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Self homotopy group of the exceptional Lie group G_2

By

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1. Introduction

Let G be a connected Lie group and $\mu: G \times G \to G$ the multiplication of G. For any space A with a base point, the based homotopy set $[A, G]$ becomes a group with respect to the binary operation $\mu_*: [A, G] \times [A, G] = [A, G \times G] \to [A, G]$. Even if A is a simple space such as the sphere, it is difficult to calculate the group $[A, G]$. A general result was given by Whitehead (p. 464 of [10]):

(1.1) $$\mathrm{nil}[A, G] \leq \mathrm{cat}\, A,$$

where nil and cat denote the nilpotency class and the Lusternik-Schnirelmann category with $\mathrm{cat}\{*\} = 0$, respectively. In [5], we determined the group structure of $[G, G]$ and proved $\mathrm{nil}[G, G] = 2$ when G is $SU(3)$ or $Sp(2)$. We want to study $\mathrm{nil}[G, G]$ for other G's. Though we have very few results, it seems reasonable to set the following:

Conjecture 1.1 If G is simple, then $\mathrm{nil}[G, G] \geq \mathrm{rank} G$.

A weaker one is

Conjecture 1.2. If G is simple and $\mathrm{rank} G \geq 2$, then $\mathrm{nil}[G, G] \geq 2$, that is, $[G, G]$ is not commutative.

Let G_2 be the exceptional Lie group of rank 2. Then the purpose of this note is to prove the following which supports 1.1.

Theorem 1.3. $\mathrm{nil}[G_2, G_2] = 3$.

Two conjectures are false in general without the assumption of simpleness of G.

Example 1.4. (1). $\mathrm{nil}[S^3 \times S^1, S^3 \times S^1] = 1$ and $\mathrm{nil}[U(2), U(2)] = 2$. Notice that $S^3 \times S^1$ and $U(2)$ are homeomorphic but not isomorphic.

(2). If $G = S^3 \times \cdots \times S^3$ (n times), then $\mathrm{rank} G = n$ and $\mathrm{nil}[G, G]$ equals 3 if $n \geq 3$

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and n if $n \leq 2$.

In §2, we indicate notation, recall some results from [3], [4], [6], [7], and state Theorem 2.2 which contains Theorem 1.3. We prove Theorem 2.2 in §3 and Example 1.4 in §4.

2. Notation and a main theorem

We do not distinguish notationally between a map and its homotopy class. Even for non-commutative group, the multiplication is denoted by $+$. For elements x, y of a group, we write $[x,y]=x+y-x-y$, the commutator. We say that a group Γ has *nilpotency class n* and write $\mathrm{nil}\Gamma=n$ if the iterated n-th commutator $[x_1,[x_2,\cdots[x_{n-1},x_n]\cdots]]$ is non zero for some n elements $x_1,\cdots,x_n$ of Γ and every iterated $(n+1)$-th commutator is zero. For a space A with a base point, $d_n: A \to A \wedge \cdots \wedge A$ (n times) denotes the diagonal map. For a topological group G, $c_2: G \wedge G \to G$ denotes the commutator map, $c_2(x \wedge y)=[x,y]$, and $\langle\, ,\, \rangle: \pi_s(G) \times \pi_t(G) \to \pi_{s+t}(G)$ is the Samelson product. For a CW complex X, $X^{(n)}$ denotes the n-skeleton of X.

As is well-known, G_2 has a cell structure:

$$G_2=S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}.$$

Let $i_n: G_2^{(n)} \subset G_2$, $i_{n,n+k}: G_2^{(n)} \to G_2^{(n+k)}$ $(k \geq 0)$ and $\bar{i}_{8,11}: S^8=G_2^{(8)}/G_2^{(6)} \to G_2^{(11)}/G_2^{(6)}$ be the inclusion maps. For $n=5$, 6, 8, 9, 11, 14, let $q_n: G_2^{(n)} \to S^n$ be the quotient map and $\rho_n: S^{n-1} \to G_2^{(n-1)}$ the attaching map of the n-cell. The cohomology structure of G_2 (Théorème 17.2 and 17.3 of [2]) implies that $\rho_5=\Sigma\eta_2$, the suspension of the Hopf map $\eta_2: S^3 \to S^2$, and

$$q_n \circ \rho_{n+1}=2\iota_n \quad \text{for} \quad n=5,\ 8, \tag{2.1}$$

where ι_n is the identity map of S^n. Let $q_{11,6}: G_2^{(11)} \to G_2^{(11)}/G_2^{(6)}$ and $\bar{q}_{11,6}: G_2^{(11)}/G_2^{(6)} \to S^{11}$ be the quotient maps. We have fibrations

$$SU(3) \xrightarrow{j} G_2 \xrightarrow{p} S^6, \qquad S^3 \xrightarrow{i'_3} SU(3) \xrightarrow{p} S^5.$$

Let $\nu_4 \in \pi_7(S^4)$ and $\mu' \in \pi_{14}(S^3)$ be the elements of [9] and set $\eta_n=\Sigma^{n-2}\eta_2 \in \pi_{n+1}(S^n)$ for $n \geq 2$ and $\nu_n=\Sigma^{n-4}\nu_4 \in \pi_{n+3}(S^n)$ for $n \geq 4$. Write $\eta_n^2=\eta_n \circ \eta_{n+1}$ and $\nu_n^2=\nu_n \circ \nu_{n+3}$.

We need

Proposition 2.1. (1)([6]). *$\pi_3(SU(3))=Z\{i'_3\}$, $\pi_{11}(SU(3))=Z_4\{[\nu_5^2]\}$ and $\pi_{14}(SU(3))=Z_4\{[\nu_5^2]\circ\nu_{11}\}\oplus Z_2\{i'_{3*}\mu'\}\oplus Z_{21}$, where $p_*[\nu_5^2]=\nu_5^2$.*

(2)([4]). *$\pi_4(G_2)=\pi_5(G_2)=\pi_7(G_2)=\pi_{10}(G_2)=\pi_{12}(G_2)=\pi_{13}(G_2)=0$, $\pi_3(G_2)=Z\{i_3\}$, $\pi_6(G_2)=Z_3$, $\pi_8(G_2)=Z_2\{[\eta_6^2]\}$, $\pi_9(G_2)=Z_2\{[\eta_6^2]\circ\eta_8\}\oplus Z_3$, $\pi_{11}(G_2)=Z\{\gamma\}\oplus Z_2\{j_*[\nu_5^2]\}$ and $\pi_{14}(G_2)=Z_8\oplus Z_2\{j_*[\nu_5^2]\circ\nu_{11}\}\oplus Z_{21}$, where $p_*[\eta_6^2]=\eta_6^2$ and $i_{3*}\mu'=j_*i'_{3*}\mu'=0$.*

(3) (Lemma 1 of [3]). *$\langle i_3, [\eta_6^2]\rangle=j_*[\nu_5^2]$ and $\langle i_3, j_*[\nu_5^2]\rangle=j_*[\nu_5^2]\circ\nu_{11}$.*

(4) (Lemma 5.8 of [7]). $[G_2^{(11)}/G_2^{(6)}, G_2]=\boldsymbol{Z}\{\gamma'\}\oplus\boldsymbol{Z}_2\{\bar{q}_{11,6}^* j_*[\nu_5^2]\}$, $\bar{q}_{11,6}^*\gamma=4\gamma'$ *and* $\bar{i}_{8,11}^*\gamma'=[\eta_6^2]$.

Given integers $m\geq 1$ and n, we denote by $\Psi(x_1, x_2, x_3; m, n)$ or simply by $\Psi(m, n)$ the group with generators x_1, x_2, x_3 and relations

$$mx_3=[x_1, x_3]=[x_2, x_3]=0, \quad [x_1, x_2]=nx_3.$$

Our main theorem is

Theorem 2.2. (1). *There exists a central extension of groups*:

$$0\rightarrow\pi_{14}(G_2)\xrightarrow{q_{14}^*}[G_2, G_2]\xrightarrow{i_{11}^*}[G_2^{(11)}, G_2]\rightarrow 0.$$

(2). $[G_2^{(11)}, G_2]=\Psi(i_{11}, q_{11,6}^*\gamma', q_{11}^* j_*[\nu_5^2]; 2,1)$.
(3). *Let* $\alpha\in[G_2, G_2]$ *be an element such that* $i_{11}^*\alpha=q_{11,6}^*\gamma'$. *Then* $[\alpha,[1,\alpha]]=0$ *and* $[1,[1,\alpha]]=q_{14}^*(j_*[\nu_5^2]\circ\nu_{11})\neq 0$ *so that* $\mathrm{nil}[G_2, G_2]=3$, *where* 1 *denotes the identity map of* G_2.
(4). *There exists* $x_0\in\pi_{14}(G_2)$ *such that* $2[1,\alpha]=2q_{14}^*(x_0)$ *and* $\langle i_3,\gamma\rangle=\pm 4x_0$.

We can show that the order of $\langle i_3,\gamma\rangle$ is odd. We omit the details.

Problem 2.3. Determine the group structure of $[G_2, G_2]$ completely.

3. Proof of Theorem 2.2

Theorem 2.2 follows from 3.4, 3.6 and 3.7 below.

Lemma 3.1. (1). $[G_2^{(6)}, G_2]=\boldsymbol{Z}\{i_6\}$.
(2). $[\Sigma G_2^{(6)}, G_2]=0$.
(3). *The following is an exact sequence of groups*:

$$0\rightarrow[G_2^{(11)}/G_2^{(6)}, G_2]\xrightarrow{q_{11,6}^*}[G_2^{(11)}, G_2]\xrightarrow{i_{6,11}^*}[G_2^{(6)}, G_2]\rightarrow 0,$$

and $[G_2^{(11)}, G_2]$ *is generated by three elements* $i_{11}, q_{11,6}^*\gamma', q_{11}^* j_*[\nu_5^2]$ *of which the last element is central.*

Proof. Since $G_2^{(5)}=\Sigma(S^2\cup_{\eta_2}e^4)$, it follows that $[G_2^{(5)}, G_2]\cong\pi_3(G_2)$ from 2.1 (2). Consider the following exact sequence of groups:

$$\pi_7(G_2)\longrightarrow[\Sigma G_2^{(6)}, G_2]\longrightarrow[\Sigma G_2^{(5)}, G_2]\xrightarrow{\Sigma\rho_6^*}$$

$$\pi_6(G_2)\xrightarrow[q_6^*]{}[G_2^{(6)}, G_2]\xrightarrow[i_{5,6}^*]{}[G_2^{(5)}, G_2]\longrightarrow\pi_5(G_2)$$

By (2.1) and 2.1(2), $(\Sigma\rho_6)^*\circ(\Sigma q_5)^*:\pi_6(G_2)\rightarrow\pi_6(G_2)$ is an isomorphism. Hence

$(\Sigma\rho_6)^*: [\Sigma G_2^{(5)}, G_2] \to \pi_6(G_2)$ is surjective, and $(\Sigma q_5)^*: \pi_6(G_2) \to [\Sigma G_2^{(5)}, G_2]$ is injective so that it is an isomorphism, since $\pi_4(G_2)=0$ by 2.1(2). By the above exact sequence, $i_{5,6}^*: [G_2^{(6)}, G_2] \cong [G_2^{(5)}, G_2]$ and $[\Sigma G_2^{(6)}, G_2]=0$. Hence we obtain (1) and (2) from which the sequence of (3) follows. By p. 465 of [10], $q_{11}^* j_*[\nu_5^2]$ is central.

The following is easy and well-known.

Lemma 3.2. *In any group,* $[x, y+z]=[x,y]+[x,z]+[[z,x],y]$.

Lemma 3.3. (1). $[i_{11}, q_{11,6}^*\gamma']=q_{11}^* j_*[\nu_5^2]$.
(2). $2q_{11,6}^*\gamma'$ *is central in* $[G_2^{(11)}, G_2]$.

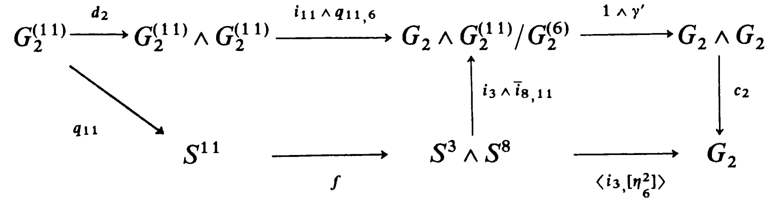
Proof. Write $x=q_{11,6}^*\gamma'$ and $y=q_{11}^* j_*[\nu_5^2]$. Let k be any integer. Then $[i_{11}, kx] \in \mathrm{Image}(q_{11,6}^*)$, since $i_{6,11}^*[i_{11}, kx]=0$. Hence there exist $m_k \in \boldsymbol{Z}$ and $n_k \in \{0,1\}$ such that $[i_{11}, kx]=m_k x+n_k y$. We have

(3.1) $[i_{11}, 2x]=2[i_{11}, x]+[[x, i_{11}], x]$ (by 3.2)

(3.2) $\qquad =2[i_{11}, x]$ (since $\mathrm{Im}(q_{11,6}^*)$ is commutative).

Inductively, we have $[i_{11}, 3x]=3[i_{11}, x]$ and $[i_{11}, 4x]=4[i_{11}, x]=4m_1 x$. Since $4x=q_{11}^*\gamma$ by 2.1(4) so that $4x$ is central by p. 465 of [10], it follows that $4m_1 x=0$ so that $m_1=0$. Therefore $[i_{11}, x]=n_1 y$ and $[i_{11}, 2x]=0$. Hence $2x$ is central in $[G_2^{(11)}, G_2]$.

The rest we must prove is the equality: $n_1=1$. There exists a map $f: S^{11} \to S^3 \wedge S^8$ which makes the following diagram commutative up to homotopy:



By using cohomology of $\boldsymbol{Z}_2$-coefficients, it follows that the degree of f is odd so that $[i_{11}, x]=c_2 \circ (1 \wedge \gamma') \circ (i_{11} \wedge q_{11,6}) \circ d=q_{11}^* \langle i_3, [\eta_6^2] \rangle=q_{11}^* j_*[\nu_5^2]$. Hence $n_1=1$ as desired.

By 2.1(4), 3.1 and 3.3(1), we have

Proposition 3.4. $[G_2^{(11)}, G_2]=\Psi(i_{11}, q_{11,6}^*\gamma', q_{11}^* j_*[\nu_5^2]; 2, 1)$ *which is of nilpotency class two.*

Lemma 3.5. $\Sigma\rho_{14}^*=0: [\Sigma G_2^{(11)}, G_2] \to \pi_{14}(G_2)$.

Proof. Write $\bar{\rho}_{14}=q_{11,6} \circ \rho_{14}: S^{13} \to G_2^{(11)}/G_2^{(6)}$. Since ρ_{14} is stably null- homotopic and $\bar{\rho}_{14}$ is in stable range, $\bar{\rho}_{14}$ is null-homotopic. It follows that

$\Sigma\rho_{14}^*\circ\Sigma q_{11,6}^*=\Sigma\bar{\rho}_{14}^*=0$ and $\Sigma\rho_{14}^*=0$, since $\Sigma q_{11,6}^*:[\Sigma G_2^{(11)}/G_2^{(6)},G_2]\to[\Sigma G_2^{(11)},G_2]$ is surjective by 3.1(2).

By applying $[-,G_2]$ to the cofibre sequence

$$S^{13}\xrightarrow{\rho_{14}}G_2^{(11)}\xrightarrow{i_{11}}G_2\xrightarrow{q_{14}}S^{14}\xrightarrow{\Sigma\rho_{14}}\Sigma G_2^{(11)}$$

we have

Proposition 3.6. *The following is a central extension of groups*:

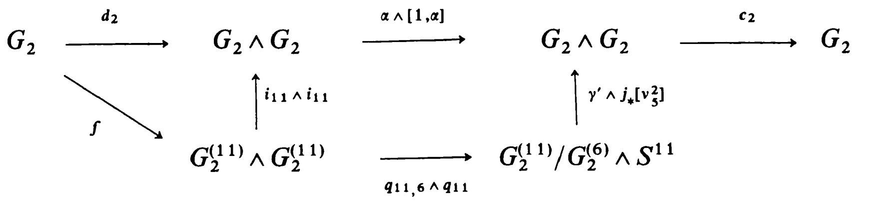
$$0\longrightarrow\pi_{14}(G_2)\xrightarrow{q_{14}^*}[G_2,G_2]\xrightarrow{i_{11}^*}[G_2^{(11)},G_2]\longrightarrow 0.$$

Proof. The exact sequence follows from 2.1(2) and 3.5. It is a central extension by p.465 of [10].

Let $\alpha\in[G_2,G_2]$ be an element such that $i_{11}^*(\alpha)=q_{11,6}^*\gamma'$. Then $[G_2,G_2]$ is generated by $\mathrm{Im}(q_{14}^*)$, 1 and α.

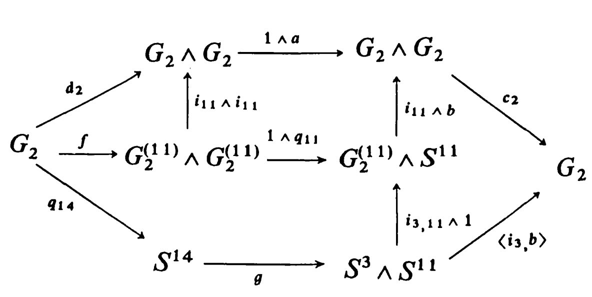
Lemma 3.7. (1). $[\alpha,[1,\alpha]]=0$.
(2). $[1,[1,\alpha]]=q_{14}^*(j_*[\nu_5^2]\circ\nu_{11})$.
(3). *There exists* $x_0\in\pi_{14}(G_2)$ *such that* $\langle i_3,\gamma\rangle=\pm 4x_0$ *and* $2[1,\alpha]=2q_{14}^*(x_0)$.

Proof. There exists a map $f:G_2\to G_2^{(11)}\wedge G_2^{(11)}$ which makes the following diagram commutative up to homotopy:



Since $(q_{11,6}\wedge q_{11})\circ f=0$, we have $[\alpha,[1,\alpha]]=c_2\circ(\gamma'\wedge j_*[\nu_5^2])\circ(q_{11,6}\wedge q_{11})\circ f=0$. This proves (1).

Let the pair (a,b) be $([1,\alpha],j_*[\nu_5^2])$ or $(4\alpha,\gamma)$. There exists a map $g:S^{14}\to S^3\wedge S^{11}$ which makes the following diagram commutative up to homotopy:



By using the integral cohomology, we have that g is a homotopy equivalence. Hence $[1,a]=\pm q_{14}^*\langle i_3,b\rangle$, that is, $[1,[1,\alpha]]=q_{14}^*\langle i_3,j_*[v_5^2]\rangle$ and

$$[1,4\alpha]=\pm q_{14}^*\langle i_3,\gamma\rangle. \tag{3.3}$$

By 2.1(3), we then have (2). Since $i_{11}^*[2\alpha,1]=0$ by 3.3(2), it follows that $[2\alpha,1]$ is central and from 3.2 that $[1,4\alpha]=2[1,2\alpha]$ and $[1,2\alpha]=2[1,\alpha]+[[\alpha,1],\alpha]=2[1,\alpha]$ by 3.7(1). Hence $[1,4\alpha]=4[1,\alpha]$ and $4[1,\alpha]=\pm q_{14}^*\langle i_3,\gamma\rangle$ by (3.3). On the other hand, since $q_{11}\circ\rho_{14}=0$, there exists a map $\tilde{q}_{11}:G_2\to S^{11}$ such that $\tilde{q}_{11}\circ i_{11}=q_{11}$. Write $\beta=j\circ[v_5^2]\circ\tilde{q}_{11}:G_2\to G_2$. Then $i_{11}^*\beta=q_{11}^*j_*[v_5^2]$ and β is of order 2 in $[G_2,G_2]$. Since $i_{11}^*[1,\alpha]=i_{11}^*\beta$, there exists $x_0\in\pi_{14}(G_2)$ such that $[1,\alpha]=\beta+q_{14}^*(x_0)$. Hence $2[1,\alpha]=2q_{14}^*(x_0)$ and $4[1,\alpha]=q_{14}^*(4x_0)$. Therefore $\langle i_3,\gamma\rangle=\pm 4x_0$. This has proved (3).

4. Proof of Example 1.4

By Theorem 4.1(1) of [8], $[S^3\times S^1,S^3\times S^1]\cong \boldsymbol{Z}\oplus\boldsymbol{Z}\oplus\boldsymbol{Z}_2$. Let $\theta\in\pi_1(U(2))\cong\boldsymbol{Z}$ and $\alpha\in\pi_3(U(2))\cong\boldsymbol{Z}$ be generators, $p:U(2)\to S^3$ the projection, and $q:U(2)\approx S^3\times S^1=(S^3\vee S^1)\cup_\rho e^4\to S^4$ the quotient map. There exists a map g which makes the following diagram commutative up to homotopy:

$$\begin{array}{ccccc}
U(2) & \xrightarrow{d_2} & U(2)\wedge U(2) & & \\
\| & & \downarrow{\scriptstyle 1\wedge p} & & \\
U(2) & & U(2)\wedge S^3 & \xrightarrow{1\wedge\alpha} & U(2)\wedge U(2) \\
{\scriptstyle q}\downarrow & & \uparrow{\scriptstyle \theta\wedge 1} & & \downarrow{\scriptstyle c_2} \\
S^4 & \xrightarrow[g]{} & S^1\wedge S^3 & \xrightarrow[\langle\theta,\alpha\rangle]{} & U(2)
\end{array}$$

By using integral cohomology, we see that g is a homotopy equivalence so that $[1,\alpha\circ p]=\pm q^*\langle\theta,\alpha\rangle$, where $\langle\theta,\alpha\rangle$ is a generator of $\pi_4(U(2))\cong\boldsymbol{Z}_2$ by [2]. Since the attaching map $\rho:S^3\to S^3\vee S^1$ of the top cell of $U(2)$ is the Whitehead product of ι_3 and ι_1, it follows that $\Sigma\rho$ is null-homotopic so that $q^*:\pi_4(U(2))\to[U(2),U(2)]$ is injective. Hence $[1,\alpha\circ p]\neq 0$ and $\mathrm{nil}[U(2),U(2)]=2$ by (1.1). Then $[U(2),U(2)]=\Psi(2,1)$ from Theorem 4.1(1) of [8]. This completes the proof of Example 1.4(1).

We write $\Pi^n S^3=S^3\times\cdots S^3$ (n times) and $\Lambda^n S^3=S^3\wedge\cdots\wedge S^3$ (n times). We define the iterated commutator map $c_n:\Lambda^n S^3\to S^3$ inductively by $c_n=c_2\circ(1\wedge c_{n-1})$ for $n\geq 3$. Then, given $f_i\in[X,S^3]$ $(1\leq i\leq n)$, we have $[f_1,[f_2,\cdots[f_{n-1},f_n]\cdots]]=c_n\circ(f_1\wedge\cdots\wedge f_n)\circ d_n\in[X,S^3]$. The following is contatined in Theorem B of [1].

Lemma 4.1. *The map $c_4:\Lambda^4S^3\to S^3$ is null-homotopic and so* $\mathrm{nil}[X,S^3]\leq 3$ *for every X.*

Proof. We have $c_4=c_2\circ(1\wedge c_2)\circ(1\wedge 1\wedge c_2)\in\pi_6(S^3)\circ\pi_9(S^6)\circ\pi_{12}(S^9)=0$ by [9].

Hence the results follow.

The following contains Example 1.4(2).

Proposition 4.2. $\mathrm{nil}[\Pi^n S^3, \Pi^n S^3] = \mathrm{nil}[\Pi^n S^3, S^3]$ *and it equals* 3 *or* n *according as* $n \geq 3$ *or* $n \leq 2$.

Proof. The case $n=1$ is trivial. Since $[X, \Pi^n S^3] \cong [X, S^3] \oplus \cdots \oplus [X, S^3]$ (n *times*), $\mathrm{nil}[X, \Pi^n S^3] = \mathrm{nil}[X, S^3]$ for any pointed space X. Hence the case $n=2$ is proved in Proposition 3.1 of [5]. Since the map $p: \Pi^n S^3 \to \Pi^{n-1} S^3$ defined by $p(x_1, \cdots, x_n) = (x_1, \cdots, x_{n-1})$ has a right inverse, $p^*: [\Pi^{n-1} S^3, S^3] \to [\Pi^n S^3, S^3]$ is a monomorphism and so $\mathrm{nil}[\Pi^{n-1} S^3 S^3] \leq \mathrm{nil}[\Pi^n S^3, S^3]$. Thus, by 4.1, it suffices to prove $\mathrm{nil}[\Pi^3 S^3, S^3] \geq 3$.

Write $G = \Pi^3 S^3$. Let $p_i: G \to S^3$ be defined by $p_i(x_1, x_2, x_3) = x_i$ $(i=1,2,3)$. There exists a map g which makes the following diagram commutative up to homotopy:

$$\begin{array}{ccccc} G & \xrightarrow{d_3} & G \wedge G \wedge G & & \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p_1 \wedge p_2 \wedge p_3} & & \\ S^9 & \xrightarrow[g]{} & S^3 \wedge S^3 \wedge S^3 & \xrightarrow[c_3]{} & S^3 \end{array}$$

By using integral cohomology, we see that g is a homotopy equivalence. Hence $[p_1, [p_2, p_3]] = \pm q^* c_3 = \pm q^* \langle \iota_3, \langle \iota_3, \iota_3 \rangle \rangle$. We have a cell-decomposition: $G = (S^3 \vee S^3 \vee S^3) \cup e^6 \cup e^6 \cup e^6 \cup_\rho e^9$. There are exact sequences:

$$[\Sigma G^{(8)}, S^3] \xrightarrow{\Sigma \rho_*} \pi_9(S^3) \xrightarrow{q^*} [G, S^3], \tag{4.1}$$

$$[S^7 \vee S^7 \vee S^7, S^3] \xrightarrow{q^*} [\Sigma G^{(8)}, S^3] \longrightarrow [S^4 \vee S^4 \vee S^4, S^3]$$

Since $\pi_7(S^3) \cong \pi_4(S^3) \cong \boldsymbol{Z}_2$ by [9], $[S^7 \vee S^7 \vee S^7, S^3] \cong [S^4 \vee S^4 \vee S^4, S^3] \cong \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$. Hence $2^2[\Sigma G^{(8)}, S^3] = 0$. On the other hand, as is well-known, $\pi_9(S^3) = \boldsymbol{Z}_3\{\langle \iota_3, \langle \iota_{3}, {}_3 \rangle \rangle\}$ (see [1]). Hence $\Sigma \rho^* = 0$ in (4.1) and the order of $[p_1, [p_2, p_3]]$ is three. Therefore $\mathrm{nil}[G, S] \geq 3$. This completes the proof.

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