof of Main Theorem. We begin with two lemmas.

Lemma 1. c.l.s.$\{h^{\phi};\phi\in\mathscr{C}\mathscr{L}_2\}=H$. *Here the term* "c.l.s." *stands for* closed *linear span.*

Proof. Let $h\in(\text{c.l.s.}\{h^{\phi};\phi\in\mathscr{C}\mathscr{L}_2\})^{\perp}$. Then for any $\phi\in\mathscr{C}\mathscr{L}_2$ and any $t\in\boldsymbol{R}$, we have

$$(h,h^{t\phi})_H=\sum_{n=1}^{\infty}\frac{t^n}{n!}(h_n,\phi^{\otimes n})_{\mathscr{L}_2^n}=0.$$

Thus we see $(h_n,\phi^{\otimes n})_{\mathscr{L}_2^n}=0$ for any $\phi\in\mathscr{C}\mathscr{L}_2$ and any $n\geq 1$.

Now, let $\psi_1, \cdots, \psi_n \in \mathscr{C}\mathscr{L}_2$, $(t_1, \cdots, t_n) \in \boldsymbol{R}^n$. By the above fact, we see

$$\left(h_n, \left(\sum_{i=1}^{n} t_i \psi_i\right)^{\otimes n}\right)_{\mathscr{L}_2^n} = 0.$$

But we can expand $\left(\sum_{i=1}^{n} t_i \psi_i\right)^{\otimes n}$ as

$$\left(\sum_{i=1}^{n} t_i \psi_i\right)^{\otimes n} = \sum_{i_1, \cdots, i_n = 1}^{n} t_{i_1} \cdots t_{i_n} \psi_{i_1} \otimes \cdots \otimes \psi_{i_n}$$

$$= \sum_{p=1}^{n} \sum_{1 \le i_1 < \cdots < i_p \le n} \sum_{\substack{a_1, \cdots, a_p \ge 1; \\ a_1 + \cdots + a_p = n}} t_{i_1}^{a_1} \cdots t_{i_p}^{a_p} \frac{n!}{a_1! \cdots a_p!}$$

$$\times \mathscr{S}(\underbrace{\psi_{i_1} \otimes \cdots \otimes \psi_{i_1}}_{a_1} \otimes \cdots \otimes \underbrace{\psi_{i_p} \otimes \cdots \otimes \psi_{i_p}}_{a_p}),$$

so that we see

$$\sum_{p=1}^{n} \sum_{1 \le i_1 < \cdots < i_p \le n} \sum_{\substack{a_1, \cdots, a_p \ge 1; \\ a_1 + \cdots + a_p = n}} t_{t_1}^{a_1} \cdots t_{i_p}^{a_p} \frac{n!}{a_1! \cdots a_p!}$$

$$\times (h_n, \underbrace{\psi_{i_1} \otimes \cdots \otimes \psi_{i_1}}_{a_1} \otimes \cdots \otimes \underbrace{\psi_{i_p} \otimes \cdots \otimes \psi_{i_p}}_{a_p})_{\mathscr{L}_2^n} = 0, \qquad {}^{\forall}(t_1, \cdots, t_n) \in \boldsymbol{R}^n.$$

Consequently, for $1 \le p \le n$, $1 \le i_1 < \cdots < i_p \le n$, $a_1, \cdots, a_p \ge 1$; $a_1 + \cdots + a_p = n$, we have

$$(h_n, \underbrace{\psi_{i_1} \otimes \cdots \otimes \psi_{i_1}}_{a_1} \otimes \cdots \otimes \underbrace{\psi_{i_p} \otimes \cdots \otimes \psi_{i_p}}_{a_p})_{\mathscr{L}_2^n} = 0.$$

In particular, when $p = n$, we have $i_1 = 1$, $i_2 = 2, \cdots, i_n = n$, $a_1 = \cdots = a_n = 1$, so that

$$(h_n, \psi_1 \otimes \cdots \otimes \psi_n)_{\mathscr{L}_2^n} = 0, \qquad {}^{\forall}\psi_1, \cdots, \psi_n \in \mathscr{C}\mathscr{L}_2, \qquad {}^{\forall}n \ge 1.$$

This means that $h = \{h_n\}_{n=1}^{\infty} = 0$.

Lemma 2. *For any* $h = \{h_n\}_{n=1}^{\infty} \in H$, $p, m \in \boldsymbol{N}$, $\alpha \in \boldsymbol{T}^{\infty}$, *we have*

$$\| Y_m(\cdot + p\alpha; h) \|^2_{L_2(\boldsymbol{T}^{\infty}; \boldsymbol{P}^{\infty})} = \sum_{n=1}^{\infty} \left(1 - \frac{1}{m}\right) \cdots \left(1 - \frac{n-1}{m}\right) \frac{1}{n!} \|h_n\|^2_{\mathscr{L}_2^n} \le \|h\|^2_H.$$

Proof. The assertion follows from the identity:

$$\boldsymbol{E}^{\infty}[\sigma_n^m(\cdot + p\alpha; h) \sigma_{n'}^{m'}(\cdot + p\alpha; h')] = \boldsymbol{E}^{\infty}[\sigma_n^m(\cdot; h) \sigma_{n'}^{m'}(\cdot; h')]$$

$$=\delta_{nn'}\binom{m\wedge m'}{n}(h,h')_{\mathscr{L}_2^n}, \quad {}^{\forall}h\in\mathscr{C}\mathscr{S}\mathscr{L}_2^n,\ {}^{\forall}h'\in\mathscr{C}\mathscr{S}\mathscr{L}_2^{n'}. \tag{10}$$

Definition 8. In the sequel, we take the system of trigonometric functions $\{\phi_k\}_{k=0}^{\infty}$ as our standard CONS of $\mathscr{L}_2$. That is,

$$\phi_0\equiv 1, \quad \{\phi_k\}_{k=1}^{\infty}=\{\sqrt{2}\cos 2\pi kx,\ \sqrt{2}\sin 2\pi lx\}_{k,l\in\boldsymbol{N}}.$$

Recall that for $x=(x_i)\in\boldsymbol{T}^{\infty}$, $m,k\in\boldsymbol{N}$,

$$\frac{1}{\sqrt{m}}\sigma_1^m(x;\phi_k)=\frac{1}{\sqrt{m}}\sum_{i=1}^{m}\phi_k(x_i).$$

Theorem 2. *If* $\alpha\in\boldsymbol{T}_{dx}^{\infty}$, *we have*

$$\left\{\frac{1}{\sqrt{m}}\sigma_1^m(x+p\alpha;\psi)\right\}_{p\in\boldsymbol{N},\psi\in\mathscr{C}\mathscr{L}_2}\ \underset{m\to\infty}{\overset{\text{f.d.}}{\Rightarrow}}\ \{I_1^{(p)}(\psi)\}_{p\in\boldsymbol{N},\psi\in\mathscr{C}\mathscr{L}_2}.$$

Proof. For each $\psi\in\mathscr{C}\mathscr{L}_2$, it follows from (10) and Proposition 1 that

$$\left\|\frac{1}{\sqrt{m}}\sigma_1^m(\cdot+p\alpha;\psi)-\sum_{l=1}^{L}(\psi,\phi_l)\frac{1}{\sqrt{m}}\sigma_1^m(\cdot+p\alpha;\phi_l)\right\|^2_{L_2(\boldsymbol{T}^{\infty};\boldsymbol{P}^{\infty})}$$

$$=\left\|I_1^{(p)}(\psi)-\sum_{l=1}^{L}(\psi,\phi_l)I_1^{(p)}(\phi_l)\right\|^2_{L_2(\Omega;P)}=\sum_{l=L+1}^{\infty}|(\psi,\phi_l)|^2\underset{L\to\infty}{\to}0.$$

Hence Theorem 2 is reduced to the following: For each $n\in\boldsymbol{N}$,

$$\left\{\frac{1}{\sqrt{m}}\sigma_1^m(x+p\alpha;\phi_k)\right\}_{1\le p,k\le n}\ \underset{m\to\infty}{\Rightarrow}\ \{I_1^{(p)}(\phi_k)\}_{1\le p,k\le n}\ \text{in law.}$$

And according to Cramér-Wold's theorem (see Theorem 29.4 of [1]), this is equivalent to the statement that for each $\{a_{pk}\}_{1\le p,k\le n}$ such that $\sum_{1\le p,k\le n}a_{pk}^2=1$,

$$\sum_{1\le p,k\le n}a_{pk}\frac{1}{\sqrt{m}}\sigma_1^m(x+p\alpha;\phi_k)\ \underset{m\to\infty}{\Rightarrow}\ N(0,1)\quad\text{in law.} \tag{11}$$

We will therefore show (11).

Put

$$X_i^{(\alpha)}(x):=\sum_{1\le p,k\le n}a_{pk}\phi_k(x_i+p\alpha_i), \qquad x=(x_i)\in\boldsymbol{T}^{\infty},\ i\in\boldsymbol{N}.$$

Then we see

$$\sum_{1\le p,k\le n} a_{pk}\frac{1}{\sqrt{m}}\sigma_1^m(x+p\alpha;\phi_k)=\sum_{1\le p,k\le n} a_{pk}\frac{1}{\sqrt{m}}\sum_{i=1}^m \phi_k(x_i+p\alpha_i)=\sum_{i=1}^m \frac{X_i^{(\alpha)}(x)}{\sqrt{m}}.$$

It is easy to see the following by the definition of $X_i^{(\alpha)}$:

$$\{X_i^{(\alpha)}\}_{i=1}^{\infty} \text{ are inependent } {}^{\forall}\alpha\in \boldsymbol{T}^{\infty}, \tag{12}$$

$$\boldsymbol{E}^{\infty}[X_i^{(\alpha)}]=0, \quad {}^{\forall}i\in \boldsymbol{N},\ {}^{\forall}\alpha\in \boldsymbol{T}^{\infty}, \tag{13}$$

$$|X_i^{(\alpha)}(x)|\le\sqrt{2}n, \quad {}^{\forall}i\in \boldsymbol{N},\ {}^{\forall}x,\alpha\in \boldsymbol{T}^{\infty}. \tag{14}$$

By (14) a triangular array $\left\{X_{mi}^{(\alpha)}:=\frac{1}{\sqrt{m}}X_i^{(\alpha)}\right\}_{1\le i\le m}$ clearly satisfies the Lindeberg condition, and so if we have

$$\lim_{m\to\infty}\sum_{i=1}^m \boldsymbol{E}^{\infty}[(X_{mi}^{(\alpha)})^2]=1, \tag{15}$$

then (11) follows from the Lindeberg-Feller theorem.

(15) is shown in the same way as in the proof of Theorem 1. Indeed we expand the sum $\sum_{i=1}^m (X_{mi}^{(\alpha)})^2$ as

$$\begin{aligned}
&\sum_{i=1}^m (X_{mi}^{(\alpha)})^2\\
&=\sum_{(p,k)} a_{pk}^2\frac{1}{m}\sum_{i=1}^m \phi_k(x_i+p\alpha_i)^2+\sum_{(p,k)\ne(q,l)} a_{pk}a_{ql}\frac{1}{m}\sum_{i=1}^m \phi_k(x_i+p\alpha_i)\phi_l(x_i+q\alpha_i)\\
&=\sum_{(p,k)} a_{pk}^2\frac{1}{m}\sum_{i=1}^m \phi_k(x_i+p\alpha_i)^2+\sum_p\sum_{k\ne l} a_{pk}a_{pl}\frac{1}{m}\sum_{i=1}^m \phi_k(x_i+p\alpha_i)\phi_l(x_i+p\alpha_i)\\
&\quad+\sum_{p\ne q}\sum_{k,l} a_{pk}a_{ql}\frac{1}{m}\sum_{i=1}^m \phi_k(x_i+p\alpha_i)\phi_l(x_i+q\alpha_i),
\end{aligned}$$

and then taking expectation, we see

$$\begin{aligned}
\boldsymbol{E}^{\infty}\left[\sum_{i=1}^m (X_{mi}^{(\alpha)})^2\right]&=\sum_{(p,k)} a_{pk}^2+\sum_{p\ne q}\sum_{k,l} a_{pk}a_{ql}\frac{1}{m}\sum_{i=1}^m\int_{\boldsymbol{T}}\phi_k(t)\phi_l(t+(q-p)\alpha_i)dt\\
&=1+\sum_{p\ne q}\sum_{k,l} a_{pk}a_{ql}\int_{\boldsymbol{T}}\phi_k(t)\frac{1}{m}\sum_{i=1}^m \phi_l(t+(q-p)\alpha_i)dt.
\end{aligned} \tag{16}$$

Since, by $\alpha\in \boldsymbol{T}_{dx}^{\infty}$,

$$\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^m \phi_l(t+(q-p)\alpha_i)=\int_{\boldsymbol{T}}\phi_l(t+(q-p)x)dx=\int_{\boldsymbol{T}}\phi_l(x)dx=0,$$

we therefore have (15). The proof is complete.

The converse of Theorem 2 also holds.

Theorem 3. *Let* $\alpha=(\alpha_i)_{i=1}^{\infty}\in \boldsymbol{T}^{\infty}$. *If we have*

$$\left\{\frac{1}{\sqrt{m}}\sigma_1^m(x+p\alpha\,;\phi)\right\}_{p\in\boldsymbol{N},\phi\in\mathscr{C}\mathscr{L}_2}\underset{m\to\infty}{\overset{\text{f.d.}}{\Rightarrow}}\{\Gamma_1^{(p)}(\phi)\}_{p\in\boldsymbol{N},\phi\in\mathscr{C}\mathscr{L}_2},$$

then $\alpha\in\boldsymbol{T}_{dx}^{\infty}$.

Proof. For $k,l\in\boldsymbol{N}$, put

$$X_i(x):=\frac{1}{\sqrt{2}}\phi_k(x_i+\alpha_i)+\frac{1}{\sqrt{2}}\phi_l(x_i+2\alpha_i),\qquad i\geq 1,\ x\in\boldsymbol{T}^{\infty}.$$

Then obviously,

$$\begin{cases}\{X_i\}_{i=1}^{\infty}\ \text{are independent},\\ \boldsymbol{E}^{\infty}[X_i]=0,\\ |X_i(x)|\leq 2.\end{cases}\tag{17}$$

As in the proof of Theorem 2

$$\boldsymbol{E}^{\infty}\left[\sum_{i=1}^{m}\left(\frac{1}{\sqrt{m}}X_i\right)^2\right]=1+\int_{\boldsymbol{T}}\phi_k(t)\frac{1}{m}\sum_{i=1}^{m}\phi_l(t+\alpha_i)dt.$$

Let $\{m'\}$ be an arbitrary subsequence of $\{1,2,\cdots\}$. Since $\left\{\frac{1}{m'}\sum_{i=1}^{m'}\delta_{\alpha_i}(dx)\right\}_{m'}$ is tight, we can take a subsequence $\{m''\}$ of $\{m'\}$ and a probability measure $\mu(dx)$ on $\boldsymbol{T}$ such that

$$\frac{1}{m''}\sum_{i=1}^{m''}\delta_{\alpha_i}(dx)\Rightarrow\mu(dx)\qquad\text{as } m''\to\infty.$$

This implies

$$\boldsymbol{E}^{\infty}\left[\sum_{i=1}^{m''}\left(\frac{1}{\sqrt{m''}}X_i\right)^2\right]=1+\int_{\boldsymbol{T}}\mu(dx)\int_{\boldsymbol{T}}\phi_k(t)\phi_l(t+x)dt\qquad\text{as } m''\to\infty.\tag{18}$$

The Lindeberg-Feller theorem together with (17) and (18) implies

$$\frac{1}{\sqrt{m''}}\sum_{i=1}^{m''}X_i\Rightarrow N\left(0,1+\int_{\boldsymbol{T}}\mu(dx)\int_{\boldsymbol{T}}\phi_k(t)\phi_l(t+x)dt\right)\qquad\text{as } m''\to\infty.$$

On the other hand, by the definition of X_i and the assumption,

$$\frac{1}{\sqrt{m}}\sum_{i=1}^{m} X_i=\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{m}}\sigma_1^m(x+\alpha;\phi_k)+\frac{1}{\sqrt{m}}\sigma_1^m(x+2\alpha;\phi_l)\right)$$

$$\Rightarrow\frac{1}{\sqrt{2}}(I_1^{(1)}(\phi_k)+I_1^{(2)}(\phi_l))\sim N(0,1)\qquad\text{as } m\to\infty.$$

Combining two convergences above we have

$$\int_T \mu(dx)\int_T \phi_k(t)\phi_l(t+x)dt=0\qquad\text{for } ^\forall k,l\geq 1.$$

Recalling the definition of ϕ_k, $k\geq 1$ (see Definition 8 above), we see that for $^\forall k\geq 1$

$$\int_T \mu(dx)\int_T \cos 2\pi kt\cos 2\pi k(t+x)dt=0,$$

$$\int_T \mu(dx)\int_T \cos 2\pi kt\sin 2\pi k(t+x)dt=0.$$

This implies

$$\hat{\mu}(k)=\int_T e^{2\sqrt{-1}\pi kx}\mu(dx)=0,\qquad ^\forall k\neq 0,$$

and hence $\mu(dx)=dx$, so that

$$\frac{1}{m''}\sum_{i=1}^{m''}\delta_{\alpha_i}(dx)\Rightarrow dx\qquad\text{as } m''\to\infty.$$

Since this holds for a subsequence $\{m''\}$ of any subsequence $\{m'\}$ of $\{1,2,\cdots\}$, we must have that $\alpha\in T_{dx}^\infty$. The proof is complete.

Lemma 3. *Let $\phi\in\mathscr{C}\mathscr{L}_2$ and $K\in N$. Put $\phi^{(K)}:=\sum_{k=1}^{K}(\phi,\phi_k)\phi_k$. Then we have*

$$\|h^\phi-h^{\phi^{(K)}}\|_H^2\leq(1+\|\phi\|_{\mathscr{L}_2}^2)e^{\|\phi\|_{\mathscr{L}_2}^2}\|\phi-\phi^{(K)}\|_{\mathscr{L}_2}^2.$$

Proof. Noting an equality

$$a_1\cdots a_n-b_1\cdots b_n=\sum_{k=1}^{n} b_1\cdots b_{k-1}(a_k-b_k)a_{k+1}\cdots a_n,\qquad a_k,b_k\in \boldsymbol{R},$$

we see, for any $\phi,\psi\in\mathscr{C}\mathscr{L}_2$, that

$$\|\phi^{\otimes n}-\psi^{\otimes n}\|_{\mathscr{L}_2^n}$$
$$=\left(\int_{[0,1]^n}(\phi(x_1)\cdots\phi(x_n)-\psi(x_1)\cdots\psi(x_n))^2dx_1\cdots dx_n\right)^{1/2}$$

$$=\left(\int_{[0,1]^n}\left(\sum_{k=1}^{n}\psi(x_1)\cdots\psi(x_{k-1})(\phi(x_k)-\psi(x_k))\phi(x_{k+1})\cdots\phi(x_n)\right)^2 dx_1\cdots dx_n\right)^{1/2}$$

$$\le\sum_{k=1}^{n}\left(\int_{[0,1]^n}\psi(x_1)^2\cdots\psi(x_{k-1})^2|\phi(x_k)-\psi(x_k)|^2\phi(x_{k+1})^2\cdots\phi(x_n)^2 dx_1\cdots dx_n\right)^{1/2}$$

$$=\|\phi-\psi\|_{\mathscr{L}_2}\sum_{k=1}^{n}\|\psi\|_{\mathscr{L}_2}^{k-1}\|\phi\|_{\mathscr{L}_2}^{n-k}$$

$$\le\|\phi-\psi\|_{\mathscr{L}_2} n\max\{\|\phi\|_{\mathscr{L}_2}\|\psi\|_{\mathscr{L}_2}\}^{n-1}.$$

Using this inequality,

$$\|h^{\phi}-h^{\phi^{(K)}}\|_H^2=\sum_{n=1}^{\infty}\frac{1}{n!}\|\phi^{\otimes n}-\phi^{(K)\otimes n}\|_{\mathscr{L}_2^n}^2$$

$$\le\sum_{n=1}^{\infty}\frac{1}{n!}\|\phi-\phi^{(K)}\|_{\mathscr{L}_2}^2 n^2\|\phi\|_{\mathscr{L}_2}^{2(n-1)}$$

$$=\sum_{n=1}^{\infty}\frac{n^2(\|\phi\|_{\mathscr{L}_2}^2)^{n-1}}{n!}\|\phi-\phi^{(K)}\|_{\mathscr{L}_2}^2.$$

Now we have only to use the following easy identity to obtain the required inequality:

$$\sum_{n=1}^{\infty}\frac{n^2x^{n-1}}{n!}=(1+x)e^x,\qquad x\in\boldsymbol{R}.$$

Before the proof of Main Theorem, we introduce an auxiliary theorem.

Theorem 4. *If* $\alpha\in\boldsymbol{T}_{dx}^{\infty}$,

$$\{1+Y_m(x+p\alpha;h^{\phi})\}_{p\in\boldsymbol{N},\phi\in\mathscr{C}\mathscr{L}_2}\underset{m\to\infty}{\overset{\text{f.d.}}{\Rightarrow}}\{e^{I_1^{(p)}(\phi)-\frac{1}{2}\|\phi\|_{\mathscr{L}_2}^2}\}_{p\in\boldsymbol{N},\phi\in\mathscr{C}\mathscr{L}_2}.$$

Proof. For $\phi\in\mathscr{C}\mathscr{L}_2$, $K\in\boldsymbol{N}$, we put $\phi^{(K)}:=\sum_{k=1}^{K}(\phi,\phi_k)\phi_k$. Then $\|\phi-\phi^{(K)}\|^2=\sum_{k=K+1}^{\infty}(\phi,\phi_k)^2\to 0$ as $K\to\infty$. By Lemma 2, we have

$$\|Y_m(\cdot+p\alpha;h^{\phi})-Y_m(\cdot+p\alpha;h^{\phi^{(K)}})\|^2=\|Y_m(\cdot+p\alpha;h^{\phi}-h^{\phi^{(K)}})\|^2$$
$$\le\|h^{\phi}-h^{\phi^{(K)}}\|_H^2.$$

Then it follows from Lemma 3 that for each $\phi\in\mathscr{C}\mathscr{L}_2$,

$$\sup_{p,m\in\boldsymbol{N}}\sup_{\alpha\in\boldsymbol{T}^{\infty}}E^{\infty}[|Y_m(\cdot+p\alpha;h^{\phi})-Y_m(\cdot+p\alpha;h^{\phi^{(K)}})|^2]\underset{K\to\infty}{\to}0.$$

Thus we have now only to prove, for each $K \in \boldsymbol{N}$, that

$$\{1+Y_m(x+p\alpha;h^{\phi^{(K)}})\}_{p\in\boldsymbol{N},\phi\in\mathscr{C}\mathscr{L}_2} \underset{m\to\infty}{\overset{\text{f.d.}}{\Rightarrow}} \{e^{\Sigma_{k=1}^K(\phi,\phi_k)I_1^{(p)}(\phi_k)-\frac{1}{2}\Sigma_{k=1}^K(\phi,\phi_k)^2}\}_{p\in\boldsymbol{N},\phi\in\mathscr{C}\mathscr{L}_2}, \quad {}^{\forall}\alpha\in\boldsymbol{T}_{dx}^{\infty}. \tag{19}$$

By definition, we have

$$m^{-\frac{n}{2}}\sigma_n^m(y;\psi^{\otimes n}) = \sum_{1\le i_1\cdots i_n\le m} \frac{\psi(y_{i_1})}{\sqrt{m}}\cdots\frac{\psi(y_{i_n})}{\sqrt{m}}, \qquad m\ge n,$$

so that

$$\begin{aligned} 1+Y_m(y;h^{\psi}) &= 1+\sum_{n=1}^{\infty} m^{-\frac{n}{2}}\sigma_n^m(y;\psi^{\otimes n}) \\ &= 1+\sum_{n=1}^{m}\sum_{1\le i_1\cdots i_n\le m}\frac{\psi(y_{i_1})}{\sqrt{m}}\cdots\frac{\psi(y_{i_n})}{\sqrt{m}} \\ &= \prod_{i=1}^{m}\left(1+\frac{\psi(y_i)}{\sqrt{m}}\right). \end{aligned}$$

This expression and $\phi^{(K)}=\sum_{k=1}^{K}c_k\phi_k$ where $c_k=(\phi,\phi_k)$, reduce (19) to

$$\left\{\prod_{i=1}^{m}\left(1+\frac{1}{\sqrt{m}}\sum_{k=1}^{K}c_k\phi_k(x_i+p\alpha_i)\right)\right\}_{p\in\boldsymbol{N},(c_1,\cdots,c_K)\in\boldsymbol{R}^K} \underset{m\to\infty}{\overset{\text{f.d.}}{\Rightarrow}} \{e^{\Sigma_{k=1}^K c_k I_1^{(p)}(\phi_k)-\frac{1}{2}\Sigma_{k=1}^K c_k^2}\}_{p\in\boldsymbol{N},(c_1,\cdots,c_K)\in\boldsymbol{R}^K}. \tag{20}$$

On the other hand, since Theorem 2 says

$$\left\{\frac{1}{\sqrt{m}}\sum_{i=1}^{m}\phi_k(x_i+p\alpha_i)\right\}_{p,k\in\boldsymbol{N}} \underset{m\to\infty}{\overset{\text{f.d.}}{\Rightarrow}} \{I_1^{(p)}(\phi_k)\}_{p,k\in\boldsymbol{N}}, \qquad {}^{\forall}\alpha\in\boldsymbol{T}_{dx}^{\infty},$$

and the law of large numbers implies

$$\frac{1}{m}\sum_{i=1}^{m}(\phi_k\phi_l)(x_i+p\alpha_i) \underset{m\to\infty}{\to} \delta_{kl} \qquad \boldsymbol{P}^{\infty}\text{-a.s.}, \qquad {}^{\forall}\alpha\in\boldsymbol{T}^{\infty},\ {}^{\forall}p,k,l\in\boldsymbol{N},$$

(20) is now reduced to

$$\prod_{i=1}^{m}\left(1+\frac{1}{\sqrt{m}}\sum_{k=1}^{K}c_k\phi_k(x_i+p\alpha_i)\right)$$

$$-e^{\sum_{k=1}^K c_k \frac{1}{\sqrt{m}} \sum_{i=1}^m \phi_k(x_i+p\alpha_i) - \frac{1}{2}\sum_{k,l=1}^K c_k c_l \frac{1}{m}\sum_{i=1}^m (\phi_k\phi_l)(x_i+p\alpha_i)} \tag{21}$$

$$\underset{m\to\infty}{\to} 0 \quad \text{in propability}, \quad {}^\forall\alpha\in \boldsymbol{T}_{dx}^\infty,\ {}^\forall p\in \boldsymbol{N},\ {}^\forall(c_1,\cdots,c_K)\in \boldsymbol{R}^K.$$

Let us show (21). Note first that

$$1+x=e^{x-\frac{x^2}{2}+r(x)}, \qquad r(x)=\int_0^x \frac{y^2}{1+y}dy \qquad ({}^\forall|x|<1),$$

$$|r(x)|\le|x|^3 \qquad \left({}^\forall|x|\le\frac{2}{3}\right).$$

Take $\delta>0$ such that $\sum_{k=1}^K |c_k|\delta<\frac{2}{3}$. Under the condition $\{\max_{1\le k\le K}\max_{1\le i\le m}|\phi_k(x_i+p\alpha_i)|\le\sqrt{m}\delta\}$, we have

$$\left|\frac{1}{\sqrt{m}}\sum_{k=1}^K c_k\phi_k(x_i+p\alpha_i)\right|\le\sum_{k=1}^K|c_k|\delta<\frac{2}{3}, \qquad 1\le{}^\forall i\le m,$$

and hence

$$\left|\prod_{i=1}^m\left(1+\frac{1}{\sqrt{m}}\sum_{k=1}^K c_k\phi_k(x_i+p\alpha_i)\right)\right.$$
$$\left.-e^{\sum_{k=1}^K c_k \frac{1}{\sqrt{m}} \sum_{i=1}^m \phi_k(x_i+p\alpha_i) - \frac{1}{2}\sum_{k,l=1}^K c_k c_l \frac{1}{m}\sum_{i=1}^m (\phi_k\phi_l)(x_i+p\alpha_i)}\right|$$
$$=e^{\sum_{k=1}^K c_k \frac{1}{\sqrt{m}} \sum_{i=1}^m \phi_k(x_i+p\alpha_i) - \frac{1}{2}\sum_{k,l=1}^K c_k c_l \frac{1}{m}\sum_{i=1}^m (\phi_k\phi_l)(x_i+p\alpha_i)}$$
$$\times\left|e^{\sum_{i=1}^m r(\frac{1}{\sqrt{m}}\sum_{k=1}^K c_k\phi_k(x_i+p\alpha_i))}-1\right|$$
$$\le e^{\sum_{k=1}^K c_k \frac{1}{\sqrt{m}} \sum_{i=1}^m \phi_k(x_i+p\alpha_i) - \frac{1}{2}\sum_{k,l=1}^K c_k c_l \frac{1}{m}\sum_{i=1}^m (\phi_k\phi_l)(x_i+p\alpha_i)}$$
$$\times\left|\sum_{i=1}^m r\left(\frac{1}{\sqrt{m}}\sum_{k=1}^K c_k\phi_k(x_i+p\alpha_i)\right)\right|e^{|\sum_{i=1}^m r(\frac{1}{\sqrt{m}}\sum_{k=1}^K c_k\phi_k(x_i+p\alpha_i))|}$$
$$\le e^{\sum_{k=1}^K c_k \frac{1}{\sqrt{m}} \sum_{i=1}^m \phi_k(x_i+p\alpha_i) - \frac{1}{2}\sum_{k,l=1}^K c_k c_l \frac{1}{m}\sum_{i=1}^m (\phi_k\phi_l)(x_i+p\alpha_i)}$$
$$\times\sum_{k=1}^K|c_k|\delta\sum_{i=1}^m\left|\frac{1}{\sqrt{m}}\sum_{k=1}^K c_k\phi_k(x_i+p\alpha_i)\right|^2 e^{\frac{2}{3}\sum_{i=1}^m|\frac{1}{\sqrt{m}}\sum_{k=1}^K c_k\phi_k(x_i+p\alpha_i)|^2}$$
$$=e^{\sum_{k=1}^K c_k \frac{1}{\sqrt{m}} \sum_{i=1}^m \phi_k(x_i+p\alpha_i) - \frac{1}{2}\sum_{k,l=1}^K c_k c_l \frac{1}{m}\sum_{i=1}^m (\phi_k\phi_l)(x_i+p\alpha_i)}$$

$$\times \delta \sum_{k=1}^{K} |c_k| \sum_{k,l=1}^{K} c_k c_l \frac{1}{m} \sum_{i=1}^{m} (\phi_k \phi_l)(x_i + p\alpha_i) e^{\frac{2}{3}\sum_{k,l=1}^{K} c_k c_l \frac{1}{m} \sum_{i=1}^{m} (\phi_k \phi_l)(x_i + p\alpha_i)}.$$

Consequently, for any $\eta > 0$, we have

$$P^{\infty}\left(\left| \prod_{i=1}^{m} \left(1 + \frac{1}{\sqrt{m}} \sum_{k=1}^{K} c_k \phi_k(x_i + p\alpha_i) \right) - e^{\sum_{k=1}^{K} c_k \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \phi_k(x_i + p\alpha_i) - \frac{1}{2}\sum_{k,l=1}^{K} c_k c_l \frac{1}{m} \sum_{i=1}^{m} (\phi_k \phi_l)(x_i + p\alpha_i)} \right| \geq \eta \right)$$

$$\leq P^{\infty}\left(\max_{1 \leq k \leq K} \max_{1 \leq i \leq m} |\phi_k(x_i + p\alpha_i)| > \sqrt{m}\delta \right)$$

$$+ P^{\infty}\left(\begin{array}{l} e^{\sum_{k=1}^{K} c_k \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \phi_k(x_i + p\alpha_i) - \frac{1}{2}\sum_{k,l=1}^{K} c_k c_l \frac{1}{m} \sum_{i=1}^{m} (\phi_k \phi_l)(x_i + p\alpha_i)} \\ \times \sum_{k=1}^{K} |c_k| \sum_{k,l=1}^{K} c_k c_l \frac{1}{m} \sum_{i=1}^{m} (\phi_k \phi_l)(x_i + p\alpha_i) \\ \times e^{\frac{2}{3}\sum_{k,l=1}^{K} c_k c_l \frac{1}{m} \sum_{i=1}^{m} (\phi_k \phi_l)(x_i + p\alpha_i)} \geq \frac{\eta}{\delta} \end{array} \right).$$

Letting $m \to \infty$, we see

$$\lim_{m \to \infty} (\text{The 1st term}) = 0, \qquad {}^{\forall}\alpha \in \boldsymbol{T}^{\infty},\ {}^{\forall}p \in \boldsymbol{N},$$

$$\overline{\lim_{m \to \infty}} (\text{The 2nd term}) \leq P\left(e^{\sum_{k=1}^{K} c_k I_1^{(p)}(\phi_k) - \frac{1}{2}\sum_{k=1}^{K} c_k^2} \sum_{k=1}^{K} |c_k| \sum_{k=1}^{K} c_k^2 e^{\frac{2}{3}\sum_{k=1}^{K} c_k^2} \geq \frac{\eta}{\delta} \right),$$

$${}^{\forall}\alpha \in \boldsymbol{T}_{dx}^{\infty},\ {}^{\forall}p \in \boldsymbol{N},\ {}^{\forall}(c_1, \cdots, c_K) \in \boldsymbol{R}^K.$$

Hence

$$\lim_{\delta \to 0} \overline{\lim_{m \to \infty}} (\text{The 2nd term}) = 0.$$

Thus (21) is proved, and hence the proof of Theorem 4 is done.

Proof of Main Theorem. Since the "only if" part was done in Theorem 3, we here prove the "if" part. So we assume $\alpha \in \boldsymbol{T}_{dx}^{\infty}$.

Take any $h=\{h_n\}_{n=1}^{\infty} \in H$ and fix it. Let $\varepsilon>0$. By Lemma 1, we can take $\psi_1, \cdots, \psi_l \in \mathscr{C}\mathscr{L}_2$ and $t_1, \cdots, t_l \in \boldsymbol{R}$ so that $\|h-\sum_{i=1}^{l} t_i h^{\psi_i}\|_H<\varepsilon$. Then Lemma 2 implies

$$\| Y_m(\cdot+p\alpha; h)-\sum_{i=1}^{l} t_i Y_m(\cdot+p\alpha; h^{\psi_i})\|_{L_2(\boldsymbol{T}^\infty;\boldsymbol{P}^\infty)}^2 \leq \|h-\sum_{i=1}^{l} t_i h^{\psi_i}\|_H^2<\varepsilon^2,$$

$$^\forall p, m \in \boldsymbol{N},\ ^\forall\alpha \in \boldsymbol{T}^{\infty}.$$

Theorem 4 implies

$$\left\{\sum_{i=1}^{l} t_i Y_m(x+p\alpha; h^{\psi_i})\right\}_{p\in\boldsymbol{N}} \underset{m\to\infty}{\overset{\text{f.d.}}{\Rightarrow}} \left\{\sum_{i=1}^{l} t_i(e^{I_1^{(p)}(\psi_i)-\frac{1}{2}\|\psi_i\|_{\mathscr{L}_2}^2}-1)\right\}_{p\in\boldsymbol{N}},$$

$$^\forall\alpha \in \boldsymbol{T}_{dx}^{\infty}.$$

On the other hand, Proposition 1 (i) shows

$$e^{I_1^{(p)}(\psi_i)-\frac{1}{2}\|\psi_i\|_{\mathscr{L}_2}^2}=1+\sum_{n=1}^{\infty}\|\psi_i\|_{\mathscr{L}_2}^n H_n\left(I_1^{(p)}\left(\frac{\psi_i}{\|\psi_i\|_{\mathscr{L}_2}}\right)\right)=1+\sum_{n=1}^{\infty}\frac{1}{n!}I_n^{(p)}(\psi_i^{\otimes n}),$$

and hence

$$\sum_{i=1}^{l} t_i(e^{I_1^{(p)}(\psi_i)-\frac{1}{2}\|\psi_i\|_{\mathscr{L}_2}^2}-1)=\sum_{n=1}^{\infty}\frac{1}{n!}I_n^{(p)}\left(\sum_{i=1}^{l} t_i\psi_i^{\otimes n}\right), \qquad ^\forall p\in\boldsymbol{N}.$$

And Proposition 1 (ii) shows

$$\left\|\sum_{n=1}^{\infty}\frac{1}{n!}I_n^{(p)}\left(\sum_{i=1}^{l} t_i\psi_i^{\otimes n}\right)-\sum_{n=1}^{\infty}\frac{1}{n!}I_n^{(p)}(h_n)\right\|_{L_2(\Omega;P)}^2$$

$$=\left\|\sum_{n=1}^{\infty}\frac{1}{n!}I_n^{(p)}\left(\sum_{i=1}^{l} t_i\psi_i^{\otimes n}-h_n\right)\right\|_{L_2(\Omega;P)}^2=\sum_{n=1}^{\infty}\frac{1}{n!}\left\|\sum_{i=1}^{l} t_i\psi_i^{\otimes n}-h_n\right\|_{\mathscr{L}_2^n}^2$$

$$=\left\|h-\sum_{i=1}^{l} t_i h^{\psi_i}\right\|_H^2<\varepsilon^2, \qquad ^\forall p\in\boldsymbol{N}.$$

From the above estimates, we can finally derive that

$$\{Y_m(x+p\alpha; h)\}_{p\in\boldsymbol{N}} \underset{m\to\infty}{\overset{\text{f.d.}}{\Rightarrow}} \left\{\sum_{n=1}^{\infty}\frac{1}{n!}I_n^{(p)}(h_n)\right\}_{p\in\boldsymbol{N}}.$$

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