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On Freudenthal's geometry and generalized adjoint varieties

By

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1. Introduction

In this paper we have several observations about Freudenthal's geometry and calculate the degrees of dual varieties of adjoint varieties. Freudenthal introduced a notion of symplectic geometry and metasymplectic geometry to study a geometric aspect of exceptional Lie groups. This is an analogy of the relation of projective geometry and projective tansformation groups. He studied several homogeneous varieties which play very important roles in his geometry. On the other hand, we obtain a homogeneous projective variety from an irreducible representation of a simple algebraic group. We call such a homogeneous variety generalized adjoint variety. In this paper we determine which representations the homogeneous varieties in Freudenthal's geometry are obtained from. The results show that there is a very interesting correspondence between the homogeneous varieties and irreducible representaions of simple algebraic groups.

It is known that the dual varieties of Freudenthal's homogeneous varieties are hypersurfaces in [KM]. The degree of such dual varieties were calculated for some cases in [M1][M2]. We give a formula for the degree of the dual variety of adjoint variety and calculate the degree of some Freudenthal's homogeneous varieties.

We calculate irreducible decompositions of representations of complex simple algebraic groups in Section3 by using a package [LiE].

When we consider a nonzero element x in a vector space as a element in a projective space, we shall use the same symbol for it.

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2. Freudenthal's geometry

We know that real number, complex numvbar, quarternion and Cayley number are Hurwitz algebras over $\boldsymbol{R}$ and that they are all of Hurwitz algebras over $\boldsymbol{R}$. Here the complexifications of them are denoted by $\boldsymbol{R}_{\boldsymbol{C}}$, $\boldsymbol{C}_{\boldsymbol{C}}$, $\boldsymbol{H}_{\boldsymbol{C}}$ and $\boldsymbol{O}_{\boldsymbol{C}}$. They have a natural conjugation as algebras over $\boldsymbol{C}$. For example a conjugation $\bar{x}$ of the element $x=a+bI+cJ+dK (a,b,c,d\in\boldsymbol{C})$ of $\boldsymbol{H}_{\boldsymbol{C}}$ is as follows,

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$$\bar{x}=a-bI-cJ-dK.$$

H. Freudenthal investigated exceptional Lie groups and their geometries according to following his magic square.

Freudenthal's magic square

elliptic geometry	B_1	A_2	C_3	F_4
projective geometry	A_2	A_2+A_2	A_5	E_6
symplectic geometry	C_3	A_5	D_6	E_7
metasymplectic geometry	F_4	E_6	E_7	E_8

Each column corresponds to $\boldsymbol{R}_C$, $\boldsymbol{C}_C$, $\boldsymbol{H}_C$ or $\boldsymbol{O}_C$ respectively. In this section we will give a brief sketch of his geometry. In Freudenthal's papers coefficient field is real number. Though we use complex number as coefficient field, most of assertions in his papers hold similarly. So we give only overview here. For detail see [Fr1, 2].

2.1. Adjoint varieties

Let G be a complex, connected, simple algebraic group with Lie algebra $\mathfrak{g}$, V a finite dimensional complex vector space, and $\rho: G\to GL(V)$ an irreducible representation of G. Then G acts on the complex projective space $\boldsymbol{P}(V)$ through the projection,

$$\pi: V\backslash 0\to \boldsymbol{P}(V).$$

If v is a highest weight vector of the representation, we obtain the unique closed G-orbit,

$$X:=\pi(G\cdot v)\subseteq \boldsymbol{P}(V).$$

The orbit is a homogeneous projective variety and we call it *generalized adjoint variety*. In paticular if the representation is the adjoint representation, its variety is called *adjoint variety*. It is shown in the paper[L] that the generalized adjoint variety is defined by a system of quadric equations in $\boldsymbol{P}(V)$.

2.2. Projective geometry

We denote by $\mathfrak{J}$ the set of all Hermitian matrices of degree 3 whose entries are elements in the complexification of a Hurwitz algebra. For $X, Y, Z\in\mathfrak{J}$, we put,

$$X\circ Y=\frac{1}{2}(XY+YX),$$

$$(X, Y):=\mathrm{tr}(X\circ Y),$$

$$X\times Y:=X\circ Y-\frac{1}{2}(\mathrm{tr}(X)Y+\mathrm{tr}(Y)X-(\mathrm{tr}(X)\mathrm{tr}(Y)-\mathrm{tr}(XY))E),$$

$$(X, Y, Z):=(X, Y\times Z),$$

$$\det(X) := \frac{1}{3}(X, X, X).$$

Then we obtain projective varieties $\mathfrak{V}$ and algebraic groups $G^{[3]}$ of type A_2, A_2+A_2, A_5 and E_6 according to Hurwitz algebras as follows,

$$\mathfrak{V} := \{X \in \boldsymbol{P}(\mathfrak{J}) | X \times X = 0\},$$
$$G^{[3]} := \{g \in GL(\mathfrak{J}) | \det(gX) = \det(X) \textit{ for all } X \in \mathfrak{J}\}.$$

Type A_2, A_2+A_2, A_5 and E_6 correspond to the second row in Freudenthal's magic square. The algebraic group $G^{[3]}$ acts on variety $\mathfrak{V}$ and the action is transitive. Moreover $\mathfrak{J}$ is an irreducible representation of $G^{[3]}$, and $\mathfrak{V}$ is the generalized adjoint variety which is obtained from $\mathfrak{J}$. For each $Y \in \mathfrak{V}$ we define the subset L_Y in $\mathfrak{V}$ as follws,

$$L_Y := \{X \in \mathfrak{V} | X \times Y = 0\}.$$

We call it the *line*. Then $\mathfrak{V}$ is naturally identified with the set of *lines* in $\mathfrak{V}$. We denote the set of *lines* in $\mathfrak{V}$ by $\mathfrak{V}^*$. Let L_Z be a *line*. For distinct $X, Y \in L_Z$ we obtain an element $X \times Y$ in $\mathfrak{V}$. This element does not depend on the choice of $X, Y \in L_Z$. Then $X \times Y$ is Z in $\mathfrak{V}$.

Remark 1. *First we define an involutive automorphism λ of $G^{[3]}$* as follows,

$$\lambda(\alpha) = {}^t\alpha^{-1} \quad \alpha \in G^{[3]}.$$

Here we denote transposed α with respect to the inner product $(X, Y) := \operatorname{tr}(X \circ Y)$ by ${}^t\alpha$. Then we obtain a following relation,

$$\alpha X \times \alpha Y = {}^t\alpha^{-1}(X \times X) \quad X \in \mathfrak{J}.$$

For $X, Y, Z \in \mathfrak{J}$ we define $\langle X, Y \rangle$ in $\operatorname{End}(\mathfrak{J})$ by

$$\langle X, Y \rangle Z := 2Y \times (X \times Z) - \frac{1}{2}(Z, Y)X - \frac{1}{6}(X, Y)Z.$$

Then the vector space which is spanned by these maps coincides with the Lie algebras of above algebraic groups $G^{[3]}$ in $\operatorname{End}(\mathfrak{J})$. We denote the Lie algebras by $\operatorname{Inv}(\det)$.

H, Freudenthal called the geometry of *points* and *lines* in $\mathfrak{V}$ *projective geometry.*

2.3. Symplectic geometry

First we define two vector spaces $\mathfrak{B}$ and $\mathfrak{L}$.

$$\mathfrak{B} := \mathfrak{J} \oplus \mathfrak{J} \oplus \boldsymbol{C} \oplus \boldsymbol{C}.$$

$$\mathfrak{L} := \operatorname{Inv}(\det) \oplus \mathfrak{J} \oplus \mathfrak{J} \oplus \boldsymbol{C}.$$

For $P_i=(X_i, Y_i, \xi_i, \omega_i)\in\mathfrak{B}(i=1,2)$, the element $P_1\times P_2$ of $\mathfrak{L}$ and skew inner product on $\mathfrak{B}$ are defined by,

$$P_1\times P_2:=\frac{1}{2}\begin{pmatrix} \langle X_1, Y_2\rangle+\langle X_2, Y_1\rangle \\ -Y_1\times Y_2+\frac{1}{2}(\xi_1 X_2+\xi_2 X_1) \\ X_1\times X_2-\frac{1}{2}(\omega_1 Y_2+\omega_2 Y_1). \\ -\frac{1}{4}((X_1, Y_2)+(X_2, Y_1)-3\xi_1\omega_2-3\xi_2\omega_1 \end{pmatrix}$$
$$\{P_1, P_2\}:=(X_1, Y_2)-(X_2, Y_1)+\xi_1\omega_2-\xi_2\omega_1$$

Then we obtain varieties $\mathfrak{M}$ and algebraic groups $G^{[2]}$ of type of C_3, A_5, D_6 and E_7 according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ and $\boldsymbol{O_C}$ as follows,

$$\mathfrak{M}:=\{P\in \boldsymbol{P}(\mathfrak{B})|P\times P=0\}.$$
$$G^{[2]}:=\{g\in GL(\mathfrak{B})|g(P\times Q)g^{-1}=gP\times gQ \textit{ for } P, Q\in\mathfrak{B}\}.$$

Type C_3, A_5, D_6 and E_7 correspond to the third row in Freudenthal's magic square. The variety $\mathfrak{M}$ is homogeneous projective veriety of $G^{[2]}$. Moreover $\mathfrak{B}$ is an irreducible representation of $G^{[2]}$, and $\mathfrak{M}$ is the generalized adjoint variety which is obtained from $\mathfrak{B}$.

For $\Theta=(\Sigma_i\langle X_i, Y_i\rangle, A, B, \rho)\in\mathfrak{L}$ and $P=(X, Y, \xi, \omega)\in\mathfrak{B}$, the element ΘP of $\mathfrak{B}$ is defined as follows,

$$\Theta P:=\begin{pmatrix} (\Sigma_i\langle X_i, Y_i\rangle+\frac{1}{3}\rho)X+2B\times Y+\omega A \\ -(\Sigma_i\langle Y_i, X_i\rangle+\frac{1}{3}\rho)Y+2A\times X+\xi B \\ (A, Y)-\rho\xi \\ (B, X)+\rho\omega. \end{pmatrix}$$

Then we can consider $\mathfrak{L}$ as a subspace of End(B). In fact $\mathfrak{L}$ coincides with the Lie algebra of $G^{[2]}$. We denote the Lie algebras by Inv$(\mathfrak{M})$. When we consider $\mathfrak{L}$ as a subspace of End$(\mathfrak{B})$, we obtain varieties $\mathfrak{N}$,

$$\mathfrak{N}:=\{\Theta\in \boldsymbol{P}(\mathrm{Inv}(\mathfrak{M}))|\Theta^2=0\}.$$

Then it is known that $\mathfrak{N}$ are the adjoint varieties of $G^{[2]}$. An element of $\mathfrak{N}$ is called a *point*. If $[\Theta_1, \Theta_2]=0$ for $\Theta_1, \Theta_2\in\mathfrak{N}$, we say that Θ_1, Θ_2 are jointed. The maximal set of *points* which are jointed each other is called *plane*. If different *planes* contain at least two *points* in common, the intersection is called *line*. If $\Theta P=0$ for $\Theta\in\mathfrak{L}$ and $P\in\mathfrak{N}$, we say that Θ is incident to P.

Proposition 1. *For each plane there exists exactly one element P of $\mathfrak{M}$ such that $\Theta P=0$ for any point Θ of the plane.*

Hence we can consider $\mathfrak{M}$ as the set of *planes* in $\mathfrak{N}$. Moreover each *plane* P has The structure of the projective geometry.

We denote the set of *lines* in $\mathfrak{N}$ by $\mathfrak{C}$. It is known that $\mathfrak{C}$ are projective homogeneous varieties. Then we expect that each $\mathfrak{C}$ is naturally embedded in an irreducible representation of $G^{[2]}$ which contains $\mathfrak{J}$ as a representation of $G^{[3]}$ and that the intersection of $P(\mathfrak{J})$ and $\mathfrak{C}$ is $\mathfrak{B}$ as the set of *lines*. We shall show it in Section 3. The dimensions of $\mathfrak{N}$, $\mathfrak{C}$ and $\mathfrak{M}$ are known as follows,

	R_C	C_C	H_C	O_C
$\mathfrak{N}$	5	9	17	33
$\mathfrak{C}$	7	12	22	42
$\mathfrak{M}$	6	9	15	27

H, Freudenthal called the geometry of *points*, *lines* and *planes* in $\mathfrak{N}$ *symplectic geometry.*

2.4. Metasymplectic geometry

2.4.1. Exceptional Lie algebras. First we define a vector space as follows,

$$\mathfrak{R}_4 := \mathrm{Inv}(\mathfrak{M}) \oplus \mathfrak{a}_1 \oplus \mathfrak{B} \oplus \mathfrak{B}.$$

Here we denote the complex simple Lie algebra of type A_1 by $\mathfrak{a}_1$ and the element Φ of $\mathfrak{R}_4$ by,

$$\Phi = \left[\begin{pmatrix} \Theta + \gamma & \underline{\delta} \\ \overline{\delta} & \Theta - \gamma \end{pmatrix}, \quad \begin{pmatrix} P \\ Q \end{pmatrix} \right].$$

Here $\Theta \in \mathrm{Inv}(\mathfrak{M})$, $\gamma, \underline{\delta}, \overline{\delta} \in \boldsymbol{C}$, $P, Q \in \mathfrak{B}$. The Lie bracket on $\mathfrak{R}_4$ is defined as follows,

$$[\Phi_1, \Phi_2] = \left[\begin{pmatrix} \Theta + \gamma & \underline{\delta} \\ \overline{\delta} & \Theta - \gamma \end{pmatrix}, \quad \begin{pmatrix} P \\ Q \end{pmatrix} \right].$$

$$\text{for } \Phi_i = \left[\begin{pmatrix} \Theta_i + \gamma_i & \underline{\delta}_i \\ \overline{\delta}_i & \Theta_i - \gamma_i \end{pmatrix}, \quad \begin{pmatrix} P_i \\ Q_i \end{pmatrix} \right] \qquad i = 1, 2.$$

Here,

$$\Theta = [\Theta_1, \Theta_2] + P_1 \times Q_2 - P_2 \times Q_1,$$

$$\gamma = \underline{\delta}_1 \overline{\delta}_2 - \underline{\delta}_2 \overline{\delta}_1 - \frac{1}{8}\{P_1, Q_2\} + \frac{1}{8}\{P_2, Q_1\},$$

$$\overline{\delta} = -2\gamma_1 \overline{\delta}_2 + 2\gamma_2 \overline{\delta}_1 - \frac{1}{4}\{Q_1, Q_2\},$$

$$\underline{\delta} = 2\gamma_1 \underline{\delta}_2 - 2\gamma_2 \underline{\delta}_1 + \frac{1}{4}\{P_1, P_2\},$$

$$P = (\Theta_1 + \gamma_1)P_2 - (\Theta_2 + \gamma_2)P_1 + \underline{\delta}_1 Q_2 - \underline{\delta}_2 Q_1,$$

$$Q=(\Theta_1-\gamma_1)Q_2-(\Theta_2-\gamma_2)Q_1+\bar{\delta}P_2-\bar{\delta}_2P_1.$$

Then each $\mathfrak{R}_4$ becomes the complex exceptional Lie algebra of type of F_4, E_6, E_7 or E_8 according to $\boldsymbol{R}_{\boldsymbol{C}}$, $\boldsymbol{C}_{\boldsymbol{C}}$, $\boldsymbol{H}_{\boldsymbol{C}}$ or $\boldsymbol{O}_{\boldsymbol{C}}$ respectively.

2.4.2. Metasymplectic geometry. We have an inportant inner product in $\mathfrak{R}_4$ as follows,

$$(\Phi_1,\Phi_2):=-\frac{1}{\epsilon_2}\operatorname{tr}(\tilde{\Phi}_1\tilde{\Phi}_2) \qquad \Phi_1,\Phi_2\in\mathfrak{R}_4$$

Here we denote $ad\Phi_i$ by $\tilde{\Phi}_i$, and $\epsilon_2:=9$, 12, 18 or 30 according to $\boldsymbol{R}_{\boldsymbol{C}}$, $\boldsymbol{C}_{\boldsymbol{C}}$, $\boldsymbol{H}_{\boldsymbol{C}}$ or $\boldsymbol{O}_{\boldsymbol{C}}$ respectively. Moreover we define a linear map $\langle\Phi_1,\Phi_2\rangle$ of $\mathfrak{R}_4$ for Φ, Φ_1, $\Phi_2\in\mathfrak{R}_4$ as follws,

$$\langle\Phi_1,\Phi_2\rangle\Phi:=\frac{1}{2}(\tilde{\Phi}_1\tilde{\Phi}_2+\tilde{\Phi}_2\tilde{\Phi}_1)\Phi-\frac{1}{2}(\Phi_1,\Phi)\Phi_2-\frac{1}{2}(\Phi_2,\Phi)\Phi_1+\epsilon(\Phi_1,\Phi_2)\Phi.$$

Here $\epsilon:=\frac{5}{26}$, $\frac{1}{6}$, $\frac{1}{7}$ or $\frac{1}{8}$ according to $\boldsymbol{R}_{\boldsymbol{C}}$, $\boldsymbol{C}_{\boldsymbol{C}}$, $\boldsymbol{H}_{\boldsymbol{C}}$ or $\boldsymbol{O}_{\boldsymbol{C}}$ respectively. Then we obtain varieties $\mathfrak{W}_4$ as follows,

$$\mathfrak{W}_4:=\{\Phi\in\boldsymbol{P}(\mathfrak{R}_4)|\langle\Phi,\Phi\rangle=0\}.$$

It is known that each $\mathfrak{W}_4$ is the adjoint variety of the adjoint group $G^{[1]}$ of $\mathfrak{R}_4$. The element of $\mathfrak{W}_4$ is called *symplecton.*

The vector space $\mathfrak{R}_1$ is defined as a linear subspace of $\operatorname{End}(\mathfrak{R}_4)$ which is spanned by $\{\langle\Phi_1,\Phi_2\rangle|\Phi_1,\Phi_2\in\mathfrak{R}_4\}$. Then we obtain varieties $\mathfrak{W}_1$ as follows,

$$\mathfrak{W}_1:=\{\langle\Phi_1,\Phi_2\rangle\in\boldsymbol{P}(\mathfrak{R}_1)|[\Phi_1,\Phi_2]=0,\Phi_1,\Phi_2\in\mathfrak{W}_4\}.$$

It is known that the adjoint group $G^{[1]}$ of $\mathfrak{R}_4$ acts on $\mathfrak{W}_1$. An element of $\mathfrak{W}_1$ is called a *point.*

For *symplecta* Φ_1, $\Phi_2\in\mathfrak{W}_4$ we define following three relations,

$$\Phi_1 \text{ is jointed with } \Phi_2 \text{ if } \langle\Phi_1,\Phi_2\rangle=0,$$

$$\Phi_1 \text{ is interwoven with } \Phi_2 \text{ if } [\Phi_1,\Phi_2]=0,$$

$$\Phi_1 \text{ is hinged with } \Phi_2 \text{ if } \{\Phi_1,\Phi_2\}=0.$$

Similarly for *points* $A,B\in\mathfrak{W}_1$ we define following three relations,

$$A \text{ is jointed with } B \text{ if } AB=0,$$

$$A \text{ is interwoven with } B \text{ if } AB-BA=0,$$

$$A \text{ is hinged with } B \text{ if } \operatorname{tr}(AB)=0.$$

Moreover we define a relation between a *point* $A\in\mathfrak{W}_1$ and a *symplecton* $\Phi\in\mathfrak{W}_4$

as follows,

A is incident to Φ if there is a *symplecton* Φ^* with $\Phi = A\Phi^*$.

The set of *points* which are incident to a fixed *symplecton* has the structure of symplectic geometry.

If A is jointed with B for $A, B \in \mathfrak{W}_1$, there exits a subset $\{\Phi \in \mathfrak{W}_4 | \Phi$ is incident to A and $B\}$ and for an element Φ of this set there is a unique *line* in the sense of symplectic geometry such that the *line* contains A and B. The set which the *line* determines in $\mathfrak{W}_1$ does not depend on the choice of Φ. So we can define the notion of the *line* in metasymplectic geometry. We denote the set of *lines* in metasymplectic geometry by $\mathfrak{W}_2$. The algebraic group $G^{[1]}$ acts on $\mathfrak{W}_4$ transitively. The algebraic group $G^{[2]}$ which is contained by the stabilizer of $\Phi \in \mathfrak{W}_4$ in $G^{[1]}$ acts on the set of lines $\mathfrak{C}$. So the algebraic group $G^{[1]}$ acts on $\mathfrak{W}_2$ transitively.

If A, B and C for A, B, $C \in \mathfrak{W}_1$ are jointed with each other and there is no *line* that contains all *points* of $\{A, B, C\}$, there exists an element $\Phi \in \mathfrak{W}_4$ such that Φ is incident to A, B and C and for each Φ there is a unique *plane* in the sense of symplectic geometry such that the *plane* contains A, B and C. The set which the *plane* determines in $\mathfrak{W}_1$ does not depend on the choice of Φ.

We denote the set of *planes* in metasymplectic geometry by $\mathfrak{W}_3$. The algebraic group $G^{[1]}$ acts on $\mathfrak{W}_4$ transitively. The algebraic group $G^{[2]}$ which is contained by the stabilizer of $\Phi \in \mathfrak{W}_4$ in $G^{[1]}$ acts on $\mathfrak{M}$. So the algebraic group $G^{[1]}$ acts on $\mathfrak{W}_3$ transitively. The dimensions of $\mathfrak{W}_1$, $\mathfrak{W}_2$, $\mathfrak{W}_3$ and $\mathfrak{W}_4$ are known as follows,

	$\boldsymbol{R}_C$	C_C	H_C	O_C
$\mathfrak{W}_1$	15	24	42	78
$\mathfrak{W}_2$	20	31	53	97
$\mathfrak{W}_3$	20	29	47	83
$\mathfrak{W}_4$	15	21	33	57

For $\Phi \in \mathfrak{W}_4$, $\{\Phi\} \subset \mathfrak{W}_1$ is defined by,

$$\{\Phi\} := \{A \in \mathfrak{W}_1 | A \text{ is incident to } \Phi\}.$$

Proposition 2. *If Φ_1 is jointed with Φ_2 for Φ_1, $\Phi_2 \in \mathfrak{W}_4$, $\{\Phi_1\} \cap \{\Phi_2\}$ is a plane. Moreover for any plane there are symplecta Φ_1, $\Phi_2 \in \mathfrak{W}_4$ such that $\{\Phi_1\} \cap \{\Phi_2\}$ is the plane.*

Proposition 3. *For symplecta Φ_1, $\Phi_2 \in \mathfrak{W}_4$, Φ_1 is jointed with Φ_2 if and only if $\mathfrak{W}_4$ contains the usual projective line over $\boldsymbol{C}$ which is spanned by Φ_1 and Φ_2 in $\boldsymbol{P}(\mathfrak{R}_4)$.*

Let $L(\mathfrak{W}_4)$ be the set of usual projective lines over $\boldsymbol{C}$ in $\boldsymbol{P}(\mathfrak{R}_4)$ which $\mathfrak{W}_4$ contains.

Proposition 4. *For a plane we define a subset in $\mathfrak{W}_4$ as the set of symplecta which contain the plane. The subset is a projective line over $\boldsymbol{C}$, and defines a element* *of* $L(\mathfrak{W}_4)$. *Then there is a one-to-one correspondence between the elements of* $L(\mathfrak{W}_4)$ *and the elements of* $\mathfrak{W}_3$.

Remark 2. Let $L(\mathfrak{M})$ be the set of usual projective lines over $\boldsymbol{C}$ in $\boldsymbol{P}(\mathfrak{B})$ which $\mathfrak{M}$ contains. Then similarly there is a one-to-one correspondence between the elements of $L(\mathfrak{M})$ and the elements of $\mathfrak{E}$.

Proposition 5. *Let* Φ_1, Φ_2, Φ_3 *be elements of* $\mathfrak{W}_4$. *If* Φ_1, Φ_2, Φ_3 *for* $i, j=1,2,3$ *are jointed with each other and* Φ_1, Φ_2, Φ_3 *are linearly independent,* $\{\Phi_1\}\cap\{\Phi_2\}\cap\{\Phi_3\}$ *is a line. Moreover for any line there are symplecta* Φ_1, Φ_2, $\Phi_3\in\mathfrak{W}_4$ *such that* $\{\Phi_1\}\cap\{\Phi_2\}\cap\{\Phi_3\}$ *is the line.*

Let $P(\mathfrak{W}_4)$ be the set of usual projective plane over $\boldsymbol{C}$ in $\boldsymbol{P}(\mathfrak{N}_4)$ which $\mathfrak{W}_4$ contains.

Proposition 6. *For a line we define a subset in* $\mathfrak{W}_4$ *as the set of symplecta which contain the line. The subset is a projective plane over* $\boldsymbol{C}$, *and defines a element of* $P(\mathfrak{W}_4)$. *Then there is a one-to-one correspondence between the elements of* $P(\mathfrak{W}_4)$ *and the elements of* $\mathfrak{W}_2$.

H, Freudenthal called the geometry of *points, lines, planes* and *symplecta* in $\mathfrak{W}_1$ *metasymplectic geometry.*

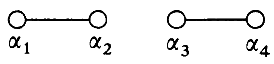
For convenience we list up symbols of spaces in the following table.

	points	*lines*	*planes*	*symplecta*
projective geometry	$\mathfrak{B}$	$\mathfrak{B}^*$		
symplectic geometry	$\mathfrak{N}$	$\mathfrak{E}$	$\mathfrak{M}$	
metasymplectic geometry	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$

3. Freudenthal's geometry and generalized adjoint varieties

3.1. Summury of results

Let G be a complex semisimple algebraic group with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra, and $\{\alpha_1, \cdots, \alpha_n\}$ simple roots with respect to $\mathfrak{h}$. In the case of $A_2 \oplus A_2$ we use the numbering of simple roots as follows.



In other cases we follow the notation of simple roots in [B].

A set of fundamental weights $\{\alpha_1^*, \cdots, \alpha_n^*\}$ is defined by,

$$\frac{2(\alpha_i^*, \alpha_j)}{(\alpha_i, \alpha_j)} = \delta_{ij}.$$

Here (,) is Killing form. If an irreducible representation of G has a highest weight λ, we denote it by V_λ.

From now on we describe several observations on Freudenthal's geometry and generalized adjoint varieties. Proofs will be given in 3.2, 3.3.

Proposition 7. *Each $\mathfrak{W}_1$ is a generalized adjoint variety which is obtained from the representation $V_{2\alpha_4^*}$, $V_{\alpha_1^*+\alpha_6^*}$, $V_{\alpha_6^*}$ or $V_{\alpha_1^*}$ of the adjoint group $G^{[1]}$ of $\mathfrak{R}_4$ according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively.*

Proposition 8. *Each $\mathfrak{W}_2$ is a generalized adjoint variety which is obtained from the representation $V_{2\alpha_3^*}$, $V_{\alpha_3^*+\alpha_5^*}$, $V_{\alpha_4^*}$ or $V_{\alpha_6^*}$ of the adjoint group $G^{[1]}$ of $\mathfrak{R}_4$ according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively.*

Proposition 9. *Each $\mathfrak{W}_3$ is a generalized adjoint variety which is obtained from the representation $V_{\alpha_2^*}$, $V_{\alpha_4^*}$, $V_{\alpha_3^*}$ or $V_{\alpha_7^*}$ of the adjoint group $G^{[1]}$ of $\mathfrak{R}_4$ according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively.*

Remark 3. From 2.4.2 each $\mathfrak{W}_4$ is a generalized adjoint variety which is obtained from the adjoint representation $V_{\alpha_1^*}$, $V_{\alpha_2^*}$, $V_{\alpha_1^*}$ or $V_{\alpha_8^*}$ of adjoint group $G^{[1]}$ of $\mathfrak{R}_4$ according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively.

For convenience we write a correspondence between metasymplectic geometry and representations in a following table.

	$\boldsymbol{R_C}$	$\boldsymbol{C_C}$	$\boldsymbol{H_C}$	$\boldsymbol{O_C}$
$\mathfrak{W}_1$	$2\alpha_4^*$	$\alpha_1^*+\alpha_6^*$	α_6^*	α_1^*
$\mathfrak{W}_2$	$2\alpha_3^*$	$\alpha_3^*+\alpha_5^*$	α_4^*	α_6^*
$\mathfrak{W}_3$	α_2^*	α_4^*	α_3^*	α_7^*
$\mathfrak{W}_4$	α_1^*	α_2^*	α_1^*	α_8^*

Proposition 10. *Each $\mathfrak{W}$ is a generalized adjoint variety which is obtained from the representation $V_{\alpha_1^*}$, $V_{\alpha_3^*}$, $V_{\alpha_5^*}$ or $V_{\alpha_7^*}$ of $G^{[2]}$ according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively. In the type D_6 we cannot distinguish $V_{\alpha_5^*}$ and $V_{\alpha_6^*}$. So in the case of $\boldsymbol{H_C}$ we can choose the representation $V_{\alpha_6^*}$ instead of $V_{\alpha_5^*}$.*

Proposition 11. *Each $\mathfrak{C}$ is a generalized adjoint variety which is obtained from the representation $V_{2\alpha_2^*}$, $V_{\alpha_2^*+\alpha_4^*}$, $V_{\alpha_4^*}$ or $V_{\alpha_6^*}$ of $G^{[2]}$ according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively.*

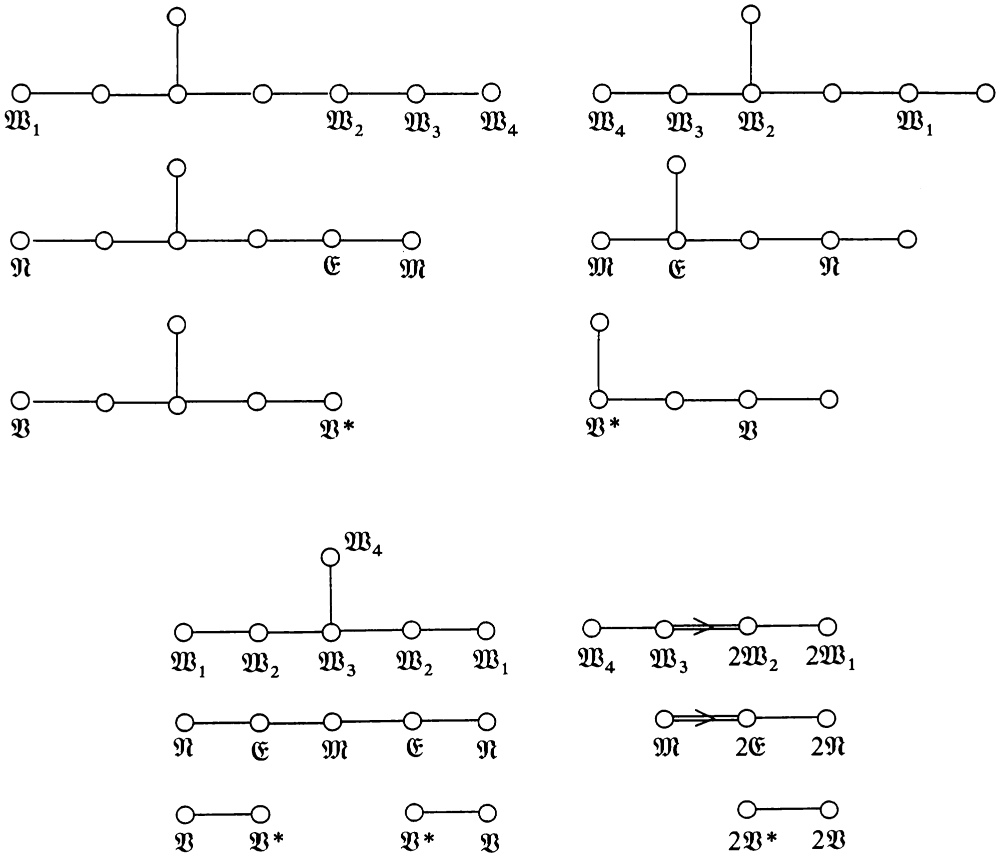
Remark 4. Each $\mathfrak{N}$ is a generalized adjoint variety which is obtained from the adjoint representation $V_{2\alpha_1^*}$, $V_{\alpha_1^*+\alpha_3^*}$, $V_{\alpha_2^*}$ or $V_{\alpha_1^*}$ of $G^{[2]}$ according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively.

For convenience we write a correspondence between symplectic geometry and representations in a following table.

	$\boldsymbol{R}_C$	$\boldsymbol{C}_C$	$\boldsymbol{H}_C$	$\boldsymbol{O}_C$
$\mathfrak{N}$	$2\alpha_1^*$	$\alpha_1^*+\alpha_5^*$	α_2^*	α_1^*
$\mathfrak{E}$	$2\alpha_2^*$	$\alpha_2^*+\alpha_4^*$	α_4^*	α_6^*
$\mathfrak{M}$	α_3^*	α_3^*	α_5^*	α_7^*

Remark 5. In projective geometry over $\boldsymbol{O}_C$ we can regard $\mathfrak{J}$ as $V_{\alpha_1^*}$ and also as $V_{\alpha_6^*}$ By Remark 1 it is better that we distinguish the projectivization of the representation in which $\mathfrak{V}$ is embedded from the projectivization of the representation in which $\mathfrak{V}^*$ is embedded. So we shall consider $\mathfrak{V}$ as the adjoint variety of $V_{\alpha_1^*}$ and $\mathfrak{V}^*$ as the adjoint variety of $V_{\alpha_6^*}$. Similarly we shall consider $\mathfrak{V}$ as a generalized adjoint variety which is obtained from the representation $V_{2\alpha_1^*}$, $V_{\alpha_1^*+\alpha_4^*}$, $V_{\alpha_2^*}$ or $V_{\alpha_1^*}$ of $G^{[3]}$ and $\mathfrak{V}^*$ as a generalized adjoint variety which is obtained from the representation $V_{2\alpha_2^*}$, $V_{\alpha_2^*+\alpha_3^*}$, $V_{\alpha_4^*}$ or $V_{\alpha_6^*}$ of $G^{[3]}$ according to $\boldsymbol{R}_C$, $\boldsymbol{C}_C$, $\boldsymbol{H}_C$ or $\boldsymbol{O}_C$ respectively.

We show results stated above in Dynkin diagrams.



When there are same symbols in a diagram, we consider that the homogeneous variety is obtained from an irreducible representation with summed highest weight. For example in metasymplectic geometry over $\boldsymbol{C_C}$ homogeneous variety $\mathfrak{W}_1$ is obtained from an irreducible representation with the highese weight $\alpha_1^* + \alpha_6^*$, and in metasymplectic geometry over $\boldsymbol{R_C}$ homogeneous variety $\mathfrak{W}_1$ is obtained from an irreducible representation with the highest weight $2\alpha_4^*$

3.2. Metasymplectic geometry and generalized adjoint varieties

In this section we prove the results in 3.1 on metasymplectic geometry. The dimension of a generalized adjoint variety is given as follows. (see e.g. [FH])

Lemma 1. *Let X be a generalized adjoint variety which is obtained from an irreducible representation V_λ of a simple algebraic group G with highest weight $\lambda = \Sigma_i n_i \alpha_i^*$. The dimension of X is equal to the cardinal number of the set $\{\alpha \in \Phi_+ \mid$ For some i with $n_i \neq 0$, α contains component of $\alpha_i\}$. Here Φ_+ is the set of positive roots.*

Each variety $\mathfrak{W}_4$ of symplecta is the adjoint variety of $G^{[1]}$.

By definition $\mathfrak{W}_1$ is a generalized adjoint variety which is obtained from an irreducible representation of $G^{[1]}$ in $\mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$. According to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ the irreducible decomposition of $\mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ is as follows,

$$\begin{aligned}
&\boldsymbol{R_C} \quad \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) = V_{\alpha_1^*} \oplus V_{\alpha_2^*} \oplus V_{2\alpha_4^*} \oplus V_{2\alpha_1^*} \oplus \boldsymbol{C},\\
&\boldsymbol{C_C} \quad \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) = V_{\alpha_2^*} \oplus V_{\alpha_4^*} \oplus V_{\alpha_1^* + \alpha_6^*} \oplus V_{2\alpha_2^*} \oplus \boldsymbol{C},\\
&\boldsymbol{H_C} \quad \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) = V_{\alpha_1^*} \oplus V_{\alpha_3^*} \oplus V_{\alpha_6^*} \oplus V_{2\alpha_1^*} \oplus \boldsymbol{C},\\
&\boldsymbol{O_C} \quad \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) = V_{\alpha_8^*} \oplus V_{\alpha_7^*} \oplus V_{\alpha_1^*} \oplus V_{2\alpha_8^*} \oplus \boldsymbol{C}.
\end{aligned}$$

Then there exists a unique irreducible representation in the irreducible decomposition such that the dimension of the generalized adjoint variety coincides with the dimension of $\mathfrak{W}_1$. The irreducible representation is $V_{2\alpha_4^*}$, $V_{\alpha_1^* + \alpha_6^*}$, $V_{\alpha_6^*}$ or $V_{\alpha_1^*}$ according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively. So each $\mathfrak{W}_1$ is a generalized adjoint variety which is obtained from $V_{2\alpha_4^*}$, $V_{\alpha_1^* + \alpha_6^*}$, $V_{\alpha_6^*}$ or $V_{\alpha_1^*}$ according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively.

The grassmanian variety which consists of all r-dimensional subspaces in a n-dimensional vector space V is denoted by $Gr(r, V)$. By Proposition 4 we can regard $\mathfrak{W}_3$ as a subset of $Gr(2, \mathfrak{g}) \subset \boldsymbol{P}(\overset{2}{\Lambda}\mathfrak{g})$. So $\mathfrak{W}_3$ is a generalized adjoint variety which is obtained from an irreducible representation of $G^{[1]}$ in $\overset{2}{\Lambda}\mathfrak{g}$. The irreducible decomposition of $\overset{2}{\Lambda}\mathfrak{g}$ is as follows according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively,

$$\begin{aligned}
&\boldsymbol{R_C} && \overset{2}{\Lambda}\mathfrak{g}=\mathfrak{g}\oplus V_{\alpha_2^*},\\
&\boldsymbol{C_C} && \overset{2}{\Lambda}\mathfrak{g}=\mathfrak{g}\oplus V_{\alpha_4^*},\\
&\boldsymbol{H_C} && \overset{2}{\Lambda}\mathfrak{g}=\mathfrak{g}\oplus V_{\alpha_3^*},\\
&\boldsymbol{O_C} && \overset{2}{\Lambda}\mathfrak{g}=\mathfrak{g}\oplus V_{\alpha_7^*},
\end{aligned}$$

Then there exists a unique irreducible representation in the irreducible decomposition such that the dimension of the generalized adjoint variety coincides with the dimension of $\mathfrak{W}_3$. Each irreducible representation is $V_{\alpha_2^*}$, $V_{\alpha_4^*}$, $V_{\alpha_3^*}$ or $V_{\alpha_7^*}$ accoding to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively. So Each $\mathfrak{W}_3$ is a generalized adjoint variety which is obtained from $V_{\alpha_2^*}$, $V_{\alpha_4^*}$, $V_{\alpha_3^*}$ or $V_{\alpha_7^*}$ according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively.

It follows from Proposition 6 that we can regard $\mathfrak{W}_2$ as a subset of $Gr(3, \mathfrak{g})\subset \boldsymbol{P}(\overset{3}{\Lambda}\mathfrak{g})$. So $\mathfrak{W}_2$ is a generalized adjoint variety which is obtained from an irreducible representation of $G^{[1]}$ in $\overset{3}{\Lambda}\mathfrak{g}$. The irreducible decomposition of $\overset{3}{\Lambda}\mathfrak{g}$ is as follows according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively,

$$\begin{aligned}
&\boldsymbol{R_C} && \overset{3}{\Lambda}\mathfrak{g}=\boldsymbol{C}\oplus V_{2\alpha_4^*}\oplus V_{\alpha_2^*}\oplus V_{2\alpha_1^*}\oplus V_{2\alpha_3^*},\\
&\boldsymbol{C_C} && \overset{3}{\Lambda}\mathfrak{g}=\boldsymbol{C}\oplus V_{\alpha_1^*+\alpha_6^*}\oplus V_{\alpha_4^*}\oplus V_{2\alpha_2^*}\oplus V_{\alpha_3^*+\alpha_5^*},\\
&\boldsymbol{H_C} && \overset{3}{\Lambda}\mathfrak{g}=\boldsymbol{C}\oplus V_{\alpha_6^*}\oplus V_{\alpha_3^*}\oplus V_{2\alpha_1^*}\oplus V_{\alpha_4^*},\\
&\boldsymbol{O_C} && \overset{3}{\Lambda}\mathfrak{g}=\boldsymbol{C}\oplus V_{\alpha_1^*}\oplus V_{\alpha_7^*}\oplus V_{2\alpha_8^*}\oplus V_{\alpha_6^*}.
\end{aligned}$$

Then there exists a unique irreducible representation in the irreducible decomposition such that the dimension of the generalized adjoint variety coincides with the dimension of $\mathfrak{W}_2$. Each irreducible representation is $V_{2\alpha_3^*}$, $V_{\alpha_3^*+\alpha_5^*}$, $V_{\alpha_4^*}$ or $V_{\alpha_6^*}$ according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively. So each $\mathfrak{W}_2$ is a generalized adjoint variety which is obtained from $V_{2\alpha_3^*}$, $V_{\alpha_3^*+\alpha_5^*}$, $V_{\alpha_4^*}$ or $V_{\alpha_6^*}$ according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively.

3.3. Symplectic geometry and generalized adjoint varieties

In this section we give a proof of results in 3.1 on symplectic geometry. The variety $\mathfrak{N}$ of points in symplectic geometry is a adjoint variety of $G^{[2]}$. The variety $\mathfrak{M}$ of planes in symplectic geometry is a generalized adjoint variety which is obtained from the irreducible representation $\mathfrak{B}$ of $G^{[2]}$. By the calculation of the dimension the representation $\mathfrak{B}$ is $V_{\alpha_3^*}$, $V_{\alpha_3^*}$, $V_{\alpha_5^*}$ or $V_{\alpha_7^*}$ as a representation of $G^{[2]}$ according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively.

By Remark 2 we can regard the variety $\mathfrak{C}$ of lines in symplectic geometry as a subset of $Gr(2, \mathfrak{R}) \subset \boldsymbol{P}(\overset{2}{\Lambda}\mathfrak{R})$. So $\mathfrak{E}$ is a generalized adjoint variety which is obtained from an irreducible representation of $G^{[2]}$ in $\overset{2}{\Lambda}\ \mathfrak{R}$. The irreducible decomposition of $\overset{2}{\Lambda}\ \mathfrak{R}$ is as follows according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively,

$$\boldsymbol{R_C} \quad \overset{2}{\Lambda}\mathfrak{R} = \boldsymbol{C} \oplus V_{2\alpha_2^*},$$

$$\boldsymbol{C_C} \quad \overset{2}{\Lambda}\mathfrak{R} = \boldsymbol{C} \oplus V_{\alpha_3^* + \alpha_5^*},$$

$$\boldsymbol{H_C} \quad \overset{2}{\Lambda}\mathfrak{R} = \boldsymbol{C} \oplus V_{\alpha_4^*},$$

$$\boldsymbol{O_C} \quad \overset{2}{\Lambda}\mathfrak{R} = \boldsymbol{C} \oplus V_{\alpha_6^*},$$

Then there exists a unique irreducible representation in the irreducible decomposition such that the dimension of the generalized adjoint variety coincides with the dimension of $\mathfrak{E}$. Each irreducible representation is $V_{2\alpha_2^*}$, $V_{\alpha_3^*+\alpha_5^*}$, $V_{\alpha_4^*}$ or $V_{\alpha_3^*}$ according to $\boldsymbol{R_C}$, $\boldsymbol{C_C}$, $\boldsymbol{H_C}$ or $\boldsymbol{O_C}$ respectively.

Remark 6. We can obtain same homogeneous varieties from highest weights which are different in constant times. But by the proof as above we consider that highest weights which are obtatined above are natural.

4. Dual varieties of adjoint varieties

Let V be a finite dimensional complex vector space, $\boldsymbol{P}(V)$ a complex projective space with a projection,

$$\pi : \mathrm{V}\backslash 0 \to \boldsymbol{P}(V)$$

and $X \subset \boldsymbol{P}(V)$ an irreducible projecrtive variety. We can define a cone variety Cone$X$ and a dual variety $\check{X}$ of X in $\boldsymbol{P}(V^*)$ as follows,

$$\mathrm{Cone}\, X := \overline{\{v \in V \mid \pi(v) \in X\}},$$

$$\check{X} := \overline{\bigcup_{x \in X} \{H \in \boldsymbol{P}(V^*) \mid H \supset T_x X\}}.$$

It is studied in [KM] which fundamental representatons give the generalized adjoint varieties whose dual varieties are hypersurfaces. In paticular it is known that the dual varieties of the adjoint varieties are hypersurfaces and that the dual varieties of $\mathfrak{W}_4$, $\mathfrak{M}$ and $\mathfrak{V}^*$ are hypersurfaces in $\boldsymbol{P}(V^*)$.

It is known in [M1,2] that for each $\mathfrak{V}^*$ the degree of the dual variety is three and that for each $\mathfrak{M}$ the degree of the dual variety is four.

Remark 7. If we identify $\mathfrak{J}^*$ with $\mathfrak{J}$ by the inner product $(X, Y) = \mathrm{tr}(X \circ Y)$, the defining equation of the dual variety of $\mathfrak{V}^*$ is as follows,

$$\det(X)=0.$$

Since X is 3×3 matrix, the degree of the dual variety of $\mathfrak{V}^*$ is three.

The variety $\mathfrak{V}^*$ is the set of *lines* in projective geometry and the variety $\mathfrak{M}$ is the set of *planes* in symplectic geometry.

In the remaining of this paper we calculate the degree of the dual variety of $\mathfrak{W}_4$ which is the set of *symplecta* in metastmplectic geometry. Each $\mathfrak{W}_4$ is a adjoint variety of an adjoint group $G^{[1]}$ of $\mathfrak{R}_4$. Here we calculate the degree of dual varieties of adjoint varieties.

Let G be a complex, connected simple algebraic group with Lie algebra $\mathfrak{g}$ and X the adjoint variety of $\mathfrak{g}$. Moreover let

$$\mathfrak{g}=\mathfrak{h}\oplus(\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha)$$

be a root decomposition with respect to a cartan subalgebra $\mathfrak{h}$. Here Φ is a root system. We fix a simple system Δ of roots. We denote the set of positive (resp. negative) roots by Φ_+(resp. Φ_-) and Weyl group by W.

For a set S we denote the cardinal number of S by $\sharp S$. When we consider a nonzero element x in a vector space as a element in a projective space, we shall use same symbol for it.

Theorem 1. *Let G be a complex, connected simple algebraic group with Lie algebra $\mathfrak{g}$, λ the highest root, W the Weyl group, X the adjoint variety of $\mathfrak{g}$, and $\check{X}$ the dual variety of X. Then we have following formula,*

$$\deg\check{X}=\sharp(W\cdot\lambda).$$

Proof. Let $x_\lambda\in\mathfrak{g}_\lambda$ be a highest root vector. By the definition the adjoint variety is

$$X=G\cdot x_\lambda\subset\boldsymbol{P}(\mathfrak{g}).$$

By using Killing form $(\ ,\ )$ we identify $\mathfrak{g}^*$ with $\mathfrak{g}$. Then

$$\check{X}=\overline{\bigcup_{x\in X}\{H\in\boldsymbol{P}(\mathfrak{g})\,|\,(H,[\mathfrak{g},x])=0\}}$$

$$=\overline{G\cdot\{H\in\boldsymbol{P}(\mathfrak{g})\,|\,(H,[\mathfrak{g},x_\lambda])=0\}}.$$

There is an element $x_{-\lambda}$ in $\mathfrak{g}_{-\lambda}$ and an element h_λ in $\mathfrak{h}$ such that following relations hold,

$$[x_\lambda,x_{-\lambda}]=h_\lambda,$$

$$[h_\lambda,x_\lambda]=2x_\lambda,$$

$$[h_\lambda, x_{-\lambda}] = -2x_{-\lambda}.$$

We have

$$[\mathfrak{g}, x_\lambda] \cap \mathfrak{h} = \boldsymbol{C} \cdot [x_\lambda, x_{-\lambda}] = \boldsymbol{C} \cdot h_\lambda,$$

$$[\mathfrak{g}, x_\lambda] \subset \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha \Big).$$

We define a variety $\check{X}_\lambda$ as follows,

$$\check{X}_\lambda := \{ H \in \boldsymbol{P}(\mathfrak{g}) \mid (H, [\mathfrak{g}, x_\lambda]) = 0 \}.$$

From above observations a following relation holds,

$$\check{X}_\lambda \supset \mathfrak{h}_\lambda := \{ h \in \mathfrak{h} \mid (h, h_\lambda) = 0 \}.$$

When a group A acts on a complex affine variety Y, we denote the ring of invariant polynomial functions by $\boldsymbol{C}[Y]^A$. By Chevalley's theorem we obtain an isomorphism,

$$\Psi^* : \boldsymbol{C}[\mathfrak{h}]^W \to \boldsymbol{C}[\mathfrak{g}]^G,$$

and the inverse morphism of the isomorphism is a restriction map. We consider the comoposition of the categorical quotient $\mathfrak{g} \to \mathfrak{g}/G$ and the isomorphism $\mathfrak{g}/G \to \mathfrak{h}/W$. We call this map the adjoint quotient and denote by

$$\Psi : \mathfrak{g} \to \mathfrak{h}/W.$$

We introduce following notation for an inclusion and a quotient,

$$i_{\check{X}} : \mathrm{Cone}\, X \hookrightarrow \mathfrak{g},$$

$$p : \mathfrak{h} \to \mathfrak{h}/W.$$

By the irreducibility of $\check{X}$,

$$\mathrm{Im}(\Psi \circ i_{\check{X}}) = p(\mathfrak{h}_\lambda) \text{ or } \mathfrak{h}/W.$$

If $\mathrm{Im}(\Psi \circ i_{\check{X}}) = \mathfrak{h}/W$, the dimension of general fibers of $\Psi \circ i_{\check{X}}$ is $\dim \mathfrak{g} - \mathrm{rank}\, \mathfrak{g} - 1$. But there exists a unique regular orbit in each fiber such that the complement of it has codimension two in the fiber. So $\Psi \circ i_{\check{X}}$ is not surjective.

$$\mathrm{Im}(\Psi \circ i_{\check{X}}) = p(\mathfrak{h}_\lambda).$$

Then we have,

$$\mathrm{Cone}\, \check{X} = \Psi^{-1} \circ p(\mathfrak{h}_\lambda).$$

We consider a W-invariant function $f(h)$ on $\mathfrak{h}$

$$f(h) = \prod_{\mu \in W \cdot \lambda} \mu(h).$$

We have following relation,

$$p^{-1}\circ p(\mathfrak{h}_{\lambda})=\{h\in\mathfrak{h}|f(h)=0\}.$$

Since the inverse of the isomorphism Ψ^* is a restriction map, there exists a unique element F in $\boldsymbol{C}[\mathfrak{g}]^G$ such that the restriction of F to $\mathfrak{h}$ equals to f and as sets

$$\mathrm{Cone}\,\check{X}=\{x\in\mathfrak{g}\,|\,F(x)=0\}.$$

Since $\check{X}$ is an irreducible hypersurface, there exists an irreducible polynomial $\tilde{F}$ on $\mathfrak{g}$ and a positive integer n such that

$$F=\tilde{F}^n.$$

Because F is G-invariant, we have

$$g\tilde{F}=\mu(g)\tilde{F}\quad g\in G, \mu(g)\in\boldsymbol{C} \text{ with } \mu(g)^n=1.$$

Since G is connected, $\mu(g)$ is constant. So $\tilde{F}$ is G-invariant too. Then $n=1$. So F is irreducible and

$$\deg(F)=\deg(f)=\sharp(W\cdot\lambda).$$

Remark 8. In the case of A_n the polynomial F is given by the discriminant of the characteristic polynomial.

Corollary 1. *The degrees of* $\mathfrak{W}_4$ are as follows.

	R_C	C_C	H_C	O_C
The degree of $\mathfrak{W}_4$	24	72	126	240

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