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Tauberian theorem of exponential type and its application to multiple convolution

By

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1. Introduction

Let $\{U_n(x)\}$ be a sequence of non-decreasing, right-continuous functions on $\mathbf{R}$ vanishing on $(-\infty, 0]$, and let $\omega_n(s)$ be the Laplace transform of $U_n(x)$. In this paper, we shall study the relationship between the asymptotic behavior of $\log \omega_n(ns)$ and that of $\log U_n(x)$. This problem is motivated by the following question: Let $X_1, X_2, \ldots$ be positive, independent random variables with common distribution. By the law of large numbers, we see that $X_1 + X_2 + \cdots + X_n \to \infty$ as $n \to \infty$, *a.s.*, and it is of interest to know how fast $P[X_1 + X_2 + \cdots + X_n \leq a](a > 0)$ tends to 0 as $n \to \infty$. In other words, we are interested in the asymptotic behavior of the multiple convolution

$$\int \cdots \int_{0 < x_1 + x_2 + \cdots + x_n \leq a} d\sigma(x_1) d\sigma(x_2) \cdots d\sigma(x_n), (a > 0)$$

as $n \to \infty$, where $\sigma(x)$ is a right-continuous non-decreasing function vanishing on $(-\infty, 0]$ and here we no longer need to assume that σ is a distribution function. This may be considered as a problem of large deviation. In a study of the local time of Gaussian processes, Kasahara, *et al.* obtained the following result (Lemma 3 of [11]). If σ varies regularly at 0 (i.e., $\lim_{\lambda \to 0} \sigma(\lambda x)/\sigma(\lambda) = x^\alpha, x > 0$, for some α; see [1]), then

$$\left(\int \cdots \int_{0 < x_1 + \cdots + x_n \leq 1} d\sigma(x_1) \cdots d\sigma(x_n) \right)^{1/n} \asymp \sigma\left(\frac{1}{n}\right), \qquad n \to \infty, \tag{1.1}$$

where $f \asymp g$ means $0 < \liminf f(x)/g(x) \leq \limsup f(x)/g(x) < \infty$. Our question is to determine the exact constant in (1.1). To this end, we first consider the case of $\sigma(x) = x_+^\alpha (\alpha > 0)$, where $x_+ = x \vee 0$. An elementary calculus provides us with

$$\int \cdots \int_{0 < x_1 + \cdots + x_n \leq 1} d\sigma(x_1) d\sigma(x_2) \cdots d\sigma(x_n) = \frac{\Gamma(\alpha + 1)^n}{\Gamma(\alpha n + 1)}, \tag{1.2}$$

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where $\Gamma(\alpha)$ is the gamma function (see Appendix for details). Therefore, by Stirling's formula, we get

$$\lim_{n\to\infty}\frac{1}{\sigma(1/n)}\left(\int\cdots\int_{0<x_1+\cdots+x_n\le 1} d\sigma(x_1)\cdots d\sigma(x_n)\right)^{1/n} = \left(\frac{e}{\alpha}\right)^{\alpha}\Gamma(\alpha+1). \tag{1.3}$$

One of the main results of this paper is that (1.3) remains valid if σ varies regularly at 0 with exponent α (Theorem 2). The idea of the proof is as follows. Put

$$U_n(x) = \frac{1}{(\sigma(1/n))^n}\int\cdots\int_{0<x_1+\cdots+x_n\le x} d\sigma(x_1)\cdots d\sigma(x_n) \tag{1.4}$$

and consider its Laplace transform. Since

$$\int_0^{\infty} e^{-nsx} dU_n(x) = \left(\frac{1}{\sigma(1/n)}\int_0^{\infty} e^{-nsx} d\sigma(x)\right)^n,$$

we have

$$\frac{1}{n}\log\int_0^{\infty} e^{-nsx} dU_n(x) = \log\frac{1}{\sigma(1/n)}\int_0^{\infty} e^{-nsx} d\sigma(x). \tag{1.5}$$

If $\sigma(x)$ is a regularly varying function, we can apply Karamata's Tauberian theorem (see [6] pp. 442–448) to the right side of (1.5) and have

$$\lim_{n\to\infty}\frac{1}{n}\log\int_0^{\infty} e^{-nsx} dU_n(x) = -\alpha\log s + \log\Gamma(\alpha+1). \tag{1.6}$$

Thus, our problem is deduced to a certain kind of Tauberian theorem of exponential type. There have been many works on this subject. For examples, Davies ([4]), Fukushima ([7]), Fukushima, Nagai and Nakao ([8]), Kasahara ([10]), Kohlbecker ([12]), Kôno ([13]), Minlos and Povzner ([14]), Nagai ([15]), and so on. Especially, Kasahara ([10]) shows that all of the above works are deduced to the relationship between measures $dU(x)$ and their Laplace transforms

$$\int_0^{\infty} \exp\{\lambda f(x/\phi(\lambda))\} dU(x), \tag{1.7}$$

where $\phi(\lambda)$ is a regularly varying function, and gives a Tauberian theorem in a most general form. From (1.7), notice that a measure dU is fixed in each Tauberian theorem of exponential type that we mentioned above, and thus we can apply none of these works to the case of (1.6), in which we have to treat the measures $dU_n(x)$ depending on n. Therefore, in the present paper we construct a Tauberian theorem which treats the case where the measures $dU_n(x)$ depend on n. Afterwards, we shall see that our theorem contains a special case of Kasahara's theorem, and the proof of our theorem becomes much easier than that of his.

We remark that, as we mentioned before, our problem may be treated in the framework of large deviations. Especially, our theorem is much silimar to the theorem due to J. Gärtner (see [9]) and R. S. Ellis (see [5]), which, however, does not contain ours.

This paper consists as follows: In Section 2, as we mentioned in the above, we give a new Tauberian theorem, and using it, we shall show that (1.3) holds in the case where σ is a regularly varying function at 0. The proof of the Tauberian theorem is given in Section 3. In section 4, we give another Tauberian theorem in which the roles of the origin and infinity are interchanged and show that our theorem includes a part of Kasahara's Tauberian theorem.

2. Main Theorem

Let $\varphi(s) \in C^1(0, \infty)$ be a decreasing convex function such that

$$\lim_{\varepsilon \to 0+} \varphi'(\varepsilon) = -\infty, \qquad \lim_{s \to \infty} \varphi'(s) = 0, \tag{2.1}$$

and define

$$\varphi^*(x) = \inf_{s>0}\{sx + \varphi(s)\}, \qquad x > 0. \tag{2.2}$$

Then $\varphi^*(x)$ is a non-decreasing concave function on $(0, \infty)$. Indeed, from (2.2), it is easy to see that φ^* is non-decreasing, and it also follows that

$$\varphi^*(tx + (1-t)y) \geq t\varphi^*(x) + (1-t)\varphi^*(y), \qquad \text{for } 0 \leq t \leq 1,$$

which proves the concavity. Since $s \mapsto sx + \varphi(s)$ attains its minimum at s such that $x + \varphi'(s) = 0$, denoting by $g(x)$ the inverse function of $-\varphi'(s)$, we have

$$\varphi^*(x) = xg(x) + \varphi(g(x)). \tag{2.3}$$

By (2.2), it follows that $\varphi^*(x) \leq sx + \varphi(s)$ for all $s > 0$ and $x > 0$, and therefore, it is easy to see that

$$\varphi(s) \geq \sup_{x>0}\{-sx + \varphi^*(x)\}.$$

However, from (2.3), we have

$$\varphi(s) = -sx + \varphi^*(x) \qquad \text{if } x = -\varphi'(s), \tag{2.4}$$

and hence, in fact it holds that

$$\varphi(s) = \sup_{x>0}\{-sx + \varphi^*(x)\}. \tag{2.5}$$

For example, put $\varphi(s) = -\alpha \log s$ $(\alpha > 0)$. Then, $g(x) = \alpha/x$ and $\varphi^*(x) = \alpha \log x + \alpha \log(e/\alpha)$. For another example, if $\varphi(s) = s^{-\alpha}$ $(\alpha > 0)$, then, $g(x) = (\alpha/x)^{1/(\alpha+1)}$ and $\varphi^*(x) = (1+\alpha)\ (x/\alpha)^{\alpha/(\alpha+1)}$.

Remark that from (2.4) and (2.5), we can see that $x \mapsto \varphi^*(x) - sx$ takes its maximal value $\varphi(s)$ at $x = -\varphi'(s)$, and furthermore, $x = -\varphi'(s)$ is the unique solution of $\varphi^*(x) - sx = \varphi(s)$, for a given $s > 0$.

Now we state our main theorem:

Theorem 1. *Let $\varphi(s), \varphi^*(x)$ be as above. Suppose $U_n(x)$ be a sequence of non-decreasing, right-continuous functions on* $\mathbf{R}$ *vanishing on* $(-\infty, 0]$. *If*

$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^\infty e^{-nsx} \mathrm{d}U_n(x) = \varphi(s), \qquad \textit{for all } s > 0, \tag{2.6}$$

then

$$\lim_{n\to\infty} \frac{1}{n} \log U_n(x) = \varphi^*(x), \qquad \textit{for all } x > 0. \tag{2.7}$$

Conversely, if

$$\limsup_{n\to\infty} \frac{1}{n} \log \int_0^\infty e^{-nsx} \mathrm{d}U_n(x) < \infty, \qquad \textit{for all } s > 0,$$

then (2.7) *implies* (2.6).

We postpone the proof of Theorem 1 and we apply the above result to multiple convolution.

Theorem 2. *Let $\alpha > 0$, and $\sigma(x)$ $(x \in \mathbf{R})$ be a non-decreasing, right-continuous function vanishing on $(-\infty, 0]$.*

(i) *If $\sigma(x)$ varies regularly at* 0 *with exponent α, then*

$$\lim_{n\to\infty} \frac{1}{\sigma(1/n)} \left(\int \cdots \int_{0 < x_1 + \cdots + x_n \leq x} \mathrm{d}\sigma(x_1) \cdots \mathrm{d}\sigma(x_n) \right)^{1/n}$$

$$= x^\alpha \left(\frac{e}{\alpha}\right)^\alpha \Gamma(\alpha + 1), \qquad \textit{for every } x > 0. \tag{2.8}$$

(ii) *Conversely, if* (2.8) *holds and if*

$$\limsup_{n\to\infty} \frac{1}{\sigma(1/n)} \int_0^\infty e^{-nsx} \mathrm{d}\sigma(x) < \infty, \qquad \textit{for all } s > 0, \tag{2.9}$$

then, $\sigma(x)$ varies regularly at 0 *with exponent α.*

Proof of Theorem 2.

(i) Recall that if we put

$$U_n(x) = \frac{1}{(\sigma(1/n))^n} \int \cdots \int_{0 < x_1 + \cdots + x_n \leq x} \mathrm{d}\sigma(x_1) \cdots \mathrm{d}\sigma(x_n),$$

then

$$\frac{1}{n}\log\int_0^\infty e^{-nsx}\mathrm{d}U_n(x)=\log\frac{1}{\sigma(1/n)}\int_0^\infty e^{-nsx}\mathrm{d}\sigma(x).$$

(See (1.4) and (1.5) in Section 1). Using Karamata's Tauberian theorem, we have

$$\lim_{n\to\infty}\frac{1}{\sigma(1/n)}\int_0^\infty e^{-nsx}\mathrm{d}\sigma(x)=s^{-\alpha}\Gamma(\alpha+1). \tag{2.10}$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\log\int_0^\infty e^{-nsx}\mathrm{d}U_n(x)=-\alpha\log s+\log\Gamma(\alpha+1). \tag{2.11}$$

As we mentioned above, if $\varphi(s)=-\alpha\log s+\log\Gamma(\alpha+1)$, then $\varphi^*(x)=\alpha\log x+\alpha\log(e/\alpha)+\log\Gamma(\alpha+1)$. Applying Theorem 1, we see that (2.11) implies

$$\lim_{n\to\infty}\frac{1}{n}\log U_n(x)=\alpha\log x+\alpha\log(e/\alpha)+\log\Gamma(\alpha+1), \tag{2.12}$$

which proves (2.8).

(ii) Assume (2.8), and put U_n as in (1.4). Then, (2.8) and (1.4) imply (2.12). By the assumption (2.9), we can apply Theorem 1 to (2.12) and thus we have (2.11). As (2.11) means (2.10), using Karamata's Tauberian theorem, we see that σ is a regularly varying function at 0 with exponent α.

Consider a positive sequence a_n which tends to ∞ as $n\to\infty$. We remark that the proof of the following theorem can be carried out completely in parallel with that of Theorem 1.

Theorem 1a. *Let $\varphi(s),\varphi^*(x)$, and $U_n(x)$ be as Theorem 1, and let a_n be a positive sequence which tends to ∞ as $n\to\infty$. If*

$$\lim_{n\to\infty}\frac{1}{a_n}\log\int_0^\infty e^{-a_nsx}\mathrm{d}U_n(x)=\varphi(s),\qquad \textit{for all } s>0, \tag{2.13}$$

then

$$\lim_{n\to\infty}\frac{1}{a_n}\log U_n(x)=\varphi^*(x),\qquad \textit{for all } x>0. \tag{2.14}$$

Conversely, if

$$\limsup_{n\to\infty}\frac{1}{a_n}\log\int_0^\infty e^{-a_nsx}\mathrm{d}U_n(x)<\infty,\qquad \textit{for all } s>0,$$

then (2.14) *implies* (2.13).

Furthermore, since

$$\frac{1}{a_n}\log\int_0^\infty e^{-b_nsx}\mathrm{d}U_n(x)$$

can be rewritten as

$$\frac{1}{a_n} \log \int_0^\infty e^{-a_n s x} \mathrm{d} U_n(a_n x/b_n),$$

we obtain

Theorem 1b. *Let $\varphi(s), \varphi^*(x)$ and $U_n(x)$ be as in Theorem* 1, *and let a_n and b_n be positive sequences, where a_n tends to ∞ as $n \to \infty$. If*

$$\lim_{n\to\infty} \frac{1}{a_n} \log \int_0^\infty e^{-b_n s x} \mathrm{d} U_n(x) = \varphi(s), \qquad \textit{for all } s > 0, \tag{2.15}$$

then

$$\lim_{n\to\infty} \frac{1}{a_n} \log U_n\left(\frac{a_n}{b_n} x\right) = \varphi^*(x), \qquad \textit{for all } x > 0. \tag{2.16}$$

Conversely, if

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \int_0^\infty e^{-b_n s x} \mathrm{d} U_n(x) < \infty, \qquad \textit{for all } s > 0,$$

then (2.16) *implies* (2.15).

According to this extention, we have the following corollaries.

Corollary 1. *Let $X_1, X_2, \ldots$ be positive independent random variables with common distribution function σ, and let β_n and γ_n be positive sequences. If*

$$\lim_{n\to\infty} \log \frac{1}{\beta_n} \int_0^\infty e^{-sx/\gamma_n} \mathrm{d}\sigma(x) = \varphi(s), \qquad \textit{for all } s > 0, \tag{2.17}$$

then for every $x > 0$,

$$\log P\left[\frac{X_1 + \cdots + X_n}{n} \leq \gamma_n x\right] = n(\log \beta_n + \varphi^*(x) + o(1)), \qquad n \to \infty. \tag{2.18}$$

Proof. Put

$$U_n(x) = \frac{1}{(\beta_n)^n} \underset{0 < x_1 + \cdots + x_n \leq x}{\int \cdots \int} \mathrm{d}\sigma(x_1) \cdots \mathrm{d}\sigma(x_n).$$

By using the same method as the proof of Theorem 2, we have

$$\frac{1}{n} \log \int_0^\infty e^{-sx/\gamma_n} \mathrm{d} U_n(x) = \log \frac{1}{\beta_n} \int_0^\infty e^{-sx/\gamma_n} \mathrm{d}\sigma(x). \tag{2.19}$$

Combining (2.17) and (2.19), we see

$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^\infty e^{-sx/\gamma_n} \mathrm{d} U_n(x) = \varphi(s). \tag{2.20}$$

Applying Theorem 1b to (2.20), we have

$$\lim_{n\to\infty}\frac{1}{n}\log U_n(n\gamma_n x)=\varphi^*(x).$$

Since

$$U_n(n\gamma_n x)=\frac{1}{(\beta_n)^n}\int_{0<(x_1-\cdots+x_n)/n\le\gamma_n x}\!\!\cdots\!\!\int d\sigma(x_1)\cdots d\sigma(x_n),$$

these two equations imply (2.18).

We remark that if $\beta_n=1$ and $\gamma_n=1$, then the above corollary is a special case of Chernoff's theorem ([3], see also [2]).

Corollary 2. *Let $X_1, X_2, \ldots$ be positive independent random variables with common distribution function σ, and let γ_n be a positive sequence such that γ_n tends to 0 as n goes to ∞. If $\sigma(x)$ varies regularly at 0 with exponent $\alpha(\alpha>0)$, then for every $x>0$,*

$$\log P\left[\frac{X_1+X_2+\cdots+X_n}{n}\le\gamma_n x\right]$$
$$=n(\log\sigma(\gamma_n)+\alpha\log x+\alpha\log(e/\alpha)+\log\Gamma(\alpha+1)+o(1)),$$
$$n\to\infty.$$

Proof. Applying Theorem 1b to the proof of Theorem 2, we see immediately

$$\lim_{n\to\infty}\frac{1}{\sigma(\gamma_n)}\left(\int_{0<(x_1+\cdots+x_n)/n\le\gamma_n x}\!\!\cdots\!\!\int d\sigma(x_1)\cdots d\sigma(x_n)\right)^{1/n}=x^\alpha\left(\frac{e}{\alpha}\right)^\alpha\Gamma(\alpha+1),$$

which proves the assertion.

3. Proof of Theorem 1

We prepare a few Lemmas to prove Theorem 1. According to the assumptions of Theorem 1, we may and do assume that

$$\limsup_{n\to\infty}\frac{1}{n}\log\int_0^\infty e^{-nsx}dU_n(x)<\infty,\qquad\text{for all } s>0,$$

throughout this section.

Lemma 3.1. *Suppose*

$$\limsup_{n\to\infty}\frac{1}{n}\log\int_0^\infty e^{-nsx}dU_n(x)\le\varphi(s),\qquad\textit{for all } s>0.$$

Then,

$$\limsup_{n\to\infty} \frac{1}{n} \log U_n(x) \le \varphi^*(x), \qquad \text{for all } x > 0.$$

Proof. We need nothing but Chebyshev's inequality. For each $y > 0$,

$$\begin{aligned}\int_0^\infty e^{-nsx} \mathrm{d}U_n(x) &\ge \int_0^y e^{-nsx} \mathrm{d}U_n(x) \\ &\ge e^{-nsy} U_n(y).\end{aligned} \tag{3.1}$$

Hence

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log U_n(y) &\le \inf_{s>0}\{sy + \varphi(s)\} \\ &= \varphi^*(y),\end{aligned}$$

which proves Lemma 3.1.

Lemma 3.2. *Suppose*

$$\liminf_{n\to\infty} \frac{1}{n} \log U_n(x) \ge \varphi^*(x), \qquad \text{for all } x > 0.$$

Then

$$\liminf_{n\to\infty} \frac{1}{n} \log \int_0^\infty e^{-nsx} \mathrm{d}U_n(x) \ge \varphi(s), \qquad \text{for all } s > 0.$$

Proof. From (3.1), we have

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log \int_0^\infty e^{-nsx} \mathrm{d}U_n(x) &\ge \sup_{y>0}\{-sy + \varphi^*(y)\} \\ &= \varphi(s),\end{aligned}$$

which proves Lemma 3.2.

Lemma 3.3. *Suppose*

$$\limsup_{n\to\infty} \frac{1}{n} \log U_n(x) \le \varphi^*(x), \qquad \text{for all } x > 0.$$

For a fixed $s > 0$, let $x_0 = -\varphi'(s)$. Then,

(i) $\displaystyle \limsup_{n\to\infty} \frac{1}{n} \log \int_\mu^\infty e^{-nsx} \mathrm{d}U_n(x) \le -s\mu + \varphi^*(\mu)$ *for each* $\mu > x_0$,

(ii) $\displaystyle \limsup_{n\to\infty} \frac{1}{n} \log \int_0^\mu e^{-nsx} \mathrm{d}U_n(x) \le -s\mu + \varphi^*(\mu)$ *for each* $0 < \mu < x_0$.

Proof. Before proving (i) and (ii), remark that $x \mapsto \varphi^*(x) - sx$ attains its maximum at x_0 as we mentioned in the previous section. Therefore, if $\mu < x_0$, then $-s\mu + \varphi^*(\mu)$ is a monotone increasing function, while $-s\mu + \varphi^*(\mu)$ is monotone decreasing if $\mu > x_0$. To prove Lemma 3.3 and 3.4, we use the following fact: If

$$\limsup_{n\to\infty} \frac{1}{n} \log a_n \le c_1,$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log b_n \le c_2,$$

then

$$\limsup_{n\to\infty} \frac{1}{n} \log(a_n + b_n) \le \max\{c_1, c_2\}.$$

(i) For each $\delta > 0$, put $\mu_k = \mu + \delta k$ $(k = 0, 1, 2, \ldots)$. Then,

$$\int_{\mu_k}^{\mu_{k+1}} e^{-nsx} \mathrm{d} U_n(x) \le e^{-ns\mu_k} U_n(\mu_{k+1}).$$

Hence

$$\limsup_{n\to\infty} \frac{1}{n} \log \int_{\mu_k}^{\mu_{k+1}} e^{-nsx} \mathrm{d} U_n(x) \le -s\mu_k + \varphi^*(\mu_{k+1}),$$

which implies

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log \int_{\mu}^{N} e^{-nsx} \mathrm{d} U_n(x) &\le \max_k \{-s\mu_k + \varphi^*(\mu_{k+1})\} \\ &\le -s\mu + \varphi^*(\mu) + s\delta,\end{aligned}$$

for each $N > \mu$. Since $\delta > 0$ is arbitrary, by putting $\delta \downarrow 0$ we get

$$\limsup_{n\to\infty} \frac{1}{n} \log \int_{\mu}^{N} e^{-nsx} \mathrm{d} U_n(x) \le -s\mu + \varphi^*(\mu).$$

On the other hand, for each $N > \mu$,

$$\begin{aligned}\int_N^\infty e^{-nsx} \mathrm{d} U_n(x) &= \int_N^\infty e^{-n(s/2)x} e^{-n(s/2)x} \mathrm{d} U_n(x) \\ &\le e^{-n(s/2)N} \int_0^\infty e^{-n(s/2)x} \mathrm{d} U_n(x).\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \log \int_N^\infty e^{-nsx} \mathrm{d} U_n(x) \le -\frac{s}{2} N + \limsup_{n\to\infty} \frac{1}{n} \log \int_0^\infty e^{-n(s/2)x} \mathrm{d} U_n(x).$$

By assumption,

$$\limsup_{n\to\infty} \frac{1}{n} \log \int_0^\infty e^{-n(s/2)x} \mathrm{d}U_n(x) < A, \qquad \text{for some } A,$$

and hence

$$\limsup_{n\to\infty} \frac{1}{n} \log \int_\mu^\infty e^{-nsx} \mathrm{d}U_n(x) \le (-s\mu + \varphi^*(\mu)) \vee \left(-\frac{s}{2}N + A\right).$$

Since N is arbitrary, by choosing sufficiently large N, we get

$$\limsup_{n\to\infty} \frac{1}{n} \log \int_\mu^\infty e^{-nsx} \mathrm{d}U_n(x) \le -s\mu + \varphi^*(\mu).$$

Similarly we can prove (ii).

Lemma 3.4. *Suppose*

$$\limsup_{n\to\infty} \frac{1}{n} \log U_n(x) \le \varphi^*(x), \qquad \textit{for all } x > 0.$$

Then

$$\limsup_{n\to\infty} \frac{1}{n} \log \int_0^\infty e^{-nsx} \mathrm{d}U_n(x) \le \varphi(s), \qquad \textit{for all } s > 0.$$

Proof. Let x_0 be as in Lemma 3.3, and choose x_1 and x_2 so that $0 < x_1 < x_0 < x_2 < \infty$. Then,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log \int_{x_1}^{x_2} e^{-nsx} \mathrm{d}U_n(x) &\le \limsup_{n\to\infty} \frac{1}{n} \log\{e^{-nsx_1} U_n(x_2)\} \\
&\le -sx_1 + \varphi^*(x_2).
\end{aligned}$$

Therefore, by Lemma 3.3,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log \int_0^\infty e^{-nsx} \mathrm{d}U_n(x) \\
&= \limsup_{n\to\infty} \frac{1}{n} \log \left(\int_0^{x_1} e^{-nsx} \mathrm{d}U_n(x) + \int_{x_1}^{x_2} e^{-nsx} \mathrm{d}U_n(x) + \int_{x_2}^\infty e^{-nsx} \mathrm{d}U_n(x) \right) \\
&\le \max\{-sx_1 + \varphi^*(x_1), -sx_1 + \varphi^*(x_2), -sx_2 + \varphi^*(x_2)\}.
\end{aligned}$$

As we mentioned in Section 2, $x_0 = -\varphi'(s)$ turns to be the unique solution of

$$\varphi^*(x) - sx = \varphi(s), \qquad \text{for a given } s > 0.$$

Thus, letting $x_1 \uparrow x_0$, $x_2 \downarrow x_0$, we see

$$\limsup_{n\to\infty} \frac{1}{n} \log \int_0^\infty e^{-nsx} \mathrm{d}U_n(x) \le -sx_0 + \varphi^*(x_0) = \varphi(s).$$

Lemma 3.5. *Suppose*

$$\limsup_{n\to\infty} \frac{1}{n} \log U_n(x) \le \varphi^*(x), \qquad \textit{for all } x > 0,$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \int_0^\infty e^{-nsx} \mathrm{d}U_n(x) \ge \varphi(s), \qquad \textit{for all } s > 0. \tag{3.2}$$

Then,

$$\lim_{n\to\infty} \frac{1}{n} \log U_n(x) = \varphi^*(x), \qquad \textit{for all } x > 0.$$

Proof. For a given $x > 0$, choose any $0 < y < x$. Since $\varphi(s)$ is a C^1 function satisfying (2.1) by assumption, we may choose $s > 0$ such that $y < -\varphi'(s) < x$. Put $x_0 = -\varphi'(s)$, then by Lemma 3.3,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log \int_0^y e^{-nsx} \mathrm{d}U_n(x) \\
&\le -sy + \varphi^*(y) \\
&< -sx_0 + \varphi^*(x_0) = \varphi(s),
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log \int_x^\infty e^{-nsx} \mathrm{d}U_n(x) \\
&\le -sx + \varphi^*(x) \\
&< -sx_0 + \varphi^*(x_0) = \varphi(s).
\end{aligned} \tag{3.4}$$

(3.2), (3.3), and (3.4) imply

$$\liminf_{n\to\infty} \frac{1}{n} \log \int_y^x e^{-nsx} \mathrm{d}U_n(x) \ge \varphi(s).$$

On the other hand, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \int_y^x e^{-nsx} \mathrm{d}U_n(x) \le -sy + \liminf_{n\to\infty} \frac{1}{n} \log U_n(x).$$

Thus, combining these two inequalities, we see

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log U_n(x) &\ge sy + \varphi(s) \\
&\ge \varphi^*(y).
\end{aligned}$$

Letting $y \uparrow x$, we obtain the assertion.

We are now ready to show Theorem 1. Assume (2.6), then combining Lemmas 3.1 and 3.5, we get (2.7). For the converse half, if we assume (2.7), then Lemmas 3.2 and 3.4 imply (2.6).

4. Another asymptotic behavior

In this section, as we mentioned in Section 1, we show that our theorem contains a part of Kasahara's Tauberian theorem. Let us see Kasahara's Tauberian theorem at first. Assume α to be a fixed positive number and $f(x)$ ($\not\equiv$ *const.*) to be a real valued non-decreasing function defined on the interval $(0, \infty)$ such that $f(\xi^{\beta})$ is concave for some $\beta(>\alpha)$. Put

$$g(x) = \sup_{\xi>0}\{f(\xi^{\alpha}) + x\xi\}, \qquad \text{for } x < 0,$$

then the following theorem holds.

Theorem A ([10]). *Suppose $\mu(\mathrm{d}x)$ be a finite Borel measure on $(0, \infty)$ and $\phi(x)$ be a regularly varying function with exponent α. Set*

$$\omega(\lambda) = \int_0^{\infty} \exp\{\lambda f(x/\phi(\lambda))\}\mu(\mathrm{d}x).$$

Then;

$$\lim_{x\to\infty} \frac{1}{x} \log \mu(\phi(x), \infty) = A(<0)$$

if and only if

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log \omega(\lambda) = g(A).$$

In the special case where $f(x) = x$, we have

Theorem B ([10]). *Set $0 < \alpha < 1$. Let $\phi(x)$ be a positive function varying regularly at ∞ with exponent α and $\psi(x)$ be the asymptotic inverse of $x/\phi(x)$. Suppose $\mu(\mathrm{d}x)$ be a finite Borel measure on $(0, \infty)$. Then,*

$$\lim_{x\to\infty} \frac{1}{x} \log \mu(\phi(x), \infty) = -A < 0$$

if and only if

$$\lim_{\lambda\to\infty} \frac{1}{\psi(\lambda)} \log \int_0^{\infty} e^{\lambda x}\mu(\mathrm{d}x) = (1-\alpha)(\alpha/A)^{\alpha/(1-\alpha)}.$$

Now we go back to our Tauberian theorem. By considering

$$\int_0^{\infty} \exp\{-nsf(x)\}\mathrm{d}U_n(x) \tag{4.1}$$

instead of $\int_0^\infty \exp\{-nsx\}\mathrm{d}U_n(x)$, we can extend Theorem 1. Here, we study the case where the roles of the origin and infinity are interchanged in Theorem 1. If we put $f(x) = -1/x$ in (4.1), then we can have the following theorem. Let $\varphi(s) \in C^1(0, \infty)$ be an increasing convex function. Suppose

$$\lim_{\varepsilon \to 0+} \varphi'(\varepsilon) = 0, \qquad \lim_{s \to \infty} \varphi'(s) = +\infty.$$

Define $\varphi_*(x)$ as

$$\varphi_*(x) = \inf_{s>0}\{\varphi(s) - sx\}.$$

From this definition, φ_* is a non-increasing concave function on $(0, \infty)$. If we denote by $g(x)$ the inverse function of $\varphi'(s)$, then we have

$$\varphi_*(x) = \varphi(g(x)) - xg(x),$$

and

$$\varphi(s) = \sup_{x>0}\{\varphi_*(x) + sx\}.$$

Furthermore, for each $s > 0$, there exists a positive unique solution of

$$\varphi_*(x) + sx = \varphi(s).$$

For example, if we put $\varphi(s) = s^\alpha$, for $\alpha > 1$, then $g(x) = (x/\alpha)^{1/(\alpha-1)}$ and $\varphi_*(x) = (1-\alpha)(x/\alpha)^{\alpha/(\alpha-1)}$.

Theorem 3. *Let $\varphi(s)$ and $\varphi_*(x)$ be as above. Suppose $\mu_n(\mathrm{d}x)$ be a sequence of Radon measures on $(0, \infty)$. If*

$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^\infty e^{nsx}\mu_n(\mathrm{d}x) = \varphi(s), \qquad \textit{for all } s > 0, \tag{4.2}$$

then

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_n(x, \infty) = \varphi_*(x), \qquad \textit{for all } x > 0. \tag{4.3}$$

Conversely, if

$$\limsup_{n\to\infty} \frac{1}{n} \log \int_0^\infty e^{nsx}\mu_n(\mathrm{d}x) < \infty, \qquad \textit{for all } s > 0,$$

then (4.3) *implies* (4.2).

Proof. Since the proof of Theorem 3 is essentially the same as that of Theorem 1, we omit the details.

Now we study the relationship between Theorem 3 and Tauberian theorems of exponential type which are already known. To see that Theorem 3 contains Theorem B, it suffices to show the following proposition.

Proposition. *Let $B > 0$, and $\phi(x)$ be a positive function varying regularly at ∞ with exponent $\alpha (0 < \alpha < 1)$. Suppose $\mu(\mathrm{d}x)$ be a Radon measure on $(0, \infty)$. If*

$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^\infty \exp\{nx/\phi(n)\}\mu(\mathrm{d}x) = B, \tag{4.4}$$

then

$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^\infty \exp\{nsx/\phi(n)\}\mu(\mathrm{d}x) = Bs^{1/(1-\alpha)}.$$

Proof. For a given $s > 0$, put $\xi = s^{1/(1-\alpha)}$. Let $\varepsilon > 0$ be arbitrary and put $\eta = (1+\varepsilon)^{-1/(1-\alpha)}\xi$. Replacing n by $n\eta$ in (4.4), we get

$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^\infty \exp\{n\eta x/\phi(n\eta)\}\mu(\mathrm{d}x) = B\eta. \tag{4.5}$$

By the property of regularly varying function, there exists an $N_\varepsilon > 0$ such that

$$(1-\varepsilon)\left(\frac{1}{\eta}\right)^\alpha \le \frac{\phi(n)}{\phi(n\eta)} \le (1+\varepsilon)\left(\frac{1}{\eta}\right)^\alpha, \qquad \text{for } n \ge N_\varepsilon. \tag{4.6}$$

Applying (4.6) to (4.5), we have

$$\begin{aligned}
&\frac{1}{n} \log \int_0^\infty \exp\left\{\frac{n\eta x}{\phi(n\eta)}\right\}\mu(\mathrm{d}x) \\
&= \frac{1}{n} \log \int_0^\infty \exp\left\{\frac{n\eta x}{\phi(n)} \frac{\phi(n)}{\phi(n\eta)}\right\}\mu(\mathrm{d}x) \\
&\le \frac{1}{n} \log \int_0^\infty \exp\left\{(1+\varepsilon)\frac{n\eta^{1-\alpha}x}{\phi(n)}\right\}\mu(\mathrm{d}x) \\
&= \frac{1}{n} \log \int_0^\infty \exp\left\{\frac{n\xi^{1-\alpha}x}{\phi(n)}\right\}\mu(\mathrm{d}x).
\end{aligned}$$

Hence,

$$\liminf_{n\to\infty} \frac{1}{n} \log \int_0^\infty \exp\left\{\frac{n\xi^{1-\alpha}x}{\phi(n)}\right\}\mu(\mathrm{d}x) \ge \frac{1}{(1+\varepsilon)^{1/(1-\alpha)}} B\xi.$$

Similarly, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \int_0^\infty \exp\left\{\frac{n\xi^{1-\alpha}x}{\phi(n)}\right\}\mu(\mathrm{d}x) \le \frac{1}{(1-\varepsilon)^{1/(1-\alpha)}} B\xi.$$

Since ε is arbitrary, letting $\varepsilon \downarrow 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^\infty \exp\left\{\frac{n\xi^{1-\alpha}x}{\phi(n)}\right\}\mu(\mathrm{d}x) = B\xi,$$

which proves our assertion.

Appendix

The following fact is well-known, but we give the proof for the convenience of the reader.

Proposition. *For* $p_1, p_2, \dots, p_n > 0$, *and* $q > 0$,

$$S := \int_K x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} (1 - x_1 - x_2 - \cdots - x_n)^{q-1} \mathrm{d}x_1 \cdots dx_n$$

$$= \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_n)\Gamma(q)}{\Gamma(p_1 + p_2 + \cdots + p_n + q)},$$

where

$$K = \{(x_1, x_2, \dots, x_n) \in \mathbf{R}^n \mid x_1, x_2, \dots, x_n > 0, \ x_1 + x_2 + \cdots + x_n \le 1\}.$$

Proof. We change the variables as follows:

$$\begin{cases} x_1 + x_2 + \cdots + x_n = u_1 \\ x_2 + \cdots + x_n = u_1 u_2 \\ \cdots \\ x_n = u_1 u_2 \cdots u_n, \end{cases}$$

or equivalently,

$$\begin{cases} x_1 = u_1(1 - u_2) \\ x_2 = u_1 u_2 (1 - u_3) \\ \cdots \\ x_n = u_1 u_2 \cdots u_n. \end{cases}$$

Then,

$$\frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)} = u_1^{n-1} u_2^{n-2} \cdots u_{n-1}^1 u_n^0.$$

Therefore,

$$S = \int_0^1 u_1^{p_1+\cdots+p_n-1}(1 - u_1)^{q-1} \mathrm{d}u_1 \int_0^1 u_2^{p_2+\cdots+p_n-1}(1 - u_2)^{p_1-1} \mathrm{d}u_2 \cdots$$

$$\times \int_0^1 u_{n-1}^{p_{n-1}+p_n-1}(1 - u_{n-1})^{p_{n-2}-1} \mathrm{d}u_{n-1} \int_0^1 u_n^{p_n-1}(1 - u_n)^{p_{n-1}-1} \mathrm{d}u_n$$

$$= B(p_1 + \cdots + p_n, q) B(p_2 + \cdots + p_n, p_1) \cdots B(p_n, p_{n-1}),$$

where $B(p,q)$ denotes the usual beta function. Using the fact that $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$, we have

$$S = \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_n)\Gamma(q)}{\Gamma(p_1+p_2+\cdots+p_n+q)},$$

which proves the assertion.

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