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Ellipticity of certain conformal immersions

By

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Abstract

We study prolongation of the conformal embedding equations. Let $(\mathscr{M}, g)$ be a C^∞ Riemannian manifold of dimension $n \geq 3$ and $(\tilde{\mathscr{M}}, \tilde{g})$ be a C^∞ Riemannian manifold of dimension $n+d$, $d < \frac{1}{2}n(n-1)$. Suppose that $f : \mathscr{M} \to \tilde{\mathscr{M}}$ is a conformal immersion with conformal factor v. If the conformal 1-nullity of f at a point $P \in \mathscr{M}$ does not exceed $n-2$, we prove that the system of conformal embedding equations admits a prolongation to a system of nonlinear partial differential equations of second order which is elliptic at the solution (f, v). In particular, if $(\mathscr{M}, g)$ and $(\tilde{\mathscr{M}}, \tilde{g})$ are analytic and f and v are of differentiability class C^2 then f and v are analytic on a neighborhood of P in $\mathscr{M}$.

0. Introduction

Let $\mathscr{M}$ be a smooth (C^∞) manifold of dimension n, $n \geq 2$, with C^∞ Riemannian metric g and let $\tilde{\mathscr{M}}$ be a C^∞ manifold of dimension $n+d$ with C^∞ Riemannian metric $\tilde{g}$. A C^1 mapping f of $\mathscr{M}$ into $\tilde{\mathscr{M}}$ is a conformal immersion if $f^*\tilde{g} = vg$, for some positive function v of $\mathscr{M}$. In terms of local coordinates, f is a conformal immersion if and only if f satisfies

$$\sum_{a,b=1}^{n+d} \tilde{g}_{ab}(u) \frac{\partial u^a}{\partial x^i} \frac{\partial u^b}{\partial x^j} = v(x) g_{ij}(x), \qquad i, j = 1, \dots, n, \tag{1}$$

where $x = (x^1, \dots, x^n)$ and $y = (y^1, \dots, y^{n+d})$ are local coordinate systems on $\mathscr{M}$ and $\tilde{\mathscr{M}}$, respectively, and

$$g_{ij}(x) = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \qquad \text{and} \qquad \tilde{g}_{ab}(y) = \tilde{g}\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right).$$

Since $g_{ij} = g_{ji}$, the number of equations in (1) is $\frac{1}{2}n(n+1)$. The unknown functions are $u = (u^1, \dots, u^{n+d})$ and v. Thus the system (1) is overdetermined if $d < \frac{1}{2}n(n-1) - 1$ and underdetermined if $d > \frac{1}{2}n(n-1) - 1$. v is the conformal factor and a solution u of (1) with $v = 1$ is an isometric immersion.

In this paper we are concerned with the ellipticity of solutions of (1). We define a conformal immersion f to be elliptic if (1) admits a prolongation which

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is elliptic at f, or equivalently, the equation for the infinitesimal conformal deformation of f satisfies a linear elliptic system (Definition 1.1). Geometric consequenses from the ellipticity are the global rigidity as studied in [16] and the regularity of the mappings as in [7] and [8].

The notion of ellipticity of immersions was first introduced in [16] by N. Tanaka to study the global rigidity of isometric immersions: He called an immersion $f : \mathcal{M}^n \to \mathbf{R}^m$ to be elliptic (we shall call T-elliptic in this paper) at $P \in \mathcal{M}$ if for every normal vector $\mathbf{n}$ to $T_P\mathcal{M}$ in $\mathbf{R}^m$, the second fundamental form associated with $\mathbf{n}$ has at least two eigenvalues of the same sign. By counting the dimensions one can easily show that if there exists a T-elliptic immersion $\mathcal{M}^n$ into $\mathbf{R}^m$ then $m \leq \frac{1}{2}n(n+1)$. Tanaka showed that if $f : \mathcal{M}^n \to \mathbf{R}^m$ is a T-elliptic immersion then the infinitesimal isometric deformations of f satisfy a system of elliptic linear partial differential equations of second order, and thus, if $\mathcal{M}$ is compact the space of the infinitesimal isometric deformations of f is finite dimensional. We say that an immersion f is infinitesimally rigid if the dimension of the space of the infinitesimal isometric deformations is equal to $\frac{1}{2}m(m+1)$, which is the dimension of the infinitesimal isometry of $\mathbf{R}^m$. We denote by $\mathrm{Imm}(\mathcal{M}, m)$ the set of all immersions of $\mathcal{M}$ into $\mathbf{R}^m$ equipped with C^3 topology.

Let $\langle\,,\rangle$ be the standard Riemannian metric of $\mathbf{R}^m$. Tanaka proved the following rigidity theorem:

Theorem 1 ([16]). *Let f be an immersion of a compact Riemannian manifold $\mathcal{M}$ into $\mathbf{R}^m$. Suppose that f is infinitesimally rigid and f is T-elliptic at every point of $\mathcal{M}$. Then there is a neighborhood $U(f)$ of f in $\mathrm{Imm}(\mathcal{M}, m)$ having the following property: If $f_1, f_2 \in U(f)$ are embeddings and $f_1^*\langle\,,\rangle = f_2^*\langle\,,\rangle$, then there is a unique euclidean transformation τ of $\mathbf{R}^m$ such that $f_1 = \tau \circ f_2$.*

A strictly convex compact surface in $\mathbf{R}^3$ is an infinitesimally rigid T-elliptic embedding, see Theorem 10 in Chapter 12 of [15]. Thus Theorem 1 is a generalization of the classical Cohn-Vossen theorem, which states that compact convex surfaces in $\mathbf{R}^3$ are rigid. It remains an open question whether an isometric immersion f of a compact Riemannian manifold $\mathcal{M}$ into $\mathbf{R}^m$ is necessarily rigid if f is elliptic (in Tanaka's sense, or more generally, elliptic in our sense). See Problem 55 of [17].

In [4] the authors showed that an isometric immersion $f : \mathcal{M}^n \to \mathbf{R}^{n+d}$ is T-elliptic if and only if the first prolongation of (1) with $v = 1$ together with the Gauss curvature equations, which come from the second prolongation of (1), form a (nonlinear) system which is elliptic at the solution f. As a result, f is analytic if $(\mathcal{M}, g)$ is analytic. This follows from the deep theory on the analyticity of elliptic solutions, see [11] and [12].

Recently, the local rigidity of conformal immersions has been studied in [2] and [5].

Let $f : \mathcal{M}^n \to \tilde{\mathcal{M}}^{n+d}$ be an immersion and let $\alpha : T_P\mathcal{M} \times T_P\mathcal{M} \to [T_{f(P)}\mathcal{M}]^\perp$ be the second fundamental form of f at the point $P \in \mathcal{M}$. Here $T_P\mathcal{M}$ is the tangent space of $\mathcal{M}$ at P and $[T_{f(P)}\mathcal{M}]^\perp$ is the normal space of f at P. Recall that the second fundamental form is a vector valued symmetric bilinear form defined by $\alpha(X, Y) = (\tilde{\nabla}_X Y)^{\perp}$, where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{\mathcal{M}}$. A conformal immersion $f : \mathcal{M}^n \to \tilde{\mathcal{M}}^{n+d}$ is said to be conformally rigid provided that for any conformal immersion $f_1 : \mathcal{M}^n \to \tilde{\mathcal{M}}^{n+d}$ there exists a conformal mapping τ of $\tilde{\mathcal{M}}^{n+d}$ into itself such that $f_1 = \tau \circ f$.

Definition 2 ([2]). Let s be an integer, $1 \le s \le d$, let $V^s \subset [T_{f(P)}\mathcal{M}]^{\perp}$ be a s-dimensional subspace. A linear subspace W of the tangent space $T_P\mathcal{M}$ is called V^s-umbilic if there exists an element $\zeta \in V^s$ satisfying

$$\pi_{V^s} \circ \alpha(X, Y) = f^*\tilde{g}(X, Y)\zeta, \qquad \text{for all } X, Y \in W,$$

where $\pi_{V^s} : [T_{f(P)}\mathcal{M}]^{\perp} \to V^s$ is the orthogonal projection. The number

$$\nu_s^c(P) = \max\{\dim W : W \text{ is } V^s\text{-umbilic for some } V^s \subset [T_{f(P)}\mathcal{M}]^{\perp}\}$$

is called the conformal s-nullity of f at P.

In [2] do Carmo and Dajczer proved the following

Theorem 3 ([2]). *Let $(\mathcal{M}^n, g)$ be a C^∞ Riemannian manifold, $(\tilde{\mathcal{M}}^{n+d}, \tilde{g})$ a simply-connected complete C^∞ Riemannian manifold of constant curvature, $d < \min\{\frac{1}{2}n - 1, 5\}$, and $f : \mathcal{M}^n \to \tilde{\mathcal{M}}^{n+d}$ a conformal immersion. Assume that for every point $P \in \mathcal{M}$ and every integer s, $1 \le s \le d$, the conformal s-nullity $\nu_s^c(P)$ satisfies $\nu_s^c(P) \le n - (2s + 1)$. Then f is conformally rigid.*

We show in this paper that if the conformal 1-nullity satisfies $\nu_1^c(P) \le n - 2$ then (1) can be prolonged to a system of second order partial differential equations which is elliptic at (f, v). In particular, if $(\mathcal{M}, g)$ and $(\tilde{\mathcal{M}}, \tilde{g})$ are analytic and (f, v) are C^2 then (f, v) are analytic.

All manifolds in this paper are assumed to be C^∞ and all immersions are assumed to be C^2 unless otherwise stated. Our main result is the following

Theorem 4. *Let $(\mathcal{M}, g)$ be a C^∞ Riemannian manifold of dimension $n \ge 3$ and let $(\tilde{\mathcal{M}}, \tilde{g})$ be a C^∞ Riemannian manifold of dimension $n + d$, $d < \frac{1}{2}n(n-1)$. Suppose that $f : \mathcal{M} \to \tilde{\mathcal{M}}$ is a conformal immersion with conformal factor v. If $\nu_1^c(P) \le n - 2$, then (1) admits a prolongation to a system of nonlinear partial differential equations of second order which is elliptic at the solution (f, v). In particular, if $(\mathcal{M}, g)$ and $(\tilde{\mathcal{M}}, \tilde{g})$ are analytic and f and v are of differentiability class C^2 then f and v are analytic on a neighborhood of P in $\mathcal{M}$.*

Conditions like $\nu_1^c(P) \le n - 2$ is necessary as the following example shows:

Let $(x(s), y(s))$ be a curve in $\mathbf{R}^2$ parametrized by arc-length s. Suppose that $(x(s), y(s))$ is C^2 but not real analytic. Then the mapping $f(s, t_1, t_2) = (x(s), y(s), t_1, t_2)$ is an isometric immersion of $\mathbf{R}^3$ into $\mathbf{R}^4$, which is C^2 but not real analytic. We see that the 2-dimensional submanifolds $s = \text{constant}$ are umbilic and thus $\nu_1^c(P) \ge 2$ at every point P.

The method of Theorem 4 is a jet-theoretic approach to differential equations: We study the prolongation of (1) and find the conditions under which (1) can be prolonged to an elliptic system. Now a conformal version of the rigidity question naturally arises: Suppose that $f : \mathcal{M} \to \tilde{\mathcal{M}}$ is a conformal immersion such that $v_1^c(P) \leq n-2$, for all $P \in \mathcal{M}$ and that $\mathcal{M}$ is compact. Then the question is whether f is necessarily conformally rigid.

Regularity of conformal mappings between equi-dimensional Riemannian manifolds is well known in the theory of G-structures: Conformal structure is a G-structure of finite type. Therefore, if the manifolds are analytic the structure-preserving mappings satisfy a complete differential system of finite order with analytic coefficients (see [9], [7] and [8]). Thus we have

Theorem 5. *Let $(\mathcal{M}, g)$ and $(\tilde{\mathcal{M}}, \tilde{g})$ be analytic Riemannian manifolds of dimension $n \geq 2$. Suppose that $f : \mathcal{M} \to \tilde{\mathcal{M}}$ is a conformal map with a conformal factor v of class C^2. Then f and v are analytic.*

A direct proof of Theorem 5 is given as a special case of Theorem 4. As another special case, we consider the case of codimension 1, and prove the following

Theorem 6. *Let $(\mathcal{M}, g)$ be an analytic Riemannian manifold of dimension $n \geq 4$ and let $(\tilde{\mathcal{M}}, \tilde{g})$ be an analytic Riemannian manifold of dimension $n+1$. Suppose that $f : \mathcal{M} \to \tilde{\mathcal{M}}$ is a conformal immersion of class C^2 with conformal factor v of class C^2. If the maximal multiplicity of principal curvatures at $f(P)$ does not exceed $n-3$ then f and v are necessarily analytic on a neighborhood of P in $\mathcal{M}$.*

In section 1 we briefly review the basic jet theory which are needed for our discussions. In section 2 we construct a system of compatibility equations by following the process described in section 1. Section 3 is devoted to proving Theorem 4, Theorem 5 and Theorem 6.

1. Preliminaries

Consider a system of r-th order partial differential equations

$$\Delta_\nu(x, u^{(r)}) = 0, \qquad 1 \leq \nu \leq l, \tag{1.1}$$

for unknown functions $u = (u^1, \ldots, u^m)$ of n variables $x = (x^1, \ldots, x^n) \in \Omega$, where Ω is an open subset of $\mathbf{R}^n$. We will assume that (1.1) is C^∞, that is, each Δ_ν is C^∞ in its arguments. In (1.1), $u^{(r)}$ denotes all the formal variables representing all the partial derivatives of unknown functions upto order r, that is,

$$u^{(r)} = (u_J^\alpha)_{1 \leq \alpha \leq m,\, 0 \leq |J| \leq r},$$

where J denotes a usual multi-index and $|J|$ denotes its order. We denote by $\mathbf{R}^{(r)}$ the $u^{(r)}$ space, which is an euclidean space of some large dimension. The product space $\Omega \times \mathbf{R}^{(r)}$ is denoted by $J^r(\Omega, \mathbf{R}^m)$ and called the r-th order jet space of the underlying space $\Omega \times \mathbf{R}^m$. Regarding (1.1) as a system of algebraic equations on $J^r(\Omega, \mathbf{R}^m)$ (1.1) defines a subset

$$\mathscr{S}_\Delta := \{(x, u^{(r)}) \in J^r(\Omega, \mathbf{R}^m) : \Delta_\nu(x, u^{(r)}) = 0, \;\; 1 \le \nu \le l\}$$

of $J^r(\Omega, \mathbf{R}^m)$, which is called the solution subvariety of (1.1). Then a C^r map $F = (F^1, \dots, F^m)$ is a solution of (1.1) if $j^r_x F \in \mathscr{S}_\Delta$ for all $x \in \Omega$, where $j^r_x F$ is the r-jet of F at x, that is,

$$j^r_x F = \left(\frac{\partial^{|J|} F^\alpha}{\partial x^J}(x) \right)_{1 \le \alpha \le m, \, 0 \le |J| \le r}.$$

The principal symbol of (1.1) is a $l \times m$ matrix $\mathbf{M}(\xi) = \mathbf{M}_\Delta(\xi; x, u^{(r)})$ whose (ν, α) entry is the homogeneous polynomial

$$\mathbf{M}_{\nu\alpha}(\xi; x, u^{(r)}) = \sum_{|J|=r} \left[\frac{\partial \Delta_\nu}{\partial u^\alpha_J}(x, u^{(r)}) \right] \cdot \xi_J,$$

of degree r in $\xi = (\xi_1, \dots, \xi_n)$.

Definition 1.1. Given a point $(x_0, u_0^{(r)}) \in \mathscr{S}_\Delta$,

(a) a non-zero vector $\xi = (\xi^1, \dots, \xi^n) \in \mathbf{R}^n$ is a noncharacteristic direction (respectively characteristic direction) to (1.1) at $(x_0, u_0^{(r)})$ if $\mathbf{M}_\Delta(\xi; x_0, u_0^{(r)})$ is of rank m (respectively rank $< m$).

(b) A hypersurface $\{x \in \Omega \mid \psi(x) = c\}$ of Ω is noncharacteristic (respectively characteristic) to (1.1) at $(x_0, u_0^{(r)})$ if $\xi = \operatorname{grad} \psi(x_0)$ is noncharacteristic (respectively characteristic) to (1.1) at $(x_0, u_0^{(r)})$.

(c) The system (1.1) is elliptic at $(x_0, u_0^{(r)})$ if there is no characteristic direction at $(x_0, u_0^{(r)})$.

On the regularity of solutions of elliptic systems we have the following theorem (see [12], [13]).

Theorem 1.2. *Suppose that* (1.1) *is determined or overdetermined, that is,* $l \ge m$, *and that* F *is a* C^r *solution. If* (1.1) *is elliptic at* $(x_0, j^r_{x_0} F)$ *then* F *is necessarily* C^∞ *on a neighborhood of* x_0 *in* Ω. *Furthermore, if the left side of* (1.1) *is analytic in its arguments then* F *is analytic.*

Prolongation of (1.1) is a process of differentiation and algebraic operations. An equation obtained by prolongation of (1.1) is called a compatibility equation of (1.1). More precisely, for each positive integer k, the k-th prolongation $\Delta^{(k)}$ of (1.1) is the ideal of the ring of C^∞ functions of $J^{r+k}(\Omega, \mathbf{R}^m)$ generated by all the partial derivatives of $\Delta_\nu, \nu = 1, \dots, l$, upto order k. If $a(x, u^{(r+k)}) \in \Delta^{(k)}$ then $a(x, u^{(r+k)}) = 0$ is called a compatibility equation of (1.1). We are particularly interested in finding a compatibility equation

$$\sum_{\nu=1}^{l} \sum_{|J|=k} b^J_\nu(x, u^{(r+k)}) \mathbf{D}_J \Delta_\nu \in \Delta^{(k)}$$

that depends only on $(x, u^{(r+k-1)})$, where $\mathbf{D}$ denotes the total differentiation. This occurs when the principal part is cancelled out in the process of prolongation. The cancellation of the principal part is due to the symmetry in (1.1) (cf. [14, Lemma 2.85]), and gives informations on the regularity of solutions that are not contained in the principal part of the original system. The curvature equations (2.3) are compatibility equations for (1) of this type.

2. Compatibility equations

In this section we construct some compatibility equations for (1). We begin with rewriting (1) as

$$\sum_{a,b=1}^{n+d} u_i^a \tilde{g}_{ab}(u) u_j^b - g_{ij}(x)v := \mathbf{F}_{ij}(x, u^{(1)}, v) = 0, \tag{2.1}$$

where the subscripts to the unknown functions u^a and v denote the partial derivatives, as in section 1. First, we consider $\frac{1}{2}(\mathbf{D}_i\mathbf{F}_{jk} - \mathbf{D}_k\mathbf{F}_{ij} + \mathbf{D}_j\mathbf{F}_{ki}) := \mathbf{H}_{ijk}$, which gives

$$\begin{aligned} \sum_{a,b=1}^{n+d} u_k^a \tilde{g}_{ab}(u) u_{ij}^b &+ \frac{1}{2} \sum_{a,b,c=1}^{n+d} u_i^a u_j^b u_k^c \left[\frac{\partial \tilde{g}_{bc}}{\partial y^a}(u) + \frac{\partial \tilde{g}_{ca}}{\partial y^b}(u) - \frac{\partial \tilde{g}_{ab}}{\partial y^c}(u) \right] \\ &- \frac{1}{2}[g_{jk}(x)v_i + g_{ki}(x)v_j - g_{ij}(x)v_k] \\ &- \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x^i}(x) + \frac{\partial g_{ki}}{\partial x^j}(x) - \frac{\partial g_{ij}}{\partial x^k}(x) \right] v \\ &= \mathbf{H}_{ijk}(x, u^{(2)}, v^{(1)}) = 0. \end{aligned} \tag{2.2}$$

Notice that $\mathbf{D}_i\mathbf{F}_{jk} = \mathbf{H}_{ijk} + \mathbf{H}_{kij}$. Thus the first prolongation $\mathbf{F}^{(1)}$ of (2.1) is generated by $\{\mathbf{F}_{ij}, \mathbf{H}_{ijk}\}_{1 \le i,j,k \le n}$. It is easy to see that the last column of the principal symbol of (2.1) is a zero vector and therefore (2.1) is not elliptic at any solution. For the same reason (2.2) is not elliptic at any solution.

Now consider $\frac{1}{2}(\mathbf{D}_{il}\mathbf{F}_{jk} + \mathbf{D}_{jk}\mathbf{F}_{il} - \mathbf{D}_{ik}\mathbf{F}_{jl} - \mathbf{D}_{jl}\mathbf{F}_{ik}) := \mathbf{K}_{ijkl}$. Then all the third order derivatives cancel out in the process of addition and subtraction and we have

$$\begin{aligned} \sum_{a,b=1}^{n+d} (u_{ik}^a u_{jl}^b - u_{il}^a u_{jk}^b)\tilde{g}_{ab}(u) &+ \frac{1}{2} \sum_{a,b,c=1}^{n+d} (u_{ik}^a u_j^b u_l^c + u_{jl}^a u_i^b u_k^c - u_{il}^a u_j^b u_k^c - u_{jk}^a u_i^b u_l^c) \\ &\cdot \left[\frac{\partial \tilde{g}_{ab}}{\partial y^c}(u) + \frac{\partial \tilde{g}_{ca}}{\partial y^b}(u) - \frac{\partial \tilde{g}_{bc}}{\partial y^a}(u) \right] \\ &+ \frac{1}{2}[g_{ik}(x)v_{jl} + g_{jl}(x)v_{ik} - g_{jk}(x)v_{il} - g_{il}(x)v_{jk}] - \frac{1}{2} \sum_{a,b,c,d=1}^{n+d} (u_i^a u_j^b u_k^c u_l^d) \\ &\cdot \left[\frac{\partial^2 \tilde{g}_{bd}}{\partial y^a \partial y^c}(u) + \frac{\partial^2 \tilde{g}_{ac}}{\partial y^b \partial y^d}(u) - \frac{\partial^2 \tilde{g}_{bc}}{\partial y^a \partial y^d}(u) - \frac{\partial^2 \tilde{g}_{ad}}{\partial y^b \partial y^c}(u) \right] \end{aligned} \tag{2.3}$$

$$+\frac{1}{2}\left[\frac{\partial g_{jl}}{\partial x^k}(x)-\frac{\partial g_{jk}}{\partial x^l}(x)\right]v_i+\frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^l}(x)-\frac{\partial g_{il}}{\partial x^k}(x)\right]v_j$$

$$+\frac{1}{2}\left[\frac{\partial g_{jl}}{\partial x^i}(x)-\frac{\partial g_{il}}{\partial x^j}(x)\right]v_k+\frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j}(x)-\frac{\partial g_{jk}}{\partial x^i}(x)\right]v_l$$

$$+\frac{1}{2}\left[\frac{\partial^2 g_{jl}}{\partial x^i\partial x^k}(x)+\frac{\partial^2 g_{ik}}{\partial x^j\partial x^l}(x)-\frac{\partial^2 g_{jk}}{\partial x^i\partial x^l}(x)-\frac{\partial^2 g_{il}}{\partial x^j\partial x^k}(x)\right]v$$

$$=\mathbf{K}_{ijkl}(x,u^{(2)},v^{(2)})=0.$$

In particular, (2.3) with $v\equiv 1$ is the classical Gauss equation [4]. Notice that (2.3) involves the derivatives of u and v up to order 2 only. In the next section we show that the second order system (2.2)–(2.3) are elliptic at certain solutions.

3. Proof of the theorems

Let $f:\mathcal{M}\to\tilde{\mathcal{M}}$ be a conformal immersion with conformal factor v. Let $P\in\mathcal{M}$ be a given reference point of $\mathcal{M}$. Let $(x^1,\dots,x^n)$ be a Riemannian normal coordinate system at P with the coordinate vector fields $\partial_{x^1},\dots,\partial_{x^n}$ and let $(y^1,\dots,y^{n+d})$ be a Riemannian normal coordinate system of $\tilde{\mathcal{M}}$ at $f(P)$ with the coordinate vector fields $\partial_{y^1},\dots,\partial_{y^{n+d}}$. There is no loss of generality in assuming that $T_{f(P)}\mathcal{M}$ is spanned by $\{\partial_{y^1},\dots,\partial_{y^n}\}$, and that $\{\partial_{y^{n+1}},\dots,\partial_{y^{n+d}}\}$ forms a basis of $[T_{f(P)}\mathcal{M}]^\perp$. For convenience, we assume that $df(\partial_{x^i}|_P)=\sqrt{v(P)}\partial_{y^i}|_{f(P)}$, $i=1,\dots,n$. In terms of these coordinates, f and v satisfy the system of partial differential equations (2.1)–(2.3) and by the assumptions on the coordinates we have

$$g(\partial_{x^i}|_P,\partial_{x^j}|_P)=g_{ij}(O)=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases};\tag{3.1}$$

$$\tilde{g}(\partial_{y^a}|_{f(P)},\partial_{y^b}|_{f(P)})=\tilde{g}_{ab}(O)=\begin{cases}1 & \text{if } a=b\\ 0 & \text{if } a\neq b\end{cases};\tag{3.2}$$

$$\frac{\partial f^a}{\partial x^i}(O)=\begin{cases}\sqrt{v(O)} & \text{if } i=a\\ 0 & \text{if } i\neq a\end{cases};\tag{3.3}$$

$$\alpha(\partial_{x^i}|_P,\partial_{x^j}|_P)=\sum_{a=n+1}^{n+d}\left[\frac{\partial^2 f^a}{\partial x^i\partial x^j}(O)\right]\partial_{y^a}|_{f(P)}.\tag{3.4}$$

To prove Theorem 4 we will show that the ellipticity of the system

$$\begin{aligned}&\mathbf{H}_{iik}(x,u^{(2)},v^{(1)})=0, && i,k=1,\dots,n;\\ &\mathbf{K}_{ijik}(x,u^{(2)},v^{(2)})=0, && i,j,k=1,\dots,n\end{aligned}\tag{3.5}$$

is equivalent to the conformal nullity condition of the theorem.

Proof of Theorem 4. It suffices to show that a hyperplane W in $T_P\mathcal{M}$ is characteristic to (3.5) at $(O, j_O^2 f, j_O^2 v)$ if and only if W is V^1-umbilic for some 1-dimensional subspace V^1 of $[T_{f(P)}\mathcal{M}]^\perp$.

Let $\xi = \xi_1 \partial_{x^1} + \cdots + \xi_n \partial_{x^n}$ be a normal vector to a characteristic hyperplane W for (3.5) at $(O, j_O^2 f, j_O^2 v)$. Then the principal symbol matrix $\mathbf{M}(\xi)$ of (3.5) at $(O, j_O^2 f, j_O^2 v)$ is not of maximal rank, that is, $\mathbf{M}(\xi)$ is of rank $< n+d+1$. We decompose $\mathbf{M}(\xi)$ into $2n$ blocks as

$$(3.6)\qquad \mathbf{M}(\xi; O, j_O^2 f, j_O^2 v) = \begin{pmatrix} \mathbf{M_H}(\xi; O, j_O^2 f, j_O^2 v) \\ \mathbf{M_K}(\xi; O, j_O^2 f, j_O^2 v) \end{pmatrix} = \begin{pmatrix} \mathbf{M}_\mathbf{H}^1(\xi; O, j_O^2 f, j_O^2 v) \\ \vdots \\ \mathbf{M}_\mathbf{H}^n(\xi; O, j_O^2 f, j_O^2 v) \\ \mathbf{M}_\mathbf{K}^1(\xi; O, j_O^2 f, j_O^2 v) \\ \vdots \\ \mathbf{M}_\mathbf{K}^n(\xi; O, j_O^2 f, j_O^2 v) \end{pmatrix},$$

where, for each $i = 1, \dots, n$, $\mathbf{M}_\mathbf{H}^i(\xi)$ is the principal symbol of the system of n equations, $\mathbf{H}_{iij}(x, u^{(2)}, v^{(2)}) = 0$, $j = 1, \dots, n$, and $\mathbf{M}_\mathbf{K}^i(\xi)$ is the principal symbol of the system of n^2 equations, $\mathbf{K}_{ijik}(x, u^{(2)}, v^{(2)}) = 0$, $j, k = 1, \dots, n$. Then, by (3.1)–(3.3), for each $i = 1, \dots, n$,

$$\mathbf{M}_\mathbf{H}^i(\xi) = \sqrt{v(O)} \begin{pmatrix} \xi_i^2 & 0\cdots 0 & \xi_1\xi_i & 0\cdots 0 & 0 & 0\cdots 0 & 0 \\ 0 & \xi_i^2\cdots 0 & \xi_2\xi_i & 0\cdots 0 & 0 & 0\cdots 0 & 0 \\ & & \vdots & & & & \\ 0 & 0\cdots 0 & \xi_n\xi_i & 0\cdots 0 & \xi_i^2 & 0\cdots 0 & 0 \end{pmatrix}.$$

$$\qquad\qquad\qquad\uparrow \text{ i-th column} \qquad \uparrow \text{ n-th column}$$

For $\mathbf{M}_\mathbf{K}^i(\xi)$, the $(n(j-1)+k)$-th row corresponds to the equation $\mathbf{K}_{ijik} = 0$. Its $(n(j-1)+k, a)$ component, $1 \le a \le n+d$, is

$$\left[\frac{\partial^2 f^a}{\partial x^j \partial x^k}(O)\right]\xi_i^2 - \left[\frac{\partial^2 f^a}{\partial x^i \partial x^k}(O)\right]\xi_i\xi_j - \left[\frac{\partial^2 f^a}{\partial x^i \partial x^j}(O)\right]\xi_i\xi_k + \left[\frac{\partial^2 f^a}{(\partial x^i)^2}(O)\right]\xi_j\xi_k,$$

for $y = (y^1, \dots, y^{n+d})$ is a normal coordinate system at the reference point and thus for each $a, b, c = 1, \dots, n+d$,

$$\left[\frac{\partial \tilde{g}_{ab}}{\partial y^c}(O) + \frac{\partial \tilde{g}_{ca}}{\partial y^b}(O) - \frac{\partial \tilde{g}_{bc}}{\partial y^a}(O)\right]$$

$$= \sum_{s=1}^{n+d} \tilde{g}^{as}(O)\left[\frac{\partial \tilde{g}_{cs}}{\partial y^b}(O) + \frac{\partial \tilde{g}_{sb}}{\partial y^c}(O) - \frac{\partial \tilde{g}_{bc}}{\partial y^s}(O)\right] = \tilde{\Gamma}_{bc}^a(O) = 0,$$

where $[\tilde{g}^{ab}]$ is the inverse matrix of $[\tilde{g}_{ab}]$ and $\tilde{\Gamma}_{bc}^a$ is the Christoffel symbol for $\tilde{\mathcal{M}}$. On the other hand, the $(n(j-1)+k, n+d+1)$ component of $\mathbf{M}_\mathbf{K}^i(\xi)$ is

(3.7) $$\tfrac{1}{2}(\xi_j\xi_k + \delta_{jk}\xi_i^2 - \delta_{ij}\xi_i\xi_k - \delta_{ik}\xi_i\xi_j).$$

For $\mathbf{M_H}(\xi)$, we note that the first n columns are linearly independent and the rest ones are zero vectors. Thus the last $d+1$ columns of $\mathbf{M_K}(\xi)$ must be linearly dependent, for (3.6) is not of maximal rank. Therefore, if we write the columns of $\mathbf{M_K}(\xi; O, j_O^2 f, j_O^2 v)$ as $A_1, \dots, A_{n+d+1}$, there exist real numbers $a_1, \dots, a_d$ and b which are not all zero so that

(3.8) $$a_1 A_{n+1} + \cdots + a_d A_{n+d} = b A_{n+d+1}.$$

Since the last column of $\mathbf{M_K}(\xi)$ is a nonzero vector, not all a_i's are zero. Consider the nonzero vector $\zeta_a = a_1\partial_{y^{n+1}} + \cdots + a_d\partial_{y^{n+d}} \in [T_{f(P)}\mathscr{M}]^{\perp}$. Then, by (3.1) and (3.4), (3.8) can be expressed for all $i, j, k = 1, \dots, n$, as

$$\tilde{g}(\alpha(\xi_i\partial_{x^j} - \xi_j\partial_{x^i}, \xi_i\partial_{x^k} - \xi_k\partial_{x^i}), \zeta_a) = \tfrac{1}{2}bg(\xi_i\partial_{x^j} - \xi_j\partial_{x^i}, \xi_i\partial_{x^k} - \xi_k\partial_{x^i}).$$

It is easy to verify that the set of vectors $\{\xi_i\partial_{x^j} - \xi_j\partial_{x^i}\}_{1\le i,j\le n}$ generates W. In fact, if $\xi_k \neq 0$, $\{\xi_k\partial_{x^j} - \xi_j\partial_{x^k}\}_{1\le j\le n}$ forms a basis of W. Hence we have

$$\tilde{g}(\alpha(X, Y), \zeta_a) = \tfrac{1}{2}bg(X, Y),$$

for all vectors X, Y contained in W, which implies that W is V^1-umbilic, where V^1 is the 1-dimensional subspace generated by ζ_a. The converse can be similarly proved.

Proof of Theorem 5. Theorem 5 is a special case of Theorem 4. This is the case $d = 0$ and we see that (3.5) is elliptic at each C^2 conformal mapping without any additional condition. In this case the principal symbol matrix $\mathbf{M}(\xi)$ of (3.5) is of the form

$$\begin{bmatrix} A & O \\ B & C \end{bmatrix},$$

where A is an $n^2 \times n$ matrix of rank n, B is a matrix of dimension $n^3 \times n$, C is a column vector of length n^3, and O is a zero column vector. By direct calculation one can show that C is not zero unless ξ is zero. In fact, the $[n(i-1) + n(j-1) + k]$-th component of C is the same as (3.7).

Proof of Theorem 6. For the case of codiminsion 1, let $f : \mathscr{M}^n \to \tilde{\mathscr{M}}^{n+1}$ be a conformal immersion and let $P \in \mathscr{M}$. Let $\mathfrak{m}^0$ be the multiplicity of zero principal curvature and let $\mathfrak{m}$ be the maximal multiplicity of nonzero principal curvatures at $f(P)$ and let $\mathfrak{p}_+$ and $\mathfrak{p}_-$ be the number of positive and negative principal curvatures respectively, counting multiplicity. Then $n = \mathfrak{p}_+ + \mathfrak{m}^0 + \mathfrak{p}_-$ and we have

(3.9) $$\nu_1^c(P) = \max\{\mathfrak{m}^0 + \min\{\mathfrak{p}_+, \mathfrak{p}_-\}, \mathfrak{m}\}.$$

Suppose the maximal multiplicity of the principal curvatures at $f(P)$ does not exceed $n-3$, that is, $\max\{\mathfrak{m}^0, \mathfrak{m}\} \le n-3$. Then we get

$$\mathfrak{m}^0 + \min\{\mathfrak{p}_+, \mathfrak{p}_-\} \le \mathfrak{m}^0 + \frac{n - \mathfrak{m}^0}{2} = \frac{\mathfrak{m}^0 + n}{2} \le \frac{(n-3)+n}{2} = n - \frac{3}{2},$$

and hence $\mathfrak{m}^0 + \min\{\mathfrak{p}_+, \mathfrak{p}_-\} \le n - 2$, which implies $\nu_1^c(P) \le n - 2$. Then by Theorem 4 we obtain Theorem 6.

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