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A necessary and sufficient condition of local integrability

By

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§ 1. Introduction

Let X_n be a nowhere-zero C^∞ complex vector field defined near a point P in $\mathbf{R}^n$. We shall say that X_n is locally integrable at P if the equation $X_n u = 0$ has C^1 solutions $u_1, u_2, \ldots, u_{n-1}$ near P such that $du_1 \wedge du_2 \wedge \cdots \wedge du_{n-1}(\mathrm{P}) \neq 0$ (see [3], [4], [5], and [12]).

Generally, the following is known: X_n is locally integrable at P if X_n is real-analytic or locally solvable at P (see [14], for instance) but there exist non-solvable vector fields which have no local integrability (due to Nirenberg [9]).

In this article we are concerned with the case where $n = 2$.

The equation $X_2 u = 0$ near P can be transformed into that of the form

$$Lu \equiv (\partial_t + ia(t,x)\partial_x)u = 0$$

near the origin in $\mathbf{R}^2$, where $a(t,x)$ is a real-valued C^∞ function.

Though there are several partial results ([7], [8], [10], [11], [12], [13], [14] for instance), the problem is open to get a necessary and sufficient condition for $Lu = 0$ to have a solution near the origin such that $\partial_x u \neq 0$:

Suppose that $a(t,x)$ is real-analytic with respect to x. Then the equation $Lu = 0$ has such a solution by the existence theorem of Cauchy-Kovalevskaya-Nagumo.

So let $a(t,x)$ be not real-analytic with respect to x. In the case where the function $t \to a(t,x)$ does not change sign in $\{t; (t,x) \in \mathcal{O}\}$ for every x by taking a neighborhood $\mathcal{O}$ of the origin, we see that the equation $Lv = -ia_x(t,x)$ has a C^∞ solution v near the origin by the local solvability of L; thus we find that the function

$$\int_0^t -ia(\xi,x)\exp\{v(\xi,x)\}d\xi + \int_0^x \exp\{v(0,\eta)\}d\eta$$

is one of the solutions satisfying the equation $Lu = 0$ with $\partial_x u \neq 0$ near the origin.

In the last case where the function $t \to a(t,x)$ changes sign in $\{t; (t,x) \in \mathcal{O}\}$ for some x by taking any small neighborhood $\mathcal{O}$ of the origin, there exists an example

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of the equation* which admits no non-constant solutions in any neighborhood of the origin ([9]). In the very case, a necessary and sufficient condition is yet to be founded.

Concerning the *Mizohata type* vector fields, however, the following results due to Treves and Sjöstrand are obtained:

Theorem A ([11]). *Assume that L satisfies $a(0,0)=0$ and $\partial_t a(0,0)\neq 0$. L is locally integrable at the origin if and only if there exists a change of local coordinates such that L becomes a non-vanishing C^∞ function multiple of the Mizohata operator $\partial_{x_1}+ix_1\partial_{x_2}$.*

Theorem B ([10]). *Assume that L satisfies $a(0,0)=0$ and $\partial_t a(0,0)\neq 0$. Then there exist C^∞ functions u^+, which is defined in $t\geq 0$, and u^-, which is defined in $t\leq 0$, such that $u^\pm(0,x)$ are real, $\partial_x u^\pm(0,x)>0$, and $Lu^\pm=0$. L is locally integrable at the origin if and only if the function ${u^+}^{-1}\circ u^-(0,x)$ is real analytic at the origin.*

Remark. X_2 is called a *Mizohata type* vector field if the following conditions hold:
(i) $X_2(0)$ and $\bar{X}_2(0)$ are C-linearly dependent.
(ii) $X_2(0)$ and $[X_2(0),\bar{X}_2(0)]$ are C-linearly independent.

In this article we give a necessary and sufficient condition of the local integrability for the class of L satisfying that

$$\min.\{k;\partial_t^k a(0,x)\neq 0\}\text{ is constant and odd,}$$

which involves the result of Theorem B.

Remark. Let us set $X_2^{(j)}=[X_2^{(j-1)},X_2]$ $(j=2,3,\dots)$, $X_2^{(1)}=\bar{X}_2$, and $X_2^{(0)}=X_2$. Suppose that $\min.\{m;X_2^{(0)}$ and $X_2^{(m)}$ are C-linearly independent$\}$ is locally constant and odd. Then X_2 becomes a non-vanishing C^∞ function multiple of the operator L satisfying the above condition.

§ 2. Result

From now on, we shall assume

$$a(t,x)=(t^{2d})'b(t,x),$$

where d is a positive integer and $b(t,x)$ a positive C^∞ function.

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* If X_n is locally integrable, then the equation $X_n u=0$ trivially has a non-trivial solution. But the converse is not necessarily true; the reason is as follows: According to Hölmander ([2], Theorem 8.9.2), there exist functions v and α belonging to $C^\infty(\mathbf{R}^2)$ and vanishing when $t\leq 0$, such that

$$v_t+\alpha(t,x)v_x=0,\qquad \operatorname{supp} v=\{t;t\geq 0\}.$$

If the equation $u_t+\alpha(t,x)u_x=0$ has a solution such that $u_x(0,0)\neq 0$ near the origin, then the function v can be expressed as a holomorphic function of u, whence v must vanish identically near the origin; this is a contradiction.

We may suppose that $b(t,x)$ has the following form ([15]):

$$b(t,x) = \alpha(t^2,x) + t\beta(t^2,x),$$

where $\alpha(t,x)$ and $\beta(t,x)$ are the real-valued C^∞ functions and $\alpha(t,x)$ is positive. Let us set L_1 and L_2 as follows:

$$L_1 = \partial_t + i\{\alpha(|t|^{1/d},x) + |t|^{1/2d}\beta(|t|^{1/d},x)\}\partial_x,$$

$$L_2 = \partial_t + i\{\alpha(|t|^{1/d},x) - |t|^{1/2d}\beta(|t|^{1/d},x)\}\partial_x.$$

Now let us assume that L is locally integrable at the origin. Then we find that there exist a positive constant T and a function $u_0(t,x) \in C^1([-T,T]\times[-T,T])$ such that $\partial_x u_0 \neq 0$ satisfying the following in $[0,T]\times[-T,T]$:

$$\{\partial_t + ib(t^{1/2d},x)\partial_x\}u_0(t^{1/2d},x) = 0$$

and

$$\{\partial_t + ib(-t^{1/2d},x)\partial_x\}u_0(-t^{1/2d},x) = 0.$$

Hence we find that the equations

(1) $$L_1u_1 = \{\partial_t + ib(t^{1/2d},x)\partial_x\}u_1(t,x) = 0 \qquad \text{in } [0,T]\times[-T,T]$$

and

(2) $$L_2u_2 = \{\partial_t + ib(-t^{1/2d},x)\partial_x\}u_2(t,x) = 0 \qquad \text{in } [0,T]\times[-T,T]$$

have C^1 solutions $u_1(t,x)$ and $u_2(t,x)$, respectively, such that $u_1(t,x)$ and $u_2(t,x)$ have the same initial value $u_0(x)$ on $t=0$ such that $u_0'(x)\neq 0$.

Conversely, assume that (1) and (2) have C^1 solutions $u_1(t,x)$ and $u_2(t,x)$, respectively, in a semi-neighborhood $U\cap\{t\geq 0\}$ of the origin, where U denotes a neighborhood of the origin, such that $u_1(0,x) = u_2(0,x)$ and $\partial_x u_1(0,x)\neq 0$. Defining the function $u(t,x)$ by $u(t,x) = u_1(t^{2d},x)$ when $t\geq 0$ and by $u(t,x) = u_2(-t^{2d},x)$ when $t\leq 0$, we easily find that $u(t,x)\in C^1(U)$ and $Lu=0$ in U.

Here, in order to make a statement simple, we introduce the following

Definition. The Cauchy problems

$$\begin{cases} L_1u = 0 \\ u|_{t=0} = u_1(x) \end{cases}$$

and

$$\begin{cases} L_2u = 0 \\ u|_{t=0} = u_2(x) \end{cases}$$

are compatible on $t=0$ if each equation of $L_1u=0$ and $L_2u=0$ has a C^1 solution in a semi-neighborhood $U\cap\{t\geq 0\}$ of the origin such that $u_1(x)=u_2(x)$ and $u_1'(x)\neq 0$.

Then we have the following

Proposition 1. *L is locally integrable at the origin if and only if the Cauchy problems*

$$\begin{cases} L_1 u = 0 \\ u|_{t=0} = u_1(x) \end{cases}$$

and

$$\begin{cases} L_2 u = 0 \\ u|_{t=0} = u_2(x) \end{cases}$$

are compatible on $t = 0$

So, we will investigate this compatibility condition.

Now, since $\alpha(t,x) > 0$, we can assume that the vector fields L_1 and L_2 are elliptic with the coefficients of Hölder continuous functions in a neighborhood of the origin with exponent $\frac{1}{2d}$. Hereafter let k denote 1 or 2. Let $\mathfrak{S}$ denote the set $\{(Z_1(t,x), Z_2(t,x), T);\ Z_k(t,x) \in C^{1+1/2d}([-T,T] \times [-T,T]),\ Z_k(0,0) = 0,$ $L_k Z_k(t,x) = 0$ in $[-T,T] \times [-T,T]$, $\Re\partial_x Z_k(t,x) > 0$, and $\Im\partial_x Z_k(t,x) > 0\}$, where T denotes a positive constant.

Here we remark that the following fact follows from a classical theorem on Beltrami equation:

Lemma 2. $\mathfrak{S} \neq \emptyset$.

Our main result is stated as follows:

Theorem 3. *L is locally integrable at the origin if and only if there exist an element* $(Z_1(t,x), Z_2(t,x), T_0) \in \mathfrak{S}$ *and a function f which is holomorphic in* $\mathfrak{Z} = \{z \in \mathbf{C}; z = Z_2(0,x), x \in (-T_0, T_0)\}$ *and satisfies* $Z_1(0,x) = f(Z_2(0,x))$.

Remark. Theorem B follows from Theorem 3.

Now, let n and p be arbitrary positive integers. Set $a_{n,p} = 1/\{(n+p-1)(n+p)\}$. Let $B_{n,p}$ be the non-overlapping open disc in the (t,x) plane with center $(p^{-1} - (a_{1,p} + a_{2,p} + \cdots + a_{n-1,p} + a_{n,p}/2), 0)$ and radius $a_{n,p}/2$ and $C_{n,p}$ the closed disc in the (t,x) plane with radius $a_{n,p}/4$ and the same center as $B_{n,p}$.

Denoting by $f_{n,p}$ any one of the non-negative C^∞ functions satisfying that $f_{n,p} = 0$ outside of $B_{n,p}$ and $f_{n,p} > 0$ inside of $C_{n,p}$, we shall define the C_0^∞ function $r(t,x)$ as follows:

(i) $r(t,x) = f_{n,p}$ in $B_{n,p}$.

(ii) For $t \geq 0$, $r(t,x)$ vanishes out side of the union of all the $B_{n,p}$.

(iii) For $t \leq 0$, $r(t,x) = r(-t,x)$.

Then we see:

Corollary 4.

$$\partial_t + i2t(1 + tr(t^2, x))\partial_x$$

is not locally integrable at the origin.

The above example is obtained by modifying an example of Nirenberg ([9], p. 8). This conclusion is contained in [9], but we will prove this by applying Theorem 3.

Next let us set

$$w = t + ix, \partial_{\bar{w}} = \frac{\partial_t + i\partial_x}{2}, \partial_w = \frac{\partial_t - i\partial_x}{2},$$

$$\mu_1(w) \equiv \mu_1(t,x) = \frac{\alpha(|t|^{1/d}, x) + |t|^{1/2d}\beta(|t|^{1/d}, x) - 1}{\alpha(|t|^{1/d}, x) + |t|^{1/2d}\beta(|t|^{1/d}, x) + 1},$$

$$\mu_2(w) \equiv \mu_2(t,x) = \frac{\alpha(|t|^{1/d}, x) - |t|^{1/2d}\beta(|t|^{1/d}, x) - 1}{\alpha(|t|^{1/d}, x) - |t|^{1/2d}\beta(|t|^{1/d}, x) + 1}.$$

Then we see that the equation $L_k u = 0$ is transcribed into the equation $\partial_{\bar{w}} u = \mu_k(w)\partial_w u$. Define the functions $\omega_n^{[k]}$ $(n = 1, 2, \dots)$ of w as follows:

$$\omega_n^{[k]}(w) = \frac{1}{2\pi i}\iint_{B_r} \frac{\mu_k(\zeta)\omega_{n-1}^{[k]}(\zeta) - \mu_k(w)\omega_{n-1}^{[k]}(w)}{(\zeta - w)^2} d\zeta \wedge d\bar{\zeta} + 1,$$

where r denotes a positive constant, and $B_r = \{\zeta \in \mathbf{C}; |\zeta| < r\}$, $\omega_0^{[k]}(w) \equiv 0$.

It is well known that, by taking a sufficiently small constant r,

$$\omega^{[k]}(w) \equiv \lim_{n\to\infty} \omega_n^{[k]}(w)$$

exists, $\neq 0$ in B_r, belongs to $C^{1/2d}(B_r)$, and satisfies

$$\omega^{[k]}(w) = \frac{1}{2\pi i}\iint_{B_r} \frac{\mu_k(\zeta)\omega^{[k]}(\zeta) - \mu_k(w)\omega^{[k]}(w)}{(\zeta - w)^2} d\zeta \wedge d\bar{\zeta} + 1.$$

So we set

$$W^{[k]}(w) = \frac{1}{2\pi i}\iint_{Br} \frac{\mu_k(\zeta)\omega^{[k]}(\zeta)}{\zeta - w} d\zeta \wedge d\bar{\zeta} + w.$$

Then we have

Corollary 5. *Assume that there exist the functions $\alpha(t,x)$ and $\beta(t,x)$ satisfying:*

$$\iint_{B_r} \frac{\mu_1(\zeta)W^{[1]}(\zeta)}{\zeta - ix} d\zeta \wedge d\bar{\zeta} = \iint_{B_r} \frac{\mu_2(\zeta)W^{[2]}(\zeta)}{\zeta - ix} d\zeta \wedge d\bar{\zeta}.$$

Then L is locally integrable at the origin.

Example. Let $b(-t,x) = b(t,x)$. Then the above assumption is satisfied.

§3. Proof of Lemma 2

Let μ be a real constant such that $0 < \mu < 1$. Let $p(\zeta)$ be a $C^{\mu}(B_r)$ function satisfying $p(0) = 0$ and $|p(\zeta)| < 1$ in B_r. Then it is known that the following result holds (see [1] for instance):

Theorem. *The equation $W_{\bar{\zeta}} = p(\zeta) W_{\zeta}$ has a $C^{1+\mu}$ solution such that $W(0) = 0$ and $W_{\zeta}(0) \neq 0$ near the origin.*

Set $\zeta = x_1 + ix_2$. For this solution W, we can easily take a real number θ such that

$$\Re \partial_{x_1}\{e^{i\theta} W\}(0) > 0 \qquad \text{and} \qquad \Im \partial_{x_2}\{e^{i\theta} W\}(0) > 0 \text{ hold.}$$

Hence, we may assume that the above solution W satisfies

$$\Re \partial_{x_1} W(0) > 0 \qquad \text{and} \qquad \Im \partial_{x_2} W(0) > 0.$$

Now, by making use of the notation in the preceding section, the equation $L_k Z_k = 0$ is transcribed into the equation $\partial_{\bar{w}} Z_k = \mu_k(w) \partial_w Z_k$ in the complex w plane. It is clear that $|\mu_k(w)| < 1$ near the origin.

Therefore we can apply Theorem above to get Lemma 2.

§4. Proof of Theorem 3

Assume that L is locally integrable at the origin. Then by Proposition 1, the Cauchy problems

$$\begin{cases} L_1 u = 0 \\ u|_{t=0} = u_1(x) \end{cases}$$

and

$$\begin{cases} L_2 u = 0 \\ u|_{t=0} = u_2(x) \end{cases}$$

are compatible on $t = 0$. So, we suppose that each equation of $L_1 u = 0$ and $L_2 u = 0$ has a C^1 solution in $[0, T] \times [-T, T]$ such that $u_1(x) = u_2(x) \equiv \gamma(x)$ and $\gamma'(x) \neq 0$.

From Lemma 2, there exists an element $(Z_1(t, x), Z_2(t, x), T_0)$ of $\mathfrak{S}$, where T_0 can be assumed to be taken sufficiently small.

Since Z_k is a solution of $L_k Z_k = 0$ such that $\Re \partial_x Z_k(t, x) > 0$, $\Im \partial_x Z_k(t, x) > 0$, there exists the holomorphic function $h_k(z)$, where $h_k(z)$ is holomorphic in $\{z \in \mathbf{C};\ z = Z_k(t, x), (t, x) \in (0, T_0] \times [-T_0, T_0]\}$, such that $u_k(t, x) = h_k(Z_k(t, x))$ in $[0, T_0] \times [-T_0, T_0]$.

We observe that $h_k(Z_k(t, x)) \in C^1([0, T_0] \times [-T_0, T_0])$ and $h_k(Z_k(0, x)) = \gamma(x)$.

So we have

(3-1) $$u_k(t,x) = \frac{1}{2\pi i}\int_{C_k} \frac{h_k(z)dz}{z - Z_k(t,x)} \quad \text{in } (t,x) \in (0,T_0) \times (-T_0, T_0),$$

where C_k denotes $\{z \in \mathbf{C}; z = Z_k(t,x), (t,x) \in \partial([0,T_0] \times [-T_0, T_0])\}$.

Divide the C_k into the two parts $J_k = \{z \in \mathbf{C}; z = Z_k(0,x), x \in [-T_0, T_0]\}$ and $J_k^c \equiv C_k \backslash J_k$. Setting $z_k = Z_k(0,x)$ and $\Gamma_k(z) = h_k(z)|J_k$, we have

(3-2) $$\Gamma_k(z_k) = \text{p.v.}\frac{1}{2\pi i}\int_{J_k} \frac{\Gamma_k(z)dz}{z - z_k} + \frac{1}{2\pi i}\int_{J_k^c} \frac{h_k(z)dz}{z - z_k}.$$

Defining the function f_k by

$$f_k(z) = \frac{1}{2\pi i}\int_{J_k^c} \frac{h_k(\zeta)d\zeta}{\zeta - z},$$

we find that $f_k(z)$ is holomorphic in $\{z; z \in \mathbf{C}\backslash J_k^c\}$. Namely, f_k is holomorphic in $\mathbf{C}\backslash\{\ z \in \mathbf{C}; z = Z_k(t,x), (t,x) \in \partial([0,T_0] \times [-T_0, T_0], \backslash\{(0,x); -T_0 \le x \le T_0\}\ \}$. From (3.2), we have

(3-3) $$\Gamma_1(z_1) - \text{p.v.}\frac{1}{2\pi i}\int_{J_1} \frac{\Gamma_1(z)dz}{z - z_1} = f_1(z_1),$$

and

(3-4) $$\Gamma_2(z_2) - \text{p.v.}\frac{1}{2\pi i}\int_{J_2} \frac{\Gamma_2(z)dz}{z - z_2} = f_2(z_2).$$

So, applying a well-known formula (we refer (107.15) in p. 330 of Muskhelishvili [6], for instance) to (3.3) and (3.4), we obtain the following:

(3.5) $$\Gamma_1(z_1) = \frac{4}{3}f_1(z_1) + \frac{2}{3\pi i}A_1(z_1)\,\text{p.v.}\int_{J_1} \frac{f_1(z)dz}{A_1(z)(z - z_1)},$$

(3.6) $$\Gamma_2(z_2) = \frac{4}{3}f_2(z_2) + \frac{2}{3\pi i}A_2(z_2)\,\text{p.v.}\int_{J_2} \frac{f_2(z)dz}{A_2(z)(z - z_2)},$$

where

$$A_1(z) = (z - Z_1(0,T_0))^{(\log 3)/2\pi i}(z - Z_1(0,-T_0))^{(-\log 3)/2\pi i},$$

$$A_2(z) = (z - Z_2(0,T_0))^{(\log 3)/2\pi i}(z - Z_2(0,-T_0))^{(-\log 3)/2\pi i}.$$

Since $\Gamma_1(z_1) = \Gamma_2(z_2) = \gamma(x)$, from (3.5), (3.6), and $\gamma'(x) \neq 0$, we obtain

(3.7) $$f_1(Z_1(0,x)) + \frac{1}{2\pi i}A_1(Z_1(0,x))\,\text{p.v.}\int_{J_1} \frac{f_1(z)dz}{A_1(z)(z - Z_1(0,x))}$$
$$= f_2(Z_2(0,x)) + \frac{1}{2\pi i}A_2(Z_2(0,x))\,\text{p.v.}\int_{J_2} \frac{f_2(z)dz}{A_2(z)(z - Z_2(0,x))}$$

and

$$\frac{d}{dx}\left\{f_1(Z_1(0,x)) + \frac{1}{2\pi i}A_1(Z_1(0,x))\,\text{p.v.}\int_{J_1}\frac{f_1(z)dz}{A_1(z)(z - Z_1(0,x))}\right\} \neq 0 \tag{3.8}$$

for $x \in (-T_0, T_0)$.

Since

$$\frac{1}{2\pi i}\text{p.v.}\int_{J_1}\frac{f_1(z)dz}{A_1(z)(z-Z_1(0,x))} = \frac{1}{2}\frac{f_1(Z_1(0,x))}{A_1(Z_1(0,x))} + \frac{\frac{f_1(Z_1(0,x))}{A_1(Z_1(0,x))}}{2\pi i} \times \log\frac{Z_1(0,T_0)-Z_1(0,x)}{Z_1(0,-T_0)-Z_1(0,x)} + \frac{1}{2\pi i}\int_{J_1}\frac{\frac{f_1(z)}{A_1(z)} - \frac{f_1(Z_1(0,x))}{A_1(Z_1(0,x))}}{z - Z_1(0,x)}dz$$

and

$$\frac{1}{2\pi i}\text{p.v.}\int_{J_2}\frac{f_2(z)dz}{A_2(z)(z-Z_2(0,x))} = \frac{1}{2}\frac{f_2(Z_2(0,x))}{A_2(Z_2(0,x))} + \frac{\frac{f_2(Z_2(0,x))}{A_2(Z_2(0,x))}}{2\pi i} \times \log\frac{Z_2(0,T_0)-Z_2(0,x)}{Z_2(0,-T_0)-Z_2(0,x)} + \frac{1}{2\pi i}\int_{J_2}\frac{\frac{f_2(z)}{A_2(z)} - \frac{f_2(Z_2(0,x))}{A_2(Z_2(0,x))}}{z - Z_2(0,x)}dz,$$

the left-hand side of (3.7) =

$$\left(\frac{3}{2} + \frac{1}{2\pi i}\log\frac{Z_1(0,T_0)-Z_1(0,x)}{Z_1(0,-T_0)-Z_1(0,x)}\right)f_1(Z_1(0,x)) + \frac{A_1(Z_1(0,x))}{2\pi i}\int_{J_1}\frac{\frac{f_1(z)}{A_1(z)} - \frac{f_1(Z_1(0,x))}{A_1(Z_1(0,x))}}{z - Z_1(0,x)}dz$$

and the right-hand side of (3.7) =

$$\left(\frac{3}{2} + \frac{1}{2\pi i}\log\frac{Z_2(0,T_0)-Z_2(0,x)}{Z_2(0,-T_0)-Z_2(0,x)}\right)f_2(Z_2(0,x)) + \frac{A_2(Z_2(0,x))}{2\pi i}\int_{J_2}\frac{\frac{f_2(z)}{A_2(z)} - \frac{f_2(Z_2(0,x))}{A_2(Z_2(0,x))}}{z - Z_2(0,x)}dz.$$

Denoting by $g_1(Z_1(0,x))$ the function

$$\frac{A_1(Z_1(0,x))}{2\pi i}\int_{J_1}\frac{\dfrac{f_1(z)}{A_1(z)}-\dfrac{f_1(Z_1(0,x))}{A_1(Z_1(0,x))}}{z-Z_1(0,x)}dz$$

and by $g_2(Z_2(0,x))$ the function

$$\frac{A_2(Z_2(0,x))}{2\pi i}\int_{J_2}\frac{\dfrac{f_2(z)}{A_2(z)}-\dfrac{f_2(Z_2(0,x))}{A_2(Z_2(0,x))}}{z-Z_2(0,x)}dz,$$

we see that g_1 is the holomorphic function of $Z_1(0,x)$ in a neighborhood of $J_1^\circ \equiv \{z \in \mathbf{C}; z = Z_1(0,x), x \in (-T_0, T_0)\}$ and g_2 is the holomorphic function of $Z_2(0,x)$ in a neighborhood of $J_2^\circ \equiv \{z \in \mathbf{C}; z = Z_2(0,x), x \in (-T_0, T_0)\}$.

Setting $F_k(Z) =$

$$\left(\frac{3}{2}+\frac{1}{2\pi i}\log\frac{Z_k(0,T_0)-Z}{Z_k(0,-T_0)-Z}\right)f_k(Z)+g_k(Z),$$

we have

$$F_1(Z_1(0,x)) = F_2(Z_2(0,x)) \qquad \text{in } (-T_0, T_0).$$

Here we note that $F_1' \neq 0$ by (3.8). So we find that there locally exists $f \equiv F_1^{-1} \circ F_2$.

Hence, we can take a positive constant which is denoted by the same letter T_0 such that the following holds:

$(Z_1(t,x), Z_2(t,x), T_0) \in \mathfrak{S}$, f is holomorphic in $\mathfrak{J} = \{z \in \mathbf{C}; z = Z_2(0,x), x \in (-T_0, T_0)\}$, and $Z_1(0,x) = f(Z_2(0,x))$.

This ends the proof of necessity.

Next we prove sufficiency.

Assume that the condition stated in Theorem 3 holds. Then we can take a positive constant T smaller that T_0 and a neighborhood $\mathfrak{U}$ of $\mathfrak{J}$ where f is holomorphic such that $\mathfrak{U} \supset Z_2([-T,T] \times [-T,T])$.

Let us define the functions $u_1(t,x)$ and $u_2(t,x)$ in $[0,T] \times [-T,T]$ as follows:

$$u_1(t,x) = Z_1(t,x) \qquad \text{and} \qquad u_2(t,x) = f(Z_2(t,x)).$$

Then it follows that $L_1u_1 = L_2u_2 = 0$, $u_1(0,x) = Z_1(0,x) = f(Z_2(0,x)) = u_2(0,x)$ and $u_1'(0,x) \neq 0$.

§5. Proof of Corollary 4

Assume the contrary. By Theorem 3, we see that there exist $(Z_1(t,x), Z_2(t,x), T)$ and a function f satisfying the following:

$$\partial_t Z_1 + i(1+|t|^{1/2}r(t,x))\partial_x Z_1 = 0,$$

$$\partial_t Z_2 + i(1-|t|^{1/2}r(t,x))\partial_x Z_2 = 0,$$

$$Z_1(0,x) = f(Z_2(0,x)),$$

$$Z_k(t,x) \in C^{1+1/2d}([-T,T] \times [-T,T]),$$

$$Z_k(0,0) = 0, \qquad \Re\partial_x Z_k(t,x) > 0, \qquad \Im\partial_x Z_k(t,x) > 0, \qquad \text{and}$$

$$f \text{ is holomorphic in } \mathfrak{U} \text{ where } \mathfrak{U} \supset Z_2([-T,T] \times [-T,T]).$$

Moreover, we may suppose that

$$\Re\{f'(Z_2(t,x))\partial_x Z_2(t,x)\} > 0, \qquad \Im\{f'(Z_2(t,x))\partial_x Z_2(t,x)\} > 0$$

in $[0,T] \times [-T,T]$, since $Z_1(0,x) = f(Z_2(0,x))$.

Since, when $t \geq 0$, $L_k = \partial_t + i\partial_x$ outside of $\bigcup_{n,p} B_{n,p}$, for non-negative t, we have

$$(\partial_t + i\partial_x)(Z_1(t,x) - f(Z_2(t,x))) = \partial_t Z_1 + i\partial_x Z_1 - f'(Z_2(t,x))(\partial_t Z_2 + i\partial_x Z_2) = 0$$

outside of $\bigcup_{n,p} B_{n,p}$. Hence, since $Z_1(0,x) - f(Z_2(0,x)) = 0$, we get

$$Z_1(t,x) = f(Z_2(t,x)) \text{ outside of } \bigcup_{n,p} B_{n,p}$$

for non-negative t by the unique continuation property. Now we can take p sufficiently large such that for every n

$$\Re\{f'(Z_2(t,x))\partial_x Z_2(t,x)\} > 0, \qquad \Im\{f'(Z_2(t,x))\partial_x Z_2(t,x)\} > 0$$

and

$$\Re\partial_x Z_2(t,x) > 0, \qquad \Im\partial_x Z_2(t,x) > 0$$

hold in $B_{n,p}$. Set $G = G(t,x) = Z_1(t,x) - f(Z_2(t,x))$. Then we have

$$\iint_{B_{n,p}} d(GdZ_1) = \int_{\partial B_{n,p}} GdZ_1 = 0.$$

Therefore it must hold that

$$\begin{aligned}
0 &= \Im \iint_{B_{n,p}} d(GdZ_1) \\
&= \Im \iint_{B_{n,p}} (G_t\partial_x Z_1 - G_x\partial_x Z_1)dtdx \\
&= \Im \iint_{B_{n,p}} f'(Z_2)L_1(G)\partial_x Z_1 dtdx \\
&= \Im \iint_{B_{n,p}} 2ir(t,x)f'(Z_2)\partial_x Z_2\partial_x Z_1 dtdx \\
&= \iint_{B_{n,p}} 2r(t,x)(\Re\{f'(Z_2)\partial_x Z_2\}\Im\partial_x Z_1 + \Im\{f'(Z_2)\partial_x Z_2\}\Re\partial_x Z_1)dtdx.
\end{aligned}$$

But

$$\iint_{B_{n,p}} 2r(t,x)(\Re\{f'(Z_2)\partial_x Z_2\}\Im\partial_x Z_1 + \Im\{f'(Z_2)\partial_x Z_2\}\Re\partial_x Z_1)dtdx$$

is positive; this is contradictory.

§6. Proof of Corollary 5

Being nearly clear, the proof is as follows:
Without loss of generality we may suppose $W^{[k]}(0) = 0$. It clearly holds that

$$\partial_{\bar{w}} W^{[k]}(w) = \mu_k(w)\partial_w W^{[k]}(w), \qquad \partial_w W^{[k]}(w) = \omega^{[k]}(w) \neq 0.$$

Multiplying a suitable complex constant by $W^{[k]}(w) \equiv W^{[k]}(t,x)$ and taking a positive constant T sufficiently small, we may further assume that $\Re\partial_t W^{[k]} > 0$ and $\Im\partial_x W^{[k]} > 0$ hold in $[-T,T] \times [-T,T]$. Thus we see

$$(W^{[1]}(t,x), W^{[2]}(t,x), T) \in \mathfrak{S}.$$

Since

$$W^{[1]}(0,x) = \frac{1}{2\pi i}\iint_{B_r} \frac{\mu_1(\zeta)W^{[1]}(\zeta)}{\zeta - ix} d\zeta \wedge d\bar{\zeta} + ix$$

and

$$W^{[2]}(0,x) = \frac{1}{2\pi i}\iint_{B_r} \frac{\mu_2(\zeta)W^{[2]}(\zeta)}{\zeta - ix} d\zeta \wedge d\bar{\zeta} + ix,$$

it follows that $W^{[1]}(0,x) = W^{[2]}(0,x)$ by our assumption.

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