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Rationality of almost simple algebraic groups

By

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Abstract

We prove the stable rationality of almost simple adjoint algebraic groups, the connected components of the Dynkin diagram of anisotropic kernel of which contain at most two vertices. The (stable) rationality of many isotropic almost simple groups with small anisotropic kernel and some related results over arbitrary fields are discussed.

Introduction

Let G be a connected linear algebraic group defined over a field k. The classical results of Chevalley (Séminaire Chevalley 1956–1958) showed that if k is an algebraically closed field then G is rational over k as k-variety, i.e., the field $k(G)$ of rational functions defined over k of G is a pure transcendental extension of k. However this is no longer true if k is not algebraically closed and one of basic geometric problems of algebraic groups over non-algebraically closed fields is the problem of rationality. A milder notion of stable rationality (and unirationality) is in sequence: An irreducible k-variety X is k-stably rational (resp. k-unirational) if there is an affine k-space A such that $X \times A$ is k-birationally equivalent to an affine k-space (resp. such that there is a surjective k-morphism $A \to X$). In general, it is difficult to verify if a given k-group (or k-variety) is rational (or irrational). We refer the readers to [Ch], [CT], [MT], [M1,2], [P], [V], [VK] and references thereof for various problems and progress related with the rationality problem.

Up to now there is no general critetion to decide which almost simple groups are stably rational over the field of definition by looking at their Tits indices, except for the trivial cases of split and quasi-split groups. Quite recently, with the papers [M1,2] it became known that

Theorem 1 ([M1, 2]). 1) *For any division algebra D of degree divisible by* 4 *over a field k, the related almost simple simply connected k-groups G with $G(k) = \mathrm{SL}_n(D)$, where $\mathrm{SL}_n(D)$ is the group of all elements of reduced norm one in the matrix algebra $\mathrm{M}_n(D)$, are not stably rational over k.*

2) *There are almost simple adjoint groups of type ${}^2\mathrm{D}_3 = {}^2\mathrm{A}_3$ which are not stably rational over the field of definition.*

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In terms of Tits indices, from Theorem 1 it follows that for any field k such that there exists a division k-algebra D of index d divisible by 4, any almost simple simply connected group G defined over a field k of type $A^{d}_{n,r}$ with Tits index

$$\overset{1}{\bullet} - \cdots - \overset{d-1}{\bullet} - \overset{d}{\odot} - \bullet - \cdots - \bullet - \overset{rd}{\odot} - \bullet - \cdots - \bullet$$

where d is divisible by 4, is not stably rational over k. In particular, it is true when $d = 4$, i.e., the maximal length of a segment consisting only of black vertices is equal to 3.

In this paper we are mainly interested in adjoint groups (which can be considered as a continuation of our previous papers [T1] and [T2]). The purpose of this note is to show that Theorem 1 can be used to get such a general criterion. Our method shows that many almost simple groups with relatively big k-rank and the degree of the related division algebra is ≤ 3 are k-stably rational. Hence in certain sense, our results are optimal. More precisely the main result of the paper is the following. We keep the notation of a classical type $X^{(d)}_{n,r}$ over a field k, as adopted by Tits, where d denotes the degree of the division algebra (associated with the group), n denotes the rank and r denotes the k-rank.

Theorem 2. *Let G be an almost simple adjoint algebraic group of type $X^{(d)}_{n,r}$ over a field k. Let $m(G)$ be the maximal number of vertices of connected components of the Dynkin diagram of anisotropic kernel of G. If $n - rd \leq 2$ and G is of classical type, or if $m(G) \leq 2$ and G is of exceptional type then G is rational over k.*

We will show in the course of the proof that for individual classes of groups, the results can be better. We remark that the number 2 in the theorem is best possible, i.e., one cannot replace 2 by any greater number. In fact, the above result of Merkurjev says that for *arbitrary* fields k, if there exists division algebra D of degree 3, then for any natural number n, there exist non stably rational groups G with $m(G) = 4n - 1$. However, it is not known if for any natural number $m \neq 4n - 1$ there are a field k and a non (stably-) rational group G over k with $m(G) = m$.

The following two key results are used here. First, it is a result of Chevalley (originally proved in [C] for fields of characterstic 0, and extended to arbitrary characteristic by Grothendieck in [SGA III, Exp. XIII, Corollaire 3.3 and Exp. XIV, Théorème 6.1]).

Theorem 3 ([C]). *Let G be a connected (linear) algebraic group defined over a field k. Then the function field $k(G)$ of a connected reductive k-group G is of the form $L(\mathbf{T})$ where L is the function field of the k-variety Tor_G of maximal tori of G (which is rational over k) and $\mathbf{T}$ is a Cartan subgroup of G defined over L.*

The second one is the following theorem of Voskresenskii (see e.g. [V]), which dated back to 1965.

Theorem 4. *Any torus of dimension ≤ 2 is rational over the field of definition.*

From these two results one deduces that, if G is connected reductive of rank ≤ 2 then it is rational over the field of definition.

It is well-known that the *almost* direct product of (stably) rational groups is far from being rational or stably rational. But it turns out that if k is one of "nice" fields such as local (p-adic or real) fields, or if the Tits index of G is "nice" (in the sense to be made clear below), then many isotropic almost simple k-group are k-(stably) rational. The method of the proof of these facts is based on a detailed analysis of the Tits index and explicit computations in the groups under consideration.

The first version of the paper (written in March 1995; see also preprint [T4], September 1995) contained two directions of study (and corresponding results obtained, the proofs of which were merely sketched):

1) Many almost simple groups are rational or stably rational over p-adic fields. It reflects the fact that except for groups of type A (naturally, due to Merkurjev's result) and certain simply connected forms of type D, any almost simple group over p-adic field are rational or stably rational.

2) Almost simple k-groups with Tits indices of their anisotropic kernels having connected components with at most two vertices (sometimes three) (called *black segments* in [T4]) (the condition again clearly necessary due to Merkurjev's result) are rational or stably rational.

Some people already worked in these two directions and for the first direction my result was completed in a (one year later) 1996 paper [CP], where it was shown that actually any almost simple group of type $\neq$ A is rational over the p-adic field. The method of proof is similar to the one we used and most of technical points are overlapped.

Also, there appeared later in 1996 a preprint [ChM], where they applied the existence of the *norm principle* for the group of R-equivalence classes (which was also studied earlier in our preprints [T5–6], where we called *corestriction principle* instead of norm principle due to its connection with Galois cohomology) to prove the rationality of special unitary groups of type ${}^2\mathrm{A}^{(d)}$ with $d \leq 3$, which is stronger the corresponding result we had in [T4]. In [T7] we investigated some relation between weak approximation and Brauer and R-equivalence relations in algebraic groups over number fields and indicated another approach to give examples of non-stably rational connected semisimple groups over p-adic fields.

Notation and convention. If k is a field, $\bar{k}$ denotes an algebraic closure of k. An almost simple group always means an absolutely almost simple group. For an almost simple group G defined over a field k which has characteristic either 0 or relatively big[1], e.g. relatively prime with the order of the center $Cent(G)$ of G let S be a maximal k-split torus of G, T a maximal k-torus of G containing S. We denote by $Ad(G) = G/Cent(G)$ the adjoint group of G. If $\dim T = n$, we denote

[1] The assumption of characteristic of k is put to simplify the arguments related with inseparability and is not essential.

by

$$\Delta = \{\alpha_1, \ldots, \alpha_n\}$$

a basis of simple roots for the root system Φ of G with respect to T. We may consider the relative root system ${}_k\Phi$ of G relatively to S and let ${}_k\Delta$ a basis of ${}_k\Phi$ compatible with Δ. If α, β are two roots, then we denote by $\langle \alpha, \beta \rangle = 2(\alpha, \beta)/(\beta, \beta)$. For $1 \leq i \leq n$ we denote by S_i the standard $\bar{k}$-split torus corresponding to the root α_i. We denote by $x_\alpha(t)$ the multiplicative one-parameter unipotent subgroup (resp. $h_\alpha(t)$ the multiplicative one-parameter diagonal subgroup) of G corresponding to a root $\alpha \in \Delta$ where we keep the same notation used in [St]. For $\alpha = \alpha_i$ we denote $x_i(t) = x_{\alpha_i}(t)$, $h_i(t) = h_{\alpha_i}(t)$, $1 \leq i \leq n$. In particular, if G is simply connected then T is the direct product of the images of $h_i := h_{\alpha_i}$, $1 \leq i \leq n$. We use intensively the notion and results of Tits' classification theory of almost simple algebraic groups as presented in [Ti1] and refer also to [B] and [BT] for other notions in algebraic groups. We often identify a simple root with the vertex representing it in the Tits index. In the Dynkin diagrams, the updown arrows, if any, indicate the vertices belonging to the same orbit under $*$-action of $Gal(k_s/k)$ on the Δ (just like in Satake diagrams, see [Sa]).

1. Some general well-known useful facts

1.1. Let G be a connected reductive k-group, S a maximal k-split torus of G. The Bruhat decomposition for G (see [BT, Section 4]) implies that

$$G \simeq Z_G(S) \times \mathbf{A}$$

as varieties, where $\mathbf{A}$ is an affine space defined over k. Thus the study of rationality of G is reduced, in certain sense, to that of $Z_G(S)$. Namely G is stably rational over k if and only if $Z_G(S)$ is and if $Z_G(S)$ is k-rational, then so is G.

However in certain cases the group $Z_G(S)$ is hard to handle with and we are forced to find a substitute, which can be studied easier. In many cases it is possible to do so. Namely let S_0 be a nontrivial k-subtorus of S. Another version of Bruhat decomposition says that

$$G \simeq Z_G(S_0) \times \mathbf{A},$$

the direct product of $Z_G(S_0)$ with an affine space $\mathbf{A}$ over k.

Therefore we are reduced to studying the connected reductive k-groups $Z_G(S_0)$. The problem here is to choose a "nice" torus S_0 so that we can prove the rationality or stable rationality of $Z_G(S_0)$, which is possible if the k-rank is relatively big. We make use frequently the following simple but very useful observation.

1.2. Proposition ([DT]). *Let S_0 be a standard k-split torus of G and $Z_G(S_0) = S_0T_0H$ (almost direct product), where T_0 is a k-torus, H a semisimple k-*

subgroup of G. Then the Tits index of H is obtained from that of G by removing all vertices not belonging to the preimage $\tilde{\theta}$ *of* $\theta \cup \{0\}$ *under the restriction map* $\Delta \to {}_k\Delta \cup \{0\}$. *Moreover* T_0 *is anisotropic and* $ST_0 = (Z_T(H))^0$.

Remarks. 1. As it was mentioned in [DT], the idea of 1.2 is due to Tits [Ti1, p. 39, lines −7 to −4, p. 45, remark d)] (see also [Se, p. 42]).

2. The equality in the last statement is not in [DT] but it is clear by comparing the dimension of both side and by making use of the previous part of the proposition. In particular it shows that if $Z_T(H)$ is connected then ST_0 contains the center of H.

Another interesting remark is the following observation due to Tits.

1.3. Proposition ([Ti2, Lemma, p. 89], [Se, Lemma 4.1.3]). *Let G be an adjoint semisimple isotropic k-groups, S a k-split torus of G. Let* $Z_G(S) = ST_0H$, *where H is the semisimple part of* $Z_G(S)$. *Then the connected center* ST_0 *of* $Z_G(S)$ *is a direct product of quasi-split (i.e. induced) k-tori. In particular, it is also cohomologically trivial.*

Remark. This proposition can be considered as a generalization of the well-known fact, that *the maximal k-tori containing a maximal k-split torus in adjoint quasi-split semisimple k-groups are induced k-tori.* For a proof of this well-known fact, see e.g. [BrT, Prop. 4.4.16], or [Ta].

The following result essentially is due to Ono—Rosenlicht (see [O]).

1.4. Proposition. *If T is a central k-torus of a connected reductive k-group G, such that for any extension* $k \subset k'$ *the canonical projection* $G(k') \to (G/T)(k')$ *is surjective (e.g. if T is quasi-split torus over k), then there is a rational k-cross section* $G/T' \to G$. *In particular the k-variety G is birationally equivalent to the product* $T' \times (G/T')$.

From above we see that it is essential to know the group $Z_G(S)/(ST_0)$, $Z_G(S)/S$ (which we call the *semisimple* and *reductive anisotropic quotient* of G, respectively) if we want to know the rationality property of G. In the next section we examine various computations of this group. The following remark is useful in the sequel.

1.5. Proposition. *Let G be a connected reductive k-group. If* $\pi : G \to G'$ *is a central k-isogeny and* $S' = \pi(S)$ *is the image in* G' *of a k-split torus S of G, then* π *induces a central isogeny*

$$Z_G(S)/S \to Z_{G'}(S')/S'$$

Moreover, if $\pi' : G' \to G''$ *is another central k-isogeny,* $S'' = \pi'(S')$ *then we have the following induced commutative diagram*

$$\begin{tikzcd} Z_G(S) \arrow[r, "\pi"] \arrow[dr, "\pi''"'] & Z_{G'}(S') \arrow[d, "\pi'"] \\ & Z_{G''}(S'') \end{tikzcd}$$

and also similar diagram for reductive quotients:

$$\begin{tikzcd} Z_G(S)/S \arrow[r, "\pi"] \arrow[dr, "\pi''"'] & Z_{G'}(S')/S' \arrow[d, "\pi'"] \\ & Z_{G''}(S'')/S'' \end{tikzcd}$$

Among others, the following general result was proved by Voskresenskiĭ and Klyachko.

1.6. Theorem ([VK]). *Let G be an adjoint k-group of type A_m, where m is even. Then G is rational over k.*

2. Rationality of groups of type A

2.1. We keep the above notation and we assume that G is an almost simple k-group. Let $P_i, 1 \leq i \leq s$, be k-groups with F_i a central k-subgroup of P_i. Assume that all F_i are k-isomorphic. By a suitable factoring out a central k-subgroup of the direct production of P_i we will obtain an almost direct product $P'_1 \dots P'_s$ with the property that the set-theoretic intersections $P'_i \cap P'_j$ are all equal and k-isomorphic to F_i. We call such a group the *product of P_i with glued central subgroups F_i*.

2.2. Proposition. *Let G be an almost simple k-group of type A_n with k-rank $r > 0$.*

a) If G is an inner form, the anisotropic semisimple quotient $Z_G(S)/S$ is the product of k-conjugate almost simple anisotropic k-groups of type ${}^1\mathrm{A}_{d-1}$ with glued center.

b) If G is an outer form, $Z_G(S)/ST_0$ is the product of anisotropic groups of type ${}^2\mathrm{A}_{d-1}$ with an anisotropic k-group of type ${}^2\mathrm{A}_{n-2rd}$ with glued central subgroup of order dividing d.

Proof. *a)* First we begin with simply connected isotropic groups of type ${}^1\mathrm{A}_n$. From [Ti1] we know that such groups have the following Tits index

$$\bullet^1 - - \cdots - - \bullet^{d-1} - - \odot^{d} - - \bullet - - \cdots - - \bullet^{rd-1} - - \odot^{rd} - - \bullet - - \cdots - - \bullet^{n}$$

Let $\tilde{S}_i$ be the standard k-split torus corresponding to the isotropic vertex i, $i = d, 2d, \dots, rd$. Here d is the index of the division k-algebra D related with this type and r denotes the k-rank of G, i.e., $G(k) = \mathrm{SL}_{r+1}(D)$. Then

$$\tilde{S} = \prod_i \tilde{S}_i$$

is a maximal k-split torus of G. We have $Z_G(\tilde{S}) = \tilde{S}\tilde{H}$, where $\tilde{H} = \prod \tilde{H}_j$ is a semisimple k-group which is a direct product of anisotropic k-groups $\tilde{H}_j$ of type ${}^1\mathrm{A}_{d-1}$ (see Proposition 1.2).

It is easy to see that all these groups $\tilde{H}_j$ are k-conjugate and k-isomorphic to the simply connected almost simple k-group G_0 of type ${}^1\mathrm{A}_{d-1}$ defined by $G_0(k) = \mathrm{SL}_1(D)$. (To see this, we may assume that in certain basis, the maximal k-split torus $S(k)$ consists of all diagonal matrices from $\mathrm{GL}_n(D)$ with coefficients from k^* and of determinant 1:

$$S(k) = \{diag(t_1, \dots, t_{r+1}) : t_1 \cdots t_{r+1} = 1,\ t_i \in k^*,\ 1 \leq i \leq r+1\},$$

thus

$$Z_G(S)(k) = \{diag(z_1, \dots, z_{r+1}) : z_i \in \mathrm{GL}_1(D),\ Nrd(z_1 \cdots z_{r+1}) = 1 \leq i \leq r+1\}.$$

It follows that all the groups H_j above are k-conjugate (i.e. conjugate by elements from $G(k)$).

Therefore if G' is a quotient of G by a central k-subgroup then we can form the corresponding centralizer of a maximal k-split torus S' of G', which is an almost direct product of S' and k-conjugate almost simple k-groups H_j' of type ${}^1\mathrm{A}_{d-1}$; and they are the homomorphic image of the simply connected almost simple k-groups $\tilde{H}_j$ of type ${}^1\mathrm{A}_{d-1}$ such that $\tilde{H}_j(k) \simeq \mathrm{SL}_1(D)$.

Some tedious computations (just like ones we have in Section 1) show that S contains the products z_iz_j of two generators z_i and z_j of centers of the groups H_i and H_j, respectively. From this and Proposition 1.5, the assertion *a*) follows.

b) Let l be the separable quadratic extension of k over which G becomes of inner type. Assume first that G is simply connected. From [Ti1] we know that the Tits index of G is as follows

$$\left.\begin{array}{c}
\bullet \cdots \bullet - - \odot - - \bullet \cdots \bullet - - \odot - - \bullet \cdots\cdots \bullet - - \odot - - \bullet \cdots \bullet \cdots \\
\qquad\quad \updownarrow \qquad\qquad\qquad\quad \updownarrow \qquad\qquad\qquad\qquad \updownarrow \qquad\qquad\qquad \\
\bullet \cdots \bullet - - \odot - - \bullet \cdots \bullet - - \odot - - \bullet \cdots\cdots \bullet - - \odot - - \bullet \cdots \bullet \cdots
\end{array}\right)$$

Denote by $G(\Psi)$ the semisimple regular subgroup of G generated by the root subgroups $X_i, \alpha_i \in \Psi$, where Ψ is a subset of Δ. By Proposition 1.2 in order to compute the intersection $ST_0 \cap H$ we are reduced to computing the intersection of the torus

$$S_dS_{n-d+1}S_{2d}S_{n-2d+1} \dots S_{rd}S_{n-rd+1}$$

with the semisimple subgroup

$$G_1' \dots G_r'A,$$

where G_i', $1 \leq i \leq r$ is the semisimple k-group $G(\Psi_i)$ with the root system generated by the basis

$$\Psi_i = \{\alpha_{(i-1)d+1}, \dots, \alpha_{id-1}, \alpha_{n-(i-1)d}, \dots, \alpha_{n-id+2}\},$$

and A is the group $G(\alpha_{rd+1}, \dots, \alpha_{n-rd})$. Then as in the part $a)$ the part $b)$ follows. The general case also follows from this as in $a)$ by making use of Proposition 1.5.

The case $d \leq 3$ is of special interest to us due to the following results which follows from Proposition 2.2.

2.3. Proposition. *Let k be a field with a division algebra D of degree $d \leq 3$, G_0 an anisotropic semisimple k-group which is an almost direct product of k-groups of type A_{d-1} isogeneous over k to either $\mathrm{SL}_{1,D}$ or $\mathrm{PGL}_{1,D}$ such that any simply connected factor contains the center of the other. Then G is rational over k. In particular, if k has a unique up to isomorphism quaternion division algebra (e.g. k is a local field), then any such almost direct product of anisotropic k-groups of type A_1 is rational over k.*

Proof. Since $\mathrm{PGL}_{1,D}$ has no center and is k-rational, we may assume that all almost simple factors of G are isomorphic to $\mathrm{SL}_{1,D}$ and they have common center (i.e. product of groups of type A_{d-1} with glued center). Let the number of almost simple components of G be r. Then from Proposition 2.2 we see that for the group G_1 with $G_1(k) = \mathrm{SL}_{r+1}(D)$ and S a maximal k-split torus of G_1 we have

$$Z_{G_1}(S) = SG_2,$$

and

$$Z_{G_1}(S)/S \simeq G.$$

Since the group $\mathrm{SL}_{1,D}$ is k-rational by assumption on d and by Theorem 4, the group $\mathrm{SL}_{n,D}$ is also for any n (see [M1], [V]), and it follows that G is rational over k.

2.4. Theorem. *Let G be an almost simple k-group of type ${}^1\mathrm{A}_{n,r}^{(d)}$, where $d \leq 3$. Then G is rational over k.*

Proof. It follows from [Ti1] that G is an almost simple k-group with the following Tits index

$$\bullet - - \odot - - \bullet - - \cdots\cdots \bullet - - \odot - - \bullet,$$

i.e. of type ${}^1\mathrm{A}_{2r+1,r}^{(2)}$, or the following

$$\bullet - - \bullet - - \odot - - \bullet - - \bullet \cdots\cdots \bullet - - \bullet - - \odot - - \bullet - - \bullet,$$

i.e. of type ${}^1\mathrm{A}_{3r+2,r}^{(3)}$.

As above let $\tilde{G}$ be the universal covering of G, with maximal k-split torus $\tilde{S}, \pi : \tilde{G} \to G$ be the corresponding projection, and $S = \pi(\tilde{S})$. Then

$$Z_{\tilde{G}}(\tilde{S}) = \tilde{S}(\tilde{G}_1 \times \cdots \times \tilde{G}_{r+1}),$$

where $\tilde{G}_i$ are anisotropic simply connected k-group of type ${}^1\mathrm{A}_{d-1}$, with $\tilde{G}_i(k) \simeq \mathrm{SL}_1(D)$, where D is a division algebra over k. We have just mentioned that $\tilde{G}_i$ is

k-rational (since $d \leq 3$) hence so is $\tilde{G}$. From the Bruhat decomposition it follows that

$$Z_{\tilde{G}}(\tilde{S}) \times \mathbf{A}^{2m}$$

is rational, where m is the dimension of the unipotent radical of a minimal parabolic k-subgroup of G. Hence from Proposition 1.4 it follows that

$$(Z_{\tilde{G}}(\tilde{S})/\tilde{S}) \times \mathbf{A}^{2m+r}$$

is also rational over k. (Notice that $\dim(\tilde{S}) = r$.) Here

$$Z_{\tilde{G}}(\tilde{S})/\tilde{S} = \tilde{G}'_1 \dots \tilde{G}'_{r+1}$$

is the product of simply connected groups of type ${}^1\mathrm{A}_{d-1}$ with glued centers (see the beginning of Section 2). The isogeny π induces a central k-isogeny

$$\tilde{G}'_1 \dots \tilde{G}'_{r+1} = Z_{\tilde{G}}(\tilde{S})/\tilde{S} \xrightarrow{\pi'} Z_G(S)/S = G_1 \dots G_{r+1},$$

with $G_i = \pi'(\tilde{G}_i)$, $1 \leq i \leq r+1$, which are also groups with glued centers. The common center of $\tilde{G}'_i$ has order 2 or 3 by assumption on d, therefore the (common) center of G_i has either the same order or just 1. In the first case π' is an isomorphism, and in the second case, $G_1 \dots G_{r+1}$ is an adjoint group hence is rational over k. In any case,

$$Z_G(S)/S \times \mathbf{A}^{2m+r}$$

is rational over k, hence so is the group G.

Now we consider almost simple groups of type ${}^2\mathrm{A}_n$.

2.5. Proposition ([ChM]). *Let G be an almost simple simply connected k-group of type ${}^2\mathrm{A}^{(d)}_{n,r}$. Then G is rational if $d \leq 3$.*

2.6. Proposition. *a) If G is adjoint and of type ${}^2\mathrm{A}^{(d)}_{n,r}$ with $n - rd \leq 2$ then G is rational over k.*

b) With notation as above, let $d = 1$, and $Z_G(S) = ST_0H$. Then ST_0 contains the center of H. Thus any group k-isogeneous to G has isomorphic semisimple anisotropic quotient to that of G.

Proof. a) Let S be a maximal k-split torus of G, $Z_G(S) = ST_0H$. Then by Proposition 1.3, ST_0 contains the center of H, and there is a birational equivalence

$$G \simeq (Z_G(S)/ST_0) \times ST_0.$$

We have a direct product over k

$$Z_G(S)/ST_0 = A_{n-rd} \times \mathrm{R}_{k'/k}(A_{d-1}) \times \dots \times \mathrm{R}_{k'/k}(A_{d-1}),$$

where A_i are adjoint k'-groups of type A_i, where k' is a separable quadratic extension of k. Here A_{d-1} is of inner type, hence rational over k', and the corresponding restriction of scalars is rational over k. Since $n - rd \leq 2$, A_{n-rd} is also rational. Therefore G is rational over k.

$b)$ We assume first that G is simply connected. The general case follows from this since if $\pi : G \to G'$ is a central k-isogeny, then $S' := \pi(S)$ is a maximal k-split torus of G' and $\pi(Z_G(S)) = Z_{G'}(S')$ (see [BT]) and we use Proposition 1.6. For simplicity we assume that $r = 1$ and we give a complete computation in this case. From Proposition 1.2 it follows that we have only to check that

$$Cent(H) \subset S_1 S_d, \tag{1}$$

where $H = G(\alpha_2, \ldots, \alpha_{n-1})$. From above we see that for an element $t \in T$,

$$t = \prod_{1 \leq i \leq n} h_i(t_i)$$

is in $Z_T(H)$ if and only if t commutes with all one-parameter unipotent subgroups X_i for $2 \leq i \leq n-1$. Hence we have the following system of equations for t_i:

$$\begin{cases} t_2^2 = t_1 t_3 \\ t_3^2 = t_2 t_4 \\ \vdots \\ t_{n-1}^2 = t_{n-2} t_n. \end{cases}$$

One checks that

$$t_i = t_2^{i-1} / t_1^{i-2}, \quad 3 \leq i \leq n,$$

while the center of H is generated by

$$h_2(\zeta) \ldots h_{n-1}(\zeta^{n-2}),$$

where ζ is a primitive $(n-1)$-th root of unity. Hence (1) is verified.

Note that (with notation as above), the isogeny π induces an isogeny (denoted by the same symbol)

$$\pi : Z_G(S)/ST_0 \to Z_{G'}(S'T_0')/S'T_0'$$

between the anisotropic semisimple quotients. Now by above ST_0 contains the center of H, the corresponding anisotropic semisimple quotient is an *adjoint* group, hence the above induced isogeny is in fact a *k-isomorphism*.

Now we assume that $k = \mathbf{R}$. Recently Chernousov [Ch] has proved that if G is an anisotropic semisimple $\mathbf{R}$-group with no factors of type E_6, E_7, E_8 then G is stably rational over $\mathbf{R}$. The main idea there (as in [M2]) is to use the group of similarity factors of the forms involved, which goes back to [T1–3], where we considered the problem of weak approximation in a close relation with the problem of rationality. In view of results above, we can state the result of Chernousov as follows.

Let G be a semisimple $\mathbf{R}$*-group with no anisotropic factors of type* E_i, $i = 6, 7, 8$. *Then G is stably rational over* $\mathbf{R}$.

3. Rationality of isotropic almost simple groups of type $\neq$A

First consider groups of classical type B – D. For groups G of type B, if G is simply connected (i.e. $G \simeq \mathrm{Spin}(f)$ for some nondegenerate quadratic form f of odd dimension) then G is rational by [Pl, Prop. 3]. (One can use our method to prove this fact easily.) If it is adjoint, then $G \simeq \mathrm{SO}(f)$ and also is rational. Thus the following is well-known, but we give a proof based on our previous consideration.

3.1. Proposition. *If G is isotropic almost simple k-group of type* B_n *then G is k-rational.*

Proof. We give a proof for the case G is simply connected only, since the special orthogonal groups are known to be rational via Cayley transformations in any characteristic (see [Di] or [W]). The Tits index of G is as follows

$$\odot^{1} - - \odot^{2} - - \cdots - - \odot^{r} - - \bullet - - \bullet - - \cdots - - \bullet \Rightarrow \bullet^{n}$$

Denote by S a maximal k-split torus of G. We find S in its standard form, i.e., writing the system of equations defining t_i

$$\begin{cases} t_{r+1}^2 = t_r t_{r+2} \\ \vdots \\ t_{n-2}^2 = t_{n-3} t_{n-1} \\ t_{n-1}^2 = t_{n-2} t_n^2 \\ t_n^2 = t_{n-1}, \end{cases}$$

thus the center of $Z_G(S)$ is

$$Cent(Z_G(S)) = \{h_1(t_1) \ldots h_r(t_r) h_{r+1}(t_r) h_{r+2}(t_r) \ldots h_{n-1}(t_r) h_n(t_n) : t_i \in \bar{k}, t_n^2 = t_r\},$$

whence $S = Cent(Z_G(S))$ and it contains the center of H, where $Z_G(S) = SH$, and $H \simeq \mathrm{Spin}(f)$ is a group of type B_{n-r}. So $Z_G(S)/S \simeq \mathrm{SO}(f)$ which is rational and so is G.

3.2. Proposition. *Let G be an isotropic almost simple k-group of type* $C_{n,r}^{(d)}$, $n - rd \leq 3$. *Then G is either rational or stably rational over k.*

Proof. We consider these case separately. The Tits index of G is as follows

$$\bullet^{1} \cdots \bullet^{d-1} - - \odot^{d} - - \bullet \cdots \bullet^{rd-1} - - \odot^{rd} - - \bullet \cdots \bullet - - \bullet \Leftarrow \bullet^{n}$$

It is well-known that G is rational if G is simply connected (by using the Cayley transformation, see [Di] or [W]). If G is adjoint, S is a maximal k-split torus of G

and $n - rd \leq 2$, then

$$Z_G(S) = SH_1H_2,$$

where H_1 is a product of k-groups of type ${}^1\mathrm{A}_{d-1}$, and H_2 is a group of type C_{n-rd} (see Prop. 1.2). We know that S contains the center of the product H_1H_2 (see Proposition 1.3), therefore

$$Z_G(S)/S \simeq Ad(H_1)Ad(H_2),$$

the direct product of adjoint groups of type ${}^1\mathrm{A}$ and C. Since adjoint k-groups of type ${}^1\mathrm{A}$ are well-known to be rational, and the adjoint k-groups of type C_n is rational if $n \leq 2$, and stably rational if $n = 3$ (see [M2, Prop. 4]). The proposition is proved.

For the case of type D, we need to distinguish two cases.

3.3. Proposition. *Let G be almost simple of type $\mathrm{D}^d_{n,r}$, $r > 0$ such that $n - rd \leq 2$.*

a) If G is adjoint, then G is rational over k.

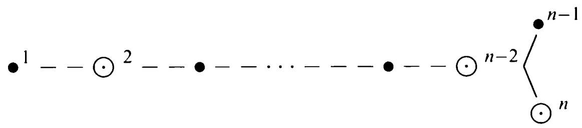
b) If $n - rd = 0$ (resp. $n - rd = 2$) and $d \leq 2$ (resp. $d = 1$) then G is rational over k.

Proof. *a)* Let S be a maximal k-split torus of G. Then

$$Z_G(S)/S = G_1 \times \cdots \times G_r \times D_{n-rd}$$

is a direct product of adjoint k-groups G_i of type ${}^1\mathrm{A}_{d-1}$ (hence rational) and a group of type D_{n-rd} which is also rational by the assumption on n, r, d. Now from results in Section 1 we conclude that G is rational over k.

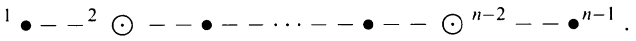
b) If $n - rd = 0$ then by [Ti1] G is of inner type and $n = r$ $(d = 1)$ (trivial case) or $n = 2r(d = 2)$. In the latter case let S be the k-split torus of G corresponding to the last circled vertex (root) in the Tits index of G:



Then

$$Z_G(S) = SA,$$

where A is an almost simple k-group of type ${}^1\mathrm{A}^{(2)}_{n-1,r-1}$ with Tits index



Then $Z_G(S)/S$ also has this type, hence from Section 2 we know that $Z_G(S)/S$ is rational, hence so is G.

If $n - rd = 2$ and $d = 1$, then the above A has rank two, so $Z_G(S)/S$ is again rational.

Now we focus our attention to exceptional groups and we have the following result. We excluded the group of type G_2 since isotropic group of this type is split.

3.4. Proposition. *Let G be an isotropic almost simple group of exceptional type over a field k.*

a) If G is of adjoint type D_4, F_4 then G is k-rational.

b) Let G be one of the following types: ${}^1E_6$; ${}^2E_{6,r}$ $r \geq 2$ and G is adjoint if G is of type ${}^2E^{16'}_{6,2}$ or ${}^2E^{16''}_{6,2}$; $E_{7,r}$, $r \geq 3$, $E_{8,r}$, $r \geq 3$. Then G is either k-rational or stably rational over k.

c) If G is of type $E^{78}_{7,1}$, then any k-group which is k-isogeneous to G is also birationally isomorphic to G over k as k-varieties.

Proof. *a)* The proof for groups of type adjoint type D_4 and F_4 was given in [T1, Prop. 2.3 and 3.3]. (There we proved G has weak approximation, but the result also holds for rationality.) It also follows from results of Section 1.

b) For the group of type ${}^1E_6$ we consider first the group ${}^1E^{(28)}_{6,2}$ with Tits index

$$\begin{array}{ccccccccc} & & & & \bullet^{2} & & & & \\ & & & & | & & & & \\ \odot^{1} & -- & \bullet^{3} & -- & \bullet^{4} & -- & \bullet^{5} & -- & \odot^{6} \end{array}$$

Let $\tilde{G}$ (resp. $\bar{G}$) be the simply connected covering (resp. adjoint group) of G and $\tilde{S}$ (resp. $\bar{S}$) a maximal k-split torus of $\tilde{G}$ (resp. $\bar{G}$). It is well-known that (see e.g. [Se])

$$Z_G(\tilde{S}) = \tilde{S}D,$$

where D is the Spin group of a quadratic form f which is a norm form of a division Cayley algebra, i.e., f is a Pfister form. By Merkurjev's result [M2, Prop. 7], the adjoint group of $SO(f)$ is stably rational over k. By Proposition 1.3, $\bar{S}$ contains the center of $Z_{\bar{G}}(\bar{S})$. Hence $\bar{G}$ is stably rational over k. For the case of simply connected G we need the following.

3.4.1. Lemma. *With above notation $\tilde{S}$ contains the center of D.*

Proof of Lemma. The Dynkin diagram of D consists of part of that of G with vertices $\{\alpha_2, \dots, \alpha_5\}$. $\tilde{S}$ is given as the connected component of identity of the group P defined by the following condition

$$P = \left\{ p = \prod_{1 \leq i \leq 6} h_i(t_i) : p x_\alpha(t) p^{-1} = x_\alpha(t), \forall \alpha = \alpha_2, \dots, \alpha_5 \right\}.$$

Since

$$h_i(t_i) x_\alpha(t) h_i(t_i)^{-1} = x_\alpha(t t_i^{\langle \alpha, \alpha_i \rangle}),$$

so we have the following system of equations for t_i:

$$\begin{cases} t_3^2 = t_1 t_4 \\ t_4^2 = t_2 t_3 t_5 \\ t_5^2 = t_4 t_6 \\ t_2^2 = t_4. \end{cases}$$

It follows that we have the following parametrization for P:

$$P = \{p = h_1(t_3^2 t_2^{-2}) h_2(t_2) h_3(t_3) h_4(t_2^2) h_5(t_3^{-1} t_2^3) h_6(t_3^{-2} t_2^4) : t_2, t_3 \in \bar{k}\},$$

which is clearly connected. Thus $S = P$. The center $Cent(D)$ of D is generated by

$$z_1 = h_2(-1) h_5(-1), \; z_2 = h_2(-1) h_3(-1),$$

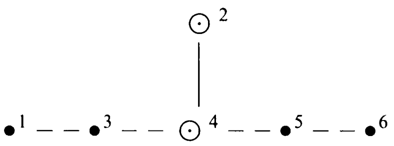
and we check immediately that

$$Cent(D) \subset S.$$

The lemma is proved.

Thus $\tilde{G}$, being k-birationally equivalent to $\tilde{S} \times Ad(D)$ which is k-birational to $\bar{G}$, is also stably rational over k.

Now we consider the case ${}^1\mathrm{E}^{16}_{6,2}$. Let G be an almost simple k-group of this type with Tits index

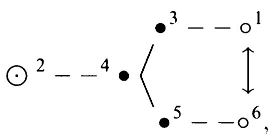


and let S be a k-split torus of G corresponding to the root α_2. Then

$$Z_G(S) = SD,$$

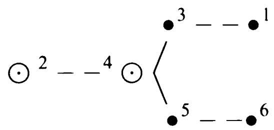
where D is of type ${}^1\mathrm{A}^3_{5,1}$ and $Z_G(S)/S$ is an almost simple k-group of the same type, hence is rational by results of Section 2. Therefore G is also k-rational.

Let G be simply connected of type $\mathrm{E}^{16'}_{6,2}$ with the following Tits index



From this we see that the anisotropic kernel of G is of type ${}^2\mathrm{A}^{(1)}_3$, which is also the anisotropic kernel A of the simply connected semisimple group H of type ${}^2\mathrm{A}_5$ with the root system spanned by $\Delta \backslash \{\alpha_2\}$. From Proposition 1.3 it follows that for a maximal k-split torus S of G, the center of the semisimple part A of $Z_G(S)$ is contained in the connected center of $Z_G(S)$, which is an induced torus. Therefore it is clear that the anisotropic semisimple quotient of G is an adjoint group of type ${}^2\mathrm{A}_3^{(1)}$. By a result of [ChM] (Prop. 2.5), adjoint groups of type ${}^2\mathrm{A}_3^{(1)}$ are rational. Hence G is also rational.

The case of type ${}^2\mathrm{E}_{6,2}^{16''}$ with Tits index



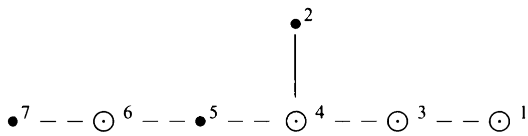
is considered in a similar way as above: If $\bar{S}$ (resp. $\bar{G} = Ad(G)$) have the meaning as above, then $\bar{S}$ contains the center of $Z_{\bar{G}}(\bar{S})$. Then

$$Z_{\bar{G}}(\bar{S})/\bar{S} \simeq \mathrm{R}_{l/k}(A),$$

where A is an adjoint l-group of type A_2 over a quadratic extension l of k, hence $\bar{G}$ is rational.

Now we assume that G is of the following types $\mathrm{E}_{7,r}$, where $r \geq 3$. We claim that they are all rational or stably rational.

Let G be a k-group of type $\mathrm{E}_{7,4}^9$, S be the standard k-split torus of G corresponding to the set of two k-roots $\{\alpha_1, \alpha_3\}$ as in the Tits index below:

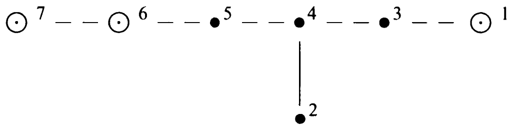


Then

$$Z_G(S) = SA,$$

where A is an almost simple k-group of type ${}^1\mathrm{A}_{5,2}^{(2)}$. Therefore $Z_G(S)/S$ has also this type, and is rational by results of Section 2 (Theorem 2.4). Hence G is rational.

Let G be of type $\mathrm{E}_{7,3}^{28}$, S the standard k-torus of G corresponding to the k-root α_7 in the following Tits index



Then

$$Z_G(S) = SA,$$

where A is a k-group of type ${}^1\mathrm{E}_{6,2}^{28}$. Any group of latter type is stably rational over k, hence the same is true for G. The case of group of type $\mathrm{E}_{8,r}$, $r \geq 3$ is similar.

c) It rests to consider the case $E_{7,1}^{78}$. Let S be a maximal k-split torus of G, where we assume that G is simply connected. One checks that S contains the center of the anisotropic kernel of type E_6 of G. Thus

$$Z_G(S)/S \simeq Z_{\bar{G}}(\bar{S})/\bar{S}$$

and we are done.

4. Concluding remarks

4.1. We say that a segment in the Tits index of an almost simple k-group G is *black* (resp. *white*) if it consists of only black (resp. white, i.e., distinguished) vertices. The *length* of a segment is the number of vertices it contains. We say that a segment is *defined over* k, if the almost simple subgroup of G with root system spanned on this segment is defined over k. In other words, the black segments are the connected components of the Dynkin diagram of anisotropic kernel of G in the Tits index of G, which are k-defined. From results proved above we derive the following main result of this paper.

4.2. Theorem. *Let G be an almost simple adjoint k-group and $m(G)$ be the maximal length of the black segments of its Tits index defined over k. If G is of classical type $X_{n,r}^{(d)}$ with $n - rd \leq 2$, or G is of exceptional type with $m(G) \leq 3$ then G is either rational or stably rational over k.*

Proof. It follows from results of Section 2 (Prop. 2.6), Section 3 (Prop. 3.1–3.4) and the Tits classification of indices [Ti1].

Remark. The number 2 in the theorem is best possible. According to [M2] there exist non stably rational adjoint groups G and fields k with $m(G) = 3$, k can even be chosen a number field. Also if k is a field such that there exist division algebras D of index $4n$ (e.g. a number field), then for the group G with $G(k) = \mathrm{SL}_m(D)$, the subgroup of reduced norm 1 of $M_m(D)$, then G is not stably rational over k and $m(G) = 4n - 1$.

5. Appendix

In this section we give some useful and frequently used formulas related with Tits index of an almost simple k-group G. We keep the same notation adopted in the Introduction.

We give here only formulas for centers $\tilde{F}$ of simply connected groups G, via generators.

$\mathbf{A}_n$:

$$\tilde{F} = \langle h_1(\zeta)h_2(\zeta^2)\dots h_n(\zeta^n)\rangle,$$

where $\zeta = (n+1)$-th primitive root of 1.

$\mathbf{B}_n$:

$$\tilde{F} = \langle h_n(-1)\rangle.$$

$\mathbf{C}_n$:

$$\tilde{F} = \langle h_1(-1)h_3(-1)\cdots h_{2[l+1/2]-1}(-1)\rangle.$$

$\mathbf{D}_n$, n even:

$$\tilde{F} = \langle z_1, z_2\rangle,$$

where

$$z_1 = h_1(-1)h_3(-1)\cdots h_{n-3}(-1)h_{n-1}(-1),$$
$$z_2 = h_1(-1)h_3(-1)\cdots h_{n-3}(-1)h_n(-1).$$

$\mathbf{D}_n$, n odd:

$$\tilde{F} = \langle h_1(-1)h_3(-1)\dots h_{n-2}(-1)h_{n-1}(-i)h_n(i)\rangle,$$

where $i = \sqrt{-1}$.

$\mathbf{E}_6$:

$$\tilde{F} = \langle h_1(\zeta)h_3(\zeta^2)h_5(\zeta)h_6(\zeta^2)\rangle,$$

where ζ is a primitive cubic root of 1.

$\mathbf{E}_7$:

$$\tilde{F} = \langle h_2(-1)h_5(-1)h_7(-1)\rangle.$$

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