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Singularities of multiplicative p-closed vector fields and global 1-forms of Zariski surfaces

By

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Abstract

We consider quotient surfaces by p-closed rational vector fields. First we show that singularities on the quotient surfaces by multiplicative p-closed vector fields are toric singularities. Then we proceed to studying global properties of Zariski surfaces. We see that non-trivial global 1-forms are related to some linear systems with base points. We also give examples of Zariski surfaces admitting non-closed regular 1-forms.

0. Introduction

Let δ be a rational vector field on a smooth surface S defined over an algebraically closed field k of characteristic $p>0$. We say that δ is p-closed if $\delta^p=\alpha\delta$ is satisfied with some rational function $\alpha\in k(S)$. Much attention has been drawn to such vector fields because of the fact that they induce quotient surfaces in the following way. For an affine open covering $S=\cup\operatorname{Spec}A_i$, a p-closed rational vector field δ induces the invariant ring $A_i^\delta:=\{a\in A_i|\delta(a)=0\}$ and form the quotient surface as $V:=\cup\operatorname{Spec}A_i^\delta$. It is then easy to see that V is normal, and since A_i^δ contains $A_i^{(p)}:=\{a^p|a\in A_i\}$, the quotient map $g:S\to V$ factors the relative Frobenius morphism of S:

$$S\xrightarrow{g}V\xrightarrow{\tilde{g}}S^{(-1)}.$$

In this paper, there are two topics we are interested in. The first one is on the singularities which appear on the quotient surface V. We prove that if the p-closed rational vector field δ satisfies certain condition, the singularities of V are toric singularities (Theorem 2.3). In general, many non-rational singularities can appear on V and we do not see any effective ways to understand their nature. However, under the condition that V admits only toric singularities, it is fairly easy to study the global as well as local properties of the quotient surface. We expect that the applications of this criterion are not only to be limited to the ones treated in this paper.

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The second topic is on Zariski surfaces (for the definition, see §1). It follows from the famous theorem by P. Deligne and L. Illusie that if a smooth projective surface X defined over k lifts to $W_2(k)$ the ring of Witt vectors of length two, the Hodge spectral sequence of X degenerates at E_1-term. So our aim is to observe the degeneration of the Hodge spectral sequence of Zariski surfaces. In Theorem 3.1, we obtain a criterion for a Zariski surface X to satisfy $H^0(\Omega_X)=0$ and $H^1(\mathcal{O}_X)=0$ involving certain linear systems with base points. It turns out that there exist examples admitting non-closed global 1-forms (Example 3.6). Consequently, such surfaces do not lift to $W_2(k)$ nor W.

In the final section we see that the Rudakov-Shafarevich theorem for $K3$ surfaces in $p=2$ can be shown as an application of Corollary 3.3.

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1. Preliminaries

The following notation is used in this paper.

$\boldsymbol{F}_p$	: the prime field of characteristic p.
k	: an algebraically closed field of characteristic p.
W, $W(k)$	: the ring of Witt vectors over k.
X	: a smooth projective surface defined over k.
$\mathcal{O}_X$	: the structure sheaf of X.
$\mathfrak{m}_{X,q}$	: the maximal ideal of the local ring $\mathcal{O}_{X,q}$ with $q\in X$.
$d_{R/k}$, d_X	: the universal derivations of a k-algebra R, and of a surface X respectively.
$q(X)$	: the dimension of the Albanese variety Alb(X) associated to X. We have the inequality: $q(X)\leq h^1(\mathcal{O}_X)$.
$\delta\in T_X\otimes k(X)$	: a p-closed rational vector field, i.e., $\delta^p=\alpha\delta$ with some $\alpha\in k(X)$.
$\rho:\Sigma_e\to\boldsymbol{P}^1$	: a Hirzebruch surface associated to the rank two locally free sheaf $\mathcal{O}_{\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1}(e)$ on $\boldsymbol{P}^1$ with $e\geq 0$. If $e\geq 1$, the unique negative section is denoted by C_0.

The relative Frobenius morphisms of X (for the definition, see [19]) are denoted by

$$\cdots\to X^{(+1)}\xrightarrow{F} X\xrightarrow{F} X^{(-1)}\to\cdots.$$

For two p-closed rational vector fields δ_1, δ_2 on X, we say that δ_1 is equivalent to δ_2 ($\delta_1\sim\delta_2$) if there exists a non-zero rational function $f\in k(X)$ such that $\delta_1=f\delta_2$. Note that δ_1 and δ_2 determine the same quotient surface if $\delta_1\sim\delta_2$.
We say that a surface X is supersingular if $H^2_{\text{ét}}(X,\boldsymbol{Q}_l)$ ($l\neq p$) is spanned by algebraic cycles, i.e., $\rho(X)=b_2(X)$.
We say that a surface X is a Zariski surface if it is dominated by $\boldsymbol{P}^2$ with a purely inseparable rational mapping of degree p.

2. Singularities by multiplicative p-closed vector fields

In this section, we study p-closed rational vector fields of multiplicative type. Let S be a smooth surface, and $\delta \in T_S \otimes k(S)$ be a p-closed rational vector field on S. Fix a point $q \in S$ and we denote its image on the quotient surface by $(V, \tilde{q})$, i.e.,

$$(S, q) \overset{\text{quotient by } \delta}{\rightarrow} (V, \tilde{q}).$$

By choosing an appropriate representative, we can assume that δ is given by $\delta = \phi\partial/\partial x + \psi\partial/\partial y$ with local coordinates x, y in $\mathcal{O}_{S,q}$, and regular functions ϕ, ψ without a common factor. Then it is known that $(V, \tilde{q})$ is a singular point of V if and only if $\phi, \psi \in \mathfrak{m}_{S,q}$. In this case, $q \in S$ is called a singular point of the p-closed rational vector field δ.

Definition 2.1. Let δ be a p-closed rational vector field expressed as above.
i) For a singular point $q \in S$ of δ, if there exists a unit $\alpha \in \mathcal{O}^*_{S,q}$ satisfying $\delta^p = \alpha\delta$, then we say that δ has a singularity of multiplicative type at $q \in S$.
ii) The multiplicity of δ at $q \in S$ is defined as $\mathrm{mult}_q \delta := \dim_k \mathcal{O}_{S,q}/(\phi, \psi)$. This definition is independent of the choices of the local coordinates x, y.

Proposition 2.2. i) *We have*:

δ is smooth at $q \in S$ if and only if $\mathrm{mult}_q \delta = 0$.
δ has a singular point of multiplicative type at q if and only if $\mathrm{mult}_q \delta = 1$.

ii) *Let D be a Weil divisor on V, then pD is a Cartier divisor. In particular, $(V, \tilde{q})$ is Q-factorial. The exceptional curves in its resolution consist of rational curves (possibly singular) and any component does not form a loop.*

iii) *If $p = 2$, $(V, \tilde{q})$ is a hypersurface singularity.*

The proof can be found, for example, in [1], [3], [6] and [16]. The following theorem shows that the singularity on the quotient surface by a p-closed rational vector field of multiplicative type is a toric singularity.

Theorem 2.3. *Suppose that a p-closed rational vector field δ on a smooth surface S has a singular point of multiplicative type at (S, q). Then there exist formal parameters $x, y \in \hat{\mathcal{O}}_{S,q}$ such that the singularity on the quotient surface $(V, \tilde{q})$ is expressed as*

$$\hat{\mathcal{O}}_{V,\tilde{q}} \cong k[[x, y]]^\delta, \quad \delta = x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}, \quad (\lambda = 1, 2, \cdots, p-1).$$

This is a toric singularity of type $\frac{1}{p}(1, \lambda)$:

$$k[[x,y]]^{\delta}=k[[x^iy^j\,|\,i,j\geq 0,\ i+\lambda j\equiv 0 \mod p]].$$

Indeed, this is a rational singularity and the exceptional divisor of the minimal resolution consists of smooth rational curves whose weighted dual graph is given by

$$(-d_1)-(-d_2)-\cdots(-d_s).$$

The integers $d_i\geq 2$, $(i=1,\cdots,s)$ are given by the continued fractional expansion:

$$\frac{p}{\lambda}=d_1-\cfrac{1}{d_2-\cfrac{1}{\ddots\ \cfrac{}{d_{s-1}-\cfrac{1}{d_s}}}}.$$

Proof. The first part is well-known, see, for example, [1], [16]. For the second part, letting n_1, n_2 be $\boldsymbol{Z}$-basis, we have the $\boldsymbol{Z}$-modules $N:=\boldsymbol{Z}n_1+\boldsymbol{Z}n_2$, $N':=\boldsymbol{Z}n_1+\boldsymbol{Z}((p-\lambda)n_1+pn_2)$, $M:=\mathrm{Hom}_{\boldsymbol{Z}}(N,\boldsymbol{Z})$ and $M':=\mathrm{Hom}_{\boldsymbol{Z}}(N',\boldsymbol{Z})$. Consider the fan Δ consisting of all the faces of the cone

$$\sigma:=\boldsymbol{R}_{\geq 0}n_1+\boldsymbol{R}_{\geq 0}((p-\lambda)n_1+pn_2).$$

Then the map of fans $(N',\Delta)\rightarrow(N,\Delta)$ determines a purely inseparable morphism: (cf. [14])

$$\mathrm{Spec}\,k[M'\cap\sigma^{\vee}]\rightarrow\mathrm{Spec}\,k[M\cap\sigma^{\vee}].$$

By choosing appropriate local parameters x, y, this morphism is described as

$$\mathrm{Spec}\,k[x,y]\rightarrow\mathrm{Spec}\,k[x^iy^j\,|\,i,j\geq 0, i+\lambda j\equiv 0 \mod p].$$

In particular, the equality $k[x,y]^{\delta}\cong k[x^iy^j\,|\,i,j\geq 0, i+\lambda j\equiv 0 \mod p]$ with $\delta=x\partial/\partial x+\lambda y\partial/\partial y$ follows from the obvious inclusion $k[x,y]^{\delta}\supset k[x^iy^j\,|\,i,j\geq 0,\ i+\lambda j\equiv 0 \mod p]$ as well as the equality of the quotient fields and the normality of these two rings. The description of the minimal resolution of the singularity follows from the general theory of toric varieties.

Remarks 2.4. i) By using the criterion in [1], the singularity on the quotient surface is Gorenstein (i.e., a rational double point) if and only if $\lambda=p-1$.
ii) A p-closed rational vector field δ is often identified with a 1-foliation $\mathscr{L}\subsetneq T_S$ which is a saturated invertible subsheaf of the tangent bundle T_S locally generated by δ. Since its cokernel $T_S/\mathscr{L}$ is torsion free, it is expressed as $I_Z\bar{\mathscr{M}}$ with some invertible sheaf $\bar{\mathscr{M}}$ and an ideal sheaf defining an effective zero cycle Z. Then δ

having a singular point (resp. singular point of multiplicative type) at $x \in S$ is equivalent to the condition $x \in \operatorname{supp} Z$ (resp. the zero cycle Z being reduced at x). In particular, the following equality holds: $\Sigma_{q \in S} \operatorname{mult}_q \delta = c_2(I_Z)$.
iii) Consider a blowing-up $\pi: S^{\sim} \to S$ at a singular point $q \in S$ of a 1-foliation $\mathscr{L} \subsetneq T_S$. Then the local generator δ of this 1-foliation is regular along the exceptional curve of the first kind $E := \pi^{-1}(q)$. Suppose that $\pi^* \delta$ vanishes along E with degree r (≥ 0), then we have an equality: $\Sigma_{\tilde{q} \in E} \operatorname{mult}_{\tilde{q}} \pi^* \delta = \operatorname{mult}_q \delta - (r^2 + r - 1)$.

Corollary 2.5. *The singularities of the quotients by p-closed vector fields of multiplicative type are classified into $(p+1)/2$ cases if $p \geq 3$, whereas into a unique case if $p=2$. For example, the exceptional curves of their minimal resolutions for $p \leq 17$ are given by the following:*

$p=2$,

$(\lambda=1)$ o

$p=3$,

$(\lambda=1)$ (-3)

$(\lambda=2)$ $o-o$

$p=5$,

$(\lambda=1)$ (-5)

$(\lambda=2,3)$ $(-3)-o$

$(\lambda=4)$ $o-o-o-o$

$p=7$,

$(\lambda=1)$ (-7)

$(\lambda=2,4)$ $(-4)-o$

$(\lambda=3,5)$ $(-3)-o-o$

$(\lambda=6)$ $o-o-o-o-o-o$

$p=11$,

$(\lambda=1)$ (-11)

$(\lambda=2,6)$ $(-6)-o$

$(\lambda=3,4)$ $(-4)-(-3)$

$(\lambda=7,8)$ $o-o-(-3)-o$

$(\lambda=5,9)$ $(-3)-o-o-o-o$

$(\lambda=10)$ $o-o-o-o-o-o-o-o-o-o$

$p=13$,

$(\lambda=1)$	(-13)
$(\lambda=2,7)$	$(-7)-o$
$(\lambda=3,9)$	$(-5)-o-o$
$(\lambda=4,10)$	$(-4)-o-o-o$
$(\lambda=5,8)$	$(-3)-(-3)-o$
$(\lambda=6,11)$	$(-3)-o-o-o-o-o$
$(\lambda=12)$	$o-o-o-o-o-o-o-o-o-o-o-o$

$p=17$,

$(\lambda=1)$	(-17)
$(\lambda=2,9)$	$(-9)-o$
$(\lambda=3,6)$	$(-6)-(-3)$
$(\lambda=4,13)$	$(-5)-o-o-o$
$(\lambda=5,7)$	$(-4)-o-(-3)$
$(\lambda=10,12)$	$o-(-4)-o-o$
$(\lambda=11,14)$	$o-(-3)-o-o-o-o$
$(\lambda=8,15)$	$(-3)-o-o-o-o-o-o-o$
$(\lambda=16)$	$o-o-o-o-o-o-o-o-o-o-o-o-o-o-o-o$

where $(-b)$ stands for a smooth rational curve whose self-intersection number is $-b$. In particular, the one with self-intersection number -2 is denoted by o.

Proposition 2.6 (Canonical Resolution in $p=2$). *Suppose $p=2$, then the singularities of a 1-foliation $\mathscr{L} \subsetneq T_S$ can be resolved by a succession of blowing-ups at singular points of this 1-foliation.*

Proof. Suppose that $\mathscr{L}$ is locally generated by a vector field $\delta=\phi\partial/\partial x+\psi\partial/\partial y$ with x,y local coordinates in $\mathcal{O}_{S,q}$, and $\phi,\psi\in\mathfrak{m}_{S,q}$ have no common factor. Letting π be a blowing-up at $q\in S$, we see by local computation that $\pi^*\delta$ vanishes along the exceptional curve $E:=\pi^{-1}(q)$ with degree $r\geq\min\{v_E(\phi),v_E(\psi)\}-1$, where v_E is the valuation associated to E. We claim that $r\geq 1$ holds in $p=2$. First we note that if δ is of multiplicative type (i.e., $\mathrm{mult}_q\delta=1$), we see $r=1$ by local computation (cf. Theorem 2.3). So it suffices to show $r\geq 1$ in non-multiplicative case (i.e., $\mathrm{mult}_q\delta\geq 2$). Suppose $r=0$. Then without loss of generality we may assume $\phi\notin\mathfrak{m}_{S,q}^2$. By an appropriate coordinate change, we can further assume $\delta=y\partial/\partial x+\psi\partial/\partial y$ with $\psi\in\mathfrak{m}_{S,q}^2$. On the other hand, we have the assumption $\delta^2=\alpha\delta$

for some $\alpha \in \mathfrak{m}_{S,q}$. This induces the equality $\psi\partial/\partial x+\delta(\psi)\partial/\partial y=\alpha(y\partial/\partial x+\psi\partial/\partial y)$, which contradicts the assumption $\psi\notin(y)$. Thus we have $r\geq 1$ and the desired result follows from the equality in Remark 2.4, iii).

The following lemma will be needed later.

Lemma 2.7. *Set* $R:=k[x^iy^j \mid i,j\geq 0, i+\lambda j\equiv 0 \mod p]$, *and let* $\pi: X\to \operatorname{Spec} R$ *be the minimal resolution. Then the* 1*-forms* $d_{R/k}(x^p)/x^p$, $d_{R/k}(y^p)/y^p$ *are regular outside the singular point at the origin. These* 1*-forms pulled-back by* π *have a pole of degree* 1 *along the exceptional curve* $E:=\pi^{-1}(0)$, *and satisfy the equality*:

$$\lambda\pi^*\left(\frac{d_{R/k}(x^p)}{x^p}\right)=\pi^*\left(\frac{d_{R/k}(y^p)}{y^p}\right).$$

Proof. The last equality follows immediately from $d_{R/k}(x^{p\lambda}y^{p(p-1)})=0$, since $x^\lambda y^{p-1}$ is an element of R. Let $E_1,\cdots,E_s$ be irreducible components of the exceptional divisor E such that $(E_i\cdot E_{i+1})=1$ for $i=1,\cdots,s-1$ (cf. Theorem 2.3). Then an elementary argument on the intersection numbers shows that the equalities

$$\pi^*\operatorname{div}(x^p)=pC+\sum_{i=1}^{s}a_iE_i,\qquad \pi^*\operatorname{div}(y^p)=pC'+\sum_{i=1}^{s}b_iE_i$$

hold with a_i, b_j integers $(1\leq a_i,\ b_j<p)$ and some effective divisors C, C' on X such that $(C\cdot E_1)=(C'\cdot E_s)=1$. Then the desired result can be obtained by local computation.

3. Applications to Zariski surfaces

It was shown by Nygaard [12] that any global 1-forms are closed for a smooth projective surface X with the Hodge-Witt cohomology $H^2(X, W\mathcal{O}_X)$ finitely generated over W. Here, we study global 1-forms of Zariski surfaces.

Let S be a smooth projective rational surface, $\mathscr{L}\subsetneq T_S$ be a 1-foliation which induces the normal quotient surface $g: S\to V$. The minimal resolution of singularities of V is denoted by $\pi: X\to V$.

(3–I)
$$\begin{array}{ccccc} & & X & & \\ & & \downarrow^{\pi} & & \\ S & \xrightarrow{g} & V & \xrightarrow{\bar{g}} & S^{(-1)} \end{array}$$

By identifying S and $S^{(-1)}$ as abstract varieties, we obtain the invertible sheaf $\mathscr{L}^{-1}$ on $S^{(-1)}$, hence $\bar{g}^*\mathscr{L}^{-1}$ and $F^*\mathscr{L}^{-1}$ are induced on V and S respectively. In particular, the pull-back by the Frobenius map satisfies $F^*\mathscr{L}^{-1}\simeq\mathscr{L}^{-p}$.

Our main theorem is:

Theorem 3.1. *We assume that the* 1*-foliation* $\mathscr{L}$ *admits only singularities of multiplicative type. Let* I_Z *(resp.* $I_{\tilde{Z}}$*) be the defining ideal of the singular points* Sing $\mathscr{L}$ *in* S *(resp.* Sing V *in* V*) with the reduced structure. Consider the natural inclusion:* $H^0(I_Z\mathscr{L}^{-1}) \subset H^0(I_{\tilde{Z}}\tilde{g}^*\mathscr{L}^{-1}) \subset H^0(I_Z\mathscr{L}^{-p})$. *Then we have the following:*
i) *If* $H^0(I_Z\mathscr{L}^{-1})=0$ *is satisfied, then* $H^1(\mathcal{O}_X)=0$ *holds,*
ii) *We have an equality* $H^0(I_{\tilde{Z}}\tilde{g}^*\mathscr{L}^{-1})=H^0(\Omega_X)$,
iii) *There exists an exact sequence*

$$0 \to H^0(I_Z\mathscr{L}^{-1}) \to H^0(\Omega_X) \xrightarrow{d_X} H^0(\Omega_X^2).$$

In particular, if $H^0(I_Z\mathscr{L}^{-1})=0$ *and* $H^0(\Omega_X)\neq 0$ *are satisfied, all the global* 1*-forms of* X *are not closed by* d_X.

Proof. i) Let $\mathscr{M}$ be the 1-foliation on V corresponding to the purely inseparable map $\tilde{g}: V \to S^{(-1)}$. Then the exact sequence

$$0 \to \mathscr{L} \to T_S \to I_Z g^*\mathscr{M} \to 0$$

is induced on S (cf. [5]). By taking its dual, we have

$$0 \to g^*\mathscr{M}^{-1} \to \Omega_S \to I_Z\mathscr{L}^{-1} \to 0. \tag{3-A}$$

Considering the universal derivation $d_S: \mathcal{O}_S \to \Omega_S$, we obtain an exact sequence:

$$0 \to \mathcal{O}_V \to g_*\mathcal{O}_S \to g_*(I_Z\mathscr{L}^{-1}).$$

From the surjection:

$$H^0(g_*\mathcal{O}_S/\mathcal{O}_V) \to H^1(\mathcal{O}_V) \to 0,$$

we see that $H^1(\mathcal{O}_V)=0$ holds under the hypothesis $H^0(I_Z\mathscr{L}^{-1})=0$. Recall that V has only rational singularities, so $H^1(\mathcal{O}_X)=0$ follows immediately.

ii) We have the following exact sequence on $V_0 := V\backslash \text{Sing}\, V$:

$$0 \to \tilde{g}^*\mathscr{L}^{-1} \to \Omega_{V_0} \to \mathscr{M}^{-1} \to 0, \tag{3-B}$$

from which the long exact sequence is induced ($\tilde{g}^*\mathscr{L}^{-1}$ is an invertible sheaf on V)

$$0 \to H^0(V, \tilde{g}^*\mathscr{L}^{-1}) \to H^0(V_0, \Omega_V) \to H^0(V_0, \mathscr{M}^{-1}).$$

We see that the last term vanishes since $H^0(g^*\mathscr{M}^{-1}) \subsetneq H^0(\Omega_S)=0$ holds by (3-A). So the natural inclusion $H^0(X, \Omega_X) \subsetneq H^0(V_0, \Omega_V) \cong H^0(V, \tilde{g}^*\mathscr{L}^{-1})$ suggests that we need to look closely at the elements of $H^0(V, \tilde{g}^*\mathscr{L}^{-1})$. Hereafter, we shall show that a local generator of $\tilde{g}^*\mathscr{L}^{-1}\otimes\mathcal{O}_{V,\tilde{q}}$ in (3-B) has a pole of degree one along the exceptional curve $E := \pi^{-1}(\tilde{q})$ on the minimal resolution.

At each singular point $q \in S$ of the 1-foliation $\mathscr{L}$, there exists a local generator δ which can be expressd as $\delta = x\partial/\partial x + \lambda y \partial/\partial y$ for some formal coordinates $x, y \in \hat{\mathcal{O}}_{S,q}$

and $\lambda \in F_p^{\times}$ (Theorem 2.3). Then by looking at the exact sequence (3-A) locally:

$$0 \to \hat{\mathcal{O}}_{S,q}(\lambda y dx - x dy) \to \hat{\mathcal{O}}_{S,q}dx \oplus \hat{\mathcal{O}}_{S,q}dy \to \hat{\mathcal{O}}_{S,q}x\frac{dx}{x} + \hat{\mathcal{O}}_{S,q}y\frac{dy}{y} \to 0.$$

we see that $I_Z \otimes \hat{\mathcal{O}}_{S,q} \cong (x,y)$ and $\mathcal{L}^{-1}$ is generated by dx/x, dy/y with the relation $\lambda dx/x = dy/y$. Applying the same argument on $S^{(-1)}$, we can take formal coordinates x', y' of $\hat{\mathcal{O}}_{S^{(-1)},q}$ with $x' = x^p$, $y' = y^p$ such that $\tilde{g}^*\mathcal{L}^{-1} \otimes \hat{\mathcal{O}}_{V,\tilde{q}}$ is generated at a singular point $\tilde{q} \in \operatorname{Sing} V$ by

$$\lambda \frac{d_{\hat{\mathcal{O}}_{V,\tilde{q}}/k}(\tilde{g}^*x')}{\tilde{g}^*x'} \left(= \frac{d_{\hat{\mathcal{O}}_{V,\tilde{q}}/k}(\tilde{g}^*y')}{\tilde{g}^*y'} \right).$$

The rationality of this singularity says $\pi^*\mathfrak{m}_{V,\tilde{q}} \cong \mathcal{O}_X(-Z)$, where Z is the fundamental cycle which, in our case, coincides with the reduced exceptional divisor $E = \pi^{-1}(\tilde{q})$. Then using Lemma 2.7, we deduce the equality $H^0(\Omega_X) = H^0(I_{\tilde{Z}}\tilde{g}^*\mathcal{L}^{-1})$.

iii) Take an affine covering $V = \cup U_i$ so that $\tilde{g}^*\mathcal{L}^{-1}$ in (3-B) is generated locally by $\{\tilde{g}^*\omega_i \mid \omega_i \in \Omega_{S^{(-1)}} \otimes k(S^{(-1)})\}$ with the transition functions $\{\tilde{g}^*\phi_{i,j} \mid \phi_{i,j} \in \mathcal{O}^*_{g(U_i \cap U_j)}\}$ such that $\tilde{g}^*\omega_i = (\tilde{g}^*\phi_{i,j})^{-1}\tilde{g}^*\omega_j$. Then an element $\phi \in H^0(I_{\tilde{Z}}\tilde{g}^*\mathcal{L}^{-1})$ is represented by $\{\phi_i \in I_{\tilde{Z}}\mathcal{O}_{U_i}\}$ such that $\phi_i = \tilde{g}^*\phi_{i,j}\phi_j$, and the corresponding global regular 1-form is given by $\{\phi_i\tilde{g}^*\omega_i\}$ with $\phi_i\tilde{g}^*\omega_i = \phi_j\tilde{g}^*\omega_j \in H^0(\Omega_V)$. Its image by d_V is given by $(d_V\phi_i) \wedge \tilde{g}^*\omega_i = (d_V\phi_j) \wedge \tilde{g}^*\omega_j \in H^0(\Omega_V^2)$. It follows from the exact suquence (3-B) that this 2-form is zero if and only if $(d_V\phi_i) \in \tilde{g}^*\mathcal{L}^{-1} \otimes k(V)$ for all i. Hereafter, we shall study when this condition is satisfied.

Consider the composition map $I_Z \xrightarrow{d_S} \Omega_S \to I_Z\mathcal{L}^{-1}$, which sends an element $t \in I_Z$ to $\delta(t) \in I_Z\mathcal{L}^{-1}$ (cf. 3-A). Taking the direct image F_* and tensoring with $\mathcal{L}^{-1}$ on $S^{(-1)}$, we have the following diagram with the exact row:

$$\begin{array}{ccccccc} & & & & F_*I_Z \otimes \mathcal{L}^{-1} & = & F_*I_Z \otimes \mathcal{L}^{-1} \\ & & & & \downarrow F_*d_S & & \downarrow F_*\delta \\ 0 & \to & F_*(g^*\mathcal{M}^{-1}) \otimes \mathcal{L}^{-1} & \to & F_*\Omega_S \otimes \mathcal{L}^{-1} & \to & F_*(I_Z \otimes \mathcal{L}^{-1}) \otimes \mathcal{L}^{-1}. \end{array}$$

Here, we have two equalities:

$$H^0(I_Z\mathcal{L}^{-1}) \cong \ker(F_*d_S : H^0(I_Z\mathcal{L}^{-p}) \to H^0(\Omega_S \otimes \mathcal{L}^{-p})),$$

$$H^0(I_{\tilde{Z}}\tilde{g}^*\mathcal{L}^{-1}) \cong \ker(F_*\delta : H^0(I_Z\mathcal{L}^{-p}) \to H^0(I_Z\mathcal{L}^{-p-1})).$$

Taking these into account, we obtain the exact sequence:

$$0 \to H^0(I_Z\mathcal{L}^{-1}) \to H^0(I_{\tilde{Z}}\tilde{g}^*\mathcal{L}^{-1}) \xrightarrow{d_S} H^0(g^*\mathcal{M}^{-1} \otimes \mathcal{L}^{-p}).$$

Then the desired result follows because we know from (3-B) that $d_S g^*\phi_i = 0$ holds if and only if $d_V\phi_i \in g^*\mathcal{L}^{-1} \otimes k(V)$ is satisfied.

Lemma 3.2. *Let X be a smooth projective surface with $H^1(\mathcal{O}_X)=0$. If $H^0(\Omega_X)=0$ is satisfied, then the Hodge spectral sequence of X degenerates at E_1-term and the crystalline cohomology $H^2_{\mathrm{crys}}(X/W)$ is torsion free.*

Proof. By the Serre duality, we have $H^1(\Omega_X^2)=H^2(\Omega_X)=0$, then the degeneration of the Hodge spectral sequence follows trivially from the Hodge diagram of X. The last assertion is a consequence of the following exact sequence (cf. [2]):

$$0 \to H^1_{\mathrm{crys}}(X/W)\otimes_W k \to H^1_{\mathrm{DR}}(X/k) \to \mathrm{Tor}_1^W(H^2_{\mathrm{crys}}(X/W), k) \to 0.$$

Corollary 3.3 (cf. [8]). *Let $\mathcal{L} \subsetneq T_S$ be a 1-foliation on a smooth rational surface S admitting singularities of multiplicative type and X be a smooth model of the quotient surface of S by this 1-foliation. The ideal sheaves I_Z, $I_{\tilde{Z}}$ are defined as in Theorem* 3.1.
i) *If $(K_S\otimes\mathcal{L})^{-1}$ is ample, then we have $H^1(\mathcal{O}_X)=0$.*
ii) *Suppose $S\cong \boldsymbol{P}^2$ or Σ_e a Hirzebruch surface. If $g^*I_{\tilde{Z}}\cong I_Z^p$ is satisfied and $\mathcal{L}^{-1}$ is ample, then $H^0(\Omega_X)=0$ holds.*

Proof. i) Consider the exact sequence (3-A):

$$0 \to K_S\otimes\mathcal{L} \to \Omega_S \to I_Z\mathcal{L}^{-1} \to 0.$$

Then we have the inclusion:

$$0 \to H^0(I_Z\mathcal{L}^{-1}) \to H^1(K_S\otimes\mathcal{L}).$$

Since $(K_S\otimes\mathcal{L})^{-1}$ is ample, we deduce $H^1(K_S\otimes\mathcal{L})=0$ by the Kodaira vanishing theorem (cf. [6, Theorem 1.6, pp. 125]) and get the desired assertion by Theorem 3.1.

ii) First consider the case $S\cong \boldsymbol{P}^2$. Suppose that $H^0(I_{\tilde{Z}}\tilde{g}^*\mathcal{L}^{-1})$, which is introduced in Theorem 3.1, is non-zero. We take a non-zero element and denote its corresponding divisor by D. Then, by Lemma 3.4 undermentioned, it is possible to choose a divisor D_0 from $H^0(I_Z\mathcal{L}^{-1}\otimes K_{\boldsymbol{P}^2}^{-1}(-1))$ such that D_0 and g^*D have no components in common. Then because of the hypothesis, we have the inequality $(g^*D\cdot D_0) \geq pc_2(I_Z)$. Therefore $(g^*D\cdot D_0)-pc_2(I_Z) = p(\mathcal{L}\cdot\mathcal{O}_{\boldsymbol{P}^2}(1))-3p \geq 0$ holds. However this is obviously a contradiction to the ampleness of $\mathcal{L}^{-1}$.

For the second case $S\cong\Sigma_e$, suppose that we can take a non-zero divisor D from $H^0(I_{\tilde{Z}}\tilde{g}^*\mathcal{L}^{-1})$. Then choose a divisor D_0 from $H^0(I_Z\mathcal{L}^{-1}\otimes K_{\Sigma_e}^{-1}(-f))$ such that D_0 and g^*D have no components in common. (Here a fiber of ρ is denoted by f.) Then because of the hypothesis, we have the inequality $(g^*D\cdot D_0)\geq pc_2(I_Z)$. Therefore $(g^*D\cdot D_0)-pc_2(I_Z)=p(\mathcal{L}\cdot\mathcal{O}_{\Sigma_e}(f))-4p\geq 0$ holds. However this is again a contradiction to the ampleness of $\mathcal{L}^{-1}$.

Lemma 3.4. *Let $\mathcal{L}$ be an invertible sheaf, and I_Z be an ideal sheaf as in Corollary* 3.3, *ii). Then we have*
i) *If $S\cong\Sigma_e$, then* $\dim H^0(I_Z\mathcal{L}^{-1}\otimes K_{\Sigma_e}^{-1}\otimes\mathcal{O}_{\Sigma_e}(-f))\geq 2$ *and the associated linear system*

is free of fixed components.
ii) *If* $S \cong \boldsymbol{P}^2$, *then* $H^0(I_Z\mathscr{L}^{-1}\otimes K_{\boldsymbol{P}^2}^{-1}\otimes\mathcal{O}_{\boldsymbol{P}^2}(-1))$ *has dimension three and this linear system is free of fixed components.*

Proof. To prove i), we recall [7] that the exact sequence $0\to\rho^*\Omega_{\boldsymbol{P}^1}\to\Omega_{\Sigma_e}\to\Omega_{\Sigma_e/\boldsymbol{P}^1}\to 0$ splits if and only if p divides e. Then consider the exact sequence

$$0\to H^0(\rho^*\Omega_{\boldsymbol{P}^1}\otimes K_{\Sigma_e}^{-1}(-f))\to H^0(\Omega_{\Sigma_e}\otimes K_{\Sigma_e}^{-1}(-f))\xrightarrow{\alpha} H^0(\mathcal{O}_{\Sigma_e}(f)).$$

Here we can check that $\dim H^0(\Omega_{\Sigma_e}\otimes K_{\Sigma_e}^{-1}(-f))\geq 2$ and α is not a zero map. Letting Γ be a prime divisor, we see that $\dim H^0(\Omega_{\Sigma_e}\otimes K_{\Sigma_e}^{-1}(-f-\Gamma)) < \dim H^0(\Omega_{\Sigma_e}\otimes K_{\Sigma_e}^{-1}(-f))$. Then the following diagram is obtained by (3-A):

$$\begin{array}{ccccccc}
0 & \to & H^0(\Omega_{\Sigma_e}\otimes K_{\Sigma_e}^{-1}(-f-\Gamma)) & \tilde{\to} & H^0(I_Z\mathscr{L}^{-1}\otimes K_{\Sigma_e}^{-1}(-f-\Gamma)) & \to & H^1(\mathscr{L}(-f-\Gamma)) \\
 & & \cap\!\!\!\!\not\ \ & & \downarrow & & \downarrow \\
0 & \to & H^0(\Omega_{\Sigma_e}\otimes K_{\Sigma_e}^{-1}(-f)) & \tilde{\to} & H^0(I_Z\mathscr{L}^{-1}\otimes K_{\Sigma_e}^{-1}(-f)) & \to & H^1(\mathscr{L}(-f)).
\end{array}$$

Here $H^1(\mathscr{L}(-f))=H^1(\mathscr{L}(-f-\Gamma))=0$ follows from the Kodaira vanishing theorem. Therefore the inequality $\dim H^0(I_Z\mathscr{L}^{-1}\otimes K_{\Sigma_e}^{-1}(-f-\Gamma))<\dim H^0(I_Z\mathscr{L}^{-1}\otimes K_{\Sigma_e}^{-1}(-f))$ holds for any prime divisor Γ.

The assertion ii) can be verified similarly using the following exact sequences:

$$0\to\Omega_{\boldsymbol{P}^2}(2)\to\mathcal{O}_{\boldsymbol{P}^2}(1)^{\oplus 3}\to\mathcal{O}_{\boldsymbol{P}^2}(2)\to 0,$$

$$0\to K_{\boldsymbol{P}^2}\otimes\mathscr{L}(2)\to\Omega_{\boldsymbol{P}^2}(2)\to I_Z\mathscr{L}^{-1}(2)\to 0.$$

Thus we conclude the proof of Lemma 3.4.

Remarks 3.5. (i) If $p=2$, the assumption $g^*I_Z\cong I_Z^p$ is automatically satisfied. In other characteristics, it is equivalent to δ a local generator of $\mathscr{L}$ being expressed locally as $\delta=x\partial/\partial x+y\partial/\partial y$ for some formal parameters x, y at each point of Z.
(ii) By Corollary 3.3, it can be observed that all the examples of Zariski surfaces given in [8, Section 3], except the ones induced by Δ_3 in $p=3$, do not have non-trivial regular 1-forms. If we drop the assumption $g^*I_Z\cong I_Z^p$ in case $p\geq 3$, the assertion is not necessarily true. In the following, we present examples of Zariski surfaces admitting non-closed regular 1-forms.
(iii) For a $K3$ surface X, the non-existence of regular vector fields (equivalently $H^0(\Omega_X)=0$) was proved by Rudakov-Shafarevich. We claim that it is possible to show this fact in $p=2$ from Corollary 3.3. This will be explained in the next section.

Example 3.6. Consider a p-closed rational vector field δ on the projective space $\boldsymbol{P}^2=\operatorname{Proj} k[X_0, X_1, X_2]$ given by

$$\delta=(x^{np}-x)\frac{\partial}{\partial x}+\lambda(y^{np}-y)\frac{\partial}{\partial y}$$
$$=\frac{1}{x_1^{pn-1}}\left[(x_1^{np}-x_1)\frac{\partial}{\partial x_1}+(\lambda y_1^{np}+(1-\lambda)x_1^{np-1}y_1-y_1)\frac{\partial}{\partial y_1}\right],$$

where $n\geq 1$, $p\geq 3$, $\lambda\in\{1,2,\cdots,p-1\}$ and $(X_0,X_1,X_2)=(x,y,1)=(1,y_1,x_1)$. Then a 1-foliation is induced on $\boldsymbol{P}^2$ with $\mathscr{L}\cong\mathcal{O}_{\boldsymbol{P}^2}(-pn+1)$ which admits only singularities of multiplicative type, and it gives the diagram (3-I). Since $\mathscr{L}^{-1}$ is ample, $H^1(\mathcal{O}_X)=0$ follows from Corollary 3.3. Then unless i) $\lambda=1$, or ii) $p\geq 3$, $(\lambda,n)=(2,1)$, consider an element in $H^0(I_Z\mathcal{O}_{\boldsymbol{P}^2}(p(pn-1)))$ given by

$$(x^{pn}-x)^{p-\lambda}(y^{pn}-y)=\frac{1}{x_1^{pn(p-\lambda+1)+p}}x_1^p(1-x_1^{pn-1})^{p-\lambda}(y_1^{pn}-x_1^{pn-1}y_1).$$

Checking the action by δ, we see that this element comes from $H^0(I_{\tilde{Z}}\tilde{g}^*\mathscr{L}^{-1})$. Thus, by Theorem 3.1, the resolution of the quotient surface X has a non-zero regular 1-form which is not closed by d_X.

Remark 3.7. Whether a Zariski surface X admits non-closed regular 1-forms or not seems to be a very subtle question. Unfortunately, the existence of such 1-forms are not well reflected on invariants such as $K_X^2/c_2(X)$. In $p=3$, for example, the Zariski surface X with $\lambda=2$, $n\geq 2$ in Example 3.6 turns out to be a minimal surface of general type and the invariant $K_X^2/c_2(X)$ ranges $7/41\leq K_X^2/c_2(X)<2/3$ (cf. [8]).

On the other hand, the Zariski surfaces induced by the above p-closed rational vector field with $(p,\lambda)=(3,1)$ and $n\geq 2$ are also studied in (loc. cit.). The minimal surfaces turn out to be of general type without a non-zero regular 1-form and satisfy $1/23\leq K_X^2/c_2(X)<1$. In both cases, the Miyaoka-Yau inequality is satisfied.

4. The Rudakov-Shafarevich theorem in $p=2$

In this section, we give a proof of the Rudakov-Shafarevich theorem in $p=2$ as an application of the results in the previous section. To prove the fundamental theorem $H^0(T_X)=0$ for a $K3$ surface X in characteristic p, they first show that there exists an elliptic fibration on any supersingular $K3$ surface, which plays the essential role. However, their proof for the case $p=2$ is very complicated and we show that it is possible to simplify it by using a quasi-elliptic fibration on a supersingular $K3$ surface X. Here we use the term supersingular $K3$ surface in the sense of Shioda.

Theorem 4.1 (Rudakov-Shafarevich [16]). *Let X be a $K3$ surface. Then we have $H^0(T_X)=0$.*

Lemma 4.2. *Any supersingular $K3$ surface in $p=2$ can be obtained as the minimal resolution of the quotient surface of $\boldsymbol{P}^2$ by a 1-foliation admitting singularities* *of multiplicative type.*

Proof. A supersingular $K3$ surface X in $p=2$ with the Artin invariant $1 \leq \sigma_0 \leq 6$ (resp. $7 \leq \sigma_0 \leq 10$) has a quasi-elliptic fibration $f: X \to \boldsymbol{P}^1$ with five degenerate fibers of type I_0^* (resp. twenty degenerate fibers of type III). This follows from ([17], §4) in case $7 \leq \sigma_0 \leq 10$, and from ([16], §5) in case $\sigma_0 = 5, 6$. The remaining case, i.e., $1 \leq \sigma_0 \leq 4$ follows from the examples in ([8], §5) and the uniqueness of NS(X) for each σ_0. See also ([17], pp. 149). Then consider the saturation of the natural exact sequence:

$$0 \to f^*\Omega_{\boldsymbol{P}^1}(2\Sigma_0 + A) \to \Omega_X \to \Omega_{X/\boldsymbol{P}^1}/\mathrm{tor.} \to 0,$$

where Σ_0 is the line of cusps, A is an effective divisor in fibers (resp. $A=0$). The last term is expressed as $\Omega_{X/\boldsymbol{P}^1}/\mathrm{tor.} \cong I_Z(f^*\Omega_{\boldsymbol{P}^1}(2\Sigma_0 + A))^{-1}$ with an ideal sheaf I_Z defining a zero cycle. However, by computing the Chern numbers, we have $c_2(I_Z) = 0$. Therefore the dual exact sequence induces a smooth 1-foliation on X. This indicates that the normalization of the fiber product $S := (X \times_{\boldsymbol{P}^1} \boldsymbol{P}^{1(+1)})^\sim$ is a smooth rational surface and the $\boldsymbol{P}^1$-fibration $S \to \boldsymbol{P}^{1(+1)}$ has twenty disjoint (-1)-curves in its singular fibers. By contracting them, we obtain a Hirzebruch surface with a 1-foliation admitting multiplicative singularities (cf. [20]). Moreover, it is possible to choose 21 disjoint (-1)-curves on S, and by Remark 2.4, ii), we obtain a birational morphism to $\boldsymbol{P}^2$ with the required property.

Proof of Theorem 4.1. By the previous lemma, we have a smooth rational surface S with the diagram:

$$\begin{array}{rcl} S & \xrightarrow[\text{p.i., flat, degree 2}]{} & X \\ {\scriptstyle \text{contraction of disjoint } (-1)-\text{curves}} \downarrow & & \downarrow {\scriptstyle \text{minimal resolution}} \\ \boldsymbol{P}^2 & \xrightarrow{\;g\;} & V. \end{array}$$

Since V admits only rational double points, the 1-foliation $\mathscr{L}$ corresponding to g satisfies $\mathscr{L} \cong \mathcal{O}_{\boldsymbol{P}^2}(-3)$, (cf. the canonical bundle formula [16, §2 Corollary 1]). Thus by Corollary 3.3 in the previous section, we obtain $H^0(\Omega_X) = 0$ for a supersingular $K3$ surface X. Then the desired assertion for a $K3$ surface follows from [16, §2 Theorem 5].

Remark 4.3. Two alternative proofs of the Rudakov-Shafarevich theorem are presented by N. Nygaard and W. Lang using the Hodge-Witt cohomologies ([11], [13]).

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