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Tauberian theorem of exponential type on limits of oscillation

By

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1. Introduction

This paper is a continuation of the author's previous paper [4], where we gave a Tauberian theorem of exponential type. In the present article we shall extend it and obtain some results on limits of oscillation. We first recall the result of [4]. Throughout this paper, let Φ denote the class of decreasing convex functions $\varphi(s) \in C^1(0, \infty)$ satisfying

$$\lim_{\varepsilon \to 0+} \varphi'(\varepsilon) = -\infty, \qquad \lim_{s \to \infty} \varphi'(s) = 0. \tag{1.1}$$

For $\varphi \in \Phi$, define

$$\varphi^*(x) = \inf_{s>0}\{sx + \varphi(s)\}, \qquad x > 0. \tag{1.2}$$

Then $\varphi^*(x)$ is a non-decreasing concave function on $(0, \infty)$, and it holds that

$$\varphi(s) = \sup_{x>0}\{-sx + \varphi^*(x)\}. \tag{1.3}$$

Theorem A ([4]). *Let $\varphi \in \Phi$ and define φ^* as in* (1.2). *Suppose a_n be a sequence of positive numbers tending to infinity as $n \to \infty$, and $U_n(x)$ be a sequence of non-decreasing, right-continuous functions on $[0, \infty)$ such that $U_n(0) = 0$.*
(i) *If*

$$\lim_{n \to \infty} \frac{1}{a_n} \log \int_0^\infty e^{-a_n s x} \mathrm{d}U_n(x) = \varphi(s), \qquad \textit{for all } s > 0, \tag{1.4}$$

then

$$\lim_{n \to \infty} \frac{1}{a_n} \log U_n(x) = \varphi^*(x), \qquad \textit{for all } x > 0. \tag{1.5}$$

(ii) *Conversely, if*

$$\limsup_{n \to \infty} \frac{1}{a_n} \log \int_0^\infty e^{-a_n s x} \mathrm{d}U_n(x) < \infty, \qquad \textit{for all } s > 0,$$

then (1.5) *implies* (1.4).

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The aim of the present paper is to extend Theorem A and construct Tauberian theorems that can treat the cases where the supremum and the infimum do not necessarily coincide. Such cases often appear in probability theory, and there have been some works on this subject (cf. [2], [3], and so on), though all of them treat fixed measures $dU(x)$. Therefore, in the present paper, we shall show theorems on limits of oscillation, that can treat the cases where the measures $dU_n(x)$ depend on n. This paper consists as follows: In Section 2, as we mentioned in the above, we give Tauberian theorems on limits of oscillation, and obtain a result on multiple convolution by applying one of the theorems. The proofs are given in Section 3. In section 4, we give other results on limits of oscillation in which the roles of the origin and infinity are interchanged and show that our theorems include Kasahara's theorem ([3]), which is a generalization of results of Davies ([2]) and Nagai ([6]).

2. Main results

We first consider the Abelian part. The following theorem is an extention of Theorem A(ii).

Theorem 2.1. *Let ψ, $\varphi \in \Phi$, and define ψ^* and φ^* as in* (1.2). *Suppose a_n be a sequence of positive numbers tending to infinity as $n \to \infty$, and $U_n(x)$ be a sequence of non-decreasing, right-continuous functions on $[0, \infty)$ such that $U_n(0) = 0$. We assume that*

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \int_0^\infty e^{-a_n s x} dU_n(x) < \infty, \qquad \textit{for all } s > 0. \tag{2.1}$$

If

$$\begin{aligned} \psi^*(x) &\le \liminf_{n\to\infty} \frac{1}{a_n} \log U_n(x) \\ &\le \limsup_{n\to\infty} \frac{1}{a_n} \log U_n(x) \le \varphi^*(x), \qquad \textit{for all } x > 0, \end{aligned} \tag{2.2}$$

then

$$\begin{aligned} \psi(s) &\le \liminf_{n\to\infty} \frac{1}{a_n} \log \int_0^\infty e^{-a_n s x} dU_n(x) \\ &\le \limsup_{n\to\infty} \frac{1}{a_n} \log \int_0^\infty e^{-a_n s x} dU_n(x) \le \varphi(s), \qquad \textit{for all } s > 0. \end{aligned} \tag{2.3}$$

Next, we consider the Tauberian part. Notice that if $\psi(s) \le \varphi(s)$ for all $s > 0$, then $\psi^*(x) \le \varphi^*(x)$ for all $x > 0$. For every $x > 0$, we determine $\xi_*(s^*(x))$ as follows: We first define

$$s^*(x) := \sup\{s \mid \psi(s) + sx \le \varphi^*(x)\}, \qquad x > 0. \tag{2.4}$$

We remark that $s_0 \le s^*(x) < \infty$, where s_0 is determined by the equation $-\varphi'(s_0) = x$. Indeed, since $\varphi(s_0) + s_0 x = \varphi^*(x)$, we have $\psi(s_0) + s_0 x \le \varphi^*(x)$, which implies $s_0 \le s^*(x)$. Next, we define

$$\xi_*(s) := \inf\{\xi | \varphi^*(\xi) - s\xi \ge \psi(s)\}, \qquad s > 0. \tag{2.5}$$

(Notice that $0 < \xi_*(s) \le -\varphi'(s)$). Thus,

$$\xi_*(s^*(x)) = \inf\{\xi | \varphi^*(\xi) - s^*(x)\xi \ge \psi(s^*(x))\}, \tag{2.6}$$

and we have

$$\xi_*(s^*(x)) \le x. \tag{2.7}$$

We stress here that for every $x > 0$, $\xi_*(s^*(x))$ and x are the smallest and the largest solution of

$$\varphi^*(\xi) - s^*(x)\xi = \psi(s^*(x)). \tag{2.8}$$

For example, let $\psi(s) = 1/(4s)$ and $\varphi(s) = 1/s$, then we have $\varphi^*(x) = 2\sqrt{x}$. From (2.4), we see that $s^*(x) = (2 + \sqrt{3})\sqrt{x}/(2x)$, and from (2.6), we have $\xi_*(s^*(x)) = (2 - \sqrt{3})^4 x$.

Theorem 2.2. *Let $\psi(s)$, $\varphi(s)$, $\varphi^*(x)$, a_n, and $U_n(x)$ be as in Theorem* 2.1. *If* (2.3) *holds, then for every* $x > 0$,

$$\varphi^*(\xi_*(s^*(x))) \le \liminf_{n\to\infty} \frac{1}{a_n} \log U_n(x) \le \limsup_{n\to\infty} \frac{1}{a_n} \log U_n(x) \le \varphi^*(x), \tag{2.9}$$

where $\xi_*(s^*(x))$ is as in the above.

We remark that if $\psi(s) = \varphi(s)$, for all $s > 0$, then $s^*(x)$ satisfies $-\varphi'(s^*(x)) = x$, and thus $\xi_*(s^*(x)) = x$, which implies that Theorems 2.1 and 2.2 include Theorem A. We postpone the proofs of Theorems 2.1 and 2.2 until the next section, and we state the following theorem, which can be obtained as a corollary of Theorem 2.2, by adopting the idea of Theorem 2 in the author's previous paper ([4]).

Theorem 2.3. *Let $\alpha > 0$, and b_n be a sequence of positive numbers tending to* 0 *as $n \to \infty$. Suppose $\sigma(x)$ be a non-decreasing, right-continuous function on $[0, \infty)$ such that $\sigma(0) = 0$. For any positive numbers C_1 and C_2, if*

$$\begin{aligned} C_1 s^{-\alpha} &\le \liminf_{n\to\infty} \frac{1}{b_n} \int_0^\infty e^{-nsx} d\sigma(x) \\ &\le \limsup_{n\to\infty} \frac{1}{b_n} \int_0^\infty e^{-nsx} d\sigma(x) \le C_2 s^{-\alpha}, \qquad \text{for all } s > 0, \end{aligned} \tag{2.10}$$

then

$$(2.11)\qquad \left(\frac{\lambda_1}{\lambda_2}\right)^{\alpha} C_2\left(\frac{e}{\alpha}\right)^{\alpha} x^{\alpha} \leq \liminf_{n\to\infty}\frac{1}{b_n}\left(\int\cdots\int_{0<x_1+\cdots+x_n\leq x} \mathrm{d}\sigma(x_1)\cdots\mathrm{d}\sigma(x_n)\right)^{1/n}$$

$$\leq \limsup_{n\to\infty}\frac{1}{b_n}\left(\int\cdots\int_{0<x_1+\cdots+x_n\leq x} \mathrm{d}\sigma(x_1)\cdots\mathrm{d}\sigma(x_n)\right)^{1/n}$$

$$\leq C_2\left(\frac{e}{\alpha}\right)^{\alpha} x^{\alpha}, \qquad \textit{for every } x>0,$$

where λ_1 $[\lambda_2]$ *is the smallest* [*largest*] *solution of*

$$(2.12)\qquad \alpha\log\lambda-\lambda=-\alpha\log(e/\alpha)+\log(C_1/C_2).$$

Proof. We refer to the proof of Theorem 2 in [4] for details, but appealing to Theorem 2.2, we can obtain

$$(2.13)\qquad C_2\left(\frac{e}{\alpha}\right)^{\alpha}(\xi_*(s^*(x))^{\alpha} \leq \liminf_{n\to\infty}\frac{1}{b_n}\left(\int\cdots\int_{0<x_1+\cdots+x_n\leq x} \mathrm{d}\sigma(x_1)\cdots\mathrm{d}\sigma(x_n)\right)^{1/n}$$

$$\leq \limsup_{n\to\infty}\frac{1}{b_n}\left(\int\cdots\int_{0<x_1+\cdots+x_n\leq x} \mathrm{d}\sigma(x_1)\cdots\mathrm{d}\sigma(x_n)\right)^{1/n}$$

$$\leq C_2\left(\frac{e}{\alpha}\right)^{\alpha} x^{\alpha}, \qquad \text{for every } x>0,$$

where $\xi_*(s^*(x))=\inf\{\xi|\alpha\log\xi-s^*(x)\xi\geq-\alpha\log s^*(x)-\alpha\log(e/\alpha)+\log(C_1/C_2)\}$. Thus it remains to show that

$$(2.14)\qquad \xi_*(s^*(x))=\left(\frac{\lambda_1}{\lambda_2}\right)x,$$

or equivalently,

$$(2.15)\qquad \frac{\xi_*(s^*(x))}{x}=\frac{\lambda_1}{\lambda_2}.$$

To see (2.15), recall that $\xi_*(s^*(x))$ and x are the smallest and the largest solution of

$$(2.16)\qquad \alpha\log\xi-s^*(x)\xi=-\alpha\log s^*(x)-\alpha\log(e/\alpha)+\log(C_1/C_2).$$

Since (2.16) can be rewritten as

$$(2.17)\qquad \alpha\log(s^*(x)\xi)-s^*(x)\xi=-\alpha\log(e/\alpha)+\log(C_1/C_2),$$

we see that the ratio of $\xi_*(s^*(x))$ to x is equal to that of the smallest to the largest solution of (2.17). Hence, by putting $\xi = \lambda/s^*(x)$ in (2.17), we can have the assertion.

3. Proofs

According to the assumptions of Theorems 2.1 and 2.2, we may and do assume that

$$\limsup_{n\to\infty}\frac{1}{a_n}\log\int_0^\infty e^{-a_n sx}\mathrm{d}U_n(x) < \infty, \qquad \text{for all } s>0,$$

throughout this section. We refer to the author's previous paper ([4]) for the following four lemmas.

Lemma 3.1. *Suppose*

$$\limsup_{n\to\infty}\frac{1}{a_n}\log\int_0^\infty e^{-a_n sx}\mathrm{d}U_n(x) \le \varphi(s), \qquad \textit{for all } s>0.$$

Then

$$\limsup_{n\to\infty}\frac{1}{a_n}\log U_n(x) \le \varphi^*(x), \qquad \textit{for all } x>0.$$

Lemma 3.2. *Suppose*

$$\liminf_{n\to\infty}\frac{1}{a_n}\log U_n(x) \ge \psi^*(x), \qquad \textit{for all } x>0.$$

Then

$$\liminf_{n\to\infty}\frac{1}{a_n}\log\int_0^\infty e^{-a_n sx}\mathrm{d}U_n(x) \ge \psi(s), \qquad \textit{for all } s>0.$$

Lemma 3.3. *Suppose*

$$\limsup_{n\to\infty}\frac{1}{a_n}\log U_n(x) \le \varphi^*(x), \qquad \textit{for all } x>0.$$

For any fixed $s>0$,

(i) $\displaystyle\limsup_{n\to\infty}\frac{1}{a_n}\log\int_0^y e^{-a_n sx}\mathrm{d}U_n(x) \le -sy+\varphi^*(y)$, *for each* $0<y<-\varphi'(s)$,

(ii) $\displaystyle\limsup_{n\to\infty}\frac{1}{a_n}\log\int_x^\infty e^{-a_n sx}\mathrm{d}U_n(x) \le -sx+\varphi^*(x)$, *for each* $x>-\varphi'(s)$.

Lemma 3.4. *Suppose*

$$\limsup_{n\to\infty}\frac{1}{a_n}\log U_n(x) \le \varphi^*(x), \qquad \textit{for all } x>0.$$

Then

$$\limsup_{n\to\infty}\frac{1}{a_n}\log\int_0^\infty e^{-a_n sx}\mathrm{d}U_n(x)\le\varphi(s)\qquad \textit{for all } s>0.$$

We now can prove Theorem 2.1. If we assume (2.2), then Lemmas 3.2 and 3.4 imply (2.3). To prove Theorem 2.2, we prepare a few more lemmas.

Lemma 3.5. *Suppose*

$$\limsup_{n\to\infty}\frac{1}{a_n}\log U_n(x)\le\varphi^*(x),\qquad \textit{for all } x>0,$$

and

$$\liminf_{n\to\infty}\frac{1}{a_n}\log\int_0^\infty e^{-a_n sx}\mathrm{d}U_n(x)\ge\psi(s),\qquad \textit{for all } s>0. \tag{3.1}$$

Then, for every $x>0$,

$$\liminf_{n\to\infty}\frac{1}{a_n}\log U_n(x)\ge\varphi^*(\xi_*(s)),\qquad \textit{for all } s>s^*(x).$$

Proof. Choose any $y<\xi_*(s)$, then by Lemma 3.3 and the definitions of $s^*(x)$ and $\xi_*(s)$, we have

$$\limsup_{n\to\infty}\frac{1}{a_n}\log\int_0^y e^{-a_n sx}\mathrm{d}U_n(x)\le -sy+\varphi^*(y)<\psi(s), \tag{3.2}$$

and

$$\limsup_{n\to\infty}\frac{1}{a_n}\log\int_x^\infty e^{-a_n sx}\mathrm{d}U_n(x)\le -sx+\varphi^*(x)<\psi(s). \tag{3.3}$$

Combining these inequalities with (3.1), we have

$$\liminf_{n\to\infty}\frac{1}{a_n}\log\int_y^x e^{-a_n sx}\mathrm{d}U_n(x)\ge\psi(s). \tag{3.4}$$

On the other hand, we have

$$\liminf_{n\to\infty}\frac{1}{a_n}\log\int_y^x e^{-a_n sx}\mathrm{d}U_n(x)\le -sy+\liminf_{n\to\infty}\frac{1}{a_n}\log U_n(x). \tag{3.5}$$

Thus, combining (3.4), (3.5), and from the assumption on y, we see

$$\liminf_{n\to\infty}\frac{1}{a_n}\log U_n(x)\ge sy+\psi(s)>\varphi^*(y).$$

Letting $y\uparrow\xi_*(s)$, we have the assertion.

Lemma 3.6. *For a fixed* $s>0$, *let* $s_n>s$ *be a sequence of positive numbers tending to* s *as* $n\to\infty$. *Then,*

$$\xi_*(s)\le\limsup_{n\to\infty}\xi_*(s_n).$$

Proof. From the definition of ξ_*, we have

$$\varphi^*(\xi_*(s_n)) - s_n\xi_*(s_n) \geq \psi(s_n). \tag{3.6}$$

Put $\limsup_{n\to\infty} \xi_*(s_n) = \eta$, then from (3.6), we have

$$\varphi^*(\eta) - s\eta \geq \psi(s), \tag{3.7}$$

which implies $\xi_*(s) \leq \eta$.

Combining Lemmas 3.5 and 3.6, we have

Lemma 3.7. *Suppose*

$$\limsup_{n\to\infty} \frac{1}{a_n} \log U_n(x) \leq \varphi^*(x), \qquad \textit{for all } x > 0,$$

and

$$\liminf_{n\to\infty} \frac{1}{a_n} \log \int_0^\infty e^{-a_n sx} \mathrm{d}U_n(x) \geq \psi(s), \qquad \textit{for all } s > 0. \tag{3.8}$$

Then, for every $x > 0$,

$$\liminf_{n\to\infty} \frac{1}{a_n} \log U_n(x) \geq \varphi^*(\xi_*(s^*(x))).$$

We are now ready to show Theorem 2.2. If we assume (2.3), then Lemmas 3.1 and 3.7 imply (2.9).

4. Another asymptotic behavior

In this section, we study the case where the roles of the origin and infinity are interchanged in Theorems 2.1 and 2.2.

Let Ψ denote the class of increasing convex functions $\varphi(s) \in C^1(0, \infty)$ satisfying

$$\lim_{\varepsilon\to 0+} \varphi'(\varepsilon) = 0, \quad \lim_{s\to\infty} \varphi'(s) = +\infty. \tag{4.1}$$

Define $\varphi_*(x)$ as

$$\varphi_*(x) = \inf_{s>0} \{\varphi(s) - sx\}, \qquad x > 0. \tag{4.2}$$

Then $\varphi_*(x)$ is non-increasing concave function on $(0, \infty)$.

Another Abelian theorem is:

Theorem 4.1. *Let* ψ, $\varphi \in \Psi$, *and define* ψ_* *and* φ_* *as in* (4.2). *Suppose* a_n *be a sequence of positive numbers tending to infinity as* $n \to \infty$, *and* $\mu_n(\mathrm{d}x)$ *be a sequence of Radon measures on* $(0, \infty)$ *such that*

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \int_0^\infty e^{a_n sx} \mu_n(\mathrm{d}x) < \infty, \qquad \textit{for all } s > 0. \tag{4.3}$$

If

$$(4.4)\qquad \psi_*(x) \le \liminf_{n\to\infty} \frac{1}{a_n} \log \mu_n(x,\infty) \le \limsup_{n\to\infty} \frac{1}{a_n} \log \mu_n(x,\infty) \le \varphi_*(x), \qquad \text{for all } x>0,$$

then

$$(4.5)\qquad \psi(s) \le \liminf_{n\to\infty} \frac{1}{a_n} \log \int_0^\infty e^{a_n s x} \mu_n(\mathrm{d}x) \le \limsup_{n\to\infty} \frac{1}{a_n} \log \int_0^\infty e^{a_n s x} \mu_n(\mathrm{d}x) \le \varphi(s), \qquad \text{for all } s>0.$$

Next, we state another Tauberian theorem. For every $x>0$, we determine $\xi^*(s_*(x))$ as follows: We first define

$$(4.6)\qquad s_*(x) := \inf\{s \mid \psi(s) - sx \le \varphi_*(x)\}.$$

Then, we define

$$(4.7)\qquad \xi^*(s) := \sup\{\xi \mid \varphi_*(\xi) + s\xi \ge \psi(s)\}.$$

Thus,

$$(4.8)\qquad \xi^*(s_*(x)) = \sup\{\xi \mid \varphi_*(\xi) + s_*(x)\xi \ge \psi(s_*(x))\}.$$

Remark that from (4.8), $\xi^*(s_*(x))$ is the largest solution of

$$(4.9)\qquad \varphi_*(\xi) + s_*(x)\xi = \psi(s_*(x)),$$

and from (4.6) and (4.8), x is the smallest solution of (4.9).

Theorem 4.2. *Let $\psi(s)$, $\varphi(s)$, $\varphi_*(x)$, a_n, and $\mu_n(\mathrm{d}x)$ be as in Theorem* 4.1. *If* (4.5) *holds, then for every $x>0$,*

$$(4.10)\qquad \varphi_*(\xi^*(s_*(x))) \le \liminf_{n\to\infty} \frac{1}{a_n} \log \mu_n(x,\infty) \le \limsup_{n\to\infty} \frac{1}{a_n} \log \mu_n(x,\infty) \le \varphi_*(x),$$

where $\xi^*(s_*(x))$ is as in the above.

Proof. Since the proofs of Theorems 4.1 and 4.2 are essentially the same as that of Theorems 2.1 and 2.2, we omit the details.

As we mentioned in section 1, we study the relationship between our theorems and some results on limits of oscillation which are already known. At first, we recall Kasahara's Tauberian theorem ([3]).

Theorem B ([3]). *Set* $0<\alpha<1$. *Let* $\phi(x)$ *be a positive function varying regularly at* ∞ *with exponent* α (*cf.* [1]) *and* $\tau(x)$ *be the asymptotic inverse of* $x/\phi(x)$. *Suppose* $\mu(\mathrm{d}x)$ *be a finite Borel measure on* $(0,\infty)$. *Then,*
(i)

$$-\infty \leq -A_1 \leq \liminf_{x\to\infty}\frac{1}{x}\log\mu(\phi(x),\infty) \leq \limsup_{x\to\infty}\frac{1}{x}\log\mu(\phi(x),\infty) \leq -A_2 \leq 0 \tag{4.11}$$

implies

$$(1-\alpha)(\alpha/A_1)^{\alpha/(1-\alpha)} \leq \liminf_{\lambda\to\infty}\frac{1}{\tau(\lambda)}\log\int_0^\infty e^{\lambda x}\mu(\mathrm{d}x) \leq \limsup_{\lambda\to\infty}\frac{1}{\tau(\lambda)}\log\int_0^\infty e^{\lambda x}\mu(\mathrm{d}x) \leq (1-\alpha)(\alpha/A_2)^{\alpha/(1-\alpha)}. \tag{4.12}$$

(ii) *Conversely, if* (4.12) *holds with* $0<A_2\leq A_1<\infty$, *then*

$$-\frac{\lambda_2}{\lambda_1}A_2 \leq \liminf_{x\to\infty}\frac{1}{x}\log\mu(\phi(x),\infty) \leq \limsup_{x\to\infty}\frac{1}{x}\log\mu(\phi(x),\infty) \leq -A_2, \tag{4.13}$$

where λ_1 $[\lambda_2]$ *is the smallest* [*largest*] *solution of*

$$\xi^\alpha - A_2\xi = (1-\alpha)(\alpha/A_1)^{\alpha/(1-\alpha)}. \tag{4.14}$$

Remark that the latter half of Kasahara's theorem is a generalization of the result of Davies ([2]). Furthermore, if we consider the special case where $A_1=A_2$, then it includes Nagai's Tauberian theorem ([6]) which was derived from Minlos-Povzner's theorem ([5]). From Proposition of the author's previous paper ([4]), we know that if the infimum and the supremum coincide, then Theorems 4.1 and 4.2 contain Theorem B. For $0<A_2<A_1$, suppose $\psi(s)=(1-\alpha)(\alpha/A_1)^{\alpha/(1-\alpha)}s^{1/(1-\alpha)}$, and $\varphi(s)=(1-\alpha)(\alpha/A_2)^{\alpha/(1-\alpha)}s^{1/(1-\alpha)}$. Then $\varphi_*(x)=-A_2x^{1/\alpha}$, and thus $\varphi_*(\xi^*(s_*(x)))=\varphi_*(x)\times(\xi^*(s_*(x))/x)^{1/\alpha}$, where $\xi^*(s_*(x))$ is as in Theorem 4.2. It is easy to see that Theorem 4.1 includes Theorem B(i). We now consider the Tauberian case. Let ξ_1 $[\xi_2]$ be the smallest [largest] solution of

$$\varphi_*(\xi)+s_*(x)\xi=\psi(s_*(x)). \tag{4.15}$$

Then, as we mentioned before, $\xi_1=x$ and $\xi_2=\xi^*(s_*(x))$. Thus, to show that Theorem 4.2 includes Theorem B(ii), it suffices to show the following proposition.

Proposition 4.3. *Let* α, λ_1 *and* λ_2 *be as in Theorem B. For a fixed* $s>0$, *solve the equation*

$$s\xi^\alpha - A_2\xi = (1-\alpha)\left(\frac{\alpha}{A_1}\right)^{\alpha/(1-\alpha)}s^{1/(1-\alpha)}, \tag{4.16}$$

and let ξ_1 $[\xi_2]$ *be the smallest* [*largest*] *solution of* (4.16). *Then*,

$$\frac{\lambda_2}{\lambda_1} = \frac{\xi_2}{\xi_1}.$$

Proof. Since our problem is to figure out the ratio of the solutions, we may put $\xi = c\zeta$ and consider the ratio of the solutions of

$$sc^{\alpha}\zeta^{\alpha} - A_2 c\zeta = (1-\alpha)\left(\frac{\alpha}{A_1}\right)^{\alpha/(1-\alpha)} s^{1/(1-\alpha)}. \tag{4.17}$$

Since (4.17) can be rewritten as

$$cA_2\left(\frac{sc^{\alpha-1}}{A_2}\zeta^{\alpha} - \zeta\right) = (1-\alpha)\left(\frac{\alpha}{A_1}\right)^{\alpha/(1-\alpha)} s^{1/(1-\alpha)}, \tag{4.18}$$

if we put $c = (s/A_2)^{1/(1-\alpha)}$, then (4.18) means

$$\zeta^{\alpha} - \zeta = (1-\alpha)\left(\frac{\alpha A_2}{A_1}\right)^{\alpha/(1-\alpha)}, \tag{4.19}$$

which proves the assertion.

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References

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels, Regular variation, Cambridge University Press, Cambridge, 1987.
[2] L. Davies, Tail probabilities for positive random variables with entire characteristic functions of very regular growth, Z. Angew. Math., **56** (1976), 334–336.
[3] Y. Kasahara, Tauberian theorems of exponential type, J. Math. Kyoto Univ., **18** (1978), 209–219.
[4] N. Kosugi, Tauberian theorems of exponential type and its application to multiple convolution, J. Math. Kyoto Univ., **39** (1999), 331–346.
[5] R. A. Minlos and A. Ja. Povzner, Thermodynamic limit for entropy, Trudy Moskov. Mat. Obsc. **17** (1968), 243–272.
[6] H. Nagai, A remark on the Minlos-Povzner Tauberian theorem, Osaka J. Math., **14**, (1977), 117–123.

Added in proof. Throughout the paper, "convex" and "concave" should be read as "strictly convex" and "strictly concave", respectively.