J. Math. Kyoto Univ. (JMKYAZ)
40-1 (2000), 185-201

Multiple fibers on elliptic surfaces in positive characteristic

By

Mitsuru KAWAZOE

Abstract

In this note, we investigate elliptic surfaces in char. $p>0$ with multiple fibers of a supersingular elliptic curve. In particular, we show that wild fibers on elliptic surfaces over $\mathbf{P}^1$ with $c_2=0$ and the general fiber being a supersingular elliptic curve can be reduced to tame fibers by taking a purely inseparable covering of degree p successively and as an application of it, we show that if such elliptic surface has only one multiple fiber, then its multiplicity is equal to p.

Introduction

In characteristic $p>0$ world, multiple fibers appeared in fibred varieties are divided into two classes. One is a tame fiber which can be treated as in characteristic zero and the other one is a wild fiber which is only appeared in positive characteristic and has curious properties (cf. [1][4]). Katsura and Ueno [1] studied multiple fibers of elliptic surfaces in positive characteristic and obtained many results. For example, they studied how to reduce a wild fiber to a tame and moreover to a non-multiple fiber by pulling back the elliptic fibration to a certain covering of a given elliptic surface and for a multiple fiber of type ${}_m\mathrm{I}_n$, $n\geq 0$ they proved that if such multiple fiber is of good type (for the definition, see [1, added in proof]), one can reduce a multiple fiber to a tame fiber by taking a certain covering successively. In particular, if the support of a multiple fiber is an ordinary elliptic curve or of type I_n, $n>0$, a covering used in the above procedure is a $\mathbf{Z}/p\mathbf{Z}$ étale covering. For a multiple fiber of a supersingular elliptic curve, their result is a little weak. For example, they could not answer whether we can take a covering of degree p in the procedure of reduction to tame fibers.

In this article, we consider elliptic surfaces in characteristic $p>0$ with multiple fibers of a supersingular elliptic curve. For the reduction of a wild fiber to a tame fiber, we show the following:

1991 *Mathematics Subject Classification.* 14J27, 14J10.

Partially supported by Grant-in-Aid for Encouragement of Young Scientists (No. 10740018), Ministry of Education, Science and Culture.
Communicated by Prof. K. Ueno, August 6, 1999

Theorem A. *Let $f: X \to \mathbf{P}^1$ be an elliptic surface in characteristic $p>0$ with general fiber being a supersingular elliptic curve and $\chi(\mathcal{O}_X)=0$. Then, we can reduce wild fibers of it to tame fibers by taking a purely inseparable covering of degree p successively.*

The situation treated in this theorem is special, but this theorem does not need the assumption that all wild fibers are of good type. After reducing wild fibers to tame fibers, we can reduce tame fibers to non-multiple fibers in the well-known manner(cf. [1]). As a corollary of this result, we have:

Corollary. *Let $f: X \to \mathbf{P}^1$ be as in Theorem A. If the multiplicity of every multiple fiber of f is a power of p, $f: X \to \mathbf{P}^1$ is obtained by a successive quotient of the trivial elliptic fibration $\mathbf{P}^1 \times E$ by p-closed derivations where E is a supersingular elliptic curve.*

For wild fibers in elliptic fibrations in characteristic $p>0$, the following question has been open for a long time (cf. [3][6][7]):

Question. Does there exist an elliptic surface over $\mathbf{P}^1$ with a multiple fiber of a supersingular elliptic curve of multiplicity p^ν $(\nu \geq 2)$?

In reference to this question, we are interested in the multiplicity of a multiple fiber with support being a supersingular elliptic curve appeared in an elliptic fibration. By applying the above results, we show the following:

Theorem B. *Let $f: X \to \mathbf{P}^1$ be an elliptic surface in characteristic $p>0$ with $\chi(\mathcal{O}_X)=0$ and the general fiber being a supersingular elliptic curve. If $f: X \to \mathbf{P}^1$ has only one multiple fiber, then the multiplicity of the multiple fiber is equal to p.*

The condition that $f: X \to \mathbf{P}^1$ has only one multiple fiber comes from only a technical reason in the proof. By virtue of this theorem, we can compute the Hodge numbers of elliptic surfaces satisfying the assumption of Theorem B.

Finally, we give a brief outline of this article. In Section 1, we shall recall the notion of a wild fiber and its property. In Section 2, we shall give a quick survey of the theory of taking quotients by derivations which is a very useful tool to understand purely inseparable maps. In Section 3, we shall consider how to reduce wild fibers to tame fibers on our surfaces by pulling-back with the Frobenius map of their Albanese varieties and prove Theorem A (Theorem 3.2) and Corollary (Corollary 3.4). In Section 4, applying the result in Section 3, we shall prove Theorem B (Theorem 4.4) and compute the Hodge numbers of elliptic surfaces satisfying the assumption of Theorem B.

Notation

We use the following notation. For an n-dimensional irreducible algebraic variety X over an algebraically closed field k of characteristic $p>0$,

$k(X)$: the function field of X,

$\mathrm{Der}_k(k(X))$: the module of derivations of $k(X)/k$,

F_X: the relative Frobenius map from X.

For a nonsingular algebraic variety X,

$\mathrm{Alb}(X)$: the Albanese variety of X,

$d\mathcal{O}_X$: the sheaf of the image of the map $d\colon \mathcal{O}_X \ni f \mapsto df \in \Omega_X^1$,

$q(X)$: the dimension of $\mathrm{Alb}(X)$,

$b_i(X)$: the i-th Betti number of X,

$c_i(X)$: the i-th Chern class of X,

$\rho(X)$: the Picard number of X,

$[D]$: the line bundle associated with D.

1. Preliminaries

In this article, we always assume that an elliptic surface $f\colon X \to C$ is minimal, that is, any fiber of f contains no exceptional curve of the first kind. For an elliptic surface $f\colon X \to C$ in characteristic $p>0$, let $\mathcal{T}$ be the torsion part of $R^1 f_* \mathcal{O}_X$ and $f^{-1}(\alpha) = mD$ a multiple fiber of f where m is the multiplicity and D is its support. Put $n = \mathrm{ord}[D]|_D$. Then there exists a non-negative integer r such that

$$m = np^r.$$

A multiple fiber $f^{-1}(\alpha) = mD$ is called a tame fiber if one of the following equivalent conditions is satisfied.

1. $\mathcal{T}_\alpha = 0$,
2. $h^0(\mathcal{O}_{mD}) = 1$,
3. $n = m$.

If a multiple fiber is not tame, we call it a wild fiber. A multiple fiber $f^{-1}(\alpha) = mD$ is wild if and only if one of the following equivalent conditions is satisfied.

1. $\mathcal{T}_\alpha \neq 0$,
2. $h^0(\mathcal{O}_{mD}) \geq 2$,
3. $1 \leq n \leq m-1$.

If D is a supersingular elliptic curve, then $\mathrm{Pic}^0(D)$ has no p-torsion point. Hence, for a multiple fiber mD of a supersingular elliptic curve D, mD is wild if and only if $(m,p) \neq 1$.

2. Quick survey of quotients by derivations

In this section, we recall some basic facts on the theory of derivations and p-closed vector fields (cf. [5]).

2.1. Let R be a ring and D a derivation of R. We denote a subring

$\{a\in R: Da=0\}$ of R by R^D. For an ideal I of R, D defines a derivation $\bar{D}$ of R/I if and only if $D(I)\subset I$.

2.2. Let K be a field of characteristic $p>0$ and D be a derivation of K. If D is a derivation of K, then D^p is also a derivation. A derivation D is called p-closed if and only if $D^p=\alpha D$ for some $\alpha\in K$. In particular, if $D^p=0$ (resp. $D^p=D$) we say that D is of additive type (resp. multiplicative type). Two derivations D, D' are called equivalent if and only if $D=fD'$ for some $f\in K$, $f\neq 0$. If D and D' are equivalent, we write $D\sim D'$.

Since the module of derivations of K/L for any subfield L of K such that K/L is an inseparable extension of degree p is one dimensional, every subfield L of K such that K/L is an inseparable extension of degree p bijectively corresponds to an equivalent class of p-closed derivations of K such that $K^D=L$ where K^D is the subfield $\{f\in K: Df=0\}$ of K.

2.3. Let $X=\bigcup \mathrm{Spec}(A_i)$ be an irreducible algebraic variety defined over an algebraically closed field k of characteristic $p>0$ and D a rational vector field on X, i.e., a derivation of $(X)/k$. As is well-known, a variety X^D defined by $\bigcup \mathrm{Spec}(A_i^D)$ where $A_i^D:=A_i\cap k(X)^D$ and a morphism $\pi_D: X\to X^D$ induced from $A_i^D \subsetneq A_i$ satisfy the following properties:

1. π_D is a finite purely inseparable morphism,
2. If X is normal, so is X^D,
3. If D is p-closed, then $\deg \pi_D=p$.

Conversely, for normal varieties X, Y and a finite purely inseparable morphism $\pi: X\to Y$ of degree p, there exists a rational p-closed vector field D on X such that $\pi=\pi_D$, $Y=X^D$ and D is uniquely determined up to a nonzero scalar multiple from $k(X)$. This fact comes from (2.2).

2.4. Let X be a nonsingular irreducible algebraic variety over k, and D a rational vector field on X. Then D can be written in a neighborhood of $x\in X$ as

$$D=h_x\left(\sum_{i=1}^{n} f_{x,i}\frac{\partial}{\partial x_i}\right)$$

where $x_1,\cdots,x_n$ are local coordinates at $x\in X$, h_x is a rational function on X and $f_{x,i}$ are regular functions at x which are relatively prime. The functions h_x determine a divisor, which we call the divisor of D and denote it by (D). Let $\langle D\rangle$ be a zero cycle on X defined by $\Sigma_{x\in X}\, m_x x$, where $m_x=\dim_k \mathcal{O}_{X,x}/(f_{x,1},\cdots,f_{x,n})$. For a rational vector field D on X, we say D has only a divisorial singularity at $x\in X$ if and only if the above ideal $(f_{x,1},\cdots,f_{x,n})$ contains the unity, that is, $\langle D\rangle=0$.

In the case of nonsingular surfaces, the following fact is well-known.

Proposition 2.1 (Rudakov-Šafarevič [5]). *Let X be a nonsingular algebraic surface, D a p-closed rational vector field on X and $x\in X$. Then the quotient surface X^D is nonsingular at $\pi_D(x)$ if and only if D has only a divisorial singularity at x.*

Finally, we recall the notion of an integral subvariety. Let V be a subvariety of X and v the general point on V. Denote $h_v^{-1}D$ by D_v, where h_v is a local equation of the divisor (D) at v. We call V an integral subvariety for D if the vector field D_v is tangent to V at v. When V is a divisor, V is an integral subvariety for D if and only if $D_v(F_v)\equiv 0 \bmod F_v$ in $\mathcal{O}_{V,v}$ for any point v of V, where F_v is a local equation of V at v. The following proposition is important.

Proposition 2.2. ([5]). *Let X be an algebraic variety which is nonsingular in codimension one and let D be a p-closed rational vector field on $X, \pi=\pi_D: X\to X^D$, V an irreducible divisor on X, V' the image of V in X^D. Then we have the following*:

1. *If V is an integral subvariety for D, then $\pi_* V=pV'$ and $\pi^* V'=V$,*
2. *If V is not an integral subvariety for D, then $\pi_* V=V'$ and $\pi^* V'=pV$.*

Let $f: X\to S$ be a proper surjective flat morphism of algebraic varieties. Let D be a p-closed derivation of $k(X)$ and V be an integral subvariety of X. If V is a fiber of f then the natural fibration $f^D: X^D\to S^D$ obtained from $f: X\to S$ has a multiple fiber over $f^D(\pi_D(V))$ of multiplicity p.

$$\begin{array}{ccc} X & \xrightarrow{\pi_D} & X^D \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^D} \\ S & \longrightarrow & S^D \end{array}$$

3. Reduction of wild fibers to tame fibers

Let $f: X\to \mathbf{P}^1$ be an elliptic surface in characteristic $p>0$. In the following, we assume that $f: X\to\mathbf{P}^1$ satisfies the following conditions $(*)$:

1. $\chi(\mathcal{O}_X)=0$,
2. the general fiber is a supersingular elliptic curve.

The condition $\chi(\mathcal{O}_X)=0$ imposes that every singular fiber is of type ${}_m\mathrm{I}_0$, that is, a multiple fiber of an elliptic curve. This can be seen from the following lemma.

Lemma 3.1 (Katsura-Ueno [1]). *For an elliptic surface $f: X\to C$, we let $\varphi: X\to$ $\mathrm{Alb}(X)$ be an Albanese map of X and $\psi: C\to J(C)$ a natural mapping into the Jacobian variety of C, with a suitable choice of base points on X and C. Then, the following conditions are equivalent.*

1. *There exists a fiber $f^{-1}(p)$, $p\in C$ such that $\varphi(f^{-1}(p))$ is a point.*
2. $\mathrm{Alb}(X)$ *is isomorphic to* $J(C)$.

Otherwise, we have $\dim \mathrm{Alb}(X)=\dim J(C)+1$.

By $c_1(X)^2=0$ and Noether's formula, $\chi(\mathcal{O}_X)=0$ if and only if $c_2(X)=0$. From the above lemma and the inequality $c_2(X)=b_0(X)-b_1(X)+b_2(X)-b_3(X)+b_4(X)\geq 2-4q(X)$, we have that if $\chi(\mathcal{O}_X)=0$, then $q(X)=1$ and every singular fiber is a multiple fiber of an elliptic curve.

Let $m_iS_i=f^{-1}(\alpha_i)$, $i=1,2,\cdots,l$ be all multiple fibers of $f\colon X\to\mathbf{P}^1$ where m_i are positive integers and S_i are their supports. Changing indices if necessary, we may assume that $m_i\le m_j$ for $i<j$. Let φ be an Albanese map of X and $E=\mathrm{Alb}(X)$. Note that E is also a supersingular elliptic curve because the general fiber of f is a supersingular elliptic curve. Moreover, since S_i is also a supersingular elliptic curve, $f^{-1}(\alpha_i)=m_iS_i$ is a wild fiber if and only if $(m_i,p)\neq 1$. For such surfaces, we can reduce wild fibers to tame fibers by pulling back an elliptic fibration to a certain covering of $f\colon X\to\mathbf{P}^1$. Indeed, we have the following theorem.

Theorem 3.2. *Let $f\colon X\to\mathbf{P}^1$ and $f^{-1}(\alpha_i)=m_iS_i$ be as above. Then there is the following diagram:*

$$\begin{array}{ccccccccccc}
E_h & \xrightarrow{F} & E_{h-1} & \xrightarrow{F} & \cdots & \xrightarrow{F} & E_j & \xrightarrow{F} & \cdots & \xrightarrow{F} & E_0:=E \\
\uparrow{\scriptstyle\varphi_h} & & \uparrow{\scriptstyle\varphi_{h-1}} & & & & \uparrow{\scriptstyle\varphi_j} & & & & \uparrow{\scriptstyle\varphi} \\
X_h & \xrightarrow{\pi_h} & X_{h-1} & \xrightarrow{\pi_{h-1}} & \cdots & \longrightarrow & X_j & \xrightarrow{\pi_j} & \cdots & \xrightarrow{\pi_1} & X_0:=X \\
\downarrow{\scriptstyle f_h} & & \downarrow{\scriptstyle f_{h-1}} & & & & \downarrow{\scriptstyle f_j} & & & & \downarrow{\scriptstyle f} \\
\mathbf{P}^1 & \xrightarrow{F} & \mathbf{P}^1 & \xrightarrow{F} & \cdots & \xrightarrow{F} & \mathbf{P}^1 & \xrightarrow{F} & \cdots & \xrightarrow{F} & \mathbf{P}^1
\end{array}$$

such that

1. *F are relative Frobenius maps,*
2. *$f_j\colon X_j\to\mathbf{P}^1$ are elliptic surfaces satisfying the condition* (∗),
3. *for the point $\alpha_i^{(j)}\in\mathbf{P}^1$, $j<h$ such that $F(\alpha_i^{(j)})=\alpha_i^{(j-1)}$ and $\alpha_i^{(0)}=\alpha_i$, the fiber $f_j^{-1}(\alpha_i^{(j)})$ is a multiple fiber of a supersingular elliptic curve with multiplicity*

$$m_i^{(j)}=\begin{cases} m_i & j=0,1,\cdots,h-r_i \\ m_i/p^{j-h+r_i} & j=h-r_i+1,h-r_i+2,\cdots,h-1 \end{cases}$$

for all $i=1,2,\cdots,l$ where r_i is the largest integer such that p^{r_i} devides m_i,

4. *φ_j are Albanese maps,*
5. *π_j are finite purely inseparable morphisms of degree p,*
6. *$f_h\colon X_h\to\mathbf{P}^1$ has only tame fibers.*

The following lemma is important in the proof of the above theorem.

Lemma 3.3. *Let $f\colon X\to\mathbf{P}^1$ be as in the Theorem* 3.2. *Then,*

1. *for the Albanese map $\varphi\colon X\to\mathrm{Alb}(X)=E$, $\varphi^*\colon H^1(E,\mathcal{O}_E)\to H^1(X,\mathcal{O}_X)$ is injective,*

2. *for a smooth fiber $f^{-1}(\beta)=D$, the natural restriction mapping $\varrho: H^1(X, \mathcal{O}_X) \to H^1(D, \mathcal{O}_D)$ is the zero mapping.*

Proof. (1) Since $q(X)=1$, $\mathrm{Alb}(X)$ is an elliptic curve. Then, by Leray's spectral sequence for $\varphi: X \to \mathrm{Alb}(X)$, we have the exact sequence,

$$0 \to H^1(E, \mathcal{O}_E) \to H^1(X, \mathcal{O}_X).$$

(2) This is the consequence of [1, Lemma 6.1 (ii)]

Proof of Theorem 3.2. Let ρ be a non-zero element of $H^1(E, \mathcal{O}_E)$. Since E is a supersingular elliptic curve, $F^*(\rho)=0$ where F^* is the induced action on $H^1(E, \mathcal{O}_E)$ from the Frobenius action $F: \mathcal{O}_E \ni g \mapsto g^p \in \mathcal{O}_E$ on $\mathcal{O}_E$.

Taking an affine open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of E, we represent ρ by a Čech cocycle $\{\rho_{\lambda\mu}\}$ with respect to this covering. By $F^*(\rho)=0$, there are $\rho_\lambda \in \Gamma(U_\lambda, \mathcal{O}_E)$ such that

$$\rho_{\lambda\mu}^p = \rho_\lambda - \rho_\mu \text{ on } U_\lambda \cap U_\mu,$$

We define the covering $\pi: E_1 \to E$ by

$$\begin{cases} u_\lambda^p = \rho_\lambda & \text{on } U_\lambda \\ u_\lambda = u_\mu + \rho_{\lambda\mu} & \text{on } U_\lambda \cap U_\mu. \end{cases}$$

$\pi: E_1 \to E$ is nothing but the relative Frobenius morphism F_{E_1}.

Since φ^* is injective, we have

$$\varphi^*\rho \neq 0, \quad F^*(\varphi^*\rho)=0$$

from the following commutative diagram:

$$\begin{array}{ccccc} 0 & \to & H^1(E, \mathcal{O}_E) & \xrightarrow{\varphi^*} & H^1(X, \mathcal{O}_X) \\ & & {\scriptstyle F^*}\downarrow & & \downarrow{\scriptstyle F^*} \\ 0 & \to & H^1(E, \mathcal{O}_E) & \xrightarrow{\varphi^*} & H^1(X, \mathcal{O}_X) \end{array}$$

Put $f_{\lambda\mu} := \varphi^*\rho_{\lambda\mu}$ on $\varphi^{-1}(U_\lambda) \cap \varphi^{-1}(U_\mu)$ and $z_\lambda := \varphi^* u_\lambda$ on $\varphi^{-1}(U_\lambda)$. We define the covering $\tilde{\pi}: \tilde{X} \to X$ by

$$\begin{cases} z_\lambda^p = \varphi^*\rho_\lambda & \text{on } \varphi^{-1}(U_\lambda) \\ z_\lambda = z_\mu + f_{\lambda\mu} & \text{on } \varphi^{-1}(U_\lambda) \cap \varphi^{-1}(U_\mu). \end{cases}$$

This is a flat covering of degree p. Let $\hat{X}$ be a minimal nonsingular model of the normalization of $\tilde{X}$. By the Stein factorization, we have the following diagram.

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{\pi}} & X \\ {\scriptstyle \hat{f}}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & \mathbf{P}^1 \end{array}$$

where C be a nonsingular complete algebraic curve. Since the restriction of $\hat{\pi}$ to the general fiber of f is trivial by Lemma 3.3, the morphism g is purely inseparable of degree p, so $C \cong \mathbf{P}^1$ and g is the Frobenius mapping. By the construction, $\varphi \circ \hat{\pi}$ factors through $F_{E_1}: E_1 \to E$, so we have the following diagram.

$$\begin{array}{ccccc} \mathbf{P}^1 & \xleftarrow{\hat{f}} & \hat{X} & \xrightarrow{\hat{\varphi}} & E_1 \\ {\scriptstyle F}\downarrow & & {\scriptstyle \hat{\pi}}\downarrow & & \downarrow{\scriptstyle F_{E_1}} \\ \mathbf{P}^1 & \xleftarrow{f} & X & \xrightarrow{\varphi} & E. \end{array}$$

It is easy to see that $\hat{\varphi}$ is an Albanese map.

In the above argument, we do not need a desingularization for the normalization of $\tilde{X}$, that is, the normalization of $\tilde{X}$ is already smooth. Let S be an exceptional curve in $\hat{X}$.

$$\begin{array}{ccc} E_1 & \xrightarrow{F_1} & E \\ {\scriptstyle \hat{\varphi}}\uparrow & & \uparrow{\scriptstyle \varphi} \\ \hat{X} & \xrightarrow{\hat{\pi}} & X \\ {\scriptstyle \hat{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{P}^1 & \xrightarrow[F]{} & \mathbf{P}^1 \end{array}$$

Since $f \circ \hat{\pi}(S)$ is a point, we see that $\hat{f}(S)$ is also a point, that is, S is contained in a fiber of $\hat{f}$. But since $q(\hat{X}) = 1$, every fiber of $\hat{f}$ has only one irreducible component and hence S must be a point. Then, we have $\hat{\pi}$ is a finite purely inseparable flat morphism of degree p and $\chi(\mathcal{O}_{\hat{X}}) = \chi(\mathcal{O}_X) = 0$.

Now we look at multiple fibers. We write $m_i = n_i p^{r_i}$ where $n_i = \mathrm{ord}[S_i]|_{S_i}$ and r_i is a non-negative integer. Since S_i is a supersingular elliptic curve, we have $(n_i, p) = 1$. Put $\hat{m}_i \hat{S}_i := \hat{f}^{-1}(\hat{\alpha}_i)$, $i = 1, 2, \cdots, l$, where $\hat{S}_i$ is the support of the fiber. For a morphism ψ, we denote the purely inseparable degree of ψ by $d_{pi}(\psi)$. For $\varphi|_{S_i}$, we have

$$d_{pi}(\varphi|_{S_i}) = p^{r_l - r_i} d_{pi}(\varphi|_{S_l}).$$

If $d_{pi}(\varphi|_{S_i}) > 1$, then $(\varphi|_{S_i})^* : H^1(S_i, \mathcal{O}_{S_i}) \to H^1(\hat{S}_i, \mathcal{O}_{\hat{S}_i})$ is the zero mapping and $\hat{\pi}|_{\hat{S}_i} : \hat{S}_i \to S_i$ is the trivial cover. Since, from Lemma 3.3, the restriction of φ to the general fiber is also the trivial cover, we have

$$\hat{m}_i = m_i = n_i p^{r_i}.$$

Furthermore, we have

$$d_{pi}(\hat{\varphi}|_{\hat{S}_i}) = d_{pi}(\varphi|_{S_i})/p$$

from the following diagram

$$\begin{tikzcd} \hat{S}_i \arrow[r, "\hat{\varphi}|_{\hat{S}_i}"] \arrow[d, "\hat{\pi}|_{\hat{S}_i}"'] & E_1 \arrow[d, "F_{E_1}"] \\ S_i \arrow[r, "\varphi|_{S_i}"'] & E \end{tikzcd}$$

Put $X_1 := \hat{X}$, $f_1 := \hat{f}$, $\pi_1 := \hat{\pi}$, $\varphi_1 := \hat{\varphi}$, $S_i^{(1)} := \hat{S}_i$, $m_i^{(1)} := \hat{m}_i$ and $X_j := \hat{X}_{j-1}$, $f_j := \hat{f}_{j-1}$, $\pi_j := \hat{\pi}_{j-1}$, $\varphi_j := \hat{\varphi}_{j-1}$ and so on. Then, for each $i = 1, 2, \cdots, l$, there is a non-negative integer j_i such that $d_{pi}(\varphi_{j_i}|_{S_i^{(j_i)}}) = 1$. From $m_1 \le m_2 \le \cdots \le m_l$ and the relation $d_{pi}(\varphi|_{S_i}) = p^{r_l - r_i} d_{pi}(\varphi|_{S_l})$, we have $j_i = \log_p d_{pi}(\varphi|_{S_l}) + r_l - r_i$ for $i = 1, 2, \cdots, l$.

For this j_i, we have $(\varphi_{j_i}|_{S_i^{(j_i)}})^* : H^1(E_{j_i}, \mathcal{O}_{E_{j_i}}) \to H^1(S_i^{(j_i)}, \mathcal{O}_{S_i^{(j_i)}})$ is an isomorphism. Constructing a covering X_{j_i+1} of X_{j_i} with respect to $\rho^{(j_i)} \in H^1(E_{j_i}, \mathcal{O}_{E_{j_i}})$ by the above procedure, we have that

1. $\pi_{j_i+1}|_{S_i^{(j_i+1)}} : S_i^{(j_i+1)} \to S_i^{(j_i)}$ *is purely inseparable of degree p*,
2. $m_i^{(j_i+1)} = m_i^{(j_i)}/p = n_i p^{r_i - 1}$,
3. $\varphi|_{S_i^{(j_i+1)}} : S_i^{(j_i+1)} \to E_{j_i+1}$ is separable, that is, $d_{pi}(\varphi_{j_i+1}|_{S_i^{(j_i+1)}}) = 1$.

So, iterating the above construction, we finally reach at (X_h, φ_h, f_h) with $f_h^{-1}(\alpha_i^{(h)}) = n_i S_i^{(h)}$ for all $i = 1, 2, \cdots, l$. Then, all multiple fibers are tame and this complets the proof.

Corollary 3.4. *Let $f: X \to \mathbf{P}^1$ be as in Theorem 3.2. If the multiplicity of every multiple fiber of f is a power of p, then $f: X \to \mathbf{P}^1$ is obtained by a successive quotient of the trivial elliptic fibration $\mathbf{P}^1 \times E$ by p-closed derivations where E is a supersingular elliptic curve.*

Proof. Under the assumption that the multiplicity of every multiple fiber is a power of p, we have $f_h : X_h : \to \mathbf{P}^1$ is free from multiple fibers, i.e., f_h is smooth. Since $\chi(\mathcal{O}_{X_h}) = 0$ and $q(X_h) = 1$, we have $b_1(X_h) = b_2(X_h) = \rho(X_h) = 2$. Furthermore, we have $X_h \cong \mathbf{P}^1 \times E_h$ with $pr_1 = f_h$ and $pr_2 = \varphi_h$. Since each π_j is a purely inseparable finite morphism of degree p by Theorem 3.2, we have that each π_j is given as a quotient map from X_j by a p-closed derivation and this completes the proof.

4. The multiplicity of a wild fiber of a supersingular elliptic Curve

In this section, we consider the case that the elliptic surface has only one multiple fiber.

Lemma 4.1. *Let $f: X \to \mathbf{P}^1$ be an elliptic surface with $\chi(\mathcal{O}_X)=0$. If $f: X \to \mathbf{P}^1$ has only one multiple fiber, then the multiple fiber is a wild fiber and its multiplicity is a power of p.*

Proof. This follows from [1, Corollary 4.2].

For a supersingular elliptic curve E, it is well-known (cf. [6]) that there is a local coordinate system $\{(U_\lambda, \eta_\lambda)\}$ such that $\eta_\lambda = \eta_\mu + b_{\lambda\mu}^p$, $b_{\lambda\mu} \in \Gamma(U_\lambda \cap U_\mu, \mathcal{O}_E)$. In the following, next two lemmas are important.

Lemma 4.2. *Let E be a supersingular elliptic curve in characteristic $p>0$ and let $\{(U_\lambda, \eta_\lambda)\}$ be a local coordinate system of E such that $\eta_\lambda = \eta_\mu + b_{\lambda\mu}^p$, $b_{\lambda\mu} \in \Gamma(U_\lambda \cap U_\mu, \mathcal{O}_E)$. Then, there does not exist a cocycle $\{c_{\lambda\mu}\} \in H^1(E, \mathcal{O}_E)$ such that $c_{\lambda\mu}^p = b_{\lambda\mu}$.*

Proof. First we note that $\{b_{\lambda\mu}\}$ is a cocycle of $H^1(E, \mathcal{O}_E)$. Suppose that there exists a cocycle $\{c_{\lambda\mu}\} \in H^1(E, \mathcal{O}_E)$ as in the assertion. Then the cocycle $\{b_{\lambda\mu}\} = \{0\}$, because the map $F^*: H^1(E, \mathcal{O}_E) \to H^1(E, \mathcal{O}_E)$ induced from the Frobenius action on $\mathcal{O}_E$ is the zero map. Hence, there exists a cochain $\{a_\lambda\} \in C^0(\{U_\lambda\}, \mathcal{O}_E)$ such that $a_\lambda - a_\mu = b_{\lambda\mu}$.

Since $\eta_\lambda = \eta_\mu + b_{\lambda\mu}^p$, we have $\eta_\lambda - a_\lambda^p = \eta_\mu - a_\mu^p$. Then, $\{\eta_\lambda - a_\lambda^p\}$ defines a global section of $\mathcal{O}_E$ over E and it must be constant. Hence, $\{d\eta_\lambda = d(\eta_\lambda - a_\lambda^p)\} = \{0\}$ in $H^0(E, \Omega_E^1)$.

But this contradicts to the assumption that $\{(U_\lambda, \eta_\lambda)\}$ is a local coordinate system of E, because if $\{(U_\lambda, \eta_\lambda)\}$ is a local coordinate system of E, then $\{d\eta_\lambda\}$ defines a non-zero holomorphic 1-form on E.

Lemma 4.3. *Let E, $\{(U_\lambda, \eta_\lambda)\}$ and $\{b_{\lambda\mu}\}$ be as in the previous lemma. Then, there exist a non-zero constant $c \in k^\times$ and a cochain $\{a_\lambda\} \in C^0(\{U_\lambda\}, \mathcal{O}_E)$ such that*

$$db_{\lambda\mu} = \omega_\lambda - \omega_\mu$$

where ω_λ is a holomorphic 1-form on U_λ given by $c\eta_\lambda^{p-1} d\eta_\lambda + da_\lambda$.

Proof. Since E is a supersingular elliptic curve, $\dim H^1(E, d\mathcal{O}_E) = 1$ and it is easy to see that

$$H^1(E, d\mathcal{O}_E) = k \cdot \{\eta_\lambda^{p-1} d\eta_\lambda - \eta_\mu^{p-1} d\eta_\mu\}.$$

We denote $\{\eta_\lambda^{p-1} d\eta_\lambda - \eta_\mu^{p-1} d\eta_\mu\}$ by ξ. Since $\{db_{\lambda\mu}\}$ is also a non-zero element of $H^1(E, d\mathcal{O}_E)$ by Lemma 4.2, there exists a non-zero constant $c \in k^\times$ such that $\{db_{\lambda\mu}\} \equiv c\xi$ in $H^1(E, d\mathcal{O}_E)$. Then, there exists $\{a_\lambda\} \in C^0(\{U_\lambda\}, \mathcal{O}_E)$ such that

$$db_{\lambda\mu}=c\xi+da_\lambda-da_\mu=(c\eta_\lambda^{p-1}d\eta_\lambda+da_\lambda)-(c\eta_\mu^{p-1}d\eta_\mu+da_\mu).$$

Now, we shall prove the following theorem.

Theorem 4.4. *Let $f\colon X\to \mathbf{P}^1$ be an elliptic surface in characteristic $p>0$ with $\chi(\mathcal{O}_X)=0$ and the general fiber being a supersingular elliptic curve. If $f\colon X\to\mathbf{P}^1$ has only one multiple fiber, then the multiplicity of the multiple fiber is equal to p.*

We prove this theorem in the following. To prove the theorem, it is sufficient to show that such surface can not have a multiple fiber of multiplicity $p^\nu(\nu\geq 2)$.

Let $f\colon X\to\mathbf{P}^1$ be an elliptic surface in characteristic $p>0$ *such that* $\chi(\mathcal{O}_X)=0$, the general fiber is a supersingular elliptic curve and f has only one multiple fiber. From Lemma 4.1, the multiple fiber is a multiple fiber of a supersingular elliptic curve with multiplicity p^l for some $l>0$.

Suppose that $l\geq 2$ and let $f^{-1}(\alpha)=p^lS$ be the only one multiple fiber of $f\colon X\to\mathbf{P}^1$. Then, from Theorem 3.2, we have the following diagram:

$$\begin{tikzcd}
E_0 \arrow[r, "F"] & E_1 \arrow[r, "F"] & E_2 \arrow[r, "F"] & \cdots \arrow[r, "F"] & E \\
X_0 \arrow[u, "\varphi_0"] \arrow[d, "f_0"'] \arrow[r, "\pi_0"] & X_1 \arrow[u, "\varphi_1"] \arrow[d, "f_1"'] \arrow[r, "\pi_1"] & X_2 \arrow[u, "\varphi_2"] \arrow[d, "f_2"'] \arrow[r, "\pi_2"] & \cdots \arrow[r] & X_n=X \arrow[u, "\varphi"] \arrow[d, "f"'] \\
\mathbf{P}^1 \arrow[r, "F"] & \mathbf{P}^1 \arrow[r, "F"] & \mathbf{P}^1 \arrow[r, "F"] & \cdots \arrow[r, "F"] & \mathbf{P}^1
\end{tikzcd}$$

such that

1. F are relative Frobenius maps,
2. $f_j\colon X_j\to\mathbf{P}^1$ are elliptic surfaces satisfying the condition $(*)$,
3. for the point $\alpha^{(j)}\in\mathbf{P}^1$, $j\geq 1$ such that $F(\alpha^{(j)})=\alpha^{(j+1)}$ and $\alpha^{(n)}=\alpha$, $f_j^{-1}(\alpha^{(j)})$ is a multiple fiber of multiplicity
$$m_j=\begin{cases} p^j & j=1,2,\cdots,l \\ p^l & j=l+1,l+2,\cdots,n,\end{cases}$$
4. φ_j are Albanese maps,
5. π_j are purely inseparable finite morphism of degree p,
6. $X_0\cong\mathbf{P}^1\times E_0$ and $f_0=\mathrm{pr}_1$ and $\varphi_0=\mathrm{pr}_2$.

Looking at the top part of the above diagram, we have the following diagram:

$$\begin{tikzcd}
S_0=E_0 \arrow[r, "\text{inclusion}"] \arrow[d] & \mathbf{P}^1\times E_0 \arrow[r] \arrow[d, "\pi_0"] & E_0 \arrow[d] \\
S_1 \arrow[r, "\text{inclusion}"] \arrow[d] & X_1 \arrow[r, "\varphi_1"] \arrow[d, "\pi_1"] & E_1 \arrow[d] \\
S_2 \arrow[r, "\text{inclusion}"] & X_2 \arrow[r, "\varphi_2"] & E_2
\end{tikzcd}$$

where S_j, $j=1, 2$, is the support of the multiple fiber of f_j and such that

1. F are relative Frobenius maps,
2. $f_j: X_j \to \mathbf{P}^1$ are elliptic surfaces satisfying the condition $(*)$,
3. $f_j^{-1}(\alpha^{(j)})=m_jS_j$ where $\alpha^{(j)}$ is the point such that $F(\alpha^{(j)})=\alpha^{(j+1)}$ and $m_j=p^j$, $j=1, 2$,
4. φ_j are Albanese maps,
5. π_j are purely inseparable finite morphism of degree p,
6. $X_0\cong\mathbf{P}^1\times E_0$ and $f_0=\mathrm{pr}_1$ and $\varphi_0=\mathrm{pr}_2$.

Since π_j, $j=1,2$, are purely inseparable finite morphisms of degree p and X_j are nonsingular, each π_j is obtained as a quotient morphism by a p-closed rational vector field on X_j with only divisorial singularities.

For $\pi_0: X_0 \to X_1$, we have the following lemma.

Lemma 4.5. *Let D_0 be a p-closed rational vector field which gives the quotient map $\pi_0:\mathbf{P}^1\times E_0 \to X_1$. Then, for a suitable choice of a local coordinate system, D_0 is written in the form*:

$$D_0=h\left(\frac{\partial}{\partial y}+f(y)\delta\right)=h\left(t^2\frac{\partial}{\partial t}+f\left(\frac{1}{t}\right)\delta\right)$$

where $\mathbf{P}^1=\mathrm{Spec}k[y]\cup\mathrm{Spec}k[t]$, $yt=1$ on $\mathrm{Spec}k[y]\cap\mathrm{Spec}k[t]$, δ is a p-closed additive regular vector field on E_0, $f\in k[y]$ and h is a rational function on $\mathbf{P}^1\times E_0$.

Proof. Take a coordinate t to satisfy that the fiber over $\{t=0\}$ is the only one multiple fiber. Since $D_0 \sim F\partial/\partial y+\delta$ where F is a rational function of $\mathbf{P}^1\times E$, D_0 is written in the form

$$D_0=h\left(F\frac{\partial}{\partial y}+\delta\right) \quad \text{on } \mathrm{Spec}k[y]\times E,$$

$$=h\left(Ft^2\frac{\partial}{\partial t}+\delta\right) \quad \text{on } \mathrm{Spec}k[t]\times E$$

for some rational function h on $\mathbf{P}^1 \times E$. Since $X_1 = X_0^{D_0}$ is nonsingular, $\langle D_0 \rangle = 0$. From this fact and f_1 has a multiple fiber over $\{t=0\}$, we have that $((t)_0.(F)_\infty) = 0$. Then, $(F)_\infty$ consists of fibers of $\mathrm{pr}_1 : \mathbf{P}^1 \times E \to \mathbf{P}^1$ and also (F) does. Hence, $F \in k(\mathbf{P}^1)$.

Since $f_1 : X_1 \to \mathbf{P}^1$ has only one multiple fiber at $t=0$, F has no zero in Spec $k[y]$. Then, $f := F^{-1} \in k[y]$ and this completes the proof.

From this lemma, we can compute the local coordinate ring of X_1.

Let $\{(U_{0,\lambda}, \eta_{0,\lambda})\}$ be a local parameter system on E_0 with respect to an affine open covering $\{U_{0,\lambda}\}$ such that

$$\eta_{0,\lambda} - \eta_{0,\mu} = b_{0,\lambda\mu}^p, \ b_{0,\lambda\mu} \in \Gamma(U_{0,\lambda} \cap U_{0,\mu}, \mathcal{O}_{E_0})$$

Then, the local coordinate ring of $(\mathrm{Spec}\, k[y] \times U_{0,\lambda})^{D_0}$ can be discribed as

$$(\Gamma(U_{0,\lambda}, \mathcal{O}_{E_0})[y])^{D_0} = \Gamma(U_{0,\lambda}, \mathcal{O}_{E_0}^p)[y^p, \tilde{H}(y) + \eta_{0,\lambda}]$$

where $H(y)$ is a polynomial and we may assume that $\tilde{H}(y) = y^{n_0}(y-a_1)^{n_1} \cdots (y-a_l)^{n_l}$ and $(p, n_0 + n_1 + \cdots + n_l) = 1$.

From the above notation, we have the following lemma.

Lemma 4.6. *Put $n := n_0 + n_1 + \cdots + n_l$ and let h, q, α, β be integers such that*

$$n + q = hp, \ 0 < q < p, \ \alpha p + \beta q = 1, \ \beta > 0.$$

Then,

$$X_1 = \bigcup_{\lambda \in \Lambda} \{\mathrm{Spec}\, \Gamma(U_{1,\lambda}, \mathcal{O}_{E_1})[x_\lambda, y_1]/(x_\lambda^p - H(y_1) - \eta_{1,\lambda})$$

$$\cup \mathrm{Spec}\, \Gamma(U_{1,\lambda}, \mathcal{O}_{E_1})[s_\lambda, t_1]/(s_\lambda^p - t_1((1-a_1^p t_1)^{n_1} \cdots (1-a_l^p t_1)^{n_l} - t_1^n \eta_{1,\lambda})^\beta)\}$$

where

$$F^* H = \tilde{H}^p, \quad s_\lambda = y_1^{-(\alpha+\beta h)} x_\lambda^\beta, \quad t_1^{\ 1} = y_1^{-1}, \quad y_1 = y_0^p, \quad t_1 = t_0^p, \quad \eta_{1,\lambda} = \eta_{0,\lambda}^p.$$

Furthermore,

$$H^0(X_1, \Omega_{X_1}^1) = f_1^* H^0\left(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}\left(\sum m_i[\beta_i]\right)\right) d\eta$$

where $\partial H(y_1)/\partial y_1 = (y_1 - \beta_1)^{m_1}(y_1 - \beta_2)^{m_2} \cdots (y_1 - \beta_r)^{m_r}$, $\beta_1, \beta_2, \cdots, \beta_r \in k$ and $d\eta$ is a regular 1-form on E_1 given by $d\eta = d\eta_{1,\lambda}$ on $U_{1,\lambda}$.

Proof. Let Y_1 be a variety defined by

$$Y_1 := \bigcup_{\lambda \in \Lambda} \{\mathrm{Spec}\, \Gamma(U_{1,\lambda}, \mathcal{O}_{E_1})[x_\lambda, y_1]/(x_\lambda^p - H(y_1) - \eta_{1,\lambda}) \cup$$

$$\mathrm{Spec}\, \Gamma(U_{1,\lambda}, \mathcal{O}_{E_1})[z_\lambda, t_1]/(z_\lambda^p - t_1^q(1-a_1^p t_1)^{n_1} \cdots (1-a_l^p t_1)^{n_l} - t_1^{hp}\eta_{1,\lambda})\}$$

where $x_\lambda = t_1^{-h} z_\lambda$ and $y_1 = t_1^{-1}$. Then, it is easy to see that X_0 is a purely inseparable covering of degree p of Y_1 with $k(Y_1) = k(X_0)^{D_0}$ and X_1 is the normalization of Y_1.

For the rest of the assertion, we can show in the following way. For the support S_1 of $f_1^{-1}(\{t_1 = 0\})$, $\mathcal{O}_{X_1}(f_1^{-1}(\beta_i)) \cong \mathcal{O}_{X_1}(pS_1)$ and from the adjunction formula, we have $\omega_{X_1} = \mathcal{O}_{X_1}((2\pi(F) - 2)S_1)$ where F is the fiber of φ_1 and $\pi(F)$ is its arithmetic genus. From an exact sequence

$$0 \to \varphi_1^* \Omega_{E_1}^1 \otimes \mathcal{O}_{X_1}\left(\sum_{i=1}^{r} m_i f_1^{-1}(\beta_i)\right) \to \Omega_{X_1}^1 \to \omega_{X_1/E_1} \otimes \mathcal{O}_{X_1}\left(-\sum_{i=1}^{r} m_i f_1^{-1}(\beta_i)\right) \to 0,$$

we have

$$0 \to \mathcal{O}_{X_1}\left(\sum_{i=1}^{r} m_i f_1^{-1}(\beta_i)\right) \to \Omega_{X_1}^1 \to \mathcal{O}_{X_1}((2\pi(F) - 2 - (n-1)p)S_1) \to 0.$$

Since an inequality $\pi(F) \leq (p-1)(n-1)/2$ holds, we have $2\pi(F) - 2 - (n-1)p \leq -n-1 < 0$ and

$$H^0(X_1, \Omega_{X_1}^1) \cong H^0\left(X_1, \mathcal{O}_{X_1}\left(\sum_{i=1}^{r} m_i f_1^{-1}(\beta_i)\right)\right) \cong H^0\left(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}\left(\sum_{i=1}^{r} m_i [\beta_i]\right)\right).$$

Corollary 4.7. *For an integer $m \leq n$, we have*

$$H^0(X_1, \Omega_{X_1}^1(mS_1)) = H^0\left(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}\left(\sum_{i=1}^{r} m_i [\beta_i] + K[\infty]\right)\right)$$

where K is the largest integer satisfying $K \leq m/p$ and $\infty = \{t_1 = 0\}$.

For $\pi_1 : X_1 \to X_2$, it is also obtained as a quotient morphism by a p-closed rational vector field on X_1.

Since X_1 does not depend on the choice of α, β, we may assume that $\alpha + \beta h \equiv 0$ mod p and put $\alpha + \beta h = \gamma p$, $\gamma \in \mathbf{Z}$. From the above lemma, we have the following relations.

$$\frac{\partial}{\partial x_\lambda} = \beta_{\lambda\mu} y_1^{-\gamma p} x_\lambda^{\beta - 1} \frac{\partial}{\partial s_\lambda} = \beta s^{n+1} (t_1^n H(t_1^{-1}) - t_1^n \eta_{1,\lambda})^{\alpha(\beta - 1)} \frac{\partial}{\partial s_\lambda}$$

$$\frac{\partial}{\partial y_1} = -H'(y_1) \frac{\partial}{\partial \eta_{1,\lambda}} = -H'(t_1^{-1}) \frac{\partial}{\partial \eta_{1,\lambda}}$$

$$\frac{\partial}{\partial x_\lambda} = \frac{\partial}{\partial x_\mu}$$

$$\frac{\partial}{\partial y_1} = \frac{\partial b_{1,\lambda\mu}}{\partial y_1} \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_1}$$

where $H'=\partial H/\partial y_1$ and $b_{1,\lambda\mu}=b_{0,\lambda\mu}^{p}$. Note that $(t_1^n H(t_1^{-1})-t_1^n\eta_{1,\lambda})^{\alpha(\beta-1)}$ is a unit element of $k[[s_1, \eta_{1,\lambda}]]$.

Now, we consider the morphism $\pi_{D_1}: X_1 \to X_2$. For a fixed λ_0, D_1 is equivalent to the following form

$$D_1 \sim F\frac{\partial}{\partial x_{\lambda_0}}+\frac{\partial}{\partial y_1}$$

for some $F\in k(X_1)$. We fix this expression in the following.

For an element Ψ of a ring of formal power series $k[[x_1, x_2]]$ and a fixed $i=1$, 2, we say Ψ is integrable on x_i if there exists $\tilde{\Psi}\in k[[x_1, x_2]]$ such that $\partial\tilde{\Psi}/\partial x_i=\Psi$. The integrability on x_i is equivalent to the condition that Ψ has no such term of the form $x_i^a x_j^b$, $a\equiv -1 \bmod p$.

Lemma 4.8. *Let D_1, F be as above and $\omega_\lambda = c\eta_{1,\lambda}^{p-1}d\eta_{1,\lambda}+da_\lambda$ as in Lemma 4.3. Then, $\{Fdy_1-db_{1,\lambda_0\lambda}-\omega_\lambda\}$ gives an element of $H^0(X_1, \Omega^1_{X_1}(nS_1))$. Moreover, F is integrable on y_1 in $k[[x_{\lambda_0}, y_1]]$, that is, the formal power expansion of F with x_{λ_0}, y_1 does not contain such terms of the form $x^i y^j$, $j\equiv -1$ mod p.*

Proof. From the above relation, on each neighborhood, D_1 is written in the following form.

$$\begin{aligned}
D_1 &= F\frac{\partial}{\partial x_{\lambda_0}}+\frac{\partial}{\partial y_1}\\
&=\left(F-\frac{\partial b_{1,\lambda_0\lambda}}{\partial y_1}\right)\frac{\partial}{\partial x_\lambda}+\frac{\partial}{\partial y_1}\\
&\sim\left(F-\frac{\partial b_{1,\lambda_0\lambda}}{\partial y_1}\right)H'(t_1^{-1})^{-1}\beta s_\lambda^{n+1}(t_1^n H(t_1^{-1})-t_1^n\eta_\lambda)^{\alpha(\beta-1)}\frac{\partial}{\partial s_\lambda}-\frac{\partial}{\partial \eta_\lambda}\\
&\sim \beta s_{\lambda_0}^{n+1}(t_1^n H(t_1^{-1})-t_1^n\eta_{\lambda_0})^{\alpha(\beta-1)}FH'(t_1^{-1})^{-1}\frac{\partial}{\partial s_{\lambda_0}}-\frac{\partial}{\partial \eta_{\lambda_0}}
\end{aligned}$$

Since X_2 has only one multiple fiber over $t_1=0$, F has no pole along fibers of f_1 over Spec $k[y_1]$. Then, since D_1 has only divisorial singularities on X_1, we have that poles of F can appear only along S_1 or fibers of φ_1 outside $\varphi_1^{-1}(U_{1,\lambda_0})$. Moreover, we have that poles of $(F-\partial b_{1,\lambda_0\lambda}/\partial y_1)H'^{-1}$ can appear only along S_1 with order at most n on $\varphi_1^{-1}(U_{1,\lambda})$, for any λ.

Put $\xi_\lambda := Fdy_1-db_{1,\lambda_0\lambda}$ on $\varphi_1^{-1}(U_{1,\lambda})$ for each λ. Then, they gives an element of $C^0(\{U_{1,\lambda}\}, \Omega^1_{X_1}(nS_1))$ and satisfy the relation $\xi_{\lambda_0}-\xi_\lambda=db_{1,\lambda_0\lambda}$. Hence, from Lemma 4.3, $\{Fdy_1-db_{1,\lambda_0\lambda}-\omega_\lambda\}_{\lambda\in\Lambda}$ gives an element of $H^0(X_1, \Omega^1_{X_1}(nS_1))$.

For the rest of the assersion, it follows from the p-closedness of D_1.

We denote the element of $H^0(X_1, \Omega^1_{X_1}(nS_1))$ given by $\{Fdy_1-db_{1,\lambda_0\lambda}-\omega_\lambda\}$ in the proof of previous lemma by ξ. From Corollary 4.7, there exists an element ψ

of $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(\Sigma m_i[\beta_i]+K[\infty]))$ where K is the largest integer satisfying $K \leq n/p$ such that

$$\xi|_{\varphi_1^{-1}(U_{1,\lambda})} = \psi d\eta_{1,\lambda}$$

for all λ. Then, for λ_0, we have

$$Fdy_1 = \omega_{\lambda_0} + \psi d\eta_{1,\lambda_0}$$

In the following, we only consider about $Fdy_1 = \omega_{\lambda_0} + \psi d\eta_{1,\lambda_0}$ and we omitt indices for the convenience. Expressing $H(y) = \Sigma_{i=1}^n h_i y^i$, $(h_n \neq 0)$, from $\eta = x^p - \mathrm{H(y)}$, we have

$$\eta^{p-1} d\eta = \left(x^p - \sum h_i y^i\right)^{p-1} \left(\sum i h_i y^{i-1}\right) dy = (n h_n^p y^{np-1} + (\text{lower terms on } y))dy$$

and from Lemma 4.3

$$\omega = c\eta^{p-1} d\eta + da = (nch_n^p \, y^{np-1} + (\text{lower terms on } y))dy + da.$$

From $H'(y) = \Pi(y-\beta_i)^{m_i}$ and Corollary 4.7, we have

$$H^0(X_1, \Omega^1_{X_1}(nS_1)) = \sum_{0 \leq j_i \leq m_i, 1 \leq i \leq r} k \cdot \frac{H'}{(y-\beta_i)^{j_i}} dy + \sum_{m=0}^{K} k \cdot H' y^m dy.$$

where K is the largest integer satisfying $K \leq n/p$. Then, for $\psi d\eta = \psi H' dy \in H^0(X_1, \Omega^1_{X_1})$, the degree of $\psi H'$ with respect to y is at most $n-1+n/p$. Hence, we have

$$\omega + \psi d\eta = (nch_n^p y^{np-1} + (\text{lower terms on } y))dy + da.$$

But this contradicts to the fact that $Fdy = \omega + \psi d\eta$ and F is integrable on y. Thus, we complete the proof of Theorem 4.4.

Corollary 4.9. *Let $f: X \to \mathbf{P}^1$ be as in Theorem* 4.4. *If f has only one multiple fiber, then we have*:

$$h^0(X, \Omega^1_X) = h^2(X, \Omega^1_X) = 1 + [\deg \varphi^* \Omega^1_E / p],$$

$$h^1(X, \Omega^1_X) = 2[\deg \varphi^* \Omega^1_E / p] + 2,$$

$$h^1(X, \mathcal{O}_X) = h^1(X, \Omega^2_X) = 2 + [(2\pi(F) - 2)/p],$$

$$h^2(X, \mathcal{O}_X) = h^0(X, \Omega^2_X) = 1 + [(2\pi(F) - 2)/p].$$

and moreover, $h^0(X, \Theta_X) = [\deg \varphi^* \Omega^1_E / p] + 1 + [2\pi(F) - 2)/p]$ *where* $\varphi: X \to E$ *is an Albanese map,* $\pi(F)$ *is the arithmetic genus of the general fiber of* φ *and the bracket* [ ] *denotes the Gauss symbol.*

Proof. By Theorem 4.4, the multiplicity of the only one multiple fiber is p. Then the local coordinate ring of X is given as Lemma 4.6. As we see in the proof of Lemma 4.6, we have

$$0 \to \mathcal{O}_X((\deg\varphi^*\Omega_E^1)S) \to \Omega_X^1 \to \mathcal{O}_X((2\pi(F)-2-\deg\varphi^*\Omega_E^1)S) \to 0$$

where S is the support of the only one multiple fiber of f. Then we have

$$\chi(X,\Omega_X^1) = \chi(X,\mathcal{O}_X((\deg\varphi^*\Omega_E^1)S)) + \chi(X,\mathcal{O}_X((2\pi(F)-2-\deg\varphi^*\Omega_E^1)S) = 0$$

by Riemann-Roch theorem. From this fact and $\chi(X,\mathcal{O}_X)=0$ and Serre duality, we can compute $h^i(\Omega_X^j)$'s. Moreover, from $\Theta_X = \Omega_X^1 \otimes \omega_X^{-1}$ and the above exact sequence, we can compute $h^0(\Theta_X)$.

Remark 4.10. Takeda [6][7] computed the Hodge numbers of an example of a false hyperelliptic surface. Corollary 4.9 gives a more general formula including Takeda's result. Note that surfaces considered in Corollary 4.9 are not false hyperelliptic surfaces in general because the general fiber of an Albanese map can have many cusps.

Department of Mathematics and Information Sciences,
College of Integrated Arts and Sciences,
Osaka Prefecture University,
e-mail: kawazoe@mi.cias.osakafu-u.ac.jp

References

[1] T. Katsura and K. Ueno, On Elliptic Surfaces in Characteristic p, Math. Ann., **272** (1985), 291-330.

[2] T. Katsura and K. Ueno, Multiple Singular Fibres of Type $\mathbf{G}_a$ of Elliptic Surfaces in Characteristic p, In: Nagata, M. (ed.) Algebraic and Topological Theories—To the Memory of Dr. Miyata, pp. 405-429. Tokyo: Kinokuniya, 1986.

[3] A. Néron, Modèles Minimaux des Espaces Principaux Homogènes sur les Courbes Elliptiques, In: Proceeding of Conference on Local Fields, NUFFIC Summer School, Driebergen 1966, pp. 66-77. Berlin, Heidelberg, New York: Springer, 1967.

[4] M. Raynaud, Surfaces elliptiques et quasi-elliptiques, manuscript (1976).

[5] A. N. Rudakov and I. R. Šafarevič, Inseparable Morphism of Algebraic Surfaces, Math. USSR Izvestija, **10** (1976), 1205-1237.

[6] Y. Takeda, False Hyperelliptic Surfaces with Section, Math. Nachr., **167** (1994), 313-329.

[7] Y. Takeda, Errata to the paper: "False hyperelliptic surfaces with section" [Math. Nachr., 167 (1994), 313-329], Math. Nachr., **182** (1996), 329.

Added in Proof:

1. In Theorem 3.2, the assumption $m_i \leq m_j$ for $i<j$ has to be corrected to the following: $r_i \leq r_j$ for $i<j$ where $m_i = n_i p^{r_i}$ and $(n_i, p) = 1$.
2. In Corollary 4.9, we need the following assumption: an Albanese map $\varphi: X \to E = \mathrm{Alb}(X)$ has a section.