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A remark on Baker operations on the elliptic cohomology of finite groups

By

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Introduction

Let G be a finite group and BG be its classifying space. In [18] it is shown that every element of $Ell^{\text{even}}(BG)$ yields a certain p-adic limit of Thompson series via elliptic character. We hope that this fact will shed light on still unknown geometric construction of elliptic cohomology and hence the study of $Ell^*(BG)$ in connection with moonshine phenomena would be important.

In theory of Thompson series we have certain Hecke operators constructed by G. Mason which are related to usual Hecke operators on modular forms (see [11]). The purpose of this note is to prove that the stable operations on elliptic cohomology constructed by A. Baker in [3] act on $Ell^{\text{even}}(BG)$ as Mason's Hecke operators act on Thompson series (Theorem 3.1). The proof of this fact is pretty easy. First we review the construction of elliptic character and Baker operations and describe the composition of these natural transformations (Proposition 1.3). By using this description and a certain explicit formula given in [18] we can easily prove Theorem 3.1 (§3).

1. Elliptic character and Baker operation

We begin by considering a general construction of natural transformations of cohomology theories obtained by Landweber exact functor theorem (Landweber exact cohomologies for short). Let $R_*(?)=R_*\otimes_{MU_*}MU_*(?)$ and $S_*(?)=S_*\otimes_{MU_*}MU_*(?)$ be Landweber exact homologies. Let

$$\Lambda : R_*\otimes_{MU_*}MU_*(MU)\rightarrow S_*$$

be a right MU_*-module map. Then we can define a natural transformation

$$R_*(X)\rightarrow S_*(X)$$

as the composite map

$$R_*\otimes_{MU_*}MU_*(X)\xrightarrow{R_*\otimes\psi_X} R_*\otimes_{MU_*}MU_*(MU)\otimes_{MU_*}MU_*(X)$$

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$$\xrightarrow{\Lambda\otimes MU_*(X)} S_*\otimes_{MU_*}MU_*(X),$$

where

$$\psi_X: MU_*(X)\to MU_*(MU)\otimes_{MU_*}MU_*(X)$$

is the coaction map and we denote a module and the identity map on it by the same notation. By using Spanier-Whitehead duality we get a (not necessarily multiplicative) natural transformation of cohomology theories

$$\Lambda(X): R^*(X)\to S^*(X)$$

on finite CW complexes. Now this natural transformation can be realized as a map of spectra which is unique up to weak homotopy (cf. [17, Theorem 9.24]). (We should remark that the extra condition in Theorem 9.30 in [17] is unnecessary.) Thus we have a (not necessarily unique) extension of $\Lambda(X)$ on any CW complexes which is still denoted by $\Lambda(X)$.

We now apply this general construction to the following two special cases. Let $K^*(?)$ be complex K-theory with coefficient ring $K^*=\mathbf{Z}[t,t^{-1}]$ ($|t|=-2$) and $Ell^*(?)$ be level 1 elliptic cohomology with coefficient ring $Ell^*=\mathbf{Z}[1/6][g_2,g_3,\Delta^{-1}]$ ($\Delta=g_2^3-27g_3^2$, $|g_2|=-8$, $|g_3|=-12$) (see [3]). Then the ring Ell^* can be viewed as the ring of modular forms on $\Gamma(1)=SL_2(\mathbf{Z})$ over $\mathbf{Z}[1/6]$, i.e., the universal ring classifying $\Gamma(1)$-test objects over $\mathbf{Z}[1/6]$-algebras with universal test object

$$(E_{\mathrm{univ}},\omega_{\mathrm{univ}})=(y^2=4x^3-g_2x-g_3, dx/y).$$

(For the description of Ell^* as the universal ring, see [9, Chapter II].) The formal group law F_K associated with $K^*(?)$ with canonical orientation x^K is given by

$$F_K(X,Y)=X+Y+tXY\in K^*[[X,Y]]$$

and the one F_{Ell} of $Ell^*(?)$ with canonical orientation x^{Ell} is given by $\hat{E}_{\mathrm{univ}}$ with local parameter $T=-2x/y$.

Let Tate(q) be the Tate curve over $\mathbf{Z}[1/6]((q))$ and $\omega_{\mathrm{can}}=dx/y$ be a canonical nowhere vanishing invariant differential on Tate(q) (see [9, Chapter II] and [18]). Then we have an injective q-expansion ring homomorphism classifying the $\Gamma(1)$-test object (Tate(q), ω_{can})

$$\lambda: Ell^*\to\mathbf{Z}[1/6]((q))$$

given by $\lambda(f)=f(q)=f(\mathrm{Tate}(q),\omega_{\mathrm{can}})$ ($\forall f\in Ell^*$). We define a ring homomorphism

$$\tilde{\lambda}: Ell^*\to K^*[1/6]((q))$$

by $\tilde{\lambda}(f)=t^{-k}\lambda(f)$ ($\forall f\in Ell^{2k}$). Then we have a unique strict isomorphism of formal group laws

$$\theta_{\text{can}}: F_K \overset{\cong}{\to} \bar{\lambda}_* F_{Ell}$$

over $K^*[1/6]((q))$ since $\lambda_* \hat{E}_{\text{univ}} = \widehat{\text{Tate}(q)}$ is canonically isomorphic to formal multiplicative group $\hat{\mathbf{G}}_m$ (see [12] and [18]). Let

$$\Theta: MU_*(MU) \to K_*[1/6]((q))$$

be the ring homomorphism classifying θ_{can}. Now the composite map

$$Ell_* \otimes_{\mathbf{Z}} MU_*(MU) \xrightarrow{\bar{\lambda}\otimes\Theta} K_*[1/6]((q)) \otimes_{\mathbf{Z}} K_*[1/6]((q)) \to K_*[1/6]((q)),$$

where the second map is the product on $K_*[1/6]((q))$, induces a right MU_*-module map

$$\bar{\lambda} = \Lambda(\bar{\lambda}, \theta_{\text{can}}): Ell_* \otimes_{MU_*} MU_*(MU) \to K_*[1/6]((q)).$$

Thus we have the following theorem proved by H. Miller [12].

Theorem 1.1 ([12]). *There is a natural transformation*

$$\bar{\lambda}(X): Ell^*(X) \to K^*(X)[1/6]((q)),$$

which is multiplicative on finite CW complexes, such that:

1. $\bar{\lambda}(pt) = \bar{\lambda}$.
2. $\bar{\lambda}(\mathbf{C}P^\infty)(x^{Ell}) = \theta_{\text{can}}(x^K)$.

The above $\bar{\lambda}(X)$ is called elliptic character.

Next we construct Baker operation. Let l be a prime. Consider the $l+1$ $\Gamma(1)$-test objects over $\mathbf{Z}[1/6l, \zeta_l]((q^{\frac{1}{l}}))$: $(\text{Tate}(q^l), l^{-1}\omega_{\text{can}})$ and $(\text{Tate}(\zeta_l^i q^{\frac{1}{l}}), \omega_{\text{can}})$ $(1 \le i \le l)$, where ζ_l denotes a primitive l-th root of unity. These $l+1$ $\Gamma(1)$-test objects yield $l+1$ ring homomorphisms

$$\lambda_i: Ell^* \to \mathbf{Z}[1/6l, \zeta_l]((q^{\frac{1}{l}}))$$

given by

$$\lambda_i(f) = \begin{cases} f(\text{Tate}(q^l), l^{-1}\omega_{\text{can}}) & (i=0) \\ f(\text{Tate}(\zeta_l^i q^{\frac{1}{l}}), \omega_{\text{can}}) & (1 \le i \le l) \end{cases} \quad (\forall f \in Ell^*).$$

We also have $l+1$ unique strict isomorphisms over $\mathbf{Z}[1/6l, \zeta_l]((q^{\frac{1}{l}}))$

$$\theta_i: \lambda_* F_{Ell} \overset{\cong}{\to} \lambda_{i*} F_{Ell}$$

since all formal group laws in question are the formal completion of the corresponding Tate curves, which is canonically isomorphic to $\hat{\mathbf{G}}_m$, with appropriate local parameter (see [3]). (All chosen local parameters are the standard one except for the one for $\lambda_{0*}F_{Ell}$ which is twisted by l^{-1}.) Then the $l+1$ pairs (λ_i, θ_i) yield $l+1$ right MU_*-module maps

$$\Lambda(\lambda_i, \theta_i): Ell_* \otimes_{MU_*} MU_*(MU) \to \mathbf{Z}[1/6l, \zeta_l]((q^{\frac{1}{l}})),$$

where the MU_*-module structure on $\mathbf{Z}[1/6l, \zeta_l]((q^{\frac{1}{l}}))$ is the one determined by the ring homomorphism classifying the formal group law $\lambda_* F_{Ell}$. Let $\bar{T}_l = \frac{1}{l}\Sigma_{0 \le i \le l} \Lambda(\lambda_i, \theta_i)$. Then we can prove that $\bar{T}_l(f \otimes 1) = (T_l f)(q)$ for all $f \in Ell^*$ and that $\mathrm{Im}\,\bar{T}_l \subseteq \mathrm{Im}\,(\lambda \otimes \mathbf{Z}[1/l])$, where

$$T_l: Ell^* \to Ell^*[1/l]$$

is Hecke operator. Therefore $\bar{T}_l$ can be viewed as a right MU_*-module map

$$\bar{T}_l: Ell_* \otimes_{MU_*} MU_*(MU) \to Ell_*[1/l].$$

Applying the above general construction to this $\bar{T}_l$ we have the following theorem proved by A. Baker [3].

Theorem 1.2 ([3]). *There is a natural transformation*

$$\bar{T}_l(X): Ell^*(X) \to Ell^*(X)[1/l]$$

such that $\bar{T}_l(pt) = T_l$.

We will call this $\bar{T}_l(X)$ Baker operation.

We end this section by describing the composition of Baker operation with elliptic character. Let

$$V_l: \mathbf{Z}((q)) \to \mathbf{Z}((q))$$

and

$$U_{l,i}: \mathbf{Z}((q)) \to \mathbf{Z}[\zeta_l]((q^{\frac{1}{l}}))$$

be the ring homomorphisms given by $V_l(q) = q^l$ and $U_{l,i}(q) = \zeta_l^i q^{\frac{1}{l}}$ $(1 \le i \le l)$ respectively. Let $U_l = \frac{1}{l}\Sigma_{1 \le i \le l} U_{l,i}$. Then V_l and U_l determine operations $V_l(X)$ and $U_l(X)$ on $K^*(X)((q))$ in the obvious way. (Here remark that $\mathrm{Im}\, U_l \subseteq \mathbf{Z}((q))$.) Let

$$\Psi^l(X): K^*(X) \to K^*(X)[1/l]$$

be Adams operation.

Proposition 1.3. *For any finite CW complex X we have a commutative diagram*

$$\begin{array}{ccc} Ell^*(X) & \xrightarrow{\tilde{T}_l(X)} & Ell^*(X)[1/l] \\ {\scriptstyle \tilde{\lambda}(X)}\downarrow & & \downarrow{\scriptstyle \tilde{\lambda}(X)} \\ K^*(X)[1/6]((q)) & \xrightarrow[\tilde{T}_l(q)(X)]{} & K^*(X)[1/6l]((q)), \end{array}$$

where $\tilde{T}_l(q)(X) = \frac{1}{l}\Psi^l(X)V_l(X) + U_l(X)$.

To prove the proposition we need the following simple lemma. Let $R^*(?)$, $S^*(?)$ and $T^*(?)$ be Landweber exact cohomologies and

$$\Lambda_1(X): R^*(X) \to S^*(X)$$

and

$$\Lambda_2(X): S^*(X) \to T^*(X)$$

be the natural transformations obtained from right MU_*-module maps

$$\Lambda_1: R_* \otimes_{MU_*} MU_*(MU) \to S_*$$

and

$$\Lambda_2: S_* \otimes_{MU_*} MU_*(MU) \to T_*$$

respectively. Let Λ_3 be the composite map

$$R_* \otimes_{MU_*} MU_*(MU) \xrightarrow{R_* \otimes \Delta} R_* \otimes_{MU_*} MU_*(MU) \otimes_{MU_*} MU_*(MU)$$

$$\xrightarrow{\Lambda_1 \otimes MU_*(MU)} S_* \otimes_{MU_*} MU_*(MU) \xrightarrow{\Lambda_2} T_*,$$

where

$$\Delta: MU_*(MU) \to MU_*(MU) \otimes_{MU_*} MU_*(MU)$$

is the coproduct. Then Λ_3 is a right MU_*-module map and yields a natural transformation

$$\Lambda_3(X): R^*(X) \to T^*(X).$$

Lemma 1.4. *For any finite CW complex X we have* $\Lambda_3(X) = \Lambda_2(X)\Lambda_1(X)$.

Proof. Work with homology and simple diagram chasing.

Proof of 1.3. First note that the natural transformation $\tilde{T}_l(q)(X)$ can be obtained from a certain right MU_*-module map

$$\tilde{T}_l(q): K_*[1/6]((q)) \otimes_{MU_*} MU_*(MU) \to K_*[1/6l]((q))$$

as described bellow. Let

$$\Psi^l : K^* \to K^*[1/l]$$

be the ring homomorphism given by $\Psi^l(t)=lt$. Then there is a unique strict isomorphism

$$\theta : F_K \overset{\cong}{\to} (\Psi^l V_l)_* F_K$$

over $K^*[1/l]((q))$ given by

$$\theta(T)=l^{-1}[l]_{F_K}(T)=l^{-1}t^{-1}[(1+tT)^l-1]\in K^*[1/l]((q))[[T]].$$

Thus the pair $(\Psi^l V_l, \theta)$ determines a right MU_*-module map

$$\Lambda(\Psi^l V_l, \theta) : K_*[1/6]((q))\otimes_{MU_*}MU_*(MU)\to K_*[1/6l]((q)).$$

Also the pair $(U_{l,i}, id_{F_K})$ determines a right MU_*-module map

$$\Lambda(U_{l,i}, id_{F_K}) : K_*[1/6]((q))\otimes_{MU_*}MU_*(MU)\to K_*[1/6l, \zeta_l]((q^{\frac{1}{l}})).$$

Let $\bar{T}_l(q)=\frac{1}{l}(\Lambda(\Psi^l V_l, \theta)+\Sigma_{1\le i\le l}\Lambda(U_{l,i}, id_{F_K}))$. Then it is easy to see that $\mathrm{Im}\,\bar{T}_l(q)\subseteq K_*[1/6l]((q))$ and that the natural transformation obtained from this $\bar{T}_l(q)$ (viewed as a map to $K_*[1/6l]((q))$) is $\frac{1}{l}\Psi^l(X)V_l(X)+U_l(X)$. (Note that $\Lambda(\Psi^l V_l, \theta)(X)$ is a ring homomorphism for finite X such that $\Lambda(\Psi^l V_l, \theta)(pt)=\Psi^l V_l$ and

$$\begin{aligned}\Lambda(\Psi^l V_l, \theta)(\mathbf{C}P^\infty)(x^K)&=\theta(x^K)\\&=l^{-1}t^{-1}[(1+tx^K)^l-1]\\&=\Psi^l(\mathbf{C}P^\infty)(x^K).)\end{aligned}$$

Therefore, by the above lemma, it is sufficient to show that

$$\bar{\lambda}(\bar{T}_l\otimes MU_*(MU))(Ell_*\otimes\Delta)=\bar{T}_l(q)(\bar{\lambda}\otimes MU_*(MU))(Ell_*\otimes\Delta)$$

and this is equivalent to

$$(\bar{\lambda}(\bar{T}_l\otimes MU_*(MU))(Ell_*\otimes\Delta))\otimes\mathbf{Q}=(\bar{T}_l(q)(\bar{\lambda}\otimes MU_*(MU))(Ell_*\otimes\Delta))\otimes\mathbf{Q}.$$

Now for any $a\otimes b\otimes 1\in MU_*\otimes_{\mathbf{Z}}MU_*\otimes_{\mathbf{Z}}\mathbf{Q}=MU_*(MU)\otimes_{\mathbf{Z}}\mathbf{Q}$ we have

$$\begin{aligned}\Delta(a\otimes b\otimes 1)&=a\otimes 1\otimes b\otimes 1\in MU_*\otimes_{\mathbf{Z}}MU_*\otimes_{\mathbf{Z}}MU_*\otimes_{\mathbf{Z}}\mathbf{Q}\\&=MU_*(MU)\otimes_{MU_*}MU_*(MU)\otimes_{\mathbf{Z}}\mathbf{Q}.\end{aligned}$$

Hence for any $f\otimes a\otimes 1\in Ell_*\otimes_{\mathbf{Z}}MU_*\otimes_{\mathbf{Z}}\mathbf{Q}=Ell_*\otimes_{MU_*}MU_*(MU)\otimes_{\mathbf{Z}}\mathbf{Q}$ we have

$\bar{\lambda}(\bar{T}_l\otimes MU_*(MU))(Ell_*\otimes\Delta)(f\otimes a)\otimes 1=a\cdot\bar{\lambda}T_l(f)\otimes 1,$

where $\cdot$ denotes the MU_*-action on $K_*[1/6l]((q))$. If $f \in Ell^{2k}$ then

$$\begin{aligned}\tilde{\lambda}T_l(f) &= t^{-k}(T_lf)(q)\\ &= t^{-k}(l^{-k-1}V_l(f(q)) + U_l(f(q))) \text{ (see [8, Chapter 1])}\\ &= \tfrac{1}{l}\Psi^l V_l(t^{-k}f(q)) + U_l(t^{-k}f(q))\\ &= (\tfrac{1}{l}\Psi^l V_l + U_l)(t^{-k}f(q))\\ &= \bar{T}_l(q)(t^{-k}f(q)\otimes 1)\\ &= \bar{T}_l(q)(\tilde{\lambda}(f)\otimes 1)\\ &= \bar{T}_l(q)(\tilde{\lambda}(f\otimes 1)\otimes 1).\end{aligned}$$

Therefore

$$\begin{aligned}a\cdot\tilde{\lambda}T_l(f)\otimes 1 &= a\cdot\bar{T}_l(q)(\tilde{\lambda}(f\otimes 1)\otimes 1)\otimes 1\\ &= \bar{T}_l(q)(\tilde{\lambda}(f\otimes 1)\otimes a)\otimes 1\\ &= \bar{T}_l(q)(\tilde{\lambda}\otimes MU_*(MU))(Ell_*\otimes\Delta)(f\otimes a)\otimes 1\end{aligned}$$

and hence we get

$$(\tilde{\lambda}(\bar{T}_l\otimes MU_*(MU))(Ell_*\otimes\Delta))\otimes\mathbf{Q} = (\bar{T}_l(q)(\tilde{\lambda}\otimes MU_*(MU))(Ell_*\otimes\Delta)\otimes\mathbf{Q}.$$

This completes the proof.

Remark 1.5. The commutativity of the diagram in Proposition 1.3 is similar to the formula (7.3) in [11].

2. Recollections on elliptic cohomology of finite groups

In this section we recall a result on the modularity of elliptic character for finite groups given in [18]. Let G be a finite group and BG be its classifying space. Since $K^*(BG) = \lim K^*(BG_i)$ for a filtration $\{BG_i\}$ on BG consisting of finite subcomplexes and $\tilde{\lambda}(BG_i)$ is multiplicative the map

$$\tilde{\lambda}(BG) : Ell^*(BG) \to K^*(BG)[1/6]((q))$$

is a ring homomorphism. For a prime p let $G_p = \{g \in G | g^{p^N} = 1 \ N \gg 0\}$ and denote by $\mathbf{C}_p$ the completion of the algebraic closure of the p-adic number field $\mathbf{Q}_p$. Then, by using the p-adic analogue of group character

$$K^0(BG)_p^\wedge \to \mathrm{Map}_G(G_p, \mathbf{C}_p)$$

based on Atiyah's isomorphism (see [5]), we have a natural ring homomorphism

$$\lambda_p(G) : Ell^*(BG) \xrightarrow{\tilde{\lambda}(BG)} K^*(BG)[1/6]((q)) \xrightarrow{\chi_p(G)} \mathrm{Map}_G(G_p, \mathbf{C}_p((q))).$$

Here $\chi_p(G)$ is the composite map

$$K^*(BG)=K^{even}(BG)\to K^0(BG)\to K^0(BG)_p^{\wedge}\to \mathrm{Map}_G(G_p,\mathbf{C}_p),$$

where the first map is given by $x\mapsto t^k x$ ($\forall x\in K^{2k}(BG)$), and we assume $p\geq 5$. (From now on we fix a prime $p\geq 5$.)

Let $V(n)=V(\mathbf{Z}_p[\zeta_{p^n}],\Gamma(1))$ be the ring of $\Gamma(1)$-generalized p-adic modular functions over $B_n=\mathbf{Z}_p[\zeta_{p^n}]$ (see [9, Chapter V] and [4, Chapter I]). The ring $V(n)$ is the universal ring classifying trivialized elliptic curves over p-adic B_n-algebras. We have an injective q-expansion homomorphism

$$\hat{\lambda}: V(n)\to \widehat{B_n((q))}$$

given by $\hat{\lambda}(f)=f(q)=f(\mathrm{Tate}(q),\varphi_{\mathrm{can}})$ ($\forall f\in V(n)$), where $\widehat{B_n((q))}$ is the p-adic completion of $B_n((q))$ and $(\mathrm{Tate}(q),\varphi_{\mathrm{can}})$ denotes the Tate curve over $\widehat{B_n((q))}$ with canonical trivialization

$$\varphi_{\mathrm{can}}=\epsilon_*\theta_{\mathrm{can}}^{-1}:\widehat{\mathrm{Tate}(q)}=\lambda_*F_{Ell}\xrightarrow{\cong}\hat{\mathbf{G}}_m=\epsilon_*F_K.$$

(Here ϵ is a ring homomorphism

$$\epsilon: K^*\to \mathbf{Z}$$

given by $\epsilon(t)=1$.) For any $f\in V(n)$ and any $a\in\mathbf{Z}_p^\times$ we define an element $[a]f\in V(n)$ by the formula

$$[a]f(E,\varphi)=f(E,a^{-1}\varphi),$$

where a^{-1} acts on φ via an automorphism of $\hat{\mathbf{G}}_m$. This gives a group action of $\mathbf{Z}_p^\times$ on $V(n)$. Let $V^k(n)=\{f\in V(n)|[a]f=a^k f\ (\forall a\in\Gamma_n)\}$, where $\Gamma_n=\{a\in\mathbf{Z}_p^\times|a\equiv 1(p^n)\}$. We also have a ring homomorphism

$$Ell^*\to V(n)$$

which preserves q-expansions and hence is injective (see [9, Chapter V] and [4, Chapter I]). When we regard Ell^* as a subring of $V(n)$ via this homomorphism E_{univ} admits a canonical trivialization over $V(n)$

$$\varphi_{\mathrm{univ}}:\hat{E}_{\mathrm{univ}}\xrightarrow{\cong}\hat{\mathbf{G}}_m$$

given by

$$\varphi_{\mathrm{univ}}(T)=\exp_{\epsilon_*F_K}\log_{F_{Ell}}(T)\in V(n)[[T]]$$

(see [18]).

Theorem 2.1 ([18]). *For any $x\in Ell^{2k}(BG)$ and $g\in G$ of order p^n there is a unique* *element* $f\in V^{-k}(n)$ *such that* $[\lambda_p(G)(x)](g)=f(\mathrm{Tate}(q),\varphi_{\mathrm{can}})$.

3. Baker operation on elliptic cohomology of finite groups and Hecke operator

Let $p\geq 5$ be a prime and l be a different prime. Let

$$T_l\colon V(n)\to V(n)$$

be Hecke operator which is an extension of Hecke operator

$$T_l\colon Ell^*\to Ell^*[1/l]$$

(see [4, Chapter II]). Then T_l induces an operation

$$T_l\colon V^k(n)\to V^k(n)$$

for every k since T_l commutes with the action of $\mathbf{Z}_p^{\times}$. By Theorem 2.1 for any $g\in G$ of order p^n we can define a ring homomorphism

$$\lambda_p(g)\colon Ell^{\mathrm{even}}(BG)\to V(n)$$

by the formula

$$[\lambda_p(G)(x)](g)=[\lambda_p(g)(x)](q)=[\lambda_p(g)(x)](\mathrm{Tate}(q),\varphi_{\mathrm{can}}).$$

With this notation we have

Theorem 3.1. *The following diagram*

$$\begin{array}{ccc} Ell^{2k}(BG) & \xrightarrow{\bar{T}_l(BG)} & Ell^{2k}(BG)[1/l] \\ {\scriptstyle\lambda_p(g)}\downarrow & & \downarrow{\scriptstyle\lambda_p(g)} \\ V^{-k}(n) & \xrightarrow[T_l]{} & V^{-k}(n) \end{array}$$

is commutative for every $g\in G$ *of order* p^n *and every* k.

Proof. For $x\in Ell^{2k}(BG)$ let

$$\bar{\lambda}(BG)(x)=t^{-k}\sum_n a_nq^n\in K^{2k}(BG)[1/6]((q))\ (a_n\in K^0(BG)[1/6]).$$

By Proposition 1.3 we have a commutative diagram

$$\begin{array}{ccc} Ell^{2k}(BG) & \xrightarrow{\bar{T}_l(BG)} & Ell^{2k}(BG)[1/l] \\ {\scriptstyle \bar{\lambda}(BG)}\downarrow & & \downarrow{\scriptstyle \bar{\lambda}(BG)} \\ K^{2k}(BG)[1/6]((q)) & \xrightarrow[\bar{T}_l(q)(BG)]{} & K^{2k}(BG)[1/6l]((q)), \end{array}$$

since $K^*(BG) = \lim K^*(BG_i)$. Thus

$$\begin{aligned} [\lambda_p(g)\bar{T}_l(BG)(x)](q) &= [\lambda_p(G)\bar{T}_l(BG)(x)](g) \\ &= [\chi_p(G)\bar{\lambda}(BG)\bar{T}_l(BG)(x)](g) \\ &= [\chi_p(G)\bar{T}_l(q)(BG)\bar{\lambda}(BG)(x)](g) \\ &= [\chi_p(G)(t^{-k}\Sigma_n(l^{-k-1}\Psi^l(BG)(a_n)q^{nl} + a_{nl}q^n))](g) \\ &= l^{-k-1}\Sigma_n a_n(g^l)q^{nl} + \Sigma_n a_{nl}(g)q^n, \end{aligned}$$

where $a(g) = [\chi_p(G)(a)](g)$ for $a \in K^0(BG)[1/6]$. On the other hand

$$[T_l\lambda_p(g)(x)](q) = \frac{1}{l}\,[l](\lambda_p(g)(x))(q^l) + \sum_n a_{nl}(g)q^n$$

since

$$[\lambda_p(g)(x)](q) = \sum_n a_n(g)q^n$$

(see [4, Chapter II]). Therefore

$$\lambda_p(g)\bar{T}_l(BG)(x) = T_l\lambda_p(g)(x)$$

is equivalent to

$$\sum_n a_n(g^l)q^{nl} = l^k([l](\lambda_p(g)(x)))(q^l).$$

Let $\phi_1(g)$ and $\phi_2(g)$ be ring homomorphisms

$$\phi_1(g), \phi_2(g) : Ell^{\text{even}}(BG) \to \mathbf{C}_p((q))$$

given by

$$\phi_1(g)(x) = \sum_n a_n(g^l)q^{nl}$$

and

$$\phi_2(g)(x) = l^k([l](\lambda_p(g)(x)))(q^l)$$

for $x \in Ell^{2k}(BG)$ with

$$\bar{\lambda}(BG)(x) = t^{-k}\sum_n a_n q^n \ (a_n \in K^0(BG)[1/6]).$$

Then we have to show that

$$\phi_1(g)(x) = \phi_2(g)(x) \tag{3.2}$$

for all $x \in Ell^{2k}(BG)$ and $g \in G$ of order p^n. To prove (3.2) first consider the case $G = \mathbf{Z}/p^n\mathbf{Z}$, $g = g_n$ (the canonical generator of $\mathbf{Z}/p^n\mathbf{Z}$) and $x = x^{Ell} \in Ell^2(B\mathbf{Z}/p^n\mathbf{Z})$. Let

$$\bar{\lambda}(B\mathbf{Z}/p^n\mathbf{Z})(x^{Ell}) = \theta_{\mathrm{can}}(x^K) = t^{-1}\sum_n a_n q^n \ (a_n \in K^0(B\mathbf{Z}/p^n\mathbf{Z})[1/6]).$$

Then, by the proof of Theorem 2.1 given in [18], the righthand side of (3.2) is

$$\begin{aligned} l([l](\lambda_p(g_n)(x^{Ell})))(q^l) &= l([l]\varphi_{\mathrm{univ}}^{-1}(\zeta_{p^n}-1))(q^l) \\ &= l(l^{-1}\varphi_{\mathrm{univ}}^{-1}(\zeta_{p^n}^l-1))(q^l) \\ &= (\varphi_{\mathrm{univ}}^{-1}(\zeta_{p^n}^l-1))(q^l) \\ &= \Sigma_n a_n(g_n^l)q^{nl}. \end{aligned}$$

(Note that $\lambda_*\varphi_{\mathrm{univ}}^{-1} = \varphi_{\mathrm{can}}^{-1} = \epsilon_*\theta_{\mathrm{can}}$.) Thus (3.2) holds in this case. Since $\phi_1(g_n)$ and $\phi_2(g_n)$ are continuous ring homomorphisms we have

$$\phi_1(g_n)(x) = \phi_2(g_n)(x)$$

for all $x \in Ell^{2k}(B\mathbf{Z}/p^n\mathbf{Z})$. Now for a general G and $g \in G$ of order p^n there is a unique homomorphism

$$\alpha : \mathbf{Z}/p^n\mathbf{Z} \to G$$

which sends g_n to g. Hence for any $x \in Ell^{2k}(BG)$

$$\begin{aligned} \phi_1(g)(x) &= \phi_1(\alpha(g_n))(x) \\ &= \phi_1(g_n)((B\alpha)^*x) \\ &= \phi_2(g_n)((B\alpha)^*x) \\ &= \phi_2(\alpha(g_n))(x) \\ &= \phi_2(g)(x). \end{aligned}$$

This completes the proof.

Remark 3.3. The commutativity of the diagram in Theorem 3.1 suggests that Baker operation could be constructed geometrically if we have a geometric construction of elliptic cohomology which explains the relations between elliptic cohomology of finite groups and theory of Thompson series appearing in Theorem 2.1.

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