J. Math. Kyoto Univ. (JMKYAZ)
39-4 (1999) 649–673

Perturbation theorems for supercontractive semigroups

By

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Abstract

Let μ be a probability measure on a Riemannian manifold. It is known that if the semigroup $e^{-t\nabla^*\nabla}$ is hypercontractive, then any function g for which $\|\nabla g\|_\infty \le 1$ will satisfy a Herbst inequality, $\int \exp(\alpha g^2)d\mu < \infty$, for small $\alpha > 0$. If the semigroup is supercontractive, then the above inequality will hold for all $\alpha > 0$. For any $\alpha > 0$ for which $Z = \int \exp(\alpha g^2)d\mu < \infty$, we define a measure μ_g by $d\mu_g = Z^{-1}\exp(\alpha g^2)d\mu$. We show that if μ is hyper- or supercontractive, then so is μ_g. Moreover, under standard conditions on logarithmic Sobolev inequalities which yield ultracontractivity of the semigroup, Gross and Rothaus have shown that $Z = \int \exp(\alpha g^2 |\log|g||^c)d\mu < \infty$ for some constants α, c. We in addition show that the perturbed measure $d\mu_g = Z^{-1}\exp(\alpha g^2|\log|g||^c)d\mu$ is ultracontractive.

1. Introduction

Let (X,μ) be a probability space and consider a sub-Markovian symmetric contraction semigroup e^{-tA} on $L^2(X,\mu)$. Recall that a semigroup e^{-tA} is **sub-Markovian** if for any $f \in L^2(X,\mu), 0 \le f \le 1$ implies $0 \le e^{-tA}f \le 1$. Then, by the Beurling-Deny conditions [D, p. 12–16], e^{-tA} is L^p-contractive from $L^2 \cap L^p$ into L^p and hence from L^p into L^p by a density argument for any $t \ge 0$ and $1 \le p \le \infty$. We say that

- e^{-tA} is **hypercontractive** if for $1 < p < q < \infty$ there is a $t_0 > 0$ such that $\|e^{-tA}\|_{L^p \to L^q} < \infty$ for any $t \ge t_0$.
- e^{-tA} is **supercontractive** if $\|e^{-tA}\|_{L^p \to L^q} < \infty$ for any $1 < p, q < \infty$ and any $t > 0$.
- e^{-tA} is **ultracontractive** if $\|e^{-tA}\|_{L^p \to L^\infty} < \infty$ for any $1 \le p < \infty$ and any $t > 0$.

There are infinitesimal versions of these various contractivities in terms of Gross's logarithmic Sobolev inequalities. The following theorem addresses the hypercontractive case.

Theorem 1.1. *Let (X,μ) be a probability space and suppose that e^{-tA} is a sub-Markovian symmetric contraction semigroup on $L^2(X,\mu)$. Then e^{-tA} is hypercontractive if and only if there exist constants $\varepsilon > 0$ and $\beta \ge 0$ such that the following (defective) logarithmic Sobolev inequality*

$$(1.1)\quad \int_X f^2 \log f^2 d\mu \le \varepsilon\mathscr{E}(f,f) + \beta\|f\|^2_{L^2(\mu)} + \|f\|^2_{L^2(\mu)}\log\|f\|^2_{L^2(\mu)} \quad (f \in \mathscr{D}(\mathscr{E}))$$

Communicated by Prof. K. Ueno, November 20, 1998

holds, where $\mathscr{E}(f,f) = (\sqrt{A}f, \sqrt{A}f), \mathscr{D}(\mathscr{E}) = \mathscr{D}(\sqrt{A})$, *is the Dirichlet form associated to* A. *Moreover, we have*

$$(1.2) \quad \|e^{-tA}\|_{L^p \to L^q} \le e^{4\beta(1/p-1/q)} \quad \text{for } 1 < p \le q < \infty \quad \text{and any } t \ge \frac{\varepsilon}{4}\log\left(\frac{q-1}{p-1}\right).$$

ε is called a logarithmic Sobolev constant and β is called a defective term. Originally, Gross [G1] proved this theorem for two important classes of Dirichlet forms, namely Dirichlet forms given by a gradient: $\mathscr{E}(f,f) = \int_X |\nabla f|^2 d\mu$ and the finite difference form on the two-point measure space. Stroock ([S], see also [DeS]) later extended Gross's proof to the general case. We refer the proof of this theorem and general background regarding logarithmic Sobolev inequalities and various contractivities of semigroups discussed here to the expository paper [G3].

Note that in (1.2), if we can reduce ε arbitrarily small by increasing β, we then have supercontractivity of the semigroup. The converse also holds by an interpolation argument.

Theorem 1.2. *Assume that the same assumption of Theorem* 1.1 *holds. Then* e^{-tA} *is supercontractive if and only if there is a function* $\beta : (0,\infty) \to [0,\infty)$ *which satisfies*

$$(1.3) \qquad \int_X f^2 \log f^2 d\mu \le \varepsilon\mathscr{E}(f,f) + \beta(\varepsilon)\|f\|^2_{L^2(\mu)} + \|f\|^2_{L^2(\mu)} \log \|f\|^2_{L^2(\mu)} \qquad (f \in \mathscr{D}(\mathscr{E}))$$

for any $\varepsilon > 0$. *Moreover, one can take* β *to be decreasing and convex.*

The only condition on β for the semigroup to be supercontractive is that $\beta(\varepsilon) < \infty$ for all $\varepsilon > 0$. One need not specify how fast $\beta(\varepsilon)$ tends to ∞ as $\varepsilon \to 0^+$. Intuitively, if $\beta(\varepsilon)$ does not grow "too fast" as ε approaches zero, the semigroup will be ultracontractive. However, a condition on $\beta(\cdot)$ which is equivalent to ultracontractivity is not known. See [C] for further discussion of this. The following theorem gives a sufficient condition for ultracontractivity. We will use it as a basis for proving results concerning ultracontractivity via logarithmic Sobolev inequalities in this paper.

Theorem 1.3 (Davies and Simon [DS]). *Assume that* (1.3) *holds and that for each* $t > 0$ *there exists a function* $c : [2,\infty) \to (0,\infty)$ *such that*

$$t = \int_2^\infty \frac{c(p)}{p} dp$$

and

$$M(t) \equiv \int_2^\infty \frac{2\beta(c(p))}{p^2} dp < \infty.$$

Then the semigroup e^{-tA} is ultracontractive and

$$\|e^{-tA}\|_{L^2\to L^\infty} \le e^{M(t)}.$$

Let us recall the ground state transform for a Schrödinger operator on $X = \mathbf{R}^n$. Denote by Δ the Laplacian in $\mathbf{R}^n$. Let V be a potential in $\mathbf{R}^n$ and assume that the form sum, $-\Delta + V$, has a form closure, H, which is a semi-bounded self-adjoint operator in $L^2(\mathbf{R}^n, dx)$ and which has a unique lowest normalized eigenfunction ϕ_0, i.e. $H\phi_0 = e\phi_0$ where $e = \inf spec(H)$. ϕ_0 can be taken to be strictly positive almost everywhere. Define a probability measure $d\mu(x) = \phi_0(x)^2 dx$ and the unitary map $M_{\phi_0} : L^2(\mathbf{R}^n, dx) \to L^2(\mathbf{R}^n, \mu)$ by $M_{\phi_0} f = f/\phi_0$. Then $A = M_{\phi_0}(H - e)M_{\phi_0}^{-1}$ is the operator on $L^2(\mathbf{R}^n, \mu)$ given by the Dirichlet form

$$\mathscr{E}(f, f) = \int_X |\nabla f(x)|^2 d\mu. \tag{1.4}$$

The domain of $\mathscr{E}$ consists of those functions f in $L^2(X, \mu)$ whose weak gradient ∇f is also in $L^2(X, \mu)$. The Schrödinger operator H on $L^2(X, dx)$ is said to be **intrinsically hyper-, super-**, or **ultracontractive** [DS] if the associated operator A on $L^2(X, \mu)$ is hyper-, super-, or ultracontractive, respectively.

We now look at the one-dimensional case. We will see in the following theorem that the type of contractivity depends on how fast the potential term grows.

Theorem 1.4 ([DS]). *Consider the Schrödinger operator $H = -\Delta + V$ on $\mathbf{R}^1$.*

(a) *If $V(x) = |x|^a$, then e^{-tH} is intrinsically ultracontractive if $a > 2$, intrinsically hypercontractive but not supercontractive if $a = 2$, and not even intrinsically hypercontractive if $a < 2$.*

(b) *If $V(x) = x^2[\log(2 + |x|^2)]^b$, then e^{-tH} is intrinsically ultracontractive if $b > 2$ and is intrinsically supercontractive (but not ultracontractive) if $0 < b \le 2$. In this case, the asymptotic behavior of the ground state ϕ_0 is given by $-\log \phi_0(x) \sim x^2(\log|x|)^{b/2}$ as $|x| \to \infty$, for $b \ge 0$.*

Motivated by the above theorem, we will find an analog of it in the abstract setting. We assume that there is a natural gradient operator, ∇, defined on a dense subset of $L^2(X, \mu)$. Let A be the nonnegative self-adjoint operator associated with the Dirichlet form on X given by ∇ as in (1.4).

In an unpublished letter to L. Gross, I. Herbst showed that for the Dirichlet form $\mathscr{E}(f, f) = \int_{\mathbf{R}} |f'(x)|^2 d\mu(x)$ on $\mathbf{R}$, a necessary condition for the measure μ to be hypercontractive is that $\int_{\mathbf{R}} e^{\alpha x^2} d\mu(x) < \infty$ for some $\alpha > 0$. Such exponential integrability under contractivity assumptions will be referred to as Herbst inequalities. See [GR] for more historical background of the subject. Herbst inequalities have been studied extensively in the hypercontractive case. See, e.g., [A, AMS, AS, ASt, BG, GR, L1, L2, R]. The supercontractive case is an immediate consequence of the hypercontractive case. We summarize these results in the following theorem.

Theorem 1.5. *Assume that* $\|\nabla g\|_\infty \le 1$.
(a) *If* e^{-tA} *is hypercontractive, then* $\int e^{\alpha g^2} d\mu < \infty$ *for small* $\alpha > 0$.
(b) *If* e^{-tA} *is supercontractive, then* $\int e^{\alpha g^2} d\mu < \infty$ *for any* $\alpha > 0$.

To assure that this is the right analog of the one-dimensional case, we will also prove the following perturbation theorem.

Theorem 1.6. *Assume that* $\|\nabla g\|_\infty \le 1$. *For any* $\alpha > 0$ *for which* $Z = \int e^{\alpha g^2} d\mu < \infty$, *define a probability measure* μ_g *on* X *by* $d\mu_g = Z^{-1} e^{\alpha g^2} d\mu$. *If* μ *is hypercontractive (supercontractive), then so is* μ_g.

Theorem 1.6 for hypercontractivity has essentially been proved in [AS, A] in a slightly different setting. This result could be regarded as an extension of the Holley-Stroock lemma [HS], which states that the perturbation of a hypercontractive measure by a bounded density is still hypercontractive. However, the Holley-Stroock lemma only requires boundedness of the density without any smoothness assumption. Theorem 1.6 allows the density to be unbounded, but requires some smoothness.

We now look at the ultracontractive case. We will take Theorem 1.3 as a basis for proving results about ultracontractivity. We assume that the family of logarithmic Sobolev inequalities (1.3) holds. In [GR], Gross and Rothaus show that, under a standard assumption on β which ensures ultracontractivity, the following Herbst-type inequality

$$\int_X e^{\gamma g^2 |\log|g||} d\mu < \infty$$

holds for some $\gamma > 0$ where g is a measurable function on X such that $\|\nabla g\|_\infty \le 1$. This result of [GR], with a slight generalization, will be stated precisely in Corollary 5.9. We also study perturbation theorems for ultracontractive semigroups. Assume that μ satisfies logarithmic Sobolev inequalities which yield ultracontractivity. Define $d\mu_g = Z^{-1} e^{\gamma g^2 |\log|g||^c} d\mu$ where Z is the normalization factor, γ and c are certain positive constants. We will show that $d\mu_g$ also satisfies logarithmic Sobolev inequalities which yield ultracontractivity.

In this paper, we consider Herbst inequalities and perturbation theorems under supercontractivity and ultracontractivity assumptions. It consists of 2 parts, which are independent of each other. Section 3 and 4 deal with generalization of Herbst inequalities to the case where the L^p-norm of the gradient of g grows polynomially in p, while Section 5 and 6 concern with perturbation theorems under the boundedness assumption on the gradient of g.

Here is an outline of this work. After setting up our notation in Section 2, we prove an estimate for the L^p-norm of a function in terms of its L^2-norm and the norms of its gradient in Section 3. This result extends those of Aida and Stroock, [ASt], to the ultracontractive case. In Section 4, we obtain a Herbst-type inequality $\int e^{\Phi(g)} d\mu < \infty$, but here we allow the L^p-norms of the gradient of g to grow polynomially in p. We return to the assumption $\|\nabla g\|_\infty \le 1$ in Section

5, where we prove perturbation theorems for hyper-, super- and ultracontractive semigroups, respectively.

There is also another point of view, [A, AMS, AS, H], in how to extend Theorem 1.4 to a general case. In these papers, their authors assume the exponential integrability of the square of the gradient of a function F, i.e. $\int e^{p|\nabla F|^2} d\mu < \infty$ for some $0 < p < \infty$. Under this assumption, it is shown that $\int e^{pF} d\mu < \infty$ for small $p > 0$ in the hypercontractive case. Then they prove that the perturbed measure $d\mu_g = e^{2F} d\mu$ is hypercontractive if μ is. We show that similar results also hold for the supercontractive case. The difference between this approach and the one in Theorems 1.5 and 1.6 is that the function g is modeled on the map $g(x) = x$ while the function F is modeled on the map $F(x) = x^2$. We will elaborate on these distinct viewpoints in Section 6.

2. Notation

We will adopt the terminology used in [AMS] throughout this work. We also refer to [BH, D, FOT, RM] for the standard theory of Dirichlet forms. Let $(X, \mathscr{B}, \mu)$ be a probability space and $\mathscr{E}$ a Dirichlet form on $L^2(X, \mu)$. Let A be a nonnegative self-adjoint operator associated with $\mathscr{E}$:

$$\mathscr{E}(f, g) = (\sqrt{A} f, \sqrt{A} g), \qquad (f, g \in \mathscr{D}(\mathscr{E}) = \mathscr{D}(\sqrt{A})).$$

We assume that there exists a dense subspace $\mathscr{D}$ in $L^2(X, \mu)$ satisfying

(A1) $\mathscr{D} \subseteq L^\infty(X, \mu) \cap \mathscr{D}(A)$,

(A2) $1 \in \mathscr{D}$ and $\mathscr{D}$ is an algebra,

(A3) $\mathscr{D}$ is a core of the Dirichlet form $\mathscr{E}$.

We define a *Carré du champ* operator $\Gamma : \mathscr{D} \times \mathscr{D} \to L^1(X, \mu)$ by

$$\Gamma(f, g) = \tfrac{1}{2}\{Af \cdot g + f \cdot Ag - A(f \cdot g)\}.$$

We assume further that

(A4) For $f, g \in \mathscr{D}, \Gamma(f, g) \in \bigcap_{1 \le p < \infty} L^p(X, \mu)$.

Notice that our definition of Γ differs from the one in [AMS] by a sign because we make a choice for A to be nonnegative here. It was shown in [AMS] that the map $(f, g) \mapsto \Gamma(f, g)$ can be extended continuously to a map $\Gamma : \mathscr{D}(\mathscr{E}) \times \mathscr{D}(\mathscr{E}) \to L^1(X, \mu)$. We also denote this continuous extension by Γ. Γ has the following properties

(i) Γ is a symmetric bilinear map from $\mathscr{D}(\mathscr{E}) \times \mathscr{D}(\mathscr{E})$ into $L^1(X, \mu)$,

(ii) $\Gamma(f, f) \ge 0 \quad \mu - a.e.$ for each $f \in \mathscr{D}(\mathscr{E})$,

(iii) $|\Gamma(f, g)|^2 \le \Gamma(f, f) \cdot \Gamma(g, g)$ for each $f, g \in \mathscr{D}(\mathscr{E})$.

We will also assume the following derivation property of Γ:

(A5) $\Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h)$ for $f, g, h \in \mathscr{D}(\mathscr{E})$.

With the assumption (A5), we have

(iv) $\mathscr{E}(f, g) = \int_X \Gamma(f, g) d\mu$ for any $f, g \in \mathscr{D}(\mathscr{E})$,

(v) if $u, v \in C_b^1(\mathbf{R}^n, \mathbf{R})$ and $f_1, \ldots, f_n \in \mathscr{D}(\mathscr{E})$, then $u(f_1, \ldots, f_n), v(f_1, \ldots, f_n) \in \mathscr{D}(\mathscr{E})$ and

$$\Gamma(u(f_1,\dots,f_n),v(f_1,\dots,f_n))$$
$$= \sum_{i,j=1}^{n} (\partial_i u)(f_1,\dots,f_n)(\partial_j v)(f_1,\dots,f_n)\Gamma(f_i,f_j) \quad \mu - a.e.$$

(vi) if $\phi : \mathbf{R} \to \mathbf{R}$ is a normal contraction, i.e. $\phi(0) = 0$ and $|\phi(s) - \phi(t)| \le |s - t|$ for all $s,\ t \in \mathbf{R}$, then $\phi \circ f \in \mathscr{D}(\mathscr{E})$ whenever $f \in \mathscr{D}(\mathscr{E})$ and

$$\Gamma(\phi \circ f, \phi \circ f) \le \Gamma(f,f) \quad \mu - \text{a.e.}$$

We write $|\nabla f| = \sqrt{\Gamma(f,f)}$ for $f \in \mathscr{D}(\mathscr{E})$.

Example 2.1. Let $X = \mathbf{R}^n$ and let $d\mu(x) = w(x)dx$ be a probability measure on $\mathbf{R}^n$. Assume that $w > 0$ on $\mathbf{R}^n$. Denote by ∇f the usual gradient of f. Let $\mathscr{D} = C_b^\infty(\mathbf{R}^n)$, the space of smooth bounded functions on $\mathbf{R}^n$ with derivatives all bounded. Then there is a unique closed quadratic form $\mathscr{E}$ with core $\mathscr{D}$ such that

$$\mathscr{E}(f,g) = \int_{\mathbf{R}^n} (\nabla f(x), \nabla g(x)) d\mu(x) \qquad (f,g \in \mathscr{D}).$$

$\mathscr{E}$ is a Dirichlet form on $(\mathbf{R}^n, \mu)$ and the associated operator A is given by

$$Af(x) = \sum_{i=1}^{n} \left(-\frac{\partial^2 f}{\partial x_i^2}(x) - \left(\frac{\partial}{\partial x_i} \log w\right) \frac{\partial f}{\partial x_i}(x) \right).$$

In particular, if $w(x) = 1/(2\pi)^{n/2} \exp\{-|x|^2/2\}$ is the Gaussian density, then

$$A = \sum_{i=1}^{n} \left(-\frac{\partial^2}{\partial x_i^2} + x_i \frac{\partial}{\partial x_i} \right).$$

In this case, Γ is simply

$$\Gamma(f,g)(x) = (\nabla f(x), \nabla g(x))_{\mathbf{R}^n}.$$

Example 2.2. Consider an abstract Wiener space (B, H, μ) and let A be the Ornstein-Uhlenbeck operator, which we denote by L. We can take $\mathscr{D}$ to be $C_b^\infty(B)$, the space of C_b^∞ cylinder functions on B. In this case, we have

$$\Gamma(f,g)(x) = (Df(x), Dg(x))_{H^*},$$

where Df denotes the H-derivative of f.

3. Moment estimates

We assume that there is a function $\beta : (0,\infty) \to [0,\infty]$ such that a *defective logarithmic Sobolev inequality*

$$\int f^2 \log \frac{f^2}{\|f\|_{L^2(\mu)}^2} d\mu \le \varepsilon \mathscr{E}(f,f) + \beta(\varepsilon) \|f\|_{L^2(\mu)}^2 \qquad (f \in \mathscr{D}(\mathscr{E})) \tag{3.1}$$

holds for any $\varepsilon > 0$. In this section, we obtain bounds on the L^p-norms of functions on X under a certain assumption on β and ε.

Lemma 3.1. *If $f \in \mathscr{D}(\mathscr{E})$ is positive everywhere and $2 \le t < \infty$, then*

$$\mathscr{E}(f^{t/2}, f^{t/2}) \le \frac{t^2}{4}\|\nabla f\|_t^2 \|f\|_t^{t-2}.$$

Proof. By property (v) in Section 2,

$$\Gamma(f^{t/2}, f^{t/2}) = \left(\frac{t}{2}\right)^2 (f^{t/2-1})^2 \Gamma(f,f) = \frac{t^2}{4} f^{t-2}|\nabla f|^2.$$

The lemma now follows from the above calculation and Hölder's inequality.

The next lemma is another form for *Gronwall's inequality.*

Lemma 3.2. *Let x, g and k be continuous real-valued functions defined on an interval $[\alpha, \beta)$ and suppose that k is differentiable on (α, β). If*

$$x(t) \le k(t) + \int_\alpha^t g(s)x(s)ds \qquad (\alpha \le t < \beta), \tag{3.2}$$

then, for $\alpha \le t < \beta$,

$$x(t) \le \exp\left(\int_\alpha^t g(s)ds\right)\left[k(\alpha) + \int_\alpha^t \exp\left(-\int_\alpha^\tau g(s)ds\right)k'(\tau)d\tau\right]. \tag{3.3}$$

Proof. Let $U(t) = k(t) + \int_\alpha^t g(s)x(s)ds$. Note that $U(\alpha) = k(\alpha)$ and $x(t) \le U(t)$. Then

$$U'(t) = k'(t) + g(t)x(t) \le k'(t) + g(t)U(t).$$

Multiplying the above inequality by $\exp(-\int_\alpha^t g(s)ds)$, we have

$$\frac{d}{dt}\left(\exp\left(-\int_\alpha^t g(s)ds\right)U(t)\right) \le \exp\left(-\int_\alpha^t g(s)ds\right)k'(t).$$

Integration from α to t gives

$$\exp\left(-\int_\alpha^t g(s)ds\right)U(t) - U(\alpha) \le \int_\alpha^t \exp\left(-\int_\alpha^\tau g(s)ds\right)k'(\tau)d\tau.$$

Hence,

$$\begin{aligned} U(t) &\le \exp\left(\int_\alpha^t g(s)ds\right)\left[U(\alpha) + \int_\alpha^t \exp\left(-\int_\alpha^\tau g(s)ds\right)k'(\tau)d\tau\right] \\ &= \exp\left(\int_\alpha^t g(s)ds\right)\left[k(\alpha) + \int_\alpha^t \exp\left(-\int_\alpha^\tau g(s)ds\right)k'(\tau)d\tau\right], \end{aligned}$$

from which (3.3) follows.

In the next theorem, we obtain bounds on L^p-norm of functions under a certain assumption on ε and β.

Theorem 3.3. *Assume that* (3.1) *holds and that* $\varepsilon : (0,\infty) \to (0,\infty)$ *is a continuous function for which the function* $p \mapsto \beta \circ \varepsilon(p)$ *is continuous. Let*

$$D(t) = \int_2^t \frac{2\beta \circ \varepsilon(s)}{s^2}\, ds \qquad (2 \le t \le \infty). \tag{3.4}$$

Then, for any $f \in \mathscr{D}(\mathscr{E})$,

$$\|f\|_p^2 \le e^{D(p)}\left[\|f\|_2^2 + \frac{1}{2}\int_2^p e^{-D(t)}\varepsilon(t)\|\nabla f\|_t^2 dt\right] \qquad (2 \le p < \infty). \tag{3.5}$$

Proof. Since $\Gamma(|f|,|f|) \le \Gamma(f,f)$ for any $f \in \mathscr{D}(\mathscr{E})$, c.f. property (vi) in Section 2, it suffices to prove the theorem for $f > 0$. By the logarithmic Sobolev inequality (3.1) and Lemma 3.1, we have

$$\begin{aligned}
\frac{d}{dt}\|f\|_t^2 &= \frac{2}{t^2}\|f\|_t^{2-t}\int_X f^t \log\frac{f^t}{\|f\|_t^t}\,d\mu \\
&\le \frac{2}{t^2}\|f\|_t^{2-t}(\varepsilon(t)\mathscr{E}(f^{t/2}, f^{t/2}) + \beta \circ \varepsilon(t)\|f\|_t^t) \\
&\le \frac{2}{t^2}\|f\|_t^{2-t}\left(\varepsilon(t)\frac{t^2}{4}\|\nabla f\|_t^2\|f\|_t^{t-2} + \beta \circ \varepsilon(t)\|f\|_t^t\right) \\
&= \frac{\varepsilon(t)}{2}\|\nabla f\|_t^2 + \frac{2\beta \circ \varepsilon(t)}{t^2}\|f\|_t^2.
\end{aligned}$$

Integration from 2 to t yields

$$\|f\|_t^2 \le \|f\|_2^2 + \int_2^t \frac{\varepsilon(s)}{2}\|\nabla f\|_s^2 ds + \int_2^t \frac{2\beta \circ \varepsilon(s)}{s^2}\|f\|_s^2 ds.$$

Applying Lemma 3.2 to the above inequality by writing

$$x(t) = \|f\|_t^2, \qquad g(t) = \frac{2\beta \circ \varepsilon(t)}{t^2} \qquad \text{and} \qquad k(t) = \|f\|_2^2 + \int_2^t \frac{\varepsilon(s)}{2}\|\nabla f\|_s^2 ds,$$

we immediately have (3.5).

Corollary 3.4. *Under the same hypothesis as in Theorem* 3.3, *suppose further that*

$$M \equiv D(\infty) = \int_2^\infty \frac{2\beta \circ \varepsilon(s)}{s^2}\, ds < \infty.$$

Then

$$\|f\|_p^2 \le e^M\left[\|f\|_2^2 + \int_2^p \frac{\varepsilon(t)}{2}\|\nabla f\|_t^2 dt\right] \qquad (f \in \mathscr{D}(\mathscr{E})). \tag{3.6}$$

Proof. It follows from the facts that $D(2) = 0$ and the map $t \to D(t)$ is an increasing function on $[2, \infty)$.

Remark. The inequality (3.6) is an extension of the inequality (3.5) in [ASt] to the case where ε and β are nonconstant.

In [GR], Gross and Rothaus have derived an inequality for the Laplace transform $E(e^{\lambda g})$, where g is a function with bounded "gradient" by adapting techniques from [L1]. We now show that our method also yields the same inequality.

Proposition 3.5. *Assume that* (3.1) *holds. Let g be a real-valued measurable function such that $\|\nabla g\|_\infty \le 1$. Then*

$$E(e^{pg})^{1/p} \le E(e^{2g})^{1/2} \exp\left[\frac{1}{2}\int_2^p \left\{\frac{\varepsilon(t)}{2} + \frac{2\beta \circ \varepsilon(t)}{t^2}\right\} dt\right], \qquad 2 \le p < \infty. \tag{3.7}$$

Proof. Let $f = e^g$. Then

$$\Gamma(f,f) = \Gamma(e^g, e^g) = e^{2g}\Gamma(g,g) = f^2\Gamma(g,g).$$

Hence, $\|\nabla g\|_\infty \le 1$ implies $\|\nabla f\|_t \le \|f\|_t$ for any $2 \le t < \infty$. It then follows from (3.5) that

$$\|f\|_t^2 \le e^{D(t)}\left[\|f\|_2^2 + \frac{1}{2}\int_2^t e^{-D(s)}\varepsilon(s)\|f\|_s^2 ds\right] \qquad (2 \le t < \infty).$$

Applying Lemma 3.2 to the above inequality with

$$x(t) = e^{-D(t)}\|f\|_t^2, \qquad k(t) = \|f\|_2^2 \qquad \text{and} \qquad g(t) = \frac{\varepsilon(t)}{2}$$

for $2 \le t < \infty$, we have

$$\|f\|_p^2 \le e^{D(p)}\|f\|_2^2 \exp\left[\int_2^p \frac{\varepsilon(t)}{2} dt\right] = \|f\|_2^2 \exp\left[\int_2^p \left\{\frac{\varepsilon(t)}{2} + \frac{2\beta \circ \varepsilon(t)}{t^2}\right\} dt\right].$$

By writing $\|f\|_p = E(e^{pg})^{1/p}$, this immediately gives (3.7).

4. Exponential integrability for a measure satisfying LSI

As is now well-known ([A, AMS, AS, ASt, BG, GR, L1, L2, R]) a probability measure μ on a space X satisfying a logarithmic Sobolev inequality has exponential integrability in the form

$$\int_X e^{\alpha f^2} d\mu < \infty \tag{4.1}$$

for α sufficiently small where f is a function for which $\|\nabla f\|_\infty \le 1$. This kind of exponential integrability under LSI assumption is known as a Herbst inequality.

In all these papers except [GR], the measure μ is assumed to satisfy an LSI for a fixed ε and with or without assuming that $\beta = 0$. In [GR], Gross and Rothaus assume that ε and β satisfy standard assumptions which ensure ultracontractivity and that the function f has bounded gradient. In the present work, we will assume standard conditions on ε and β for ultracontractivity, but we allow the L^p-norms of the gradient of f to grow polynomially in p. With explicit forms of the function $\varepsilon(\cdot)$, we can improve inequality (4.1). The techniques here are those of Aida and Stroock's, [ASt, Cor. 3.7].

Proposition 4.1. *Let $f \in \mathscr{D}(\mathscr{E})$ be such that there exist $C > 0$ and $\lambda \geq 0$ such that $\|\nabla f\|_p \leq Cp^\lambda$ for all $2 \leq p < \infty$. Assume further that*

(i) (3.1) *holds,*

(ii) *there exist a function $\alpha : [2, \infty) \to (0, \infty)$ and constants $\gamma > 0, B > 0$ such that*

$$\alpha(s) \leq B/s^\gamma \qquad \text{for any } s \geq 2,$$

and $M = \int_2^\infty 2\beta(\alpha(s))/s^2 ds < \infty$

(iii) $2\lambda - \gamma + 1 > 0$.

Let $b = 2/(2\lambda - \gamma + 1)$. Then, for any $k > 0$ such that $keb\left(\frac{e^M BC^2 b}{2}\right)^{b/2} < 1$,

$$\text{(4.2)} \qquad \int_X e^{k|f|^b} d\mu \leq \exp[k((2e^M)^{1/2}\|f\|_{L^2(\mu)})^b] + S\left[1 - keb\left(\frac{e^M BC^2 b}{2}\right)^{b/2}\right]^{-1}$$

where $S = \sup_{n \geq 0} \frac{1}{n!}\left(\frac{n}{e}\right)^n$. In particular, $\int_X e^{k|f|^b} d\mu < \infty$.

Proof. By the assumptions above,

$$\int_2^p \alpha(t)\|\nabla f\|_t^2 dt \leq BC^2 \int_2^p t^{2\lambda-\gamma} dt \leq BC^2 \frac{p^{2\lambda-\gamma+1}}{2\lambda - \gamma + 1} = \frac{b}{2} BC^2 p^{2\lambda-\gamma+1}.$$

Hence, by Corollary (3.4),

$$\|f\|_{nb}^2 \leq e^M \left[\|f\|_2^2 + \frac{BC^2 b}{4}(nb)^{2\lambda-\gamma+1}\right] \qquad \text{if } nb \geq 2.$$

The inequality $(a + b)^k \leq 2^k(a^k + b^k)$, for any $a, b \geq 0, k > 0$, implies that

$$\|f\|_{nb}^{nb} \leq (2e^M)^{nb/2}\left[\|f\|_2^{nb} + \left(\frac{BC^2 b}{4}\right)^{nb/2} (nb)^n\right] \qquad \text{for } nb \geq 2.$$

Moreover, we know that $\|f\|_{nb} \leq \|f\|_2$ for $nb < 2$. Using the preceeding estimates in the power series expansion of the left-hand-side of (4.2), we have

$$\int_X e^{k|f|^b} d\mu = \sum_{n=0}^{\infty} \frac{1}{n!} k^n \int_X |f|^{nb} d\mu$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{n!} k^n (2e^M)^{nb/2} \|f\|_2^{nb} + \sum_{n=0}^{\infty} \frac{1}{n!} k^n \left(2e^M \frac{BC^2 b}{4}\right)^{nb/2} (nb)^n.$$

$$\leq \exp(k(2e^M)^{b/2} \|f\|_2^b) + \left\{\sup_{n \geq 0} \frac{1}{n!} \left(\frac{n}{e}\right)^n\right\} \sum_{n=0}^{\infty} \left[ebk\left(\frac{e^M BC^2 b}{2}\right)^{b/2}\right]^n.$$

This immediately yields (4.2).

Next, we consider the case when $\alpha(s)$ tends to infinity slower than those in the previous proposition.

Proposition 4.2. *Assume the same hypothesis as in Proposition* 4.1 *except that here we assume that there exist constants* $\gamma > 0$, $A, B \geq 0$ *such that*

$$\alpha(s) \leq \frac{B}{(A + \log s)^\gamma} \qquad \textit{for all } s \geq 2.$$

Let $b = \dfrac{2}{2\lambda + 1}$. *Then*

$$\int_X e^{k|f|^b} d\mu < \infty \qquad \textit{for any } k > 0. \tag{4.3}$$

Proof. Since it is not our purpose to get a good estimate for the bound of the integral in (4.3) here, we may and will assume that $B = C = 1$ for simplicity in the calculation. For any $\delta > 0$, there is a constant C_δ such that

$$\int_2^p \frac{t^{2\lambda}}{(A + \log t)^\gamma} dt \leq (1 + \delta) \frac{p^{2\lambda+1}}{(A + \log p)^\gamma} + C_\delta \tag{4.4}$$

for any $p \geq 2$. For the idea how to obtain this estimate, see, e.g., [GR, Cor. 4.7]. Therefore,

$$\|f\|_{nb}^{nb} = (\|f\|_{nb}^2)^{nb/2}$$

$$\leq (2e^M)^{nb/2} \left[\|f\|_2^{nb} + \left\{\frac{(1+\delta)(nb)^{2\lambda+1}}{(\log nb)^\gamma} + C_\delta\right\}^{nb/2}\right]$$

$$\leq (2e^M)^{nb/2} \left[\|f\|_2^{nb} + 2^{nb/2} \left\{\frac{(1+\delta)^{nb/2} (nb)^n}{(\log nb)^{nb\gamma/2}} + C_\delta^{nb/2}\right\}\right]$$

for $nb \geq 2$. Let n_o be the smallest integer such that $n_o \geq 2\lambda + 1 = 2/b$. Since $\|f\|_{nb}^{nb} \leq \|f\|_2^{nb}$ for $nb < 2$, we have

$$\int_X e^{k|f|^b} d\mu = \sum_{n=0}^{\infty} \frac{1}{n!} k^n \int_X |f|^{nb} d\mu$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{n!} k^n (2e^M)^{nb/2} \|f\|_2^{nb} + \sum_{n=n_0}^{\infty} \frac{1}{n!} \frac{k^n (4e^M)^{nb/2} (1+\delta)^{nb/2} (nb)^n}{(\log nb)^{nb\gamma/2}}$$

$$+ \sum_{n=0}^{\infty} \frac{1}{n!} k^n (4e^M)^{nb/2} C_\delta^{nb/2}.$$

Clearly, the first and the third sums are finite for any $k > 0$. We now consider the second sum.

$$\sum_{n=n_0}^{\infty} \frac{1}{n!} \frac{k^n (4e^M)^{nb/2} (1+\delta)^{nb/2} (nb)^n}{(\log nb)^{nb\gamma/2}} \leq \left\{ \sup_{n \geq 0} \frac{1}{n!} \left(\frac{n}{e}\right)^n \right\} \sum_{n=n_o}^{\infty} \left(\frac{kbe(4e^M(1+\delta))^{b/2}}{(\log nb)^{b\gamma/2}} \right)^n.$$

For a fixed $k > 0$, given $0 < a < 1$, we can find n_1 such that

$$A_n \equiv \frac{kbe(4e^M(1+\delta))^{b/2}}{(\log nb)^{b\gamma/2}} < a < 1$$

for any $n \geq n_1$. Consequently, $\sum_{n_1}^{\infty} (A_n)^n \leq \sum_{n_1}^{\infty} a^n < \infty$. From this, we have (4.3).

5. Perturbation of supercontractive semigroups I

In this section, we perturb the measure μ by densities of the form $e^{\Phi(g)}$ where $\|\nabla g\|_\infty \leq 1$ and $\Phi : \mathbf{R} \to \mathbf{R}$ is a real-valued function on $\mathbf{R}$. We will see that the stronger contractivity the semigroup has, the faster growth of $\Phi(u)$, for large u, we can allow on the perturbation and still preserve the same contractivity.

First we look at the hypercontractive case.

Theorem 5.1. *Let (X, μ) be a probability space. Assume that there exist constants $\varepsilon > 0$ and $\beta \geq 0$ for which the following logarithmic Sobolev inequality*

$$\text{(5.1)} \qquad \int f^2 \log \frac{f^2}{\|f\|_{L^2(\mu)}^2} d\mu \leq \varepsilon \mathscr{E}(f, f) + \beta \|f\|_{L^2(\mu)}^2 \qquad (f \in \mathscr{D}(\mathscr{E}))$$

holds. If $g : X \to \mathbf{R}$ is a measurable function such that $\|\nabla g\|_\infty \leq 1$, then $Z = \int_X e^{2\alpha g^2} d\mu < \infty$ for any α for which $2\alpha < 1/\varepsilon$.

Proof. See, e.g., [GR, Example 4.1].

Under the assumption of Theorem 5.1, we fix α such that $2\alpha < 1/\varepsilon$ and define a probability measure μ_g by $d\mu_g = Z^{-1} e^{2\alpha g^2} d\mu$. Then $\mathscr{D}$ is dense in $L^2(X, \mu_g)$. See [AS, Lemma 2.3] for the proof of this fact. Denote by $\mathscr{E}_g$ the Dirichlet form associated to the measure μ_g, (5.2) $$\mathscr{E}_g(f,f) = \int_X \Gamma(f,f) d\mu_g.$$

By assumption (A4), $\mathscr{E}_g$ is well-defined on $\mathscr{D}$, so we take $\mathscr{D}$ to be a predomain of $\mathscr{E}_g$. In fact, $(\mathscr{E}_g, \mathscr{D})$ is closable. We defer the proof of this fact until the end of Section 6.

Theorem 5.2. *Assume that μ satisfies the logarithmic Sobolev inequality* (5.1) *for some constants ε and β. Then there are constants $\tilde{\varepsilon}$ and $\tilde{\beta}$ such that*

$$\int f^2 \log \frac{f^2}{\|f\|^2_{L^2(\mu_g)}} d\mu_g \le \tilde{\varepsilon}\mathscr{E}_g(f,f) + \tilde{\beta}\|f\|^2_{L^2(\mu_g)} \qquad (f \in \mathscr{D}).$$

We can restate this theorem in terms of semigroups associated with Dirichlet forms. Let $\{P_t\}$ and $\{P_t^g\}$ be the semigroups associated with the Dirichlet forms $\mathscr{E}$ and $\mathscr{E}_g$, respectively. If $\{P_t\}$ is hypercontractive, then $\{P_t^g\}$ is also hypercontractive.

Example 5.3. Gaussian case: $X = \mathbf{R}^1$ and $d\mu = (1/\sqrt{2\pi})e^{-x^2/2}dx$. In this case, it is well-known that the logarithmic Sobolev inequality (5.1) holds with $\varepsilon = 2$ and $\beta = 0$. We can perturb μ by any density e^{cx^2} where $c < 1/2$. Hence, the above theorems are sharp in this case.

Proof of Theorem 5.2. The proof here is essentially the same as [AS, Theorem 3.1] adapted to the above setting. Let $f \in \mathscr{D}$. Then we have $fe^{\alpha g^2} \in \mathscr{D}(\mathscr{E})$ and

$$\Gamma(fe^{\alpha g^2}, fe^{\alpha g^2}) = \Gamma(f,f)e^{2\alpha g^2} + 4\alpha fge^{2\alpha g^2}\Gamma(f,g) + 4\alpha^2 f^2 g^2 e^{2\alpha g^2}\Gamma(g,g).$$

Note that

$$4\alpha fg\Gamma(f,g) \le 2(2\alpha fg)\sqrt{\Gamma(f,f)}\sqrt{\Gamma(g,g)} \le \frac{\Gamma(f,f)}{\delta_1} + \delta_1 4\alpha^2 f^2 g^2 \Gamma(g,g).$$

The second inequality follows from the inequality $2ab \le a^2/\varepsilon + \varepsilon b^2$. Therefore,

$$\mathscr{E}(fe^{\alpha g^2}, fe^{\alpha g^2}) \le \left(1 + \frac{1}{\delta_1}\right)\mathscr{E}_g(f,f) + 4\alpha^2(1+\delta_1)\int f^2 g^2 d\mu_g.$$

Note that $\Gamma(g,g) \le \|\nabla g\|^2_\infty \le 1$. Now, substituting $fe^{\alpha g^2}$ into (5.1) and using the above calculations, we have

(5.3) $$\int f^2 \log f^2 d\mu_g + \int 2\alpha f^2 g^2 d\mu_g \le \varepsilon\left(1 + \frac{1}{\delta_1}\right)\mathscr{E}_g(f,f) + 4\varepsilon\alpha^2(1+\delta_1)\int f^2 g^2 d\mu_g$$
$$+ \beta\|f\|^2_{L^2(\mu_g)} + \|f\|^2_{L^2(\mu_g)} \log \|f\|^2_{L^2(\mu_g)}.$$

By Young inequality, $st \le s\log s - s + e^t$ for $s \ge 0, t \in \mathbf{R}$,

$$f^2 g^2 = (\delta_2 f^2)(g^2/\delta_2) \le \delta_2 f^2 \log f^2 + (\delta_2 \log \delta_2 - \delta_2)f^2 + e^{g^2/\delta_2}.$$

Choose $\delta_2 > 0$ large enough so that $2\alpha + 1/\delta_2 < 1/\varepsilon$. Hence,

$$\|e^{g^2/\delta_2}\|_{L^1(\mu_g)} = \int e^{g^2/\delta_2} d\mu_g = \int e^{g^2(2\alpha+1/\delta_2)} d\mu < \infty.$$

Now, (5.3) becomes

$$\begin{aligned}(5.4)\quad \int f^2 \log f^2 d\mu_g \le \varepsilon\left(1+\frac{1}{\delta_1}\right)\mathscr{E}_g(f,f) + (4\varepsilon\alpha^2(1+\delta_1)-2\alpha)\delta_2 \int f^2 \log f^2 d\mu_g \\ + \{\beta + (4\varepsilon\alpha^2(1+\delta_1)-2\alpha)(\delta_2\log\delta_2 - \delta_2)\}\|f\|^2_{L^2(\mu_g)} \\ + (4\varepsilon\alpha^2(1+\delta_1)-2\alpha)\|e^{g^2/\delta_2}\|_{L^1(\mu_g)} + \|f\|^2_{L^2(\mu_g)}\log\|f\|^2_{L^2(\mu_g)}.\end{aligned}$$

Note that $(4\varepsilon\alpha^2(1+\delta_1)-2\alpha)\delta_2 = 2\alpha\delta_2(2\alpha\varepsilon\delta_1 + 2\alpha\varepsilon - 1)$ and $2\alpha\varepsilon - 1 < 0$ by the assumption. So we can choose δ_1 such that $(4\varepsilon\alpha^2(1+\delta_1)-2\alpha)\delta_2 < 1$. The rest of the proof follows from the following Lemma.

Lemma 5.4. *Let μ be a probability measure and Q a densely-defined quadratic form on $L^2(\mu)$. Suppose that there exist real numbers a, b and c such that for any f in the domain of $Q, \mathscr{D}(Q)$,*

$$\int f^2 \log f^2 d\mu \le Q(f) + a\|f\|^2 + b + c\|f\|^2 \log\|f\|^2.$$

Then, for $f \in \mathscr{D}(Q)$,

$$(5.5)\qquad \int f^2 \log(f^2/\|f\|^2) d\mu \le Q(f) + (a+b)\|f\|^2.$$

Proof. For $f \in \mathscr{D}(Q)$ with $\|f\| = 1$, we have

$$(5.6)\qquad \int f^2 \log f^2 d\mu \le Q(f) + a + b.$$

For any nonzero $f \in \mathscr{D}(Q)$, putting $f/\|f\|$ in (5.6) and multiplying everything by $\|f\|^2$, we obtain (5.5).

Remark. Although this Lemma does not give the best constant, it is enough for our purpose. See, e.g., Lemma 2.2 in [G2] for a sharper result.

Next, we look at the supercontractive case. The following Corollary is immediate from Theorem 5.1.

Corollary 5.5. *Assume that there exists a function $\beta : (0,\infty) \to [0,\infty)$ such that the following logarithmic Sobolev inequality*

$$(5.7)\qquad \int f^2 \log\frac{f^2}{\|f\|^2_{L^2(\mu)}} d\mu \le \varepsilon\mathscr{E}(f,f) + \beta(\varepsilon)\|f\|^2_{L^2(\mu)} \qquad (f \in \mathscr{D}(\mathscr{E}))$$

holds for all $\varepsilon > 0$. If $\|\nabla g\|_\infty \le 1$, then $\int_X e^{2\alpha g^2} d\mu < \infty$ for any $\alpha \in \mathbf{R}$.

Theorem 5.6. *Under the same assumption as in Corollary* 5.5, *fix* $\alpha \in \mathbf{R}$ *and let* $d\mu_g = e^{2\alpha g^2} d\mu / \int_X e^{2\alpha g^2} d\mu$. *Then there exists a function* $\tilde{\beta} : (0, \infty) \to [0, \infty)$ *such that*

$$\int f^2 \log \frac{f^2}{\|f\|^2_{L^2(\mu_g)}} d\mu_g \le \tilde{\varepsilon}\mathscr{E}_g(f, f) + \tilde{\beta}(\tilde{\varepsilon})\|f\|^2_{L^2(\mu_g)} \qquad (f \in \mathscr{D}) \tag{5.8}$$

for any $\tilde{\varepsilon} > 0$. *In other words, if the semigroup* $\{P_t\}$ *is supercontractive, then the perturbed semigroup* $\{P_t^g\}$ *is also supercontractive.*

Proof. We will only sketch the proof here. We follow the steps in the proof of Theorem 5.2 but we will simplify the constants in the supercontractive case. We can take $\delta_1 = 1$ and $\delta_2 = 1$ since $e^{g^2} \in L^p(\mu)$ for any $p > 0$. Moreover, we can drop the term $\int 2\alpha f^2 g^2 d\mu_g$ from the inequality (5.3) to simplify the calculation. (We cannot drop it in the hypercontractive case because it gives a critical value for α.) The inequality (5.4) now becomes

$$\int f^2 \log f^2 d\mu_g \le 2\varepsilon\mathscr{E}_g(f, f) + 8\varepsilon\alpha^2 \int f^2 \log f^2 d\mu_g + \beta(\varepsilon)\|f\|^2_{L^2(\mu_g)}$$
$$+ 8\varepsilon\alpha^2 \|e^{g^2}\|_{L^1(\mu_g)} + \|f\|^2_{L^2(\mu_g)} \log \|f\|^2_{L^2(\mu_g)}.$$

Choose $\varepsilon_0 > 0$ so that $0 < 8\varepsilon_0\alpha^2 \le 1/2$. Then $1/2 \le 1 - 8\varepsilon\alpha^2 < 1$ for $0 < \varepsilon \le \varepsilon_0$. It follows that

$$\int f^2 \log f^2 d\mu_g \le 4\varepsilon\mathscr{E}_g(f, f) + 2\beta(\varepsilon)\|f\|^2_{L^2(\mu_g)} + 16\varepsilon\alpha^2 \|e^{g^2}\|_{L^1(\mu_g)}$$
$$+ 2\|f\|^2_{L^2(\mu_g)} \log \|f\|^2_{L^2(\mu_g)}$$

for any $\varepsilon \in (0, \varepsilon_0]$. By Lemma 5.4, we have

$$\int f^2 \log f^2 d\mu_g \le 4\varepsilon\mathscr{E}_g(f, f) + \{2\beta(\varepsilon) + 16\varepsilon\alpha^2 \|e^{g^2}\|_{L^1(\mu_g)}\}\|f\|^2_{L^2(\mu_g)}$$
$$+ \|f\|^2_{L^2(\mu_g)} \log \|f\|^2_{L^2(\mu_g)}.$$

For $0 < \varepsilon \le \varepsilon_0$, we write $A(\varepsilon) = 2\beta(\varepsilon) + 16\varepsilon\alpha^2 \|e^{g^2}\|_{L^1(\mu_g)}$. Let $\tilde{\varepsilon} = 4\varepsilon$ and define

$$\tilde{\beta}(\tilde{\varepsilon}) = \begin{cases} A(\tilde{\varepsilon}/4) = A(\varepsilon) & \text{if } 0 < \tilde{\varepsilon} \le 4\varepsilon_0; \\ A(\varepsilon_0) & \text{if } \tilde{\varepsilon} > 4\varepsilon_0. \end{cases}$$

We now have (5.8).

We can see from the formulas defining $\tilde{\varepsilon}$ and $\tilde{\beta}$ that the perturbation of μ by the density $e^{2\alpha g^2}$ also holds for the ultracontractive case as summarized in the following Corollary.

Corollary 5.7. *Under the assumption of Theorem* 5.6, *if, for each* $t > 0$, *there is a function* $\eta : [2, \infty) \to (0, \infty)$ *such that*

$$t = \int_2^\infty \eta(p)/p\,dp \qquad \text{and} \qquad 2\int_2^\infty \beta(\eta(p))/p^2 dp < \infty$$

then, for any $t > 0$, *we can find a function* $\tilde{\eta} : [2, \infty) \to (0, \infty)$ *such that*

$$t = \int_2^\infty \tilde{\eta}(p)/p\,dp \qquad \text{and} \qquad 2\int_2^\infty \tilde{\beta}(\tilde{\eta}(p))/p^2 dp < \infty$$

In particular, the semigroup P_t^g *associated with* $\mathscr{E}_g$ *is ultracontractive.*

We wish next to prove stronger perturbation theorems for a subclass of supercontractive semigroups. So far, we have only used the fact that $\beta(\varepsilon) < \infty$ for all $\varepsilon > 0$ for an arbitrary supercontractive semigroup. However, if we know that β does not grow too fast as ε approaches zero, we then can perturb the measure μ by densities $e^{\Phi(g)}$ for some functions Φ that grow faster than the previous cases.

The following Proposition and Corollary are slight generalizations of Corollaries 4.4 and 4.7 in [GR].

Proposition 5.8. *Let* g *be a real-valued function on* X. *Suppose that there exist constants* K, L, M *and* b *such that*

$$E(e^{\lambda g}) \le e^{K\lambda + L\lambda^2/(M+\log\lambda)^b} \qquad \text{for any } \lambda \ge 1$$

and that $-g$ *satisfies the same inequality. Then*

$$E(e^{\gamma g^2 \|\log|g|\|^b}) < \infty \qquad \text{if } 0 \le \gamma < (4L)^{-1}.$$

Proof. For details of the proof, see [GR, Cor. 4.4]. We will only sketch the proof here. Define $\Phi(s) = \gamma s^2(\log_+ s)^b$ for $s \ge 0$ and let $\phi = \Phi'$. Then $\phi(s) = 0$ for $0 \le s \le 1$ and

$$\phi(s) = \gamma s(\log s)^{b-1}(b + 2\log s) = \gamma(bs(\log s)^{b-1} + 2s(\log s)^b) \qquad \text{for } s > 1.$$

Define the functions ψ and Ψ from $[0, \infty)$ to $[0, \infty]$ by

$$\psi(\lambda) = \inf\{s \ge 0 : \lambda \le \phi(s)\} \qquad \text{and} \qquad \Phi(\lambda) = \int_0^\lambda \psi(v)dv.$$

Then ψ is a nondecreasing, left continuous function, $\psi(0) = 0$ and $\psi(\phi(s)) = s$ for any $s > 1$. Hence, the functions Φ and Ψ form complementary Young's functions [Z, p. 76–78], which yield Young's inequality, $\lambda s \le \Phi(s) + \Psi(\lambda)$, with the equality if and only if $\lambda = \phi(s)$.

Choose $s_0 > 1$, e.g. $s_0 = 2$. Let $s \ge s_0$ and write $\lambda = \phi(s)$. Then

$$\psi'(\lambda) = (\phi'(s))^{-1} = \gamma^{-1}[2(\log s)^b + 3b(\log s)^{b-1} + b(b-1)(\log s)^{b-2}]^{-1}.$$

Hence, ψ' exists and is bounded on $[R, \infty)$ where $R = \phi(s_0) > 0$. Thus, the assumptions b. of [GR, Theorem 4.3] are satisfied. At $\lambda = \phi(s)$, for $s > 1$,

$$\Psi(\lambda) = \frac{1}{\gamma} \frac{\lambda^2}{(\log \lambda)^b} \frac{(b + \log s)(\log \phi(s))^b}{(\log s)^{b-1}(b + 2\log s)^2}.$$

Set $u(\lambda) = \Psi(\lambda)/[(4\gamma)^{-1}\lambda^2/(M + \log\lambda)^b]$. It is easy to verify that $u(\lambda) \to 1$ as $\lambda \to \infty$. Note that

$$\frac{L\lambda^2}{(M + \log\lambda)^b} - \Psi(\lambda) = \frac{\lambda^2}{(M + \log\lambda)^b}\left[L - \frac{u(\lambda)}{4\gamma}\right].$$

Therefore, $\int_1^\infty E(e^{\lambda g})e^{-\Psi(\lambda)}d\lambda < \infty$. The same argument applies to $-g$. So the hypothesis of [GR, Theorem 4.3] are satisfied.

Corollary 5.9. *Assume that* (5.7) *holds and that there exist a function* $\alpha : [2, \infty) \to (0, \infty)$ *and constants* $B, b > 0$ *and* $C \geq 0$ *such that*

$$\alpha(s) \leq \frac{B}{(C + \log s)^b}, \qquad s \geq 2 \tag{5.9}$$

and

$$M \equiv \int_2^\infty \frac{2\beta(\alpha(\tau))}{\tau^2} d\tau < \infty. \tag{5.10}$$

If $\|\nabla g\|_\infty \leq 1$, *then*

$$E(e^{\rho g^2 |\log|g||^b}) < \infty \qquad \text{for } 0 \leq \rho < (4B)^{-1}. \tag{5.11}$$

Proof. See [GR, Cor. 4.7] for the proof. Recall from the inequality (4.4) that for any $\delta > 0$, there is a constant C_δ such that

$$\int_2^\lambda \frac{1}{(C + \log\tau)^b} d\tau \leq (1 + \delta) \frac{\lambda}{(C + \log\lambda)^b} + C_\delta$$

for any $\lambda \geq 2$.

Remark. Conditions (5.9) and (5.10) for $b > 1$ guarantee ultracontractivity by Theorem 1.3.

Remark. The conclusion of Corollary 5.9 is independent of C. We put C instead of 1, as in [GR, Cor. 4.7], in the hypothesis to increase flexibility of B.

Now assume that $\|\nabla g\|_\infty \leq 1$. Fix $a > 1$ and let γ be an arbitrary real number. It is a consequence of Corollary 5.9 that $Z = \int_X e^{2\gamma g^2 |\log(|g|+a)|^{b/2}} d\mu < \infty$. Define $d\mu_g = Z^{-1}e^{2\gamma g^2|\log(|g|+a)|^{b/2}}d\mu$. Then μ_g is a probability measure on X. Denote by $\mathscr{E}_g$ the Dirichlet form associated with μ_g as in (5.2). Note that the power of $\log(|g| + a)$ is $b/2$ and not b. According to the one-dimensional case (Theorem 1.4), we expect to perturb the measure μ by a density $e^{2\gamma g^2|\log(|g|+a)|^b}$. However, the author has not been able to prove it in this case because of a certain technicality in the method of the proof.

Theorem 5.10. *Assume that there are functions* $\alpha : [2, \infty) \to (0, \infty)$ *and* $\beta : (0, \infty) \to [0, \infty)$ *satisfying conditions* (5.9) *and* (5.10) *and that the following logarithmic Sobolev inequalities*

$$\int f^2 \log \frac{f^2}{\|f\|^2_{L^2(\mu)}} d\mu \le \varepsilon \mathscr{E}(f, f) + \beta(\varepsilon)\|f\|^2_{L^2(\mu)} \qquad (f \in \mathscr{D}(\mathscr{E})) \tag{5.12}$$

hold for any $\varepsilon > 0$. *Then there exist functions* $\tilde{\alpha} : [2, \infty) \to (0, \infty)$ *and* $\tilde{\beta} : (0, \infty) \to [0, \infty)$ *satisfying conditions* (5.9) *and* (5.10) *(with new constants B and C) which make the perturbed logarithmic Sobolev inequalities*

$$\int f^2 \log \frac{f^2}{\|f\|^2_{L^2(\mu_g)}} d\mu_g \le \tilde{\varepsilon} \mathscr{E}_g(f, f) + \tilde{\beta}(\tilde{\varepsilon})\|f\|^2_{L^2(\mu_g)} \qquad (f \in \mathscr{D})$$

hold for all $\tilde{\varepsilon} > 0$.

Proof. Without loss of generality, we may assume that $g \ge 0$. Let $f \in \mathscr{D}$. It can be verified that

$$\begin{aligned}
&\Gamma(g^2(\log(g+a))^{b/2}, g^2(\log(g+a))^{b/2}) \\
&\qquad = g^2(\log(g+a))^{b-2}\left(2\log(g+a) + \frac{b}{2}\right)^2 \Gamma(g, g). \\
&\qquad \le \left(2 + \frac{b}{2\log a}\right)^2 g^2(\log(g+a))^b \Gamma(g, g).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Gamma(fe^{\gamma g^2(\log(g+a))^{b/2}}, fe^{\gamma g^2(\log(g+a))^{b/2}}) \\
&\quad \le 2e^{2\gamma g^2(\log(g+a))^{b/2}}[\Gamma(f, f) + \gamma^2 f^2 \Gamma(g^2(\log(g+a))^{b/2}, g^2(\log(g+a))^{b/2})] \\
&\quad \le 2e^{2\gamma g^2(\log(g+a))^{b/2}}\left[\Gamma(f, f) + \gamma^2\left(2 + \frac{b}{2\log a}\right)^2 f^2 g^2(\log(g+a))^b \Gamma(g, g)\right].
\end{aligned}$$

Here, we use the inequality $\Gamma(fe^F, fe^F) \le 2[e^{2F}\Gamma(f, f) + f^2 e^{2F}\Gamma(F, F)]$. Write $C = 2\gamma^2\left(2 + \frac{b}{2\log a}\right)^2$ and recall that $\Gamma(g, g) \le \|\nabla g\|^2_\infty \le 1$. Substitute $fe^{\gamma g^2(\log(g+a))^{b/2}}$ into (5.12). It follows from the above calculations that

$$\begin{aligned}
&\int f^2 \log f^2 d\mu_g + \int f^2 \cdot 2\gamma g^2(\log(g+a))^b d\mu_g \\
&\qquad \le 2\varepsilon\mathscr{E}_g(f, f) + C\varepsilon \int f^2 g^2(\log(g+a))^b d\mu_g + \beta(\varepsilon)\|f\|^2_{L^2(\mu_g)} \\
&\qquad\quad + \|f\|^2_{L^2(\mu_g)} \log \|f\|^2_{L^2(\mu_g)}.
\end{aligned} \tag{5.13}$$

Since the second term on the left-hand-side is nonnegative, we can drop it and the inequality still holds. By the Young inequality,

$$f^2g^2(\log(g+a))^b \le \delta f^2 \log f^2 + (\delta\log\delta - \delta)f^2 + e^{(1/\delta)g^2(\log(g+a))^b}.$$

Hence, (5.13) becomes

$$\int f^2 \log f^2 d\mu_g \le 2\varepsilon\mathscr{E}_g(f,f) + C\delta\varepsilon \int f^2 \log f^2 d\mu_g$$
$$+ \{\beta(\varepsilon) + C\varepsilon(\delta\log\delta - \delta)\}\|f\|^2_{L^2(\mu_g)}$$
$$+ C\varepsilon\|e^{(1/\delta)g^2(\log(g+a))^b}\|_{L^1(\mu_g)} + \|f\|^2_{L^2(\mu_g)}\log\|f\|^2_{L^2(\mu_g)}.$$

Note that

$$N \equiv \|e^{(1/\delta)g^2(\log(g+a))^b}\|_{L^1(\mu_g)} = \int e^{(1/\delta)g^2(\log(g+a))^b}\cdot e^{2\gamma g^2(\log(g+a))^{b/2}}d\mu$$
$$\le \left\{\int e^{(p/\delta)g^2(\log(g+a))^b}d\mu\right\}^{1/p}\left\{\int e^{2\gamma q g^2(\log(g+a))^{b/2}}d\mu\right\}^{1/q}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. By (5.11), the second integral on the right converges for any $q > 0$. To make the first integral on the right converge, choose $p > 1$ and $\delta > 0$ so that $\dfrac{p}{\delta} < \dfrac{1}{4B}$. Now, choose $\varepsilon_0 > 0$ such that $\frac{1}{2} \le 1 - C\delta\varepsilon < 1$ for any $0 < \varepsilon \le \varepsilon_0$. Therefore,

$$\int f^2 \log f^2 d\mu_g \le 4\varepsilon\mathscr{E}_g(f,f) + 2\{\beta(\varepsilon) + C\varepsilon(\delta\log\delta - \delta)\}\|f\|^2_{L^2(\mu_g)}$$
$$+ 2\|f\|^2_{L^2(\mu_g)}\log\|f\|^2_{L^2(\mu_g)} + 2C\varepsilon N$$

for any $0 < \varepsilon \le \varepsilon_0$. Write $A(\varepsilon) = 2\{\beta(\varepsilon) + C\varepsilon(\delta\log\delta - \delta) + C\varepsilon N\}$ for $0 < \varepsilon \le \varepsilon_0$. Let $\tilde{\varepsilon} = 4\varepsilon$ and define

$$\tilde{\beta}(\tilde{\varepsilon}) = \begin{cases} A(\tilde{\varepsilon}/4) = A(\varepsilon) & \text{if } 0 < \tilde{\varepsilon} \le 4\varepsilon_0; \\ A(\varepsilon_0) & \text{if } \tilde{\varepsilon} > 4\varepsilon_0. \end{cases}$$

We have

$$\int f^2 \log \frac{f^2}{\|f\|^2_{L^2(\mu_g)}} d\mu_g \le \tilde{\varepsilon}\mathscr{E}_g(f,f) + \tilde{\beta}(\tilde{\varepsilon})\|f\|^2_{L^2(\mu_g)} \qquad (f \in \mathscr{D}).$$

By choosing $\tilde{\alpha} = 4\alpha$, we have proved the theorem.

6. Perturbation of supercontractive semigroups II

In this section, we present an alternative setting used in [AMS, AS, H, A]. Let F be a measurable function and consider the weighted measure

$$d\mu_F = e^{2F} d\mu \Big/ \int_X e^{2F} d\mu,$$

assuming that e^{2F} is integrable. Then μ_F is a probability measure on X. We will prove perturbation theorems for hyper- and supercontractive semigroups for μ_F. The difference between this setting and the one in Section 5 is that we regard a measurable function g for which $\|\nabla g\|_\infty \le 1$ in Section 5 as a generalization of the linear function $g(x) = x$ on $\mathbf{R}$, while we regard the function F above as a generalization of the quadratic function $F(x) = x^2$ on $\mathbf{R}$.

First, we look at conditions which ensure the integrability of the function e^{2F}. The following theorem, which could be viewed as a kind of Herbst-type inequality under the exponential integrability of the gradient, is essentially Theorem 3.1 in [AMS].

Theorem 6.1 (hypercontractive case). *Assume that μ satisfies the following logarithmic Sobolev inequality*

$$\int f^2 \log \frac{f^2}{\|f\|_{L^2(\mu)}^2} d\mu \le \varepsilon \mathscr{E}(f,f) + \beta \|f\|_{L^2(\mu)}^2 \qquad (f \in \mathscr{D}(\mathscr{E})). \tag{6.1}$$

For any $p > 0$, if $e^{|\nabla F|^2} \in L^p(X,\mu)$, then $e^{|F|} \in L^{2a}(X,\mu)$ for any $a > 0$ for which $a^2 < p/\varepsilon$.

Example 6.2. Let $X = \mathbf{R}^1$ and $d\mu = (1/\sqrt{2\pi})e^{-x^2/2}dx$, the standard Gaussian measure. In this case, μ satisfies a logarithmic Sobolev inequality with Log-Sobolev constant $\varepsilon = 2$. Let $F(x) = x^2$. Then $e^{p|\nabla F|^2} = e^{4px^2} \in L^1(\mu)$ if $p < p_0 := 1/8$. Also, $\int e^{2aF} d\mu = \int e^{2ax^2} d\mu < \infty$ for any $0 < a < 1/4$, that is, $a^2 < 1/16 = p_0/\varepsilon$. This shows that the above theorem is sharp in this case.

Corollary 6.3 (supercontractive case). *Assume that μ satisfies a family of logarithmic Sobolev inequalities*

$$\int_X f^2 \log \frac{f^2}{\|f\|_{L^2(\mu)}^2} d\mu \le \varepsilon \mathscr{E}(f,f) + \beta(\varepsilon) \|f\|_{L^2(\mu)}^2 \qquad (f \in \mathscr{D}(\mathscr{E})) \tag{6.2}$$

where $\beta(\varepsilon) < \infty$ for all $\varepsilon > 0$. Then $e^{|\nabla F|^2} \in L^p(X,\mu)$ for some $0 < p < \infty$ implies $e^{|F|} \in L^p(X,\mu)$ for all $0 < p < \infty$.

We now turn to the perturbation theorems in this setting. First, let us look at the hypercontractive case. We assume that

$$e^{|\nabla F|^2} \in L^p(X,\mu) \qquad \text{for some } p > \varepsilon. \tag{6.3}$$

Theorem 6.1 guarantees the integrability of e^{2F}. By replacing F by $F - \frac{1}{2}\log\int_X e^{2F}d\mu$, we can assume that $\int_X e^{2F}d\mu = 1$. It can be verified that $\mathscr{D}$ is dense in $L^2(X,\mu_F)$. Denote by $\mathscr{E}_F$ the Dirichlet form whose underlying measure is μ_F. By assumption (A4), $\mathscr{D} \subseteq \mathscr{D}(\mathscr{E}_F)$, so we will take it as a predomain for $\mathscr{E}_F$.

Theorem 6.4. *Assume that μ satisfies the logarithmic Sobolev inequality* (6.1). *Then there are constants $\tilde{\varepsilon}$ and $\tilde{\beta}$ such that*

$$\int_X f^2 \log f^2 d\mu_F \leq \tilde{\varepsilon}\mathscr{E}_F(f,f) + \tilde{\beta}\|f\|^2_{L^2(\mu_F)} + \|f\|^2_{L^2(\mu_F)} \log\|f\|^2_{L^2(\mu_F)} \qquad (f \in \mathscr{D}).$$

In the language of semigroups, if $\{P_t\}$ and $\{P_t^F\}$ are the semigroups associated with the Dirichlet forms $\mathscr{E}$ and $\mathscr{E}_F$, respectively and if $\{P_t\}$ is hypercontractive, then $\{P_t^F\}$ is also hypercontractive.

For the proof, see [AMS, Lemma 3.1] and [A, Lemma 4.1].

By using Theorem 6.4, Aida and Shigekawa prove the closability of $\mathscr{E}_F$.

Proposition 6.5 ([AS], Prop. 3.2). *$(\mathscr{E}_F, \mathscr{D})$ is closable.*

Now we look at the perturbation theorem under the supercontractivity assumption. Assume that F is a measurable function such that

$$e^{|\nabla F|^2} \in L^p(X,\mu) \qquad \text{for some } 0 < p < \infty. \tag{6.4}$$

By Corollary 6.3, e^{2F} is integrable, so again we assume that $\int_X e^{2F}d\mu = 1$. Moreover, $\mathscr{D}$ is dense in $L^2(X,\mu_F)$ and $\mathscr{D} \subseteq \mathscr{D}(\mathscr{E}_F)$.

Theorem 6.6. *Assume that μ satisfies the family of logarithmic Sobolev inequalities* (6.2). *Then there is a function $\tilde{\beta} : (0,\infty) \to [0,\infty)$ such that*

$$\int_X f^2 \log f^2 d\mu_F \leq \tilde{\varepsilon}\mathscr{E}_F(f,f) + \tilde{\beta}(\tilde{\varepsilon})\|f\|^2_{L^2(\mu_F)} + \|f\|^2_{L^2(\mu_F)} \log\|f\|^2_{L^2(\mu_F)} \qquad (f \in \mathscr{D})$$

for any $\tilde{\varepsilon} > 0$. In other words, if the semigroup $\{P_t\}$ is supercontractive, then the perturbed semigroup $\{P_t^F\}$ is also supercontractive.

Proof. The proof here is an extension of [AS, Lemma 3.1] and [A, Lemma 4.1] and is very similar to the proof of Theorem 5.6. Let $f \in \mathscr{D}$. Then we have $fe^F \in \mathscr{D}(\mathscr{E})$ and

$$\Gamma(fe^F, fe^F) = e^{2F}\Gamma(f,f) + 2fe^{2F}\Gamma(f,F) + f^2e^{2F}\Gamma(F,F).$$

Note that

$$f\Gamma(f,F) \leq f\Gamma(f,f)^{1/2}\Gamma(F,F)^{1/2} \leq \tfrac{1}{2}\Gamma(f,f) + \tfrac{1}{2}f^2\Gamma(F,F).$$

Here, we use the inequality $ab \leq \frac{1}{2}(a^2+b^2)$. It follows that

$$\Gamma(fe^F, fe^F) \leq 2e^{2F}\Gamma(f,f) + 2f^2e^{2F}\Gamma(F,F).$$

Substituting fe^F into (6.2) and using the above calculations, we have

$$\int f^2 \log f^2 d\mu_F \le -\int f^2 \cdot 2F d\mu_F + 2\varepsilon\mathscr{E}_F(f,f) + 2\varepsilon\int f^2 \Gamma(F,F) d\mu_F + \beta(\varepsilon)\|f\|^2_{L^2(\mu_F)} + \|f\|^2_{L^2(\mu_F)} \log \|f\|^2_{L^2(\mu_F)}. \tag{6.5}$$

By the Young inequality, $st \le s\log s - s + e^t$ $(s \ge 0, t \in \mathbf{R})$,

$$\begin{aligned} f^2|F| = (\delta_1 f^2)(|F|/\delta_1) &\le \delta_1 f^2 \log \delta_1 f^2 - \delta_1 f^2 + e^{|F|/\delta_1} \\ &= \delta_1 f^2 \log f^2 + (\delta_1 \log \delta_1 - \delta_1) f^2 + e^{|F|/\delta_1}. \end{aligned}$$

Hence,

$$\int f^2|F| d\mu_F \le \delta_1 \int f^2 \log f^2 d\mu_F + (\delta_1 \log\delta_1 - \delta_1)\|f\|^2_{L^2(\mu_F)} + \|e^{|F|/\delta_1}\|_{L^1(\mu_F)}. \tag{6.6}$$

By the Young inequality again,

$$\begin{aligned} f^2\Gamma(F,F) &\le \delta_2 f^2 \log(\delta_2 f^2) - \delta_2 f^2 + e^{\Gamma(F,F)/\delta_2} \\ &= \delta_2 f^2 \log f^2 + (\delta_2 \log\delta_2 - \delta_2) f^2 + e^{\Gamma(F,F)/\delta_2}. \end{aligned}$$

Thus,

$$\int f^2\Gamma(F,F) d\mu_F \le \delta_2 \int f^2 \log f^2 d\mu_F + (\delta_2 \log\delta_2 - \delta_2)\|f\|^2_{L^2(\mu_F)} + \|e^{\Gamma(F,F)/\delta_2}\|_{L^1(\mu_F)} \tag{6.7}$$

Putting (6.6) and (6.7) in (6.5), we have

$$\begin{aligned} \int f^2 \log f^2 d\mu_F \le{}& (2\delta_1 + 2\varepsilon\delta_2)\int f^2 \log f^2 d\mu_F \\ &+ 2\varepsilon\mathscr{E}_F(f,f) + \|f\|^2_{L^2(\mu_F)} \log\|f\|^2_{L^2(\mu_F)} \\ &+ \{2(\delta_1\log\delta_1 - \delta_1) + 2\varepsilon(\delta_2\log\delta_2 - \delta_2) + \beta(\varepsilon)\}\|f\|^2_{L^2(\mu_F)} \\ &+ 2\|e^{|F|/\delta_1}\|_{L^1(\mu_F)} + 2\varepsilon\|e^{\Gamma(F,F)/\delta_2}\|_{L^1(\mu_F)}. \end{aligned} \tag{6.8}$$

The next step is to choose δ_1, δ_2 and ε_0 so that $0 < 2\delta_1 + 2\varepsilon\delta_2 < 1$ and $1 - (2\delta_1 + 2\varepsilon\delta_2)$ is bounded away from zero uniformly on some interval $(0, \varepsilon_0]$ of ε. Choose c such that $0 < c < p$ and let $\delta = 2/(p+c)$ so that $0 < c < 1/\delta < p$. Now let $\delta_1 = c\delta/2$ and $\delta_2 = \delta$ and choose $\varepsilon_0 < (p-c)/4$. Hence, $2\delta_1 + 2\varepsilon\delta_2 = \delta(c + 2\varepsilon)$. It is easy to verify that $0 < \alpha \le 1 - \delta(c+2\varepsilon) < 1$ for each $\varepsilon \in (0, \varepsilon_0]$ where $\alpha = 1 - \delta(c + 2\varepsilon_0) > 0$. From these choices, (6.8) becomes

$$(6.9)\quad \int f^2 \log f^2 d\mu_F \le \alpha^{-1}[2\varepsilon\mathscr{E}_F(f,f) + \|f\|^2_{L^2(\mu_F)} \log \|f\|^2_{L^2(\mu_F)} + \{2(\delta_1 \log\delta_1 - \delta_1) + 2\varepsilon(\delta_2 \log\delta_2 - \delta_2) + \beta(\varepsilon)\}\|f\|^2_{L^2(\mu_F)} + 2\|e^{|F|/\delta_1}\|_{L^1(\mu_F)} + 2\varepsilon\|e^{\Gamma(F,F)/\delta_2}\|_{L^1(\mu_F)}]$$

for any $\varepsilon \in (0, \varepsilon_0]$. By Lemma 5.4, we obtain

$$\int f^2 \log(f^2/\|f\|^2_{L^2(\mu_F)}) d\mu_F \le \frac{2\varepsilon}{\alpha}\mathscr{E}_F(f,f) + A(\varepsilon)\|f\|^2_{L^2(\mu_F)}$$

for any $\varepsilon \in (0, \varepsilon_0]$, where

$$(6.10)\quad A(\varepsilon) = \frac{1}{\alpha}\{2(\delta_1 \log\delta_1 - \delta_1) + 2\varepsilon(\delta_2 \log\delta_2 - \delta_2) + \beta(\varepsilon) + 2\|e^{|F|/\delta_1}\|_{L^1(\mu_F)} + 2\varepsilon\|e^{\Gamma(F,F)/\delta_2}\|_{L^1(\mu_F)}\}.$$

We note here that

$$\|e^{|F|/\delta_1}\|_{L^1(\mu_F)} = \int e^{2|F|/c\delta} \cdot e^{2F} d\mu = \int e^{2(1+1/c\delta)|F|} d\mu < \infty$$

by Corollary 6.3, and

$$\|e^{\Gamma(F,F)/\delta_2}\|_{L^1(\mu_F)} = \int e^{|\nabla F|^2/\delta} \cdot e^{2F} d\mu \le \left\{\int e^{p|\nabla F|^2} d\mu\right\}^{1/p\delta} \left\{\int e^{2q|F|} d\mu\right\}^{1/q} < \infty$$

by Hölder's inequality, where $1/q + 1/p\delta = 1$. Since $p\delta > 1$, we have $1 < q < \infty$ and thus $\|e^{2|F|}\|_{L^q(\mu)} < \infty$ by Corollary 6.3. Now, let $\tilde{\varepsilon} = 2\varepsilon/\alpha$ and

$$\tilde{\beta}(\tilde{\varepsilon}) = \begin{cases} A(\alpha\tilde{\varepsilon}/2) = A(\varepsilon) & \text{if } 0 < \tilde{\varepsilon} \le 2\varepsilon_0/\alpha; \\ A(\alpha\tilde{\varepsilon}_0/2) = A(\varepsilon_0) & \text{if } \tilde{\varepsilon} > 2\varepsilon_0/\alpha. \end{cases}$$

We then have $0 < \tilde{\beta}(\tilde{\varepsilon}) < \infty$ for all $\tilde{\varepsilon} > 0$. This finishes the proof of the theorem.

The following Corollary is immediate from the formulas of $\tilde{\varepsilon}$, $\tilde{\beta}$ and $A(\varepsilon)$:

Corollary 6.7 (ultracontractive case). *Under the assumption of Theorem* 6.6, *if, for each* $t > 0$, *there is a function* $\eta : [1, \infty) \to (0, \infty)$ *such that*

$$\int_1^\infty \eta(t)/t\,dt < \infty \qquad \textit{and} \qquad \int_1^\infty \beta(\eta(t))/t^2 dt < \infty,$$

then, for any $t > 0$, *there exists a function* $\tilde{\eta} : [1, \infty) \to (0, \infty)$ *such that*

$$\int_1^\infty \tilde{\eta}(t)/t\,dt < \infty \qquad \textit{and} \qquad \int_1^\infty \tilde{\beta}(\tilde{\eta}(t))/t^2 dt < \infty.$$

In particular, the semigroup $\{P_t^F\}$ *associated with* $\mathscr{E}_F$ *is ultracontractive.*

Finally, we indicate the relation between the two settings. In this section, we assume the exponential integrability of $|\nabla F|^2$ and conclude the integrability of $e^{|F|}$. In Section 5, we assume that $\|\nabla g\|_\infty \le 1$ and conclude the integrability of e^{g^2}. As mentioned at the beginning of this section, modulo constants, the function F is a generalization of $F(x) = x^2$ and $\mathbf{R}$ and g is a generalization of $g(x) = x$ on $\mathbf{R}$. Hence, if we assume that $\|\nabla g\|_\infty \le 1$ and write $F = \alpha g^2$, then we should be able to obtain results in Section 5 from the corresponding ones in this section. This is indeed the case due to the following lemmas.

Lemma 6.8 (hypercontractive case). *Assume that μ satisfies the logarithmic Sobolev inequality* (6.1). *Let g be a measurable function such that $\|\nabla g\|_\infty \le 1$. Fix a positive real number α for which $2\alpha < 1/\varepsilon$ and write $F = \alpha g^2$. Then the hypothesis* (6.3) holds, i.e. $e^{|\nabla F|^2} \in L^p(X, \mu)$ *for some* $p > \varepsilon$.

Proof. Note that $|\nabla F|^2 = 4\alpha^2 g^2 |\nabla g|^2 \le 4\alpha^2 g^2$. If $p = 1/2\alpha > \varepsilon$, then $\int e^{p|\nabla F|^2} d\mu \le \int e^{2\alpha g^2} d\mu < \infty$ by Theorem 5.1.

Lemma 6.9 (supercontractive case). *Assume that μ satisfies the logarithmic Sobolev inequality* (6.2). *Let g be a measurable function such that $\|\nabla g\|_\infty \le 1$. Fix a real number $\alpha > 0$ and write $F = \alpha g^2$. Then the hypothesis* (6.4) *holds, i.e. $e^{|\nabla F|^2} \in L^p(X, \mu)$ for some $p > 0$.*

By using the above Lemmas, we see that Theorem 5.2 and 5.6 are immediate consequences of Theorem 6.4 and 6.6, respectively. Moreover, we can deduce the closability of the Dirichlet form $\mathscr{E}_g$ from Proposition 6.5 and Lemma 6.8. However, we do not have an analog for Theorem 5.10 for the perturbed measure μ_F.

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Acknowledgments. This work is based on the author's Ph.D. dissertation written under the supervision of Professor Leonard Gross at Cornell University. The author gratefully acknowledges his invaluable advice and encouragement. It is also a pleasure to acknowledge useful discussion with S. Aida and E. Bueler. The author is indebted to the Ananda Mahidol Foundation from Thailand for providing sponsorship throughout this graduate studies at Cornell University.

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