

On regular surfaces of general type with $p_g=3$ and non-birational bicanonical map

By

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Abstract

In this paper we classify all regular surfaces of general type with $p_g=3$ and non-birational bicanonical map which do not have a pencil of curves of genus 2.

Introduction

The bicanonical map of a smooth curve of genus $g \geq 2$ fails to be birational if and only if $g=2$. This phenomenon has an echo in dimension 2, namely if a smooth surface S of general type has a pencil of curves of genus 2, i.e. it has a rational map to a curve whose general fibre F is a smooth irreducible curve of genus 2, then the line bundle $\mathcal{O}_S(K_S) \otimes \mathcal{O}_F$ is special on F , and therefore the bicanonical map ϕ of S cannot be birational.

We shall refer to the above exception to the birationality of the bicanonical map ϕ as to the *standard case*. A *non-standard case* will be the one of a surface of general type S for which ϕ is not birational, but there is no pencil of curves of genus 2. Bombieri (see [B], thm. 5) proved that if $K^2 \geq 10$ and $p_g \geq 6$, then all surfaces for which ϕ is not birational present the standard case. He also gave an example (see [B], pg. 194), already found by Du Val (see [DV]), of a minimal surface with $K^2=9$, $p_g=6$ exhibiting the non-standard case. Later on I. Reider proved in [R] that the hypothesis $K^2 \geq 10$ alone ensures that one has a standard case if ϕ is not birational. From these results it follows that there is only a finite number of families of minimal surfaces of general type presenting the non-standard case for the non birationality of the bicanonical map.

In the case of regular surfaces, i.e. those surfaces S for which $q:=h^1(S, \mathcal{O}_S)=0$, the problem of classifying the non-standard cases with $p_g \geq 2$ has been considered by Du Val in the quoted paper [DV]. He implicitly makes a few rather restrictive assumptions, under which however he is able to come to a detailed classification (see theorem (1.2) below).

Later on Xiao Gang studied the same problem in the paper [X1]. He mainly took the point of view of the projective study of the image of the bicanonical

map. He thus found a list of numerical possibilities for the invariants of the cases which might occur, determined some properties of the surfaces in question and gave some examples.

In recent years the non-standard cases with $p_g \geq 4$ have been classified in full detail in the paper [CFM], whose results essentially confirm the classification proposed by Du Val. In particular, all the non-standard cases with $p_g \geq 4$ are regular.

The irregular surfaces with $p_g = 3$ presenting the non-standard case have been studied in the paper [CCM]. It turns out that if S is a minimal irregular surface with $p_g = 3$ presenting the non-standard case for the non-birationality of the bicanonical map then S is isomorphic to the symmetric product of a smooth irreducible curve of genus 3, thus $p_g = q = 3$ and $K^2 = 6$.

More information on the problem and on the results quoted above are contained in the expository paper [C].

The purpose of this article is to finish the classification of the non-standard cases with $p_g = 3$, by dealing with the case of regular surfaces. We thus complete here the full classification of surfaces with $p_g \geq 3$ and with non birational bicanonical map.

Briefly, what we prove here is that, for minimal surfaces with $p_g = 3$ and $q = 0$, in addition to the non-standard cases obtained by specialization from the families described in [CFM] and besides the well-known case of double planes branched along octic curves, there is only one more family of non-standard cases, having $K^2 = 8$. All these cases are either the surfaces in Du Val's classification or specializations of those surfaces.

The present paper is organized as follows. In section 1 we describe in some detail the surfaces appearing in Du Val's classification theorem (1.2) and state our classification theorem (1.6), whose proof occupies the rest of the paper. Section 2 and 3, which are rather technical, are devoted to establishing various properties of the canonical curves of minimal regular surfaces of general type with non birational bicanonical map, and canonical system not composed with a pencil. In section 4 we prove that, in fact, the canonical system of a regular surface with $p_g = 3$ presenting the non-standard case is not composed with a pencil, so that the results from section 2 and 3 can be safely applied. Finally, in sections 5 and 6 we show that all surfaces with $p_g = 3$ and $q = 0$ presenting the non-standard case belong to the families of Du Val's examples described in section 1.

Various technical results, that are used repeatedly throughout the paper and can be of independent interest, are proved in an appendix which we put, for the reader's convenience, in section 7 at the end of the paper.

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0. Notation and conventions

We will denote by S a projective algebraic surface over the complex field. Usually S will be smooth, minimal, of general type.

We will say that S presents the *non-standard case* for non birationality of the bicanonical map, or simply that S presents the non-standard case, if S is a surface of general type with non birational bicanonical map and containing no pencil of curves of genus 2.

We denote by K_S , or simply by K if there is no possibility of confusion, a canonical divisor on S . We denote, as usual, by p_g the *geometric genus* of S and by $q:=q(S)=h^1(S, \mathcal{O}_S)$ the *irregularity* of S .

By a *curve* on S we mean an effective, non zero divisor on S . We denote by $p_a(C)$ the arithmetic genus of a curve C . Also $C \cdot D$ will denote the intersection number of the divisors C, D on S , and C^2 the self-intersection of the divisor C . We denote by \equiv the linear equivalence for divisors on S and by \sim the numerical equivalence. $|D|$ will be the complete linear system of the effective divisors $D' \equiv D$, and $\phi_D: S \rightarrow \mathbf{P}(H^0(S, \mathcal{O}_S(D)^\vee)) = |D|^\vee$ the natural rational map defined by $|D|$.

The bicanonical map ϕ_{2K} will be usually simply denoted by ϕ . If the bicanonical map is of degree 2 onto its image, there is an involution $\iota: S \rightarrow S$, such that for a general point $x \in S$, one has $\phi(x) = \phi(\iota(x))$. We will refer to ι as to the *bicanonical involution* of S .

If C is a curve on S and m is an integer, one says that C is *m-connected* if, for every decomposition $C = A + B$ with A and B curves, one has $A \cdot B \geq m$. Notice that this definition makes sense even if m is non-positive.

If x is a point of S , D is a divisor as above and n is a positive integer, we will denote by $|D - np|$ the linear subsystem of $|D|$ formed by all divisors in $|D|$ having at p a point of multiplicity at least n .

In general, if V is a complete variety and \mathcal{L} is a line bundle on V , one can consider the complete linear system $|\mathcal{L}|$ determined by \mathcal{L} . If V is a curve and \mathcal{L} has degree d and $\dim |\mathcal{L}| = r$, one says, as usual, that $|\mathcal{L}|$ is a g_d^r on V . An irreducible curve C of arithmetic genus $p_a(C) \geq 2$ is called *hyperelliptic* if it possesses a g_2^1 , which is then uniquely determined by the fact that the canonical series of C is composed with it.

We will say that a singularity of a curve lying on a smooth surface is of *type* $[n, n]$, or briefly a $[n, n]$ -point, if it is a n -tuple point which has another n -tuple point infinitely near to it. More generally we will speak of $[n, \dots, n]$ -points for an n -tuple point which has another n -tuple point infinitely near to it and so on, all the consecutive points lying on a given linear branch.

Finally, the symbol \simeq denotes in general an isomorphism between objects under consideration.

1. Surfaces with non birational bicanonical map

In this paragraph we will describe the surfaces with non birational bicanonical map, not possessing a pencil of curves of genus 2, which appear in our classification

theorem (1.6).

Du Val is probably the first author who dealt, in [DV], in a systematic way with the problem under consideration. He supposes that the bicanonical map is not birational for the surface S on which he implicitly makes the following assumptions:

- (a) the surface S is regular, i.e. $q=0$;
- (b) the general canonical curve $C \in |K|$ is smooth and irreducible.

(1.1) **Remark.** (i) Notice that if (a) and (b) above occur, then C is hyperelliptic. Conversely if the general canonical curve of a surface S is (smooth and) hyperelliptic, then clearly the bicanonical map cannot be birational, and, if $q=0$, its degree must be 2.

(ii) There is a typical situation in which the bicanonical map is not birational. Consider a double cover $\pi: S \rightarrow X$, with S, X smooth, irreducible, complete surfaces, with branch curve B on X . Then we have:

$$\pi_* \mathcal{O}_S \simeq \mathcal{O}_X \oplus \mathcal{O}_X(-\xi), \quad \mathcal{O}_S(K_S) \simeq \pi^*(\mathcal{O}_X(K_X + \xi))$$

where $\mathcal{O}_X(2\xi) \simeq \mathcal{O}_X(B)$. Hence:

$$\pi_* \mathcal{O}_S(2K_S) \simeq \mathcal{O}_X(2(K_X + \xi)) \oplus \mathcal{O}_X(2K_X + \xi)$$

Therefore $|2K_S|$ factors through p if $h^0(X, \mathcal{O}_X(2K_X + \xi)) = 0$. We will see various examples of this situation later on.

Du Val's result from [DV] is as follows:

(1.2) **Theorem.** *Let S be a smooth minimal surface of general type. Assume that $p_g \geq 3$, $q=0$, the general canonical curve $C \in |K|$ is smooth and irreducible and the bicanonical map on S is not birational. Then either we are in the standard case, i.e. S has a pencil of curves of genus 2, or S is one of the surfaces, so-called Du Val's examples, described below in the list of examples (1.4). Their invariants are shown in the following table:*

				K^2			
p_g	6	8*		9*			
		↓	↙	↓			
	5	7		8			
		↓	↙	↓			
	4	6		7	8*		
		↓	↙	↓	↘	↓	
	3	5		6	6	7	8* 2*

For each pair of invariants (p_g, K^2) in the above table, the corresponding Du Val's examples fill up an irreducible family of double covers of which we will describe the general member in (1.4) below. The explanation for the arrows in the table (1.3) is the following:

↓ means that one imposes a [3,3]-point (i.e. a triple point with an infinitely near triple point, namely with three coincident tangent lines) to the branch curve of the double cover. As it is well known, this operation drops the geometric genus and K^2 both by 1;

✓ means that one imposes a 4-tuple point to the branch curve of the double cover, which drops the geometric genus by 1 and K^2 by 2.

This also explains why we put the asterisks in the above table: the surfaces marked by them are the ones we really need to describe, since the others are obtained by imposing the aforementioned singularities to the branch curve. We will call the surfaces of these families *Du Val's ancestors*, because they generate all the other examples.

Now we come to the description of the Du Val's ancestors:

(1.4) **Examples.** All the examples S we are going to introduce now are double covers and present the non-standard case. Indeed by their description it is easy to rule out the presence of a genus 2 pencil (compare [C], pg. 63, where "line" has to be replaced by "conic"). Furthermore, by taking into account the double cover representation, it is easy to see that, for all the examples, one has $q=0$. We leave all this to the reader and refer to [CFM] for the details. For further information about these examples see also [C].

(i) $p_g=6, K^2=8$.

This is the double cover of the plane $\pi: S \rightarrow \mathbf{P}^2$, branched along a smooth curve B of degree 10. One has:

$$\mathcal{O}_S(K) \simeq \pi^*(\mathcal{O}_{\mathbf{P}^2}(2)), \quad \pi_*\mathcal{O}_S \simeq \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(-5)$$

and therefore

$$\pi_*\mathcal{O}_S(2K) \simeq \mathcal{O}_{\mathbf{P}^2}(-1) \oplus \mathcal{O}_{\mathbf{P}^2}(4)$$

Hence we are in the situation described in remark (1.1, ii).

(ii) $p_g=6, K^2=9$.

This example can also be found in [B], pg. 193. Consider the diagram:

$$\begin{array}{ccc} S' & \xrightarrow{p} & S \\ f \downarrow & & \downarrow g \\ \mathbf{F}_2 & \xrightarrow{\varphi} & Q_0 \end{array}$$

Here \mathbf{F}_2 is the Hirzebruch surface $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-2))$. We denote by C_0 the (-2)-section of \mathbf{F}_2 , by F the ruling and by $H=C_0+2F$ the divisor class which maps \mathbf{F}_2 to \mathbf{P}^3 as a quadric cone Q_0 . Furthermore:

(a) $f: S' \rightarrow \mathbf{F}_2$ is the double cover branched along a smooth curve $B=C_0+B'$ where B'

is a general member in $|7H|$;

(b) $p: S' \rightarrow S$ is the contraction to a point x of the exceptional curve E of S' which lies above C_0 in the cover $f: S' \rightarrow F_2$;

(c) f induces the double cover $g: S \rightarrow Q_0$ of the quadric cone Q_0 , which is branched at the vertex v and along a general septic surface section of Q_0 .

One has:

$$\mathcal{O}_{S'}(K) \simeq f^*(\mathcal{O}_{F_2}(2C_0 + 3F)), \quad f_*\mathcal{O}_{S'} \simeq \mathcal{O}_{F_2} \oplus \mathcal{O}_{F_2}(-4C_0 - 7F)$$

and therefore:

$$f_*\mathcal{O}_{S'}(2K) \simeq \mathcal{O}_{F_2}(4C_0 + 6F) \oplus \mathcal{O}_{F_2}(-F)$$

i.e. we are in the situation described in remark (1.1, ii), hence the bicanonical map of S' factors through f . Notice that the pull-back to S of the lines of the ruling of Q_0 vary in a linear pencil $|D|$ of hyperelliptic curves of genus 3, with the base point x , such that $3D$ is linearly equivalent to K_S . The point x is a Weierstrass point of the g_2^1 on C and therefore $|2K|$ cuts out on C the triple of the g_2^1 which is still composed with the g_2^1 itself. Therefore the presence of the pencil $|C|$ is responsible for the non birationality of the bicanonical map of S (compare proposition (4.4) below).

(iii) $p_g = 4$, $K^2 = 8$.

This surface S is extensively described in [CFM], to which we defer the reader for further information about it. The surface S is defined by the following commutative diagram:

$$\begin{array}{ccc} S' & \xrightarrow{p} & S \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{\pi} & Q \simeq \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3 \end{array}$$

where:

(a) $\pi: Y \rightarrow Q$ is the double cover of a smooth quadric $Q \subset \mathbf{P}^3$ branched along a curve B of type (10,10) with four ordinary 6-tuple points, of the form $B = \eta_1 + \eta_2 + \eta'_1 + \eta'_2 + B'$, where η_1, η_2 are two distinct lines of one ruling of Q , η'_1, η'_2 are two distinct lines of the other ruling, and B' is a curve of type (8,8) with 4 ordinary quadruple points at the points $x_{ij} = \eta_i \cap \eta'_j$, $i, j = 1, 2$, and smooth elsewhere;

(b) Y has four singular points y_{ij} over the points x_{ij} , $i, j = 1, 2$. The surface S' is the minimal resolution of the singularities of Y , but it is not a minimal surface. It has four exceptional divisors corresponding to the lines $\eta_1, \eta_2, \eta'_1, \eta'_2$ of Q . By blowing them down to four points p_1, p_2, p'_1, p'_2 , one has the surface S .

Let Z be the cycle of degree 12 defined by the intersection of the square of the ideals of the points x_{ij} , $i, j = 1, 2$ in Q . Standard properties of double covers yield that the image of $|K_S|$ on Q via $\pi \circ f$ is the linear system $|\mathcal{L}_{Z,Q}(3)| = \eta_1 + \eta_2 + \eta'_1 + \eta'_2 + |\mathcal{O}_Q(1)|$. This means that $|K_S|$ has dimension 3 and four base points at the points

p_1, p_2, p'_1, p'_2 . Furthermore the rational map $g = \pi \circ f \circ p^{-1}$ is the canonical map of S . Hence the general canonical curve of S is hyperelliptic, and therefore the bicanonical map is not birational (see remark (1.1, i)).

A few remarks are in order. First, there is a degeneration of the surface S which has not been mentioned by Du Val. It is obtained by letting, say, η_1 and η_2 come together, in which case the canonical system of S acquires a fixed part (see [CFM] for details).

Secondly, one proves that $Tors(S) = Tors(H_1(S, \mathbf{Z})) \simeq \mathbf{Z}_2$. One way of seeing this is to realize S as a double cover of the plane $q: S \rightarrow \mathbf{P}^2$, by composing the canonical map g with the stereographic projection of the quadric to the plane from one of the points x_{ij} . The branch curve G is then of the form $G = a_1 + a_2 + G'$, where a_1 and a_2 are distinct lines, and G' is a curve of degree 12. The singularities of G' are as follows: G has a 4-tuple point at $\gamma = a_1 \cap a_2$, and two [4,4]-points $\alpha_i \in a_i$, different from each other and from γ , where the infinitely near 4-tuple point to α_i lies on the line a_i , $i=1, 2$. The aforementioned degeneration of S for which the canonical system has a fixed part corresponds to the possibility one of the points α_i becomes infinitely near to γ along a_i .

This description will remind the reader of the double plane representation of a general Enriques surface (see [E] or, for a modern reference, [CD]), and therefore will suggest to him the assertion about the torsion. Indeed, from this double plane representation, one sees that there is on S a base point free pencil $|D|$ of curves of genus 3, i.e. the pull-back on S via q of the pencil of lines through γ , which has two double fibres, which are the pull-back to S of the lines a_i , $i=1, 2$. The difference of the two double fibres is a torsion element of order 2 in $\text{Pic}(S)$. The torsion group of the surface cannot be bigger than \mathbf{Z}_2 (compare proposition (1.5) below).

We notice that the derived case $p_g=3, K^2=6$ in table (1.3) has been studied in detail by Bartalesi and Catanese [BC].

(iv) $p_g=3, K^2=8$.

This surface S is described by the following commutative diagram:

$$\begin{array}{ccc} S' & \xrightarrow{p} & S \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{\varphi} & \mathbf{P}^2 \end{array}$$

where :

(a) $\pi: Y \rightarrow \mathbf{P}^2$ is a double cover with branch curve $B = \sum_{i=1}^6 L_i + B'$, where $L_i, i=1, \dots, 6$ are the six sides of a complete quadrangle with A_1, \dots, A_4 vertices of multiplicity 3 and A_5, A_6, A_7 vertices of multiplicity 2, and where B' is a general curve in the linear system of plane curves of degree 14 having 5-uple points at each of the points A_1, \dots, A_4 and 4-uple points at each of the points A_5, A_6, A_7 , and no other

singularity. Such a curve is easily seen to exist by Bertini's theorem;

(b) $f: S' \rightarrow Y$ is a minimal resolution of the singularities of Y , which are the points a_i over the points A_i , $i=1, \dots, 7$. The surface S' is not minimal: it has six exceptional divisors corresponding to the lines L_i , $i=1, \dots, 6$. By blowing them down to six points l_i , $i=1, \dots, 6$, one has the surface S .

One sees that the image of $|K_{S'}|$ on \mathbf{P}^2 via $\pi \circ f$ is the linear system $\sum_{i=1}^6 L_i + |\mathcal{O}_{\mathbf{P}^2}(1)|$. Hence $|K_S|$ has dimension 2 and six base points at the points l_i , $i=1, \dots, 6$. The rational map $g = \pi \circ f \circ p^{-1}$ is therefore the canonical map for S . Again the general canonical curve of S is hyperelliptic, and therefore the bicanonical map is not birational.

Although the present example may seem very different from the previous one, they are in fact similar. This can be seen in the following way. Make a quadratic transformation $\sigma: \mathbf{P}^2 \rightarrow \mathbf{P}^2$ based at A_2, A_3, A_4 . Then one has a cartesian square:

$$\begin{array}{ccc} Y' & \xrightarrow{s} & Y \\ f' \downarrow & & \downarrow f \\ \mathbf{P}^2 & \xrightarrow{\sigma} & \mathbf{P}^2 \end{array}$$

where $f': Y' \rightarrow \mathbf{P}^2$ is a new double cover and $s: Y' \rightarrow Y$ is a birational map. The branch curve G of $f': Y' \rightarrow \mathbf{P}^2$ is the proper transform of B via σ , hence it is of the form $G = a_1 + a_2 + a_3 + G'$, where a_1, a_2, a_3 are three distinct lines passing through a point γ , and G' is a curve of degree 13 whose only singularities are: a 5-uple point at γ and three [4,4]-points $\alpha_i \in a_i$ distinct from γ , where the infinitely near 4-ple point to α_i lies on the line a_i , $i=1, 2, 3$.

Now we see the similarity with the example (iii), and we can also see that there is a pencil $|D|$ of curves on S with $D^2=0$, with arithmetic genus 3 and with 3 double curves. This shows that there is an embedding $\mathbf{Z}_2^2 \subset \text{Tors}(S)$ and we will prove in proposition (1.5) below that actually $\text{Tors}(S) \simeq \mathbf{Z}_2^2$.

We want to make a final remark concerning these surfaces which also shows their similarity with the ones in the previous example. By letting, in the late double plane representation, the points α_i , $i=1, 2, 3$, become collinear along a line L , the canonical system acquires a fixed component, i.e. the pull-back on S of the line L , which is an elliptic curve with self-intersection -1. A similar situation takes place if one of the points α_i becomes infinitely near to γ along a_i . This degeneration, again, has not been mentioned by Du Val.

(v) $p_g = 3$, $K^2 = 2$.

This is the double cover $\pi: S \rightarrow \mathbf{P}^2$ branched along a smooth curve B of degree 8. The analysis is similar to the one of the example (i) above.

Before stating our classification theorem, which extends Du Val's one (1.2), we owe to the reader a proposition announced in the course of the discussion of examples (1.4, iii, iv):

(1.5) **Proposition.** (i) Let S be a minimal surface with $p_g=3$, $q=0$, $K^2 < 8$. If $\mathbf{Z}_2 \hookrightarrow \text{Tors}(S)$ then $\text{Tors}(S) \simeq \mathbf{Z}_2$.

(ii) Let S be a minimal surface with $p_g=3$, $q=0$, $K^2=8$ and with no pencil of curves of genus 2. If $\mathbf{Z}_2^2 \hookrightarrow \text{Tors}(S)$ then $\text{Tors}(S) \simeq \mathbf{Z}_2^2$.

Proof. Suppose we are in case (i). Then we have an irreducible étale double cover $f: S' \rightarrow S$. Since $K_{S'}^2 < 2\chi(\mathcal{O}_{S'})$, S' is regular (see [B], lemma 14, pg. 212). So we may apply [X2], corollary 4, pg. 141, which says that if $K_{S'}^2 < \frac{8}{3}(\chi(\mathcal{O}_{S'}) - 2)$ and S' is regular, then S' is algebraically simply connected. This ends the proof.

As for (ii), we proceed in a similar way. The subgroup \mathbf{Z}_2^2 of $\text{Tors}(S)$ determines an étale quadruple cover $f: S' \rightarrow S$. In this case we have $K_{S'}^2 = 2\chi(\mathcal{O}_{S'}) < \frac{8}{3}(\chi(\mathcal{O}_{S'}) - 2)$. Now S' is regular. Otherwise, by [H], S' would possess a pencil of curves of genus 2, and the same would happen for S , a contradiction. Then we can conclude as in (i).

We finish this section by stating our classification theorem, which we will prove in the following paragraphs.

(1.6) **Theorem.** Let S be a smooth minimal surface of general type. Assume $p_g \geq 3$, $q=0$, and the bicanonical map of S not birational. Then either we are in the standard case or S is obtained as in table (1.3) as a specialization of a Du Val ancestor.

(1.7) **Remarks.** (i) We want to stress that the difference between this statement and Du Val's one (1.2) lies in the basic fact that we do not make any assumption at all on the canonical system. This makes our work technically much more complicated than Du Val's.

(ii) The cases $p_g \geq 4$, $q=0$ in the statement of theorem (1.6) have been worked out in [CFM]. Therefore in what follows we will only deal with the case $p_g=3$, $q=0$.

(iii) There are no surfaces with $p_g \geq 4$, $q > 0$, non-birational bicanonical map and no pencil of curves of genus 2. This has been proved in [CFM]. The case $p_g=3$, $q > 0$ has been worked out in [CCM]. Thus the present paper concludes the classification of surfaces with $p_g \geq 3$, non-birational bicanonical map and no pencil of curves of genus 2.

2. Non birational bicanonical map: general properties of canonical divisors

Let S be a minimal *regular* surface of general type and let $|K| = |M| + F$ where $|M|$ is the moving part of $|K|$ and F the fixed part. If $\text{Im } \phi_K$ is a surface then, by Bertini's theorem, the general curve in $|M|$ is irreducible. If $p_g=2$ and $q=0$, the general curve M in $|M|$ is also irreducible: otherwise M would be composed of $k \geq 2$ curves of a rational pencil, and therefore $p_g = h^0(S, \mathcal{O}_S(M)) = k + 1 \geq 3$, a contradiction.

(2.1) **Proposition.** Let S be a minimal surface of general type with $p_g \geq 2$, $q=0$. As above we let $|K| = |M| + F$. Assume that the general curve M in $|M|$ is irreducible,

and that the bicanonical map of S is not birational. Let $S := P_g - 2$. Then:

- (i) the general curve M in $|M|$ is hyperelliptic;
- (ii) if $F \neq 0$, then there is a decomposition $F = F_1 + \dots + F_n$, where F_1, \dots, F_n are 1-connected curves such that $\mathcal{O}_M(F_i) \simeq \gamma$ (where $|\gamma|$ is the g_2^1 on M), $\mathcal{O}_{F_i}(F_k) \simeq \mathcal{O}_{F_i}$ and such that either $F_i \leq F_k$ or $F_i \cap F_k = \emptyset$, for $i, k \in \{1, \dots, n\}$, $k < i$;
- (iii) the linear system $|M|$ cuts out on M a complete special linear series of dimension s which is $|\mathcal{O}_M(M)| = |\gamma^{\otimes s}| + D$, $|\gamma^{\otimes s}|$, the movable part of the series, is composed with $|\gamma|$, and D , the fixed divisor of the series, has degree $d = M^2 - 2s = M^2 - (2p_g - 4) = p_a(M) - 2s - n - 1$ and it is such that $\mathcal{O}_M(2D) = \gamma^{\otimes d}$;
- (iv) if, in addition, $\text{Im } \varphi_K$ is a surface (i.e. $p_g \geq 3$) then $\deg \phi_K = 2$.

Proof. Since ϕ_{2K} separates different curves in $|K|$ and $|K+M| \subset |2K|$, φ_{K+M} is not birational on M . Since $q=0$, the restriction map $H^0(S, \mathcal{O}_S(K+M)) \rightarrow H^0(M, \omega_M)$ is surjective and therefore M is hyperelliptic. This proves assertion (i).

If $F \neq 0$, we can apply theorem (7.1) to two smooth points x, y of $K = M + F$ such that $x, y \in M$ and $\varphi_{2K}(x) = \varphi_{2K}(y)$, and we obtain (ii).

The first assertion of (iii) follows from $q=0$. In particular $|\mathcal{O}_M(M)| = |\gamma^{\otimes s}| + D$, where D is some fixed divisor of degree $d = M^2 - 2s = M^2 - (2p_g - 4)$.

If $F=0$, i.e. $M=K$, we have $\mathcal{O}_M(2K) = \omega_M = \gamma^{\otimes n}$, with $n = K^2$, hence $\mathcal{O}_M(2D) = \gamma^{\otimes d}$, with $d = K^2 - 2s = p_a(M) - 1 - 2s$, concluding the proof of (iii) in this case.

Let $F \neq 0$. By part (ii) and by the adjunction formula, one has:

$$\begin{aligned} \gamma^{p_a(M)-1} &\simeq \omega_M \simeq \mathcal{O}_M(2M) \otimes \mathcal{O}_M(F) \simeq \\ &\simeq \gamma^{2s} \otimes \mathcal{O}_M(2D) \otimes \mathcal{O}_M(F_1) \otimes \dots \otimes \mathcal{O}_M(F_n) \simeq \gamma^{s+n} \otimes \mathcal{O}_M(2D) \end{aligned}$$

whence (iii) easily follows.

Assertion (iv) is obvious.

(2.2) Proposition. *Under the same assumptions, the general curve M in $|M|$ is smooth. Furthermore $D = p_1 + \dots + p_d$ where p_1, \dots, p_d are distinct Weierstrass points of M .*

Proof. If M is not smooth then there exist multiple base points for the linear system $|M|$. Let then p be a multiple base point of $|M|$ and let α be its multiplicity on the general curve M in $|M|$. Let $\sigma: \tilde{S} \rightarrow S$ be the blow-up of S at p , E the exceptional divisor and \tilde{M} the strict transform of M . Since $|K_{\tilde{S}} + \tilde{M} + E|$ is contained in $|\sigma^*(K+M)| = |K_{\tilde{S}} + \tilde{M} + (\alpha-1)E|$, $|K_{\tilde{S}} + \tilde{M} + E|$ cuts the complete canonical system on $\tilde{M} + E$ and φ_{K+M} is not birational, we can apply theorem (7.1) to $\omega_{\tilde{M}+E}$. Since for the general curve M in $|M|$ the curve $\tilde{M} + E$ has exactly two components and $\tilde{M} \cdot E = \alpha$, theorem (7.1) implies that $\alpha=2$ and that $\mathcal{O}_{\tilde{M}}(E)$ is a g_2^1 on \tilde{M} . Denote this g_2^1 by $|\eta|$. From $q=0$ and $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{M} + E)) = h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{M})) = p_g$, one has $h^0(\tilde{M} + E, \mathcal{O}_{\tilde{M}+E}(\tilde{M} + E)) = h^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{M})) = p_g - 1$. Since \tilde{M} is hyperelliptic we have $\mathcal{O}_{\tilde{M}}(\tilde{M}) \simeq \eta^{\otimes (p_g-2)} \otimes \mathcal{F}$, with \mathcal{F} an invertible sheaf on \tilde{M} such that $h^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\mathcal{F})) = 1$. Now the exact sequence:

$$0 \rightarrow \mathcal{O}_E(E) \rightarrow \mathcal{O}_{\tilde{M}+E}(\tilde{M}+E) \rightarrow \mathcal{O}_{\tilde{M}}(\tilde{M}+E) \rightarrow 0$$

gives rise to an isomorphism:

$$H^0(\tilde{M}+E, \mathcal{O}_{\tilde{M}+E}(\tilde{M}+E)) \xrightarrow{r} H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{M}+E))$$

since:

$$H^0(E, \mathcal{O}_E(E)) = H^1(E, \mathcal{O}_E(E)) = 0.$$

But this is a contradiction, because:

$$\mathcal{O}_{\tilde{M}}(\tilde{M}+E) \simeq \mathcal{O}_{\tilde{M}}(\tilde{M}) \otimes \mathcal{O}_{\tilde{M}}(E) \simeq \eta^{\otimes (p_g-1)} \otimes \mathcal{F},$$

thus r cannot be an isomorphism. Therefore $|M|$ has no multiple base points and so by Bertini's theorem the general element of the linear system $|M|$ is smooth. The remainder of the assertion follows by (2.1, iii).

(2.3) Proposition. *Let S be as above. Assume that $p_g \geq 3$ and let F_i be a fixed part of $|K|$ as in (2.1, ii). Then :*

- (i) $h^0(S, \mathcal{O}_S(M-F_i)) = p_g - 2$;
- (ii) $\text{Im } \varphi_K(F_i)$ is a line;
- (iii) any curve C in $|M-F_i|$ is such that $3 \geq h^0(C, \mathcal{O}_C) \geq 2$.

Proof. Since F_i is 1-connected and S is regular, one has $h^1(S, \mathcal{O}_S(-F_i)) = 0$. Considering the long exact sequence obtained from

$$0 \rightarrow \mathcal{O}_S(-F_i) \rightarrow \mathcal{O}_S(M-F_i) \rightarrow \mathcal{O}_M(M-F_i) \rightarrow 0$$

one has $h^0(S, \mathcal{O}_S(M-F_i)) = h^0(M, \mathcal{O}_M(M-F_i))$. By (2.1, ii, iii) one has $h^0(M, \mathcal{O}_M(M-F_i)) = s = p_g - 2$, hence (i).

Part (ii) is an immediate consequence of (i).

To prove part (iii) consider the long exact sequence

$$\begin{aligned} 0 \rightarrow H^0(S, \mathcal{O}_S(-(M-F_i))) \rightarrow H^0(S, \mathcal{O}_S(F_i)) \rightarrow H^0(M, \mathcal{O}_M(F_i)) \rightarrow \\ \rightarrow H^1(S, \mathcal{O}_S(-(M-F_i))) \rightarrow H^1(S, \mathcal{O}_S(F_i)) \rightarrow H^1(M, \mathcal{O}_M(F_i)) \rightarrow \dots \end{aligned}$$

Since $\mathcal{O}_M(F_i)$ is the g_2^1 on M and by the theorem of Riemann-Roch $h^1(S, \mathcal{O}_S(F_i)) = 1$, we obtain $1 \leq h^1(S, \mathcal{O}_S(-(M-F_i))) \leq 2$.

Let C be any curve in $|M-F_i|$. From $q=0$, we have $h^1(S, \mathcal{O}_S(-(M-F_i))) = h^0(C, \mathcal{O}_C) - 1$, proving (iii).

3. Further properties of canonical divisors

Let S be a minimal surface, with $p_g=3$ and $q=0$, and non birational bicanonical map. We will keep notation and assumptions on S from §2. In particular, we still assume that the canonical system is not composed with a pencil. In the present paragraph we collect some technical lemmas concerning this case.

- (3.1) **Lemma.** *Let S be as above and let $p := p_1$ be a base point of $|M|$. Then:*
- (i) *the linear system $|M - 3p| = |M - 2p|$ is not empty. More precisely there is one single curve $M_p \in |M - 3p|$ and p is exactly of multiplicity three for M_p ;*
 - (ii) *any other base point of $|M|$ is a non-singular point of M_p and any curve $M \in |M|$ not containing any component of M_p intersects M_p only at the base points $p = p_1, \dots, p_d$ with intersection multiplicities $m_1 = 3, m_2 = \dots = m_d = 1$;*
 - (iii) *if $\sigma: S' \rightarrow S$ is the blow-up of S at p , then the strict transform M' of M_p on S' is such that $h^0(M', \mathcal{O}_{M'}) = 3$.*

Proof. Remark first that, since p is a base point of $|M|$, and $h^0(S, \mathcal{O}_S(M)) = 3$, the linear system $|M - 2p|$ is not empty. By (2.2) one has $|M - 2p| = |M - 3p|$. Furthermore (2.2) and $p_g = 3$ imply that there is a unique curve M_p in $|M - 3p|$.

Let M be any curve in $|M|$ which does not contain any component of M_p . Let $m_i, i = 1, \dots, d$, be the intersection multiplicity of M_p at the base point p_i of $|M|$. By (2.1, iii), we have $M^2 = m_1 + \dots + m_d \geq 3 + m_2 + \dots + m_d \geq 3 + (d - 1) = M^2$. Hence $m_1 = 3$ and $m_2 = \dots = m_d = 1$, finishing the proof of (i) and proving (ii).

As for (iii), let E be the exceptional divisor of the blow-up at p . We set $M'' = \sigma^*(M_p) - 2E$, whereas $M' = \sigma^*(M_p) - 3E$. Since p is not a base point of ω_M for the general curve M in $|M|$ and $q = 0$, then p is not a base point of $|K + M|$. By a theorem of Francia (see [F] or theorem (1.1) of [M]) we have $h^0(M'', \mathcal{O}_{M''}) = 1$.

Since p is a Weierstrass point of the general curve in $|M|$, we have $|K + M - p| = |K + M - 2p|$. By theorem (1.3) of [M], this implies that $h^0(M', \mathcal{O}_{M'}) = h^0(M'', \mathcal{O}_{M''}) + 2$, proving (iii).

- (3.2) **Lemma.** *Let S be as above and let F_i be a fixed component of $|K|$ as in (2.1, ii). Let C be the unique curve in $|M - F_i|$ (see (2.3, i)). Then:*

- (i) $h^0(C, \mathcal{O}_C) = 3$;
- (ii) C is 0-connected;
- (iii) either $F_i^2 = -1$ or $F_i^2 = -2$ (and thus $K \cdot F_i = 1$ or $K \cdot F_i = 0$ respectively);
- (iv) all base points of $|M|$ lie on C and are non-singular points of C and therefore the rational map determined by $|M|$ contracts every component of C .

Proof. Suppose first that $M^2 = 2$, i.e. the linear system $|M|$ has no base points. Notice that $h^0(S, \mathcal{O}_S(2M)) = 6$ because $|\mathcal{O}_M(2M)| = |\gamma^{\otimes 2}|$. Since $M \cdot F_i = 2$, then $M \cdot C = M \cdot (M - F_i) = 0$ and thus $\mathcal{O}_C(M) \simeq \mathcal{O}_C$.

Consider now the restriction sequence obtained from

$$0 \rightarrow \mathcal{O}_S(M + F_i) \rightarrow \mathcal{O}_S(2M) \rightarrow \mathcal{O}_C \rightarrow 0.$$

Since $h^0(S, \mathcal{O}_S(M + F_i)) = 3$ and $h^0(S, \mathcal{O}_S(2M)) = 6$, we get $h^0(C, \mathcal{O}_C) \geq 3$. Therefore $h^0(C, \mathcal{O}_C) = 3$ by proposition (2.3, iii). So we proved part (i) of the assertion for $M^2 = 2$.

Let now $M^2 > 2$, i.e. $|M|$ has d base points p_1, \dots, p_d , which also lie on the curve particular curve $C + F_i \in |M|$. Since by (2.1, ii) $|\mathcal{O}_M(F_i)|$ is the $g - \frac{1}{2}$ on a general curve

of $|M|$, and by (3.1), a base point of $|M|$ has either multiplicity 1 or 3 on a curve of $|M|$, all the points p_1, \dots, p_d lie on C , and they are non-singular points of C . This proves (iv).

Let $\pi: \tilde{S} \rightarrow S$ be the blow-up of S at p_1, \dots, p_d . Let \tilde{M} be the strict transform of the general curve in $|M|$, and let \tilde{C} be the strict transform of C . We can then apply the same reasoning as in the case $M^2=2$ to $|\tilde{M}|$ and \tilde{C} thus obtaining $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}})=3$. Since the points p_1, \dots, p_d are non-singular points of C , we have $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}})=h^0(C, \mathcal{O}_C)$, concluding the proof of part (i).

It is sufficient to prove assertion (ii) for \tilde{C} . Suppose that \tilde{C} decomposes as $\tilde{C}=G+H$, with $G \cdot H = \alpha < 0$. By 1-connectedness of $\tilde{M}=G+H+\pi^*(F_i)$ (see lemma 2.6 of [M]), we have $G \cdot \pi^*(F_i) \geq 1 - \alpha$, $H \cdot \pi^*(F_i) \geq 1 - \alpha$ and hence $\pi^*(F_i) \cdot \tilde{C} \geq 2 - 2\alpha \geq 4$. Now remark that $-2 \leq F_i^2$, since $F_i \cdot K = F_i \cdot M + F_i^2 = 2 + F_i^2$ and K is nef, and remark also that $2 = M \cdot F_i$ implies that $F_i \cdot C = 2 - F_i^2$. Therefore, from $\pi^*(F_i) \cdot \tilde{C} = 2 - 2\alpha \geq 4$, we obtain $\pi^*(F_i)^2 = -2$, $\alpha = -1$, $\tilde{C} \cdot \pi^*(F_i) = 4$, $H \cdot \pi^*(F_i) \geq 2$ and $G \cdot \pi^*(F_i) \geq 2$ and therefore $H \cdot \pi^*(F_i) = G \cdot \pi^*(F_i) = 2$. This yields $G \cdot (\tilde{M} - G) = H \cdot (\tilde{M} - H) = 1$ and thus $G^2 = H^2 = -1$, since every component of \tilde{C} is contracted by ϕ_M . Also for this reason and since $G \cdot H = G^2 = H^2 = -1$, (7.4) implies that $H = G$ and G is 1-connected. So one has that the curve $\tilde{M} - G = G + \pi^*(F_i)$ is such that $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{M} - G)) = 2$, because G is contracted by $\phi_{\tilde{M}}$. But this is a contradiction since $|K_{\tilde{S}}|$ is not composed with a pencil and $\pi^*(K_S) \geq \tilde{M} + \pi^*(F_i) = 2(G + \pi^*(F_i)) = 2(\tilde{M} - G)$. This proves (ii).

To prove (iii), it suffices to notice that, since C is 0-connected and $h^0(C, \mathcal{O}_C) = 3$, C breaks in three curves (see (7.3)) which all meet F_i positively because M is 1-connected. Therefore $F_i \cdot C \geq 3$. We have seen above that $F_i^2 \geq -2$ and $F_i \cdot C = 2 - F_i^2$ and therefore we have (iii).

(3.3) **Notation.** In what follows, we will keep the notation used in the proof of (3.2), i.e. $\pi: \tilde{S} \rightarrow S$ will be the blow-up of S at the d base points p_1, \dots, p_d of $|M|$. We will denote by \tilde{M} the strict transform of the general curve in $|M|$, whilst we will denote by $\tilde{\Gamma}$ the strict transform of a curve Γ under π .

We keep the notation of proposition (3.1) and we prove now the:

(3.4) **Lemma.** *Let p be a base point of $|M|$, let M_p be the unique curve in $|M - 3p|$ and let M' be the strict transform on S' , the blow-up of S at p , of M_p . Then, if the curve M' is 0-connected, the curve M_p decomposes as:*

- (I) $M_p = A_1 + A_2 + A_3$, where A_1, A_2, A_3 are 1-connected curves such that:
 - (i) $A_1 \cdot A_2 = A_1 \cdot A_3 = A_2 \cdot A_3 = 1$;
 - (ii) $p \in A_1 \cap A_2 \cap A_3$ and p is a non-singular point of A_i ;
 - (iii) $h^0(S, \mathcal{O}_S(M - A_i)) = 2$, i.e. the rational map determined by $|M|$ contracts each A_i , $i = 1, 2, 3$ to one point.

If M' is not 0-connected, then M_p decomposes as:

- (II) $M_p = 2G + D$, where D, G are 1-connected curves such that:
 - (i) p is a non-singular point of D and G ;

(ii) $G^2=0$, $G \cdot D=1$;

(iii) $\mathcal{O}_G(G) \neq \mathcal{O}_G$;

(iv) $h^0(S, \mathcal{O}_S(M-D))=h^0(S, \mathcal{O}_S(M-G))=2$, i.e. the rational map determined by $|M|$ contracts each of D and G to a point.

Furthermore case (II) happens if and only if the curve M_p has a decomposition $M_p=B_1+B_2$ with $B_1 \cdot B_2=1$, $p \in B_1 \cap B_2$. In particular if $|K|$ has no fixed part the decomposition of M_p is of type (I), since in that case M_p is 2-connected.

Proof. By proposition (3.1), from which we keep the notation, the curve M' is such that $h^0(M', \mathcal{O}_{M'})=3$ and therefore it is not 1-connected. Since M_p is 1-connected, if $M'=A'+B'$ with $A'>0$, $B'>0$ and $A' \cdot B' \leq 0$, then there exist curves A, B such that $M_p=A+B$, $A'=\sigma^*A-E$, $B'=\sigma^*B-2E$ and either $A \cdot B=1$ and $A' \cdot B'=-1$, or $A \cdot B=2$ and $A' \cdot B'=0$, (cf. [B], pg. 183).

Assume that M' is not 0-connected. We will prove we are in case (II). Let $M_p=A+B$ with $A \cdot B=1$ and $p \in A \cap B$. Since p is a triple point of M_p and $A \cdot B=1$, A and B must have common components. Thus we can write $A=G+D$, $B=G+F$ where G is a curve and D and F are effective divisors without common components. Now $1=A \cdot B=G^2+G \cdot D+G \cdot F+D \cdot F$ and $1 \leq G \cdot (M-G)=G^2+G \cdot D+G \cdot F$ imply that $D \cdot F=0$ and $G^2+G \cdot D+G \cdot F=1$. Therefore $D \cap F=\emptyset$. Since p is a triple point of M , then $p \in G$, p is a simple point of G and, say, $p \in D$, $p \notin F$. In particular $D \neq 0$. Since $p \in G$, we have $1 \leq M \cdot G=2G^2+G \cdot D+G \cdot F=G^2+(G^2+G \cdot D+G \cdot F)=G^2+1$ implying $G^2 \geq 0$. Now $G^2+G \cdot D+G \cdot F=1$ and $G^2 \geq 0$ imply $G \cdot D+G \cdot F \leq 1$. Since $A=G+D$ and $B=G+F$ are 1-connected (by lemma (A.4) of [CFM]) and $D \neq 0$, we must have $F=0$ and $G^2=0$, $D \cdot G=1$.

Notice that, again by lemma (A.4) of [CFM], both G and D are 1-connected. Hence we have a decomposition as in (II) satisfying properties (i), (ii).

As for property (iii) we recall that $q=h^1(S, \mathcal{O}_S)=0$, and so we have the exact sequence:

$$0 \rightarrow H^0(S, \mathcal{O}_S) \rightarrow H^0(S, \mathcal{O}_S(G)) \rightarrow H^0(G, \mathcal{O}_G(G)) \rightarrow 0$$

If $\mathcal{O}_G(G) \simeq \mathcal{O}_G$, then we would have $h^0(S, \mathcal{O}_S(G))=2$, hence $|M|$ would be composite with the pencil $|G|$, a contradiction.

Now we turn to property (iv). Since p is a non-singular point of D and G , then \tilde{D} and \tilde{G} , as well as D and G , are 1-connected (see (3.3) for notation). From $\tilde{M} \cdot \tilde{D}=\tilde{M} \cdot \tilde{G}=0$, which is an immediate consequence of (3.1, i), we have immediately (iv).

Assume now that M' is 0-connected. So we can apply proposition (7.3) to M' obtaining a decomposition $M'=A'_1+A'_2+A'_3$, with $A'_i+A'_j=0$, for $1 \leq i < j \leq 3$.

Using the same trick from [B], pg. 183, we mentioned before, we see that there are curves A_i on S such that $A'_i=\sigma^*(A_i)-E$, $i=1, 2, 3$. Then $M_p=A_1+A_2+A_3$ is a decomposition of type (I) and satisfying properties (i), (ii). Now for property (iii), it is enough, as usual, to show that A_i is a 1-connected curve for $i=1, 2, 3$. Suppose otherwise. Then $A_i=C+D$, where $C>0$, $D>0$ and $C \cdot D \leq 0$. Since $A_i \cdot (A_j+A_k)=2$,

if $j, k \neq i$, and, by 1-connectedness of M , $C \cdot (D + A_j + A_k) \geq 1$, $D \cdot (C + A_j + A_k) \geq 1$, we must have $C \cdot D = 0$ and $C \cdot (A_j + A_k) = D \cdot (A_j + A_k) = 1$. Now p is a non-singular point of A_i and so, say, $p \in C$, $p \notin D$. If we let C' be the strict transform of C by σ , we have $C' \cdot (M' - C') = (\sigma^*(C) - E) \cdot (\sigma^*(M - C) - 2E) = C \cdot (M - C) - 2 = -1$, contrary to the assumption that M' is 0-connected. This proves that the curves A_i , $i = 1, 2, 3$, are 1-connected and (iii) also follows.

Finally, the last assertion is now immediate.

(3.5) **Lemma.** *Suppose that F_i is a fixed component of $|K|$ as in proposition (2.1, ii), with $F_i^2 = -1$ and let C be the unique curve in $|M - F_i|$. Then C decomposes as $C = A_1 + A_2 + A_3$ where A_1, A_2, A_3 are 1-connected curves such that:*

- (i) $A_1 \cdot A_2 = A_1 \cdot A_3 = A_2 \cdot A_3 = 0$;
- (ii) $A_1 \cdot F_i = A_2 \cdot F_i = A_3 \cdot F_i = 1$;
- (iii) $A_k^2 \leq 0$ and $M \cdot A_k \leq 1$, for $k = 1, 2, 3$, and if, in addition, $A_k < C - A_k$, then $M \cdot A_k = 0$;
- (iv) $h^0(S, \mathcal{O}_S(M - A_k)) = 2$, i.e. the rational map determined by $|M|$ contracts each A_k , $k = 1, 2, 3$ to one point.

Proof. Since $M \cdot F_i = 2$, then $C \cdot F_i = 3$. Now, by lemma (3.2), C is 0-connected and $h^0(C, \mathcal{O}_C) = 3$, and so we can apply proposition (7.3) to obtain a decomposition $C = A_1 + A_2 + A_3$ where A_1, A_2, A_3 satisfy (i).

Since every curve in $|M|$ is 1-connected (by lemma (2.6) of $[M]$), we have $A_k \cdot (M - A_k) = A_k \cdot F_i \geq 1$, for $k = 1, 2, 3$. From $F_i \cdot (A_1 + A_2 + A_3) = 3$, we have assertion (ii). The fact that the curves A_k are 1-connected comes from lemma (A.4) of $[CFM]$, since the curves in $|M|$ are 1-connected and $A_k \cdot (M - A_k) = 1$, for $k = 1, 2, 3$.

Now we turn to assertion (iii). Notice that $M \cdot A_k$, for $k = 1, 2, 3$, is equal to the number of base points of $|M|$ lying on A_k which are all non-singular points for C and therefore also non-singular for each curve A_k , $k = 1, 2, 3$ see (3.2, iv)). Thus the second part of (iii) follows right away.

As for the first part of (iii), we notice that $A_k \cdot (M - A_k) = A_k \cdot F_i = 1$ and $M \cdot (M - A_k) \geq M \cdot F_i = 2$, hence $(M - A_k)^2 \geq 1$. If, for some $k \in \{1, 2, 3\}$, $A_k^2 > 0$, then, by the index theorem we have $A_k^2 = 1$, $M \sim 2A_k$ and $M^2 = 4$ (cf. lemma (2.6) of $[M]$). This cannot occur. Indeed, since $M^2 = 4$, there is some base point p for $|M|$. By applying lemma (3.4), we would have a decomposition $M_p = Z_1 + Z_2 + Z_3$ with $M \cdot Z_i > 0$, $i = 1, 2, 3$, and $4 = M^2 = M \cdot Z_1 + M \cdot Z_2 + M \cdot Z_3$. This implies that one of the numbers $M \cdot Z_i$ is odd, and therefore M cannot be numerically divisible by 2. Therefore, for $k = 1, 2, 3$, $A_k^2 \leq 0$ and so $M \cdot A_k = A_k^2 + 1 \leq 1$.

Assertion (iv) follows by (3.2, iv) and 1-connectedness of the curves A_k .

(3.6) **Lemma.** *Suppose that F_i is a fixed component of $|K|$ as in proposition (2.1, ii), with $F_i^2 = -2$ and let C be the unique curve in $|M - F_i|$. Then C decomposes as $C = A_1 + A_2 + A_3$ where A_1, A_2, A_3 are curves such that:*

- (i) $A_1 \cdot A_2 = A_1 \cdot A_3 = A_2 \cdot A_3 = 0$;

- (ii) $A_j \cdot F_i = A_k \cdot F_i = 1$ and $A_l \cdot F_i = 2$, for $\{j, k, l\} = \{1, 2, 3\}$;
 (iii) if $A_j \cdot F_i = 1$, then $M \cdot A_j \leq 1$ and if $A_k \leq C - A_k$ then $M \cdot A_k = 0$;
 (iv) $h^0(S, \mathcal{O}_S(M - A_k)) = 2$, i.e. the rational map determined by $|M|$ contracts each A_k , $k = 1, 2, 3$ to one point.

Proof. Since C is 0-connected, we can apply again (7.3) to obtain a decomposition of C as the sum of three curves $C = A_1 + A_2 + A_3$ where

- (a) $h^0(A_i, \mathcal{O}_{A_i}) = 1$, for $i \in \{1, 2\}$;
 (b) $\mathcal{O}_{A_1}(A_2 + A_3) \simeq \mathcal{O}_{A_1}$ and $\mathcal{O}_{A_2}(A_3) \simeq \mathcal{O}_{A_2}$;
 (c) $A_1 \cdot A_2 = A_2 \cdot A_3 = A_1 \cdot A_3 = 0$.

The same arguments as above and $C \cdot F_i = 4$ imply that $A_j \cdot F_i = A_k \cdot F_i = 1$ and $A_l \cdot F_i = 2$, for $\{j, k, l\} = \{1, 2, 3\}$ and therefore, by lemma (A.4) of [CFM], A_j and A_k are 1-connected. Therefore we have $h^0(A_i, \mathcal{O}_{A_i}) = 1$, for $i = 1, 2, 3$, unless, possibly $A_3 \cdot F_i = 2$. In this case we have $A_1 \cdot (M - A_1) = A_2 \cdot (M - A_2) = 1$, thus $\bar{A}_1^2 = \bar{A}_2^2 = -1$ (see (3.3) for notation). Hence, by lemma (7.4) we conclude that either, $A_1 \leq A_2$ or $A_1 \cap A_2 = \emptyset$. In the later case we have $\mathcal{O}_{A_1}(A_2) \simeq \mathcal{O}_{A_1}$, but this is also true in the former case. This is a trivial consequence of $\mathcal{O}_{A_1}(A_2 + A_3) \simeq \mathcal{O}_{A_1}$ and $\mathcal{O}_{A_2}(A_3) \simeq \mathcal{O}_{A_2}$. Thus by the last part of (7.3), the image of the restriction map $r: H^0(C, \mathcal{O}_C) \rightarrow H^0(A_3, \mathcal{O}_{A_3})$ is 1-dimensional. By taking into account (3.2, iv), one sees that this implies that the image of the restriction map $r': H^0(S, \mathcal{O}_S(M)) \rightarrow H^0(A_3, \mathcal{O}_{A_3}(M))$, which is not the zero map, has dimension at most 1, hence it has exactly dimension 1. Thus we proved assertions (i), (ii) and (iv).

The proof of assertion (iii) is identical to the proof of assertion (iii) of lemma (3.5).

(3.7) Lemma. *If A is a curve such that $M \cdot A = A^2 + 1$ and the image of the restriction map $r: H^0(S, \mathcal{O}_S(M)) \rightarrow H^0(A, \mathcal{O}_A(M))$ is 1-dimensional then:*

- (i) $0 \leq M \cdot A \leq 1$ (and thus $-1 \leq A^2 \leq 0$);
 (ii) if $M \cdot A = 1$, there is a unique base point p of M lying on A and the curve M_p has a decomposition of type (II) as in lemma (3.4), with $M_p = 2A + H$;
 (iii) A is 1-connected and for every fixed component F_i of $|K|$ as in (2.1, ii), one has $0 \leq A \cdot F_i \leq 1$; furthermore if $A \cdot F_i = 1$, then $A \leq C$ where C is the unique curve in $|M - F_i|$.

Proof. For (i) notice first that since $|M|$ has no fixed components, $M \cdot A \geq 0$. Now, since A is contracted by ϕ_M , $|M - A|$ is a linear system of dimension 1. Therefore $M \cdot (M - A) \geq 2$, hence $(M - A)^2 = M \cdot (M - A) - A \cdot (M - A) = M \cdot (M - A) - 1 \geq 1$.

Now, by the index theorem, we have that either $A^2 \leq 0$, hence $M \cdot A \leq 1$ or $A^2 = 1$, $A \sim M - A$ yielding $M \sim 2A$ and $M^2 = 4$. This last possibility cannot occur; this can be proved with an argument we already made in the proof of part (iii) of (3.5). Therefore we have $M \cdot A \leq 1$, proving (i).

Since A is contracted by ϕ_M , $M \cdot A$ is exactly the number of base points of M which lie on A and so, if $M \cdot A = 1$, there is a unique base point p of $|M|$ lying on A . Then, since $h^0(S, \mathcal{O}_S(M - A)) = 2$, there exists a curve D in $|M - A|$ such that $p \in D$, and thus $D + A \in |M - 2p|$. By lemma (3.1, i), $D + A = M_p$, and so $A \leq M_p$.

Since p is a non-singular point of A , and $A \cdot (M - A) = 1$, by lemma (3.4) the curve M_p has a type (II) decomposition, proving (ii).

As for (iii), remark first that $\tilde{A}^2 = -1$ (for notation see (3.3)). Hence by lemma (7.4), the curve \tilde{A} is 1-connected and therefore so is A . Furthermore $\tilde{A} \cdot \tilde{M} = 0$. Suppose that $A \cdot F_i \geq 2$. Then, since $\tilde{M} = \tilde{C} + \pi^*(F_i)$ with C as in (3.2), we get $\alpha := \tilde{A} \cdot \tilde{C} \leq -2$ and therefore $(\tilde{C} - 2\tilde{A})^2 = \tilde{C}^2 - 4\alpha - 4 \geq \tilde{C}^2 + 4$. Now, since the intersection form on curves contracted by $\phi_{\tilde{M}}$ is negative definite, we get $\tilde{C} \leq -4$. On the other hand, since $\tilde{C} \cdot \tilde{M} = 0$, we have $-\tilde{C}^2 = \tilde{C} \cdot \pi^*(F_i)$, and thus $-\tilde{C}^2 = C \cdot F_i \leq 4$ (see (2.1, ii) and (3.2, iii)). Therefore $(\tilde{C} - 2\tilde{A})^2 = 0$ and thus, by (7.4), $\tilde{C} = 2\tilde{A}$. Thus \tilde{C} is not 0-connected, and therefore also C is not 0-connected, contradicting (3.2, ii). Therefore $A \cdot F_i \leq 1$.

Suppose now that $\beta := A \cdot F_i \leq -1$. Notice that $\tilde{A} \cdot \pi^*(F_i) = A \cdot F_i = \beta$. Then $\pi^*(F_i)$ and \tilde{A} have common components. Write then $\tilde{A} = D + F$ and $\pi^*(F_i) = D + G$, where F and G have no common components. From $\tilde{M} \cdot \tilde{A} = 0 \neq \tilde{M} \cdot \pi^*(F_i)$, we have $G \neq 0$. Then $F^2 - 2F \cdot G + G^2 = (\tilde{A} - \pi^*(F_i))^2 = -1 - 2\beta + \pi^*(F_i)^2 \geq 1 + \pi^*(F_i)^2$, and therefore $F^2 + G^2 \geq \pi^*(F_i)^2 + 1 + 2F \cdot G \geq F_i^2 + 1$.

If $F_i^2 = -1$, we obtain $F^2 + G^2 \geq 0$ and if $F_i^2 = -2$, we obtain $F^2 + G^2 \geq -1$, with equality holding if $\beta = -1$. Since any curve contained either in \tilde{A} or in $\pi^*(F_i)$ has negative self-intersection, the former case is impossible. We claim that also the latter case cannot occur. In fact, if $F^2 + G^2 \geq -1$, we must have $F^2 + G^2 = -1$, and therefore $G^2 = -1$ and $F = 0$, thus $\tilde{A} < \pi^*(F_i)$. This is impossible. In fact $\tilde{A}^2 = -1$ and $\tilde{A} \cdot \pi^*(F_i) \leq -1$ imply that $\pi^*(F_i)$ is not 1-connected. On the other hand F_i is 1-connected by (2.1, ii), hence also $\pi^*(F_i)$ is 1-connected, which proves our claim. In conclusion, we have proved that $0 \leq A \cdot F_i \leq 1$.

In order to prove the last part of (iii), it suffices to remark that if $A \cdot F_i = 1$, then $\tilde{A} \cdot (\tilde{M} - \pi^*(F_i)) = -1$. If $F_i^2 = -1$, then by lemma (3.5) we have $\tilde{C} = \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3$, where C is the curve in $|M - F_i|$. Notice that $C^2 = M^2 - 5$, hence $\tilde{C}^2 = -3$ because we blow-up the $d = M^2 - 2$ base points of $|M|$. Since $\tilde{A}_j^2 \leq -1$, for $j = 1, 2, 3$, because the curves \tilde{A}_j are contracted by $|\tilde{M}|$, we have $\tilde{A}_j^2 = -1$, for $j = 1, 2, 3$. Applying (7.4) we have that $A = A_j$ for one of the curves A_j and so $A \leq M - F_i$. If $F_i^2 = -2$, $\tilde{M} - \pi^*(F_i) = \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3$, where $\tilde{A}_j^2 = \tilde{A}_k^2 = -1$, and $\tilde{A}_l^2 = -2$, for $\{j, k, l\} = \{1, 2, 3\}$. Notice that the proof of (3.6) implies that $A_l \cdot F_i = 2$, whereas $A_j \cdot F_i = A_k \cdot F_i = 1$. If $\tilde{A} \neq \tilde{A}_j$ and $\tilde{A} \neq \tilde{A}_k$, one has, by (7.4), $\tilde{A} \cdot \tilde{A}_j = \tilde{A} \cdot \tilde{A}_k = 0$. Therefore $\tilde{A} \cdot \tilde{A}_l = -1$, implying again by (7.4), that either $\tilde{A} \leq \tilde{A}_l$ or $\tilde{A}_l \leq \tilde{A}$. But in this last case one has $\tilde{A} = \tilde{A}_l + B$. Then $-1 = \tilde{A} \cdot \tilde{A}_l = \tilde{A}_l^2 + \tilde{A}_l \cdot B = -2 + \tilde{A}_l \cdot B$, hence $\tilde{A}_l \cdot B = 1$, which yields $B^2 = -1$. Furthermore we have $\tilde{M} \cdot B = 0$ and $B \cdot \pi^*(F_i) = \tilde{A} \cdot \pi^*(F_i) - \tilde{A}_l \cdot \pi^*(F_i) = 1 - 2 = -1$, which is impossible by the first part of assertion (iii).

(3.8) **Lemma.** *Suppose M has a type (II) decomposition at two distinct base points p, q of $|M|$, say $M_p = 2A_1 + H_1$ and $M_q = 2A_2 + H_2$. Then:*

- (i) $A_1 \cap A_2 = \emptyset$
- (ii) $H_1 = H_2$

(iii) $A_1 \not\equiv A_2$ but $2A_1 \equiv 2A_2$.

Proof. Since $M \cdot A_1 = M \cdot A_2 = 1$ and $p \in A_1$, $q \in A_2$, one has $A_1 \not\leq A_2$ and $A_2 \not\leq A_1$. Then (i) follows from $\tilde{A}_1^2 = \tilde{A}_2^2 = -1$ and lemma (7.4).

Notice now that, by (3.1) and (3.4), all base points of $|M|$ lie on H_i and are non singular points of H_i , for $i=1, 2$. So, since $M \cdot H_1 = 2 + H_1^2$, we have $\tilde{H}_1^2 = -2$, $i=1, 2$. Since $A_1 \cdot A_2 = 0$, we have also:

$$\begin{aligned} 2 + H_1^2 &= M \cdot H_1 = (2A_2 + H_2) \cdot H_1 = 2A_2 \cdot H_1 + H_1 \cdot H_2 = \\ &= 2A_2 \cdot (2A_1 + H_1) + H_1 \cdot H_2 = 2A_1 \cdot M + H_1 \cdot H_2 = 2 + H_1 \cdot H_2 \end{aligned}$$

Therefore one has $H_1 \cdot H_2 = H_1^2$, thus also $\tilde{H}_1 \cdot \tilde{H}_2 = -2$. Then, by (7.4), we have $\tilde{H}_1 = \tilde{H}_2$ implying $H_1 = H_2$, i.e. (ii).

Part (ii) yields $2A_1 \equiv 2A_2$. Since by (3.4) we have $\mathcal{O}_{(A_1)} \neq \mathcal{O}_{A_1}$ and by (i) we know that $A_1 \cap A_2 = \emptyset$, we see that $A_1 \not\equiv A_2$, thus proving (iii).

4. Exclusion of $|K|$ composed with a pencil

In this section we are going to prove the following theorem, which will enable us to apply the properties of §2 and §3 to the canonical system of the surfaces we want to describe.

(4.1) Theorem. *Let S be a minimal surface of general type with $p_g=3$, $q=0$, non birational bicanonical map, presenting the non-standard case. Then $|K|$ is not composed with a pencil.*

From [R] and [CCM], we know that if $K^2 \geq 9$, $p_g \leq 5$ and the bicanonical map is not birational, then S presents the standard case. Therefore we may assume $K^2 \leq 8$. The proof of this theorem consists of several steps. First we need the following numerical data:

(4.2) Lemma. *Let S be a minimal surface of general type with $p_g=3$ and $K^2 \leq 8$, and assume that $|K|$ is composed with a pencil \mathcal{P} of curves of genus $g \geq 3$. Then:*

(a) \mathcal{P} is a rational pencil and thus $K = 2C + Z$ where $C \in \mathcal{P}$ and Z is an effective divisor.

(b) *The only numerical possibilities are:*

(P_1) $K^2 = 8$, $C^2 = 0$, $K \cdot C = 4$, $K \cdot Z = 0$ and $Z^2 = -8$.

(P_2) $K^2 = 8$, $C^2 = 2$, $K \cdot C = 4$, $Z = 0$ and \mathcal{P} has two simple (either proper or infinitely near) base points.

(P_3) $C^2 = 1$, $K \cdot C = 3$, $C \cdot Z = 1$, $Z^2 = 0$ or $Z^2 = -1$ or $Z^2 = -2$ (giving respectively $K^2 = 8$, $K^2 = 7$ and $K^2 = 6$) and the pencil \mathcal{P} has one simple base point o .

Furthermore in case (P_3) the curve Z is 2-connected.

Proof. Since $|K|$ is composite with a pencil \mathcal{P} , $K \sim aC + Z$, where C is a curve in \mathcal{P} and Z is the fixed part of $|K|$. We have $p_g = 3 \leq a + 1$, hence $a \geq 2$. Notice that one has $a = 2$ if and only if \mathcal{P} is a rational pencil and $K \equiv 2C + Z$. Since $\mathcal{O}_S(K)$ is nef, one has

$$(†) \quad K \cdot aC = a^2 C^2 + aC \cdot Z \leq K^2 \leq 8.$$

Suppose first that $C^2=0$. Then, from the assumption $g=p_a(C) \geq 3$, one has $C \cdot Z \geq 4$ and thus, by (†) the only possibility is case (P_1) .

Suppose now that $C^2 > 0$. Then the pencil \mathcal{P} having base points, is rational and therefore $a=2$. Hence, by (†), necessarily $C^2 \leq 2$. If $C^2=2$, then $C \cdot Z=0$, implying, by 2-connectedness of K , that $Z=0$ and one has possibility (P_2) . Let $C^2=1$. Since $K \cdot C + C^2 = 2C^2 + C \cdot Z + 1 \equiv 0 \pmod{2}$ we must have $C \cdot Z \equiv 1 \pmod{2}$ and thus $Z \neq 0$ and (†) yields $C \cdot Z=1$. Since $K \cdot Z \geq 0$, we have the possibilities in (P_3) .

For the last assertion, suppose that, in case (P_3) , one has $Z=A+B$, with A, B curves such that $A \cdot B \leq 1$. Then, by 2-connectedness of the curves in $|K|$, we have $A \cdot 2C \geq 1$, $B \cdot 2C \geq 1$ and so $A \cdot C \geq 1$, $B \cdot C \geq 1$, contradicting $C \cdot Z=1$.

(4.3) **Lemma.** *With the same assumptions and notation of (4.2), suppose that the bicanonical map is not birational. Then the general curve C in \mathcal{P} is hyperelliptic. Furthermore, letting η be the line bundle on C such that $|\eta|$ is the g_2^1 on C , one has:*

Case (P_1) : $Z=2Z_1$, where $Z_1^2=-2$ and $\mathcal{O}_C(Z_1) \simeq \eta$.

Case (P_2) : $\mathcal{O}_C(C) \simeq \eta$ and $q > 0$.

Case (P_3) : $\mathcal{O}_C(C) \simeq \mathcal{O}_C(o) \simeq \mathcal{O}_C(Z)$ and $\mathcal{O}_C(2o) \simeq \eta$. Furthermore in this case o is a simple point of Z , there exists $C' \in \mathcal{P}$ such that $Z \leq C'$ and every fibre of \mathcal{P} is 2-connected, with the possible exception of C'

Proof. Let x, y be two general points such that $\varphi(x)=\varphi(y)$. Then x, y are on the same fibre C of \mathcal{P} and therefore we can apply theorem (7.1) to the curve $D=2C+Z$, obtaining a decomposition $D=D_1+D_2$, where D_1, D_2 are curves such that $D_1 \cdot D_2=2$ and $D_1 \leq D_2$. In case (P_1) , since $C \leq D_1 \cap D_2$, we must have $K \cdot D_1 = K \cdot D_2 = 4$ and, by the index theorem, $D_1 \sim D_2$. Therefore $D_1 = D_2$ and thus $Z=2Z_1$, as stated.

To prove the remainder of the statement we look at the linear system $|K+C+A|$, where $A:=C+Z_1$ in case (P_1) , $A:=C$ in case (P_2) and $A:=C+Z$ in case (P_3) . Consider the exact sequence

$$0 \rightarrow \mathcal{O}_S(K+A) \rightarrow \mathcal{O}_S(K+C+A) \rightarrow \omega_C(A) \rightarrow 0.$$

In all three cases $A \cdot (K-A)=2$, implying, by 2-connectedness of the canonical divisors and lemma (A.4) of [CFM], that the curve A is 1-connected. Then, since $A^2 > 0$, one has $h^1(S, \mathcal{O}_S(K+A))=0$ and so $|K+C+A|$ cuts out a complete linear series of degree $2g$ on C . Since $|K+C+A| \subset |2K|$, ϕ_{K+C+A} is not birational. Therefore C is hyperelliptic and $\mathcal{O}_C(K+C+A) \simeq \eta^{\otimes g}$, where $|\eta|=g_2^1$ on C . We also have $\mathcal{O}_C(A) \simeq \eta$.

Now we consider the three cases separately.

In case (P_1) , in order to conclude, it suffices to remark that $\mathcal{O}_C(C) \simeq \mathcal{O}_C$.

In case (P_2) the assertion about the irregularity of S follows immediately by considering the long exact sequence obtained from

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(C) \rightarrow \mathcal{O}_C(C) \rightarrow 0$$

and using $\mathcal{O}_C(C) \simeq \eta$.

In case (P_3) , since $K+C=3C+Z$, $\mathcal{O}_C(K+C) \simeq \eta^{\otimes 2}$ and $\mathcal{O}_C(C+Z) \simeq \eta$ we obtain $\mathcal{O}_C(2C) \simeq \mathcal{O}_C(C+Z) \simeq \eta$. Therefore the base point o of $|C|$ is a Weierstrass point of C and Z intersects a general curve of \mathcal{P} at o . Since Z is 2-connected and o is a simple point of Z , Z is contained in some curve C' of \mathcal{P} .

Notice that every curve in \mathcal{P} is 1-connected because \mathcal{P} has a base point (cf. lemma (2.6) of [M]). Suppose that C'' in \mathcal{P} decomposes as $C''=A_1+A_2$, with A_1, A_2 curves such that $A_1 \cdot A_2=1$. Since C is nef and $C^2=1$ we will have, say, $C \cdot A_1=1$ and $C \cdot A_2=0$ and thus o lies on A_1 . But, by 2-connectedness of the canonical divisors, we must have $A_2 \cdot (K-A_2)=A_2 \cdot (A_1+C+Z) \geq 2$ and thus $A_2 \cdot Z \geq 1$, implying $A_1 \cdot Z \leq 0$. Since $o \in A_1 \cap Z$, necessarily A_1 and Z have common components and thus $C''=C'$.

Now we can give the:

Proof of (4.1) By (4.3), possibility (P_2) can only occur if $q>0$. To prove the theorem we have to show that also possibilities (P_1) and (P_3) do not occur. First let us suppose that (P_3) happens.

In this case by upper semi-continuity we have $h^0(C, \mathcal{O}_C(2o)) \geq 2$, for every curve C in \mathcal{P} . Recall that every curve $C \in \mathcal{P}$ different from C' is 2-connected. Then for every $C \neq C'$ in \mathcal{P} , we have $h^0(C, \mathcal{O}_C(2o))=2$ and $\mathcal{O}_C(2o)$ is generated by its global sections (see [CFM], propositions (A.5) and (A.6)). Now we want to show that this also happens for C' .

Notice that, since C' is 1-connected, then the restriction map $H^0(C', \mathcal{O}_{C'}(2o)) \rightarrow H^0(Z, \mathcal{O}_Z(2o))$ is injective by lemma (A.1) of [CFM]. Since $|2C|$ induces on C' a section vanishing only at o , then $|\mathcal{O}_{C'}(2o)|$ has no fixed components on C' . Since Z is 2-connected, by lemma (A.5) and (A.6) of [CFM] one has necessarily $h^0(C', \mathcal{O}_{C'}(2o))=2$ and $\mathcal{O}_{C'}(2o)$ generated by its global sections unless possibly if Z is irreducible and rational, i.e. unless $K^2=6$ and Z is an irreducible (-2) -curve.

We are going to see that also in this case $h^0(C', \mathcal{O}_{C'}(2o))=2$ and $\mathcal{O}_{C'}(2o)$ is generated by its global sections. Suppose then that $K^2=6$. Let $C'=Z+B$ and notice that $Z \cdot B=3$. One has $3C \equiv K+B$. Consider the linear system $|K+C|$. Since $|K+C|=|Z+3C|$, the base points of $|K+C|$ can only lie on Z . Since S is regular, $|K+C|$ cuts the complete canonical series $|\omega_C|$ on every curve C in \mathcal{P} and therefore o cannot be a base point of $|K+C|$. On the other hand, since $(K+C)^2=13$ and the map determined by $|K+C|$ is not birational, then $|K+C|$ has at least a base point $p \in Z$, without however having Z as a fixed component, because $o \in Z$. Since $(K+C) \cdot Z=1$, then $|K+C|$ has a unique simple base point $p \in Z$.

Consider the exact sequence

$$0 \rightarrow H^0(S, \mathcal{O}_S(3C)) \rightarrow H^0(S, \mathcal{O}_S(K+C)) \xrightarrow{r} H^0(Z, \mathcal{O}_Z(K+C)).$$

As we saw above, $\text{Im } r$ is 1-dimensional. Since, by the Riemann-Roch theorem, $h^0(S, \mathcal{O}_S(K+C))=6$, then $h^0(S, \mathcal{O}_S(3C))=5$, implying, by Riemann-Roch, that $h^1(S, \mathcal{O}_S(K+B))=h^1(S, \mathcal{O}_S(3C))=1$. Since $q=0$, we have then $h^0(B, \mathcal{O}_B)=2$.

Consider the long exact sequence obtained from

$$0 \rightarrow \mathcal{O}_Z(2o - B) \rightarrow \mathcal{O}_C(2o) \rightarrow \mathcal{O}_B \rightarrow 0.$$

Since $\mathcal{O}_Z(2o - B)$ has degree -1 on Z , we see that $h^0(C', \mathcal{O}_{C'}(2o)) = 2$. Suppose that $\mathcal{O}_C(2o)$ is not generated by its global sections. Then, since $|2C|$ induces on C' a section vanishing only at o , the point o is a common zero of the global sections of $\mathcal{O}_C(2o)$ and so $h^0(C', \mathcal{O}_{C'}(o)) = 2$. But this would imply that o is a base point of $|\omega_{C'}|$, and we are going to see that this is a contradiction. In fact $|K + C|$ cuts the complete canonical system on every curve C in \mathcal{P} , included C' . Since, as we saw before, o is not a base point of $|K + C|$, then o cannot be a base point for $|\omega_{C'}|$.

In conclusion, we have seen that for every C in \mathcal{P} one has $h^0(C, \mathcal{O}_C(2o)) = 2$ and $\mathcal{O}_C(2o)$ is generated by its global sections. Hence we can now imitate an argument from [CFM], proposition (1.7).

Let $\sigma: \tilde{S} \rightarrow S$ be the blow-up of S at o and let E be the exceptional divisor. We denote by $p: \tilde{S} \rightarrow \mathbf{P}^1$ the morphism determined by the strict transform on \tilde{S} of the pencil \mathcal{P} . Since $h^0(C, \mathcal{O}_C(2o)) = 2$ for every curve $C \in \mathcal{P}$, the sheaf $p^*(\mathcal{O}_{\tilde{S}}(2E))$ is locally free of rank 2. Moreover we have $h^0(\mathbf{P}^1, p_*(\mathcal{O}_{\tilde{S}}(2E))) = 1$, hence $p_*(\mathcal{O}_{\tilde{S}}(2E)) \simeq \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-e)$, where e is a positive integer. As usual, we set $\mathbf{F}_e = \text{Proj}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-e))$, we denote by F a fibre of the structure map $\mathbf{F}_e \rightarrow \mathbf{P}^1$, and by C_0 the section of $\mathbf{F}_e \rightarrow \mathbf{P}^1$ such that $C_0^2 = -e$.

Since for every curve $C \in \mathcal{P}$ the sheaf $\mathcal{O}_C(2o)$ is generated by global sections, the natural map $p^*p_*(\mathcal{O}_{\tilde{S}}(2E)) \rightarrow \mathcal{O}_{\tilde{S}}(2E)$ is surjective and defines a morphism $\pi: \tilde{S} \rightarrow \mathbf{F}_e$ over \mathbf{P}^1 which is generically finite of degree 2 and such that $\pi^*(C_0) = 2E$. Then C_0 is a component of the branch locus of π and $e = -C_0^2 = -2E^2 = 2$.

Now

$$\sigma^*(3C) - E = \pi^*(C_0 + 3F)$$

and this is a contradiction since

$$h^0(\mathbf{F}_2, \mathcal{O}_{\mathbf{F}_2}(C_0 + 3F)) = 6$$

whereas

$$h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\sigma^*(3C) - E)) < 6$$

In fact $h^0(S, \mathcal{O}_S(K + C)) = 6$, $K + C - Z \equiv 3C$ and Z , as we saw, is not a fixed part of the linear system $|K + C|$. In conclusion, we have proved that possibility (P_3) cannot occur.

Next we consider possibility (P_1) . Since $g = 0$, we have $h^0(C + Z_1, \mathcal{O}_{C+Z_1}(C + Z_1)) = 1$. Moreover, by the adjunction formula, $\mathcal{O}_{C+Z_1}(C + Z_1)^{\otimes 3} \simeq \omega_{C+Z_1}$. Then we can apply lemma (7.2) to $C + Z_1$ and conclude that, for the general curve C in \mathcal{P} , $C \cap Z_1 = \{P, Q\}$, where P, Q are distinct points of C such that $\mathcal{O}_C(P + Q) \simeq \eta$, where $|\eta|$ is the g_2^1 on C .

Since $K = 2(C + Z_1)$, every divisor on S has even self-intersection. Thus the curves of \mathcal{P} are either 2-connected or are multiple curves.

Suppose that every curve in \mathcal{P} is 2-connected. Then Z_1 induces on every such

a curve a base point-free g_2^1 (cf. the discussion of case (P_3)). Using a similar argument as above, we realize S as a double cover $p: S \rightarrow \mathbf{F}_1$, which sends Z_1 generically 2 to 1 to the section C_0 of \mathbf{F}_1 with self-intersection -1 . But this, as above, leads to a contradiction. In fact $\mathcal{O}_S(K+C)$ would then be the pull back of $\mathcal{O}_{\mathbf{F}_1}(2C_0+3F)$, but $h^0(S, \mathcal{O}_S(K+C))=6$ whilst $h^0(\mathbf{F}_1, \mathcal{O}_{\mathbf{F}_1}(2C_0+3F))=9$.

Thus we may assume there is at least a curve \tilde{C} of \mathcal{P} which is not 2-connected, hence $\tilde{C}=2F$, where F is a 2-connected genus 2 divisor. Notice that the existence of this double curve and the 2-connectedness of the canonical divisor on S easily imply that Z_1 is irreducible. Since S is regular, the restriction map $H^0(S, \mathcal{O}_S(K+F)) \rightarrow H^0(F, \omega_F)$ is surjective and therefore $h^0(S, \mathcal{O}_S(K+F))=5$. On the other hand $\mathcal{O}_S(K+F-C)=\mathcal{O}_S(C+F+2Z_1)$ and thus $h^0(S, \mathcal{O}_S(K+F-C))=2$, implying that for any curve C in \mathcal{P} the restriction map $H^0(S, \mathcal{O}_S(K+F)) \rightarrow H^0(C, \omega_C)$ is surjective. Since $(K+F) \cdot Z_1=1$ and for C general in \mathcal{P} the rational map determined by $|K+F|$ identifies the two distinct points in which Z_1 intersects C , the restriction map $H^0(S, \mathcal{O}_S(K+F)) \rightarrow H^0(Z_1, \mathcal{O}_{Z_1}(K+F))$ is not surjective. Therefore $|K+F|$ will have a base point p on Z_1 . Since F is 2-connected, p does not lie on F (see [CFM], proposition (A.7)). This implies that there is a curve $C'' \in \mathcal{P}$, different from \tilde{C} , such that $|\omega_{C''}|$ is not generated by its global sections. Thus there exists another double fibre F' . But then $\mu := \mathcal{O}_S(F-F')$ is a non-zero 2-torsion element in $\text{Pic}(S)$. By the Riemann-Roch theorem we have:

$$2h^0(S, \mathcal{O}_S(C+Z_1) \otimes \mu) = 3 + h^1(S, \mathcal{O}_S(C+Z_1) \otimes \mu)$$

Hence $h^0(S, \mathcal{O}_S(C+Z_1) \otimes \mu) \geq 2$. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_S(F+F') \rightarrow \mathcal{O}_S(C+Z_1) \otimes \mu \rightarrow \mathcal{O}_{Z_1} \rightarrow 0.$$

Since $h^0(S, \mathcal{O}_S(F+F'))=1$, the restriction map $H^0(S, \mathcal{O}_S(C+Z_1) \otimes \mu) \rightarrow H^0(Z_1, \mathcal{O}_{Z_1})$ is not the zero map. This implies that also the restriction map $H^0(S, \mathcal{O}_S(K)) \rightarrow H^0(Z_1, \mathcal{O}_{Z_1})$ is not the zero map, which is a contradiction to $|K|$ having Z_1 as a fixed component. Therefore also case (P_1) is ruled out.

We finish this paragraph with a proposition which will be useful later on in order to recognize specializations of the Du Val ancestor with $p_g=6$, $K^2=9$ which we met in example (1.4, ii). Its proof uses some ideas from [CFM], which are also essentially contained in the proof of theorem (4.1) above.

(4.4) Proposition. *Let S be a surface with $p_g \geq 3$, $q=0$, whose bicanonical map is not birational. Suppose there is on S an irreducible curve C such that $C^2=1$, $K \cdot C=3$ and $|C|$ is a pencil. Then S is a specialization of the Du Val ancestor with $p_g=6$, $K^2=9$ (example (1.4, ii)). If S presents the non-standard cases the specialization in question is as indicated in table (1.3).*

Proof. To prove the proposition it suffices to show that:

(i) every smooth curve in the pencil $|C|$ is hyperelliptic and the base point p is a Weierstrass point of the g_2^1 on it;

(ii) $h^0(S, \mathcal{O}_S(2C))=4$ and the linear system $|2C|$ is base point free.

Having (i) and (ii), we can apply the same reasoning as in lemma (2.6) of [CFM] to obtain the description of S as a double cover of \mathbf{F}_2 , which shows that it is a specialization of the Du Val ancestor with $p_g=6$, $K_2=9$.

As for the type of singularities of the branch locus that can occur, let m be the maximum multiplicity of a singularity of the branch locus. Since $p_g \geq 3$ we have $m \leq 7$. If $m=6, 7$ one has $p_g=3$, and it is easy to see that the *minimal* surface corresponding to the double cover has $K^2=2$ and no pencil $|C|$ as in the statement of the present proposition: we leave the details to the reader who is advised to project down to \mathbf{P}^2 from the point of multiplicity m , thus realizing the surface in question as a double plane with a branch curve of degree 8 (see proposition (5.1) below). Therefore we may assume $m \leq 5$. Actually the case $m=5$ is a specialization in moduli of the case $m=4$ (cf. remark (2.7) of [CFM]). Finally one notices that, if the branch curve has more than one 4-uple (or 5-uple) point, the surface S contains a genus 2 pencil, i.e. the one which is the pull-back on S of the pencil of plane sections through such two singular points of the branch curve.

Finally we have to prove (i) and (ii) above By Riemann-Roch we have:

$$2 + h^0(S, \mathcal{O}_S(K-C)) = p_g + h^1(S, \mathcal{O}_S(K-C)).$$

Since $p_g \geq 3$, we have $h^0(S, \mathcal{O}_S(K-C)) > 0$, i.e. there is a curve $C' \equiv K-C$. Notice that, since $(K-C) \cdot C = 2$ then every curve $C' \in |K-C|$ is 1-connected by lemma (A.4) of [CFM]. At this point, with the same reasoning as in lemma (2.5) of [CFM], one proves (i).

Consider then the restriction map $r: H^0(S, \mathcal{O}_S(K)) \rightarrow H^0(C, \mathcal{O}_C(K))$. Notice that, by adjunction, $\mathcal{O}_C(K) \simeq \mathcal{O}_C(p) \otimes \eta$, where $|\eta|$ is the g_2^1 on C . From the above propositions (for $p_g=3$) and from proposition (1.7) of [CFM] (for $p_g \geq 4$) we know that the canonical system of S is not composed with the pencil $|C|$. Therefore the rank of r is 2, hence $h^0(S, \mathcal{O}_S(K-C)) = p_g - 2$ and (ii) again can be proved as in lemma (2.5) of [CFM].

(4.5) **Remark.** Suppose S is a minimal surface of general type with $K^2 \geq 4$ and C is a curve on it such that $h^0(S, \mathcal{O}_S(C)) \geq 2$ and $K \cdot C \leq 3$. By the index theorem, one has that the movable part of $|C|$ is either a pencil of curves of genus 2 or a pencil of curves of genus 3 with a base point, like in the statement of proposition (4.4).

5. The case $p_g=3$, $q=0$ and the canonical system without fixed components

Let S be a minimal surface, with $p_g=3$ and $q=0$, and non birational bicanonical map. We will keep the notation as in the previous paragraphs. We will also assume that $|K|$ is not composite with a pencil, which, as we saw in §4, will always be the case if S does not contain a pencil of curves of genus 2. We will suppose here that $|K|$ has no fixed component, i.e., with the above notation, $M=K$. Let p_1, \dots, p_d be the base points of $|K|$, which are smooth, transversal base points. Furthermore one has $d=K^2-2$ (see proposition (2.2)).

Let $\pi: \tilde{S} \rightarrow S$ be the blow-up of S at p_1, \dots, p_d , and let E_1, \dots, E_d be the exceptional divisors of the blow-up. The base point free linear system $|\tilde{M}| = |\pi^*(M) - E_1 - \dots - E_d|$ determines a *double plane*, i.e. a morphism $\psi := \phi_{\tilde{M}}: \tilde{S} \rightarrow \mathbf{P}^2$, which is generically finite of degree two. We will call such a double plane the *canonical double plane* of S .

In the present paragraph we will describe the double planes arising in this way, by describing their branch curves B in \mathbf{P}^2 . Notice that the curves B are reduced, and we will in fact describe their isolated singularities.

We start with the following proposition whose immediate proof we omit:

(5.1) **Proposition.** *Let S be as above. Then $|K|$ is base point free (i.e. $d=0$), if and only if $K^2=2$. In this case the corresponding canonical double plane has a branch curve B which has degree 8 and has at most irrelevant singularities.*

Thus, from now on we will assume that $|K|$ is not base point free, i.e. $d \geq 1$ and therefore $K^2 \geq 3$. In the following lemma we collect some basic information about the branch curve of the canonical double plane of S :

(5.2) **Lemma.** *Let S be as above and let B be the branch curve of the corresponding canonical double plane. Then:*

- (i) B has degree $2d+8$;
- (ii) the exceptional curves E_1, \dots, E_d are mapped by ϕ to d distinct lines L_1, \dots, L_d in \mathbf{P}^2 , which are contained in B ;
- (iii) the essential singular points of B all lie on $L_1 \cup \dots \cup L_d$ and on each line L_i there are at most three singular points of B ;
- (iv) every intersection point of two of the lines L_1, \dots, L_d is a relevant singular point of B .

Proof. Assertion (i) follows by Hurwitz's formula and by adjunction.

Assertion (ii) is a consequence of proposition (2.2).

Let $z \in B$ be an essential singular point. Then there is some irreducible curve Γ on \tilde{S} contracted by ϕ to z . Then $\tilde{M} \cdot \Gamma = 0$. If $\Gamma \cdot E_i = 0$ for all $i=1, \dots, d$, then π isomorphically maps Γ to a rational curve on S with self-intersection -2 . This situation would only produce non essential singularities of B at z . So we may assume that there is a $i \in \{1, \dots, d\}$ such that $\Gamma \cdot E_i > 0$, which implies that $z \in L_i$. This proves the first assertion in (iii). The remainder of (iii) follows from lemma (3.1, iii).

Let $z = L_i \cap L_j$, with $1 \leq i < j \leq d$. Since L_i [resp. L_j] is the image via ϕ of E_i [resp. E_j], and since $E_i \cap E_j = \emptyset$, there must be a curve A on \tilde{S} contracted by ϕ to z , not containing E_i and E_j but meeting both E_i and E_j . The image of such a curve on S via π has positive intersection with K . Therefore the resulting singularity of B at z cannot be irrelevant.

Now we want to explain which kind of singularities of B may occur along one of the lines L_i , $i=1, \dots, d$. Let L be such a line. It is the image of an exceptional divisor E on \tilde{S} , which in turn corresponds to a base point p of $|K|$ on S .

Let us go back to lemma (3.4) and let us summarize the information it gives us in the present situation. First we notice we are in case (I) of that lemma. Let

$\tilde{M}_p \in |\tilde{M}|$ the curve on \tilde{S} corresponding to M_p . Since \tilde{M}_p contains $2E$, the image via ψ of \tilde{M}_p is just the line L . Therefore the strict transform \tilde{A} on \tilde{S} of a curve $A := A_i$ of the decomposition of type (I) is contracted by ψ to a point $z \in L$. Accordingly A is contracted by the map ϕ_M (see lemma (3.4, I, iii)), therefore if M is the general curve in $|M|$, the points $M \cap A$ are exactly the base points of $|M|$ on A . Lemma (3.1) yields that every such a base point is smooth for A , and M and A intersect transversally there. Hence the number of base points of $|M|$ on A is just $M \cdot A$, i.e. $K \cdot A$.

We have the following lemma:

(5.3) **Lemma.** *In the above situation, the point z belongs to A^2+2 lines among L_1, \dots, L_d . Furthermore it has multiplicity $2A^2+6$ for B . Finally if L_1, \dots, L_d pass through the same point z , then S has a pencil of curves of genus two.*

Proof. One has $M \cdot A = K \cdot A = A^2 + 2$. Hence $|M|$ has $A^2 + 2$ distinct base points on A , proving the first assertion. Notice now that $\tilde{A}^2 = -2$. The arithmetic genus of A is $g = A^2 + 2$. Since the base points we blow-up are smooth points of A , this is also the arithmetic genus of \tilde{A} . Hence the second assertion easily follows.

As for the last assertion, the curve A contracted by ϕ_M to z is such that $A^2 = d - 2$, hence z is a point of multiplicity $2d + 2$ for B . Therefore the pull-back to \tilde{S} of the pencil of lines through z is a pencil of curves of genus two.

According to lemma (5.2), we can write $B = L_1 + \dots + L_d + B'$. The next proposition gives us information about the configuration of lines $L_1 + \dots + L_d$ occurring in B .

(5.4) **Proposition.** *Let L_1, \dots, L_d be a set of $d, d \geq 3$, distinct lines in \mathbf{P}^2 such that each line meets the union of the other lines in at most three distinct points. Then either all lines pass through the same point, or $d \leq 6$ and one has the following possibilities:*

- (C₃): $d=3$ and the three lines form a triangle;
- (C_{4A}): $d=4$ and the four lines are the edges of a quadrilateral;
- (C_{4B}): $d=4$ and three of the lines meet in one point which does not belong to the fourth line;
- (C₅): $d=5$ and four of the lines are the edges of a quadrilateral and the fifth line joins two opposite vertices;
- (C₆): $d=6$ and the six lines are the sides of a complete quadrangle.

Proof. Assume that not every line passes through the same point. Let h be the maximum number such that h lines L_1, \dots, L_h with $h < d$ meet at a point q . Then $L_k, k > h$, does not pass through q , hence $L_k \cap (L_1 \cup \dots \cup L_h)$ consists of h distinct points and therefore $h \leq 3$.

Suppose first that $h=3$. Then L_4 meets L_1, L_2, L_3 , in three other distinct points q_1, q_2, q_3 and thus, if there is a fifth line L_5 , it necessarily meets L_4 in one

of these three points. On the other hand L_5 meets L_1, L_2, L_3 in three distinct points s_1, s_2, s_3 , respectively one of which coincides with a point q_i , say $q_3 = s_3$. Then, if there is a sixth line L_6 , it meets L_1 in, say, q_1 and therefore meets L_2 in s_2 and L_3 in another point r_3 . So the six lines are the sides of the quadrangle determined by q, q_1, s_2, s_3 . If $d \geq 7$ then L_7 would necessarily have to pass through the non collinear points q_2, s_1 and r_3 a contradiction.

Suppose now that $h = 2$. If the number of lines d is 3 we have case (C_3) , whilst, if $d \geq 4$, clearly the only possibility is case (C_{4A}) .

We are finally in a position to describe also the behaviour of the part B' of the branch curve B of our canonical double planes. This is done in the following theorem, which is, in practice, a new, more precise version of Du Val's statement (1.2):

(5.5) Theorem. *Let S be a minimal surface, with $p_g = 3$, $q = 0$, and non birational bicanonical map, which presents the non-standard case. Suppose $|K|$ has no fixed components, but only $d > 0$ base points. Then one has $5 \leq K^2 \leq 8$ and accordingly $3 \leq d \leq 6$ and one has the following possibilities:*

$K^2 = 5, d = 3$: $L_1 + L_2 + L_3$ is a triangle, B' has degree 11, has 4-uple points at the vertices of the triangle, three 3-ple points one on each line L_i , and B has no other relevant singularities;

$K^2 = 6, d = 4$ (case (A)): $L_1 + L_2 + L_3 + L_4$ form a quadrilateral, B' has degree 12 and has six 4-uple points at the vertices of the quadrilateral, and B has no other relevant singularities;

$K^2 = 6, d = 4$ (case (B)): $L_1 + L_2 + L_3$ pass through a point z whereas L_4 does not; B' has degree 12 and has a 5-uple point at z and 4-uple points at the intersections $L_i \cap L_4$, $i = 1, 2, 3$, and three more 3-ple points along the lines L_i , $i = 1, 2, 3$, and B has no other relevant singularities;

$K^2 = 7, d = 5$: $L_1 + L_2 + L_3 + L_4$ form a quadrilateral, and L_5 joins two opposite vertices z_1, z_2 , of it; B' has degree 13, has 5-uple points at z_1, z_2 , has 4-uple points at the remaining vertices of the quadrilateral and a 3-ple point on L_5 , and B has no other relevant singularities;

$K^2 = 8, d = 6$: $L_1 + L_2 + L_3 + L_4 + L_5 + L_6$ form a complete quadrangle; B' has degree 14, has 5-uple points at the four triple vertices of the complete quadrangle, has three 4-uple points at the double vertices of the complete quadrangle, and B has no other relevant singularities.

The surfaces one thus finds are exactly the Du Val examples with $p_g = 3$ appearing in table (1.3).

Proof. The line configurations and the multiplicities of B' are dictated by (5.3) and (5.4). The fact that there are no further relevant singularities easily follows with a direct double plane computation since $p_g = 3$, $q = 0$.

The last assertion is clear for the case $K^2 = 8$ by taking into account example (1.4, iv). Consider now the case $K^2 = 7$. Let us make a quadratic transformation of the plane based at z_1, z_2 and at one of the remaining vertices of the quadrilateral,

for instance at $z=L_1 \cap L_2$. This transformation contracts L_1 and L_2 to distinct points α_1, α_2 , and L_5 to a point α . Furthermore L_3 and L_4 are mapped to different lines a_1, a_2 passing through α_1 and α_2 . We let $\gamma=a_1 \cap a_2$. The curve B' is in turn mapped to a curve G' of degree 12 with two [4,4]-points at $\alpha_i \in a_i$, where the infinitely near 4-uple point to α_i lies on the line a_i , $i=1, 2$, another 4-uple point at γ , and a point [3.3] at α . The original double plane is therefore birationally equivalent to the new double plane with branch curve given by $G=a_1+a_2+G'$. By comparing this description with the double plane description of the Du Val ancestor with $p_g=4, q=0, K^2=8$, given in example (1.4, iii), we conclude that our surface is a specialization of the aforementioned Du Val ancestor as shown in table (1.3).

A similar analysis works for the case $K^2=6$, case (A). We leave the details to the reader.

Again a similar argument would easily work also for the case $K^2=5$, showing that it is a specialization of the Du Val ancestor with $p_g=6, K^2=8$ as shown in table (1.3). However case (B) with $K^2=6$ is more difficult to analyse in this way. So we present another argument which works for both cases $K^2=5$ and $K^2=6$, case (B). The observation here is that in both cases the surface possesses a pencil $|C|$ of curves of genus 3 with a base point. Such a pencil is provided by the pull-back on the surface of the pencil of lines through the point $L_1 \cap L_2$. Then, applying proposition (4.4), we see that both surfaces are specializations of the Du Val ancestor with $p_g=6, K^2=9$.

(5.6) **Remark.** As a consequence of our analysis, we have the following fact: *if S is a minimal surface, with $p_g=3, q=0$ and non birational bicanonical map, such that $|K|$ has no fixed components and is not composite with a pencil, then either $K^2 \leq 8$ and we have one of the cases listed in theorem (5.5) or S has a pencil of curves of genus two.*

6. The case $p_g=3, q=0$ and the canonical system with fixed components

In order to finish our classification theorem, we study in this section the minimal regular surfaces S with non-birational bicanonical map, presenting the non-standard case, whose canonical system has fixed components. The result we will prove here is the following theorem:

(6.1) **Theorem.** *Suppose S is a minimal surface with $p_g=3, q=0$, non-birational bicanonical map, presenting the non-standard case, and having canonical system with fixed components. Then S is a specialization of a Du Val ancestor as indicated in table (1.3).*

This result will follow as a consequence of propositions (6.5) and (6.6) below. At the end we will explain how the specialization of the Du Val ancestors takes place in order to produce fixed components of the canonical system (see remark (6.6) below).

By §4, we know that $|K|$ is not composed with a pencil, and therefore we can use the properties of the canonical divisors studied in §§2 and 3. Also, as we

already remarked, from [R] and [CCM], we know that if $K^2 \geq 9$, $p_g \leq 5$ and the bicanonical map is not birational, then S presents the standard case. Thus we may assume $K^2 \leq 8$. Since in proposition (4.4) we described the minimal surfaces with non birational bicanonical map, containing a pencil $|C|$ such that $C_2 = 1$, $K \cdot C = 3$, to keep the proofs shorter we will assume also that S does not contain such a pencil. We start by showing the following proposition which makes more precise in our case the result of proposition (2.1):

(6.2) **Proposition.** *Suppose S is a minimal surface with $p_g = 3$, $q = 0$, $K^2 \leq 8$, non-birational bicanonical map, presenting the non-standard case, and having canonical system with fixed components. Assume furthermore that S has no pencil $|C|$ with $C^2 = 1$ and $K \cdot C = 3$. Then $|K| = |M| + F$, with $|M|$ the movable part which is irreducible and has $d \geq 0$ simple base points, and F is a 1-connected fixed curve such that $M \cdot F = 2$ and either $F^2 = -1$ or $F^2 = -2$.*

Proof. Let us first notice that if $K = M + F_1 + \dots + F_n$, then $K^2 = M^2 + 2n + K \cdot F_1 + \dots + K \cdot F_n$. By hypothesis $K^2 \leq 8$, and therefore $n \leq 3$, with equality occurring only if $K \cdot F_1 = K \cdot F_2 = K \cdot F_3 = 0$ and $M^2 = 2$.

Suppose there is a fixed component F_1 with $F_1^2 = -1$. Since $K \cdot F_1$ is then odd, we cannot have $n = 3$. Now we will see that also $n = 2$ cannot occur. Assume in fact that $n = 2$. Consider a decomposition $M - F_1 = A_1 + A_2 + A_3$, as in lemma (3.5). One has $M \cdot A_i = A_i^2 + 1$, hence $K \cdot A_i + A_i^2 = M \cdot A_i + F_1 \cdot A_i + F_2 \cdot A_i + A_i^2 = 2A_i^2 + 2 + A_i \cdot F_2$, for $i = 1, 2, 3$. Therefore $A_i \cdot F_2$ must be even for all $i = 1, 2, 3$. Since $F_2 \cdot F_1 = 0$ and $F_2 \cdot M = 2$, there is $i \in \{1, 2, 3\}$ such that $A_i \cdot F_2 = 2$. This contradicts lemma (3.7, iii).

Therefore if $n \geq 2$ all the fixed components F_i have self-intersection -2 .

Suppose that $n = 2$ and let $M - F_1 = A_1 + A_2 + A_3$ be a decomposition as in lemma (3.6). Then we have $A_i \cdot F_1 = 1$, $A_j \cdot F_1 = 1$ and $A_k \cdot F_1 = 2$, for $\{i, j, k\} = \{1, 2, 3\}$. Furthermore, one has $A_h \cdot M = A_h^2 + A_h \cdot F_1$, for $h = 1, 2, 3$. Notice also that $K^2 = K \cdot M = K \cdot A_1 + K \cdot A_2 + K \cdot A_3$. By lemma (3.6, iv), we have $h^0(S, \mathcal{O}_S(M - A_h)) = 2$, for $h = 1, 2, 3$. Consider the pencil $|C|$ corresponding to $h^0(S, \mathcal{O}_S(M - A_k))$. Then $K \cdot C \leq K \cdot A_i + K \cdot A_j$. The same argument used above in the case of a fixed F with $F^2 = -1$, shows that $A_i \cdot F_2 = A_j \cdot F_2 = 0$, $A_k \cdot F_2 = 2$, and therefore $K \cdot A_i = A_i^2 + 2$, $K \cdot A_j = A_j^2 + 2$. By lemma (3.7), we have $0 \leq M \cdot A_i = A_i^2 + 1 \leq 1$, hence $-1 \leq A_i^2 \leq 0$, and similarly $-1 \leq A_j^2 \leq 0$. Accordingly $1 \leq K \cdot A_i \leq 2$ and $1 \leq K \cdot A_j \leq 2$. Since we are excluding the existence of pencils $|C|$ with $K \cdot C \leq 3$, we must have $K \cdot A_i = 2$, $K \cdot A_j = 2$ and $A_i^2 = A_j^2 = 0$. By lemma (3.7) and lemma (3.2, iii), there are two distinct base points p, q of $|M|$ which lie in A_i and A_j respectively and the curves M_p and M_q have type (II) decomposition $M_p = 2A_i + H$, $M_q = 2A_j + H$ where $|2A_j| = |2A_i|$ is a base point free genus 3 pencil.

Consider now the two divisors $G = A_i + A_j + F_1$ and $G' = A_k + F_1 + F_2$. One has $G^2 = 2$, $G'^2 = A_k^2 + 4$, $G \cdot G' = 2$. By the index theorem we must have $G'^2 \leq 2$, i.e. $A_k^2 \leq -2$. On the other hand, we have $0 \leq M \cdot A_k = A_k^2 + 2$. Hence we have $A_k^2 = -2$, $M \cdot A_k = 0$, which yields $M^2 = 4$, $K^2 = 8$ and $G \sim G'$. This implies that $A_i + A_j \sim A_k + F_2$.

On the other hand since clearly $A_i \sim A_j$ we have $\mathcal{O}_S(2A_i) \sim \mathcal{O}_S(A_k + F_2)$. Since $|\mathcal{O}_M(2A_i)|$ and $|\mathcal{O}_M(A_k + F_2)|$ are the same g_2^1 on a general curve of $|M|$, and therefore $\mathcal{O}_M(2A_i - A_k - F_2)$ is trivial on a general curve of $|M|$ we have, by proposition (1.6) of [CFM], $\mathcal{O}_S(2A_i) \simeq \mathcal{O}_S(A_k + F_2)$. So $|K|$ is the sum of the two pencils $|A_k + F_1 + F_2|$ and $|A_i + A_j + F_1|$, a contradiction. Therefore the case $n=2$ does not occur.

Finally we turn to the case $n=3$, in which, as we saw already, one has $M^2=2$, i.e. $|M|$ is base point free. Consider again the decomposition of the curve $M - F_1 = A_1 + A_2 + A_3$, where $A_i \cdot F_1 = 1$, $A_j \cdot F_1 = 1$ and $A_k \cdot F_1 = 2$, for $\{i, j, k\} = \{1, 2, 3\}$. In this case, since $|M|$ has no base points, we have $M \cdot A_h = 0$, for $h=1, 2, 3$, hence $A_i^2 = A_j^2 = -1$, $A_k^2 = -2$.

Since $K \cdot (M - A_k) = K \cdot A_i + K \cdot A_j$ we must either have $K \cdot A_i > 1$ or $K \cdot A_j > 1$, because otherwise, the pencil corresponding to $|\mathcal{O}_S(M - A_k)|$ would be a genus 2 pencil. Suppose $K \cdot A_i > 1$. Since $K \cdot A_i = 1 + F_2 \cdot A_i + F_3 \cdot A_i$, $F_2 \cdot A_i + F_3 \cdot A_i$ is positive and even, and so, by lemma (3.7, iii), we must have $A_i \cdot F_2 = A_i \cdot F_3 = 1$, yielding $K \cdot A_i = 3$.

Consider the pencil $|\mathcal{O}_S(M - A_i)|$ and let $|V|$ be its movable part. We have then $M = V + A$ where $|V|$ is a pencil without fixed components, $A_i \leq A$ and $A \cdot V \geq 1$, by lemma (2.6) of [M]. Since $M^2=2$ and M is nef and not composed with a pencil we have that $M \cdot V = 2$, $M \cdot A = 0$ and therefore either:

(i) $V^2=0$, $A^2=-2$ and $V \cdot A=2$;

or

(ii) $V^2=1$, $A^2=-1$ and $V \cdot A=1$.

Remark that $|\mathcal{O}_M(V)|$ is the g_2^1 on the general curve $M \in |M|$ and similarly $|\mathcal{O}_V(M)|$ is a g_2^1 on the general curve $V \in |V|$.

Consider case (i). Since $8 \geq K^2 = K \cdot M = K \cdot V + K \cdot A$, $K \cdot A \geq K \cdot A_i = 3$ and we are assuming that $|V|$ is not a genus 2 pencil, we must have $K \cdot A = 4$ and $K \cdot V = 4$. Thus:

$$F_1 \cdot A + F_2 \cdot A + F_3 \cdot A = K \cdot A - M \cdot A = K \cdot A = 4$$

$$F_1 \cdot V + F_2 \cdot V + F_3 \cdot V = K \cdot V - M \cdot V = 2 \tag{*}$$

In particular we have $0 \leq V \cdot F_l \leq 2$ and $0 \leq A \cdot F_l \leq 2$, $l=1, 2, 3$. First we see that the case $V \cdot F_l = A \cdot F_l = 1$ cannot occur. Suppose in fact that $V \cdot F_l = A \cdot F_l = 1$. Then we claim that there is a curve in the pencil $A + |V|$ containing F_l , but F_l is not contained in A . Suppose in fact that there is no such a curve in $A + |V|$ and let Γ be the unique curve in $|M - F_l|$. Then $|M|$ is spanned by the pencil $A + |V|$ and by $\Gamma + F_l$. Since $A \cdot F_l = 1$, then $|M|$ certainly has some base points on F_l a contradiction. However A cannot contain F_l because, as we recalled already, $h^0(S, \mathcal{O}_S(M - F_l)) = 1$ by proposition (2.3, i). This proves the claim.

Since $V \cdot F_l = 1$, there is an irreducible component Δ of F_l , which is a smooth, rational curve, such that $V \cdot \Delta = 1$, whereas for any other irreducible component Δ' of F_l , one has $V \cdot \Delta' = 0$. Let us prove that $A = 2\Delta + A'$, where A' is effective.

First of all we prove that A contains Δ . Otherwise, in fact, there would be no curve in $A + |V|$ containing Δ hence containing F_l , contrary to what we saw before. Now we prove that, in fact, A contains 2Δ . Let V be the general curve

in $|V|$ and let p be its intersection point with Δ . Since, of course, the bicanonical involution fixes V and F_b , it fixes p , which is therefore a Weierstrass point of the g_2^1 on V . On the other hand, since $|\mathcal{O}_V(M)|$ is the g_2^1 on V , then $|\mathcal{O}_V(A)|$ is also the g_2^1 . Hence A has multiplicity of intersection 2 with V at p , therefore p lies on $A - \Delta$. Since, as V moves in $|V|$, the point p describes the whole of Δ , we have the assertion.

With the same argument one proves that a curve in $|2K|$ containing Δ contains Δ with multiplicity at least 2, i.e. Δ is a fixed component of $|2K - \Delta|$.

Now, by taking into account (*), we see that if $V \cdot F_l = 1$, there is another curve F_h such that $V \cdot F_h = 1$, with $h \neq l$. Since Δ is the only irreducible component of A meeting the curves in $|V|$, we see that Δ is contained in both F_h and F_l . Suppose that $1 \leq l < h \leq 3$. Then, by (2.1, ii), we see that $F_h < F_l$ and $\mathcal{O}_{F_h}(F_l) \simeq \mathcal{O}_{F_h}$.

Let $k \neq h, l$. We notice that $\mathcal{O}_{F_k}(F_h) \simeq \mathcal{O}_{F_k}$. The assertion trivially holds if $F_k \cap F_h = \emptyset$. Otherwise, since F_k cannot contain Δ , by (2.1, ii) we see that $F_k < F_h$ and the assertion follows. Similarly one proves that $\mathcal{O}_{F_k}(F_l) \simeq \mathcal{O}_{F_k}$.

Now we consider the exact sequence:

$$0 \rightarrow H^0(S, \mathcal{O}_S(K + M + F_k)) \rightarrow H^0(S, \mathcal{O}_S(2K - F_l)) \rightarrow H^0(F_h, \mathcal{O}_{F_h}).$$

Since $|2K - F_l|$ is contained in $|2K - \Delta|$, but F_l contains Δ as a simple component, because $V \cdot F_l = 1$, we see that Δ has to be a fixed component for $|2K - F_l|$. Therefore the last map to the right in the above sequence is the zero map, because $\Delta < F_h$. Hence $h^1(S, \mathcal{O}_S(K + M + F_k)) > 0$, i.e. $h^0(M + F_k, \mathcal{O}_{M + F_k}) > 1$. But this is clearly contradicted by the exact sequence:

$$0 \rightarrow \mathcal{O}_M(-F_k) \rightarrow \mathcal{O}_{M + F_k} \rightarrow \mathcal{O}_{M + F_k} \otimes \mathcal{O}_{F_k} \rightarrow 0$$

since $\mathcal{O}_{M + F_k} \otimes \mathcal{O}_{F_k} \simeq \mathcal{O}_{F_k}$. In fact $M + F_k \equiv K - (F + F_h)$ and $\mathcal{O}_{F_k}(K) \simeq \mathcal{O}_{F_k}(F_l + F_h) \simeq \mathcal{O}_{F_k}$.

In conclusion, we have proved that we may assume that $V \cdot F_l = 2$, $V \cdot F_h = V \cdot F_k = 0$, i.e. $A \cdot F_l = 0$, $A \cdot F_h = A \cdot F_k = 2$, $\{l, h, k\} = \{1, 2, 3\}$. Remark now that one has $(V + F_l)^2 = (A + F_h + F_k)^2 = (V + F_l) \cdot (A + F_h + F_k) = 2$. Hence by the index theorem we have $V + F_l \sim A + F_h + F_k$. Since $\mathcal{O}_M(V + F_l) \simeq \mathcal{O}_M(A + F_h + F_k)$, as before proposition (1.6) of [CFM] implies that $\mathcal{O}_S(V + F_l) \simeq \mathcal{O}_S(A + F_h + F_k)$. Therefore we would have $K \equiv V + A + F_l + F_h + F_k \simeq 2(V + F_l)$, hence $|K|$ would be composite with the pencil $|V|$, a contradiction.

Consider now case (ii). As above, we see that in this case $K \cdot V \leq 5$ and $K \cdot A \geq 3$. But, by assumption, we must have $K \cdot V > 3$, which yields $K \cdot V = 5$ and $K \cdot A = 3$. Then:

$$F_1 \cdot A + F_2 \cdot A + F_3 \cdot A = F_1 \cdot V + F_2 \cdot V + F_3 \cdot V = 3. \quad (**)$$

Since $0 = M \cdot A = A^2 + 1$ and A is 1-connected, by lemma (3.7, iii) we have $0 \leq A \cdot F_l \leq 1$, $l = 1, 2, 3$, and so by (**) we must have $A \cdot F_1 = A \cdot F_2 = A \cdot F_3 = 1$.

Consider now the restriction map $r: H^0(S, \mathcal{O}_S(2K)) \rightarrow H^0(V, \mathcal{O}_V(2K))$. We have $\text{Ker } r \simeq H^0(S, \mathcal{O}_S(K + A + F_1 + F_2 + F_3))$. Notice that for any irreducible component θ of F_l , $l = 1, 2, 3$, one has $K \cdot \theta = 0$ and $V \cdot \theta \geq 0$, hence $\theta \cdot (K + A + F_1 + F_2 + F_3) = \theta \cdot$

$(2K - V) \leq 0$. Since $F_1 \cdot (K + A + F_1 + F_2 + F_3) = -1$ and F_1 is 1-connected, by corollary (A.3) of [CFM] we see that $\mathcal{O}_{F_1}(K + A + F_1 + F_2 + F_3)$ has no global sections, hence $|K + A + F_1 + F_2 + F_3| = |K + A + F_2 + F_3| + F_1$. By (2.1, ii), $\mathcal{O}_{F_2}(F_1) \simeq \mathcal{O}_{F_2}$ and therefore, by the above argument, for any irreducible component θ of F_2 , we have $K \cdot \theta = F_1 \cdot \theta = 0$, hence $\theta \cdot (K + A + F_2 + F_3) = \theta \cdot (2K - V - F_2 + F_3) \leq 0$, implying as above that $|K + A + F_1 + F_2 + F_3| = |K + A + F_3| + F_1 + F_2$. Finally in the same way we see that $|K + A + F_1 + F_2 + F_3| = |K + A| + F_3 + F_1 + F_2$, thus $h^0(S, \mathcal{O}_S(K + A + F_1 + F_2 + F_3)) = h^0(S, \mathcal{O}_S(K + A)) = 5$, since A is 1-connected (by lemma (A.4) of [CFM]) of arithmetic genus 2. Then $|2K|$ cuts on a general curve of $|V|$, which is a smooth curve of genus 4, a complete g_{10}^6 . This contradicts the fact that the bicanonical map is not birational.

Therefore also case (ii) does not occur and we proved the proposition.

Next we prove the following:

(6.3) **Lemma.** *Under the same assumptions as in (6.2), either $M^2=4$ or $M^2=5$.*

Proof. Suppose that $F^2 = -1$, i.e. $K \cdot F = 1$. By lemma (3.5) we have $M - F = A_1 + A_2 + A_3$, where $A_i \cdot F = 1$, $A_1 \cdot A_2 = A_2 \cdot A_3 = A_3 \cdot A_1 = 0$, hence $M \cdot A_i = A_i^2 + 1$, $i=1, 2, 3$, and therefore $K \cdot A_i = M \cdot A_i + F \cdot A_i = A_i^2 + 2$, $i=1, 2, 3$.

By lemma (3.5, iv) we know that $h^0(S, \mathcal{O}_S(M - A_i)) = 2$, $i=1, 2, 3$. Let $|C_i|$ be the corresponding pencil. As usual, we may assume $K \cdot C_i \geq 4$, $i=1, 2, 3$. This immediately implies $A_j^2 + A_k^2 \geq -1$ for $j \neq k$, $j \neq i$ and $k \neq i$. So we see that at most one of the curves A_l , $l=1, 2, 3$, has negative self-intersection. This implies that $M^2 = M \cdot F + M \cdot A_1 + M \cdot A_2 + M \cdot A_3 = 2 + 3 + A_1^2 + A_2^2 + A_3^2 \geq 4$, since, by lemma (3.7), $A_l^2 \geq -1$ for $l=1, 2, 3$.

Suppose now that $F^2 = -2$, hence $K \cdot F = 0$. We apply lemma (3.6) and we have $M - F = A_1 + A_2 + A_3$, where $A_i \cdot F = 1$, $A_j \cdot F = 1$ and $A_k \cdot F = 2$, for $\{i, j, k\} = \{1, 2, 3\}$. Notice that we have $M \cdot A_i = A_i^2 + 1$ and $M \cdot A_j = A_j^2 + 1$, and so $K \cdot A_i = A_i^2 + 2$ and $K \cdot A_j = A_j^2 + 2$. By lemma (3.7, i) one has $0 \geq A_i^2 \geq -1$ and $0 \geq A_j^2 \geq -1$.

By lemma (3.6, iv), $h^0(S, \mathcal{O}_S(M - A_k)) = 2$. If we consider the corresponding pencil $|C|$, we may assume $K \cdot C \geq 4$. This implies $A_i^2 + A_j^2 \geq 0$ and therefore $A_i^2 = A_j^2 = 0$ yielding $M^2 \geq 4$.

On the other hand $8 \geq K^2 \geq K \cdot M = M^2 + F \cdot M = M^2 + 2$, proving that $M^2 \leq 6$. Now we will see that the case $M^2 = 6$ cannot occur.

Suppose that $M^2 = 6$. From the assumption $K^2 \leq 8$ and $K^2 = K \cdot M + K \cdot F = M^2 + 2 + K \cdot F$, we necessarily have $K^2 = 8$, and $K \cdot F = 0$, implying that $F^2 = -2$. Then we have $M - F = A_1 + A_2 + A_3$, where $A_i \cdot F = 1$, $A_j \cdot F = 1$ and $A_k \cdot F = 2$, for $\{i, j, k\} = \{1, 2, 3\}$. So $6 = M^2 = M \cdot F + M \cdot A_1 + M \cdot A_2 + M \cdot A_3 = 2 + 4 + A_1^2 + A_2^2 + A_3^2$, and therefore $A_1^2 + A_2^2 + A_3^2 = 0$. As above, because we are assuming that S does not contain a pencil $|C'|$ with $K \cdot C' \leq 3$, we have $A_i^2 = A_j^2 = 0$, hence also $A_k^2 = 0$. Thus $(A_i + A_j + F)^2 = (A_k + F)^2 = (A_i + A_j + F) \cdot (A_k + F) = 2$. By the index theorem we have then $A_i + A_j + F \sim A_k + F$ and therefore $A_i + A_j \sim A_k$. Now lemmas (3.7) and (3.8) imply that $|2A_i| = |2A_j|$ is a genus 3 pencil without fixed components or base points and $A_k \sim C$. Since $C \cdot A_k = 0$, every component of A_k is contained in a fibre of the

pencil $|C|$. From $h^0(S, \mathcal{O}_S(A_k)) = 1$, $A_k^2 = 0$, $K \cdot A_k = 4$ and $A_k \sim C$, we see that the only possibility is $A_k = D_1 + D_2$, where $2D_1 \equiv 2D_2 \equiv C$ and $D_1 \neq D_2$ (see [BPV], pg. 90). Notice that D_1 and D_2 are different from A_i and A_j , otherwise we would have a contradiction to lemma (3.2, iv). On the other hand being D_1, D_2, A_i, A_j pairwise distince contradicts both lemma (3.2, i) and proposition (1.5, ii). Therefore the case $M^2 = 6$ is excluded.

In the next two propositions we will freely use some elementary facts about resolutions of double planes singularities and actions of Cremona transformations to double planes and their branch curves, which the reader will certainly be able to easily work out on his own. However, for specific and complete references on this subject, we refer to [Fe] and [Ca].

(6.4) Proposition. *Under the same assumptions as in (6.2), suppose that $M^2 = 4$. Then $K^2 = 6, 7$ and S is a specialization of the Du Val ancestor with $K^2 = 8$, $p_g = 4$.*

Proof. Suppose that $M^2 = 4$. Then from $K^2 = K \cdot M + K \cdot F = M^2 + 2 + K \cdot F$, we have the cases: (a) $K^2 = 7$ if $F^2 = -1$, $K \cdot F = 1$, (b) $K^2 = 6$ if $F^2 = -2$, $K \cdot F = 0$.

Suppose we are in case (a). Then the analysis made in the previous proof implies that $M - F = A_1 + A_2 + A_3$, where $A_i^2 = A_j^2 = 0$, $A_k^2 = -1$, for $\{i, j, k\} = \{1, 2, 3\}$. Also we have $M \cdot A_h = A_h^2 + 1$, $h = 1, 2, 3$. Hence, by lemma (3.2, iv), the two base points p, q , of $|M|$ lie one on A_i , the other on A_j . By lemmas (3.7) and (3.8) the curves M_p, M_q , as in lemma (3.4), are such that $M_p = 2A_i + H$, $M_q = 2A_j + H$, and $|2A_i| = |2A_j|$ is a pencil of genus 3 without base points or fixed components.

Suppose we are in case (b). As in the proof of (6.3), we have that $M - F = A_1 + A_2 + A_3$, where $A_i \cdot F = A_j \cdot F = 1$ and $A_k \cdot F = 2$ for $\{i, j, k\} = \{1, 2, 3\}$, and $A_i^2 = A_j^2 = 0$, $A_k^2 = -2$, and also $M \cdot A_i = M \cdot A_j = 1$, $M \cdot A_k = 0$. Again by lemma (3.2, iv), the two base points p, q , of $|M|$ lie one on A_i , the other on A_j and we reach the same conclusion as before, i.e. the curves M_p, M_q are such that $M_p = 2A_i + H$, $M_q = 2A_j + H$, and $|2A_i| = |2A_j|$ is a pencil of genus 3 without base points or fixed components.

Now we consider, in either one of the above cases, the canonical double plane $\psi: \tilde{S} \rightarrow \mathbf{P}^2$ of S , or, equivalently, the rational map $\phi_M: S \rightarrow \mathbf{P}^2$ (see §2). The branch curve G has, in both cases, degree 14: in fact $(K + M) \cdot M = 10$, i.e. the general curve M in $|M|$ has genus 6. The curve G contains two distinct lines a_1 and a_2 corresponding to the base points p and q of $|M|$ which are blown up on \tilde{S} (this can be seen as in lemma (5.2, ii)). The curves A_i and A_j , which are 1-connected, are contracted to points $\alpha_1 \in a_1$ and $\alpha_2 \in a_2$. Furthermore H , which is also 1-connected is contracted to the point $\gamma = a_1 \cap a_2$.

Let $G = a_1 + a_2 + G'$. Since $H^2 = 0$, $K \cdot H = 2$, and the two base points which are blown up in passing from S to \tilde{S} are on H , the point γ has to be a point of mutiplicity at least 6 for G , hence of multiplicity at least 4 for G' . This can be seen, by the way, also in a different manner: in fact the pencil of lines through γ

pulls back on S to the base point free pencil of genus 3 curves $|2A_i|=|2A_j|$. Then the assertion follows by the Hurwitz's formula, which actually shows that the multiplicity in question is either 6 or 7.

Since $A_i^2=A_j^2=0$, since the genus of both A_i and A_j is two, and $M_p=2A_i+H$, $M_q=2A_j+H$, and the two base points which are blown up are one on A_i , the other on A_j , we see that α_1 and α_2 are at least [5,5]-points for G with tangent lines given, of course, by a_1 and a_2 respectively, i.e. they are [4,4]-points for G' with tangent lines given by a_1 and a_2 . Since a_1 and a_2 cannot be contained in G' , we see that the indicated multiplicities for α_1 , α_2 and γ are the effective ones. Now a standard double plane analysis show that G has to have another essential singularity x which can either be a 4-uple point, yielding $K^2=6$, or a [3,3]-point, yielding $K^2=7$ (or specializations of these: e.g. a 5-uple point instead of a 4-uple point is admitted, etc.). Of course then x must be the image of A_3 and therefore x must be on the same line r with α_1 and α_2 . Such a line r is in fact the image of F . Since F is not contained in any curve of the pencil $|2A_i|=|2A_j|$, then r does not pass through γ and therefore α_1 , α_2 are both distinct from γ . Of course also x is distinct from α_1 , α_2 on r .

In order to finish our proof, it suffices to recall the double plane description of the Du Val ancestor with $p_g=4$, $K^2=8$ given in example (1.4, iii).

(6.5) Proposition. *Under the same assumptions as in (6.2), suppose that $M^2=5$. Then either $K^2=7$ and S is a specialization of the Du Val ancestor with $K^2=8$, $p_g=4$, or $K^2=8$ and S is a (particular) Du Val ancestor with $K^2=8$, $p_g=3$.*

Proof. Suppose that $M^2=5$, and so $|M|$ has three base points p , q , r . Then from $K^2=K \cdot M + K \cdot F = M^2 + 2 + K \cdot F$, we have the two cases: (a) $K^2=7$ if $F^2=-2$, $K \cdot F=0$, (b) $K^2=8$ if $F^2=-1$, $K \cdot F=1$.

Suppose we are in case (a). Then, as usual, we have that $M-F=A_1+A_2+A_3$, where $A_i \cdot F=A_j \cdot F=1$, $A_k \cdot F=2$ for $\{i, j, k\}=\{1, 2, 3\}$, and $A_i^2=A_j^2=0$. Since $M^2=5$ we deduce that $A_k^2=-1$. By lemma (3.2, iv), two of the base points of $|M|$, say p , q , lie one on A_i , the other on A_j . Then, as before, the curves M_p , M_q are such that $M_p=2A_i+H$, $M_q=2A_j+H$, and $|2A_i|=|2A_j|$ is a pencil $|C|$ of genus 3 without base points or fixed components.

Since $A_i \cdot A_k=0$, we see that A_k is contained in curves of $|C|$. Then the intersection form is negative semidefinite on the space spanned by the irreducible components of A_k . If it is negative definite, then lemma (7.4) implies that A_k is 1-connected and therefore it is contained in a unique curve of $|C|$. Otherwise there is a curve Z contained in A_k such that $Z^2=0$ (cf. [BPV], pg 90). Then, of course, $2Z \in |C|$. By using lemma (3.2, iv) as we already did in the previous proof, we see that Z should be different from A_i and A_j , hence the pencil $|2A_i|$ would have three double curves, and therefore $Z_2^2 \hookrightarrow \text{Tors}(S)$, which, by proposition (1.5), contradicts $K^2=7$. In conclusion we see that there is a 1-connected curve C of the pencil $|2A_i|$ such that $C=A_k+D$. Since $A_k^2=-1$, one has $A_k \cdot D=1$ yielding that A_k and D are 1-connected. Moreover $D^2=-1$, $K \cdot D=1$.

Notice that $M_p = 2A_i + H$ and, since A_i contains only the base point p of $|M|$, one has $r \in H$. Furthermore r is a simple point of H by lemma (3.1, ii). Since $M - F = A_1 + A_2 + A_3$, A_i only contains the base point p and A_j only q , by lemma (3.2, iv) we see that $r \in A_k$ and again it is a simple point of A_k . Hence r is in $A_k \cap H$, and therefore it is a singular point of $A_k + D + H$, implying that $M_r = A_k + D + H$ (see lemma (3.1, i)). Since r is only a double point of $A_k + H$, it must lie on D and it is a smooth point of D by lemma (3.1, i). Notice that $M \cdot D = 1$ and actually the general curve in $|M|$ meets D only at the base point r . Hence D , as well as A_i , A_j , A_k and H is contracted by the rational map ϕ_M .

As in the previous proof, we consider the canonical double plane $\psi: S \rightarrow \mathbf{P}^2$ and we proceed as above to describe the branch curve B of the double plane and its singular points. Since $K \cdot M = 7$, the degree of B is 16. In the present case B contains three distinct lines l_p , l_q and l_r corresponding to the base points p , q , r of $|M|$. We will call B' the residual curve of B with respect to l_p , l_q and l_r , whose degree is 13.

The lines l_p , l_q and l_r pass through the same point χ which comes from the contraction of the curve H . Furthermore the contractions of the curves A_i , A_j , A_k , D give rise to singular points $\xi_i \in l_p$, $\xi_j \in l_q$, $\xi_k \in l_r$, $\delta \in l_r$. Notice that F is also mapped to a line f which contains ξ_i , ξ_j , ξ_k and that the pencil of lines through χ corresponds to the genus 3 pencil $|C|$ on S . Since F is not contained in any curve of $|C|$, then f does not pass through χ and we see that ξ_i , ξ_j , ξ_k are points of l_p , l_q , l_r distinct from χ .

As for the singularities presented by B and B' at the aforementioned points, arguments which are similar to the ones used in the proof of proposition (6.4) show that:

- (a) χ is a point of multiplicity 8 for B , hence it is a point of multiplicity 5 for B' ;
- (b) ξ_i is a $[5,5]$ -point with tangent line l_p and ξ_j is a $[5,5]$ -point with tangent line l_q , hence they are $[4,4]$ -points for B' , with the aforementioned tangent lines;
- (c) ξ_k is a point of multiplicity 6 for B , hence of multiplicity 5 for B' ;
- (d) δ is a point of multiplicity 4 for B , hence it is a point of multiplicity 3 for B' .

Now, in order to see that such a surface is a specialization of the Du Val ancestor with $K^2 = 8$, $p_g = 4$ as stated, we perform the quadratic transformation ω based at the points χ , ξ_j , ξ_k . This gives rise to a new double plane $\pi: S' \rightarrow \mathbf{P}^2$ which is birational to the original one, i.e. there is a birational transformation $\sigma: \tilde{S} \rightarrow S'$ such that $\omega \circ \psi = \sigma \circ \pi$. Let G be its branch curve. It is no more than an exercise to verify that:

- (i) G has degree 14, and more precisely $G = a_1 + a_2 + G'$ where a_1 , a_2 are distinct lines and G' is a residual curve of degree 12. Actually G' is the proper transform via ω of B' , whereas a_1 corresponds to l_p and a_2 corresponds to the fundamental point ξ_j of ω ;
- (ii) the intersection point γ of a_1 and a_2 , which comes from the contraction via ω of the line f , has multiplicity 6 for G and 4 for G' . However the image via ω of the point ξ_i , becomes infinitely near to γ along a_1 . Hence G' has a $[4,4,4]$ -point at γ with tangent line a_1 ;

(iii) there is a further [3,3]-point for both B and B' at the point z corresponding to δ along the line a_3 through γ corresponding to the fundamental point ξ_k of ω .

Now the assertion follows just by comparing with the example (1.4, iii). The process of performing the quadratic transformation ω shows that a direct, and alternative, way of obtaining the double plane $\pi: S' \rightarrow \mathbf{P}^2$ from S is the following. One considers the 2-dimensional linear system $|\Gamma| = |2M - A_k - A_j - H| = |M - A_k + A_i|$. Then the rational map $\phi_\Gamma: S \rightarrow \mathbf{P}^2$ gives rise to the double plane in question.

Finally, we suppose that $K^2=8$. As usual, we have $M - F = A_1 + A_2 + A_3$, where $1 = M \cdot A_i = A_i^2 + 1 = A_i \cdot F$, for $i \in \{1, 2, 3\}$, implying $A_i^2=0$, $K \cdot A_i=2$, for $i \in \{1, 2, 3\}$. By lemma (3.7) the three base points p, q, r of $|M|$ will lie in A_i, A_j, A_k respectively, and the curves M_p, M_q, M_r , as in lemma (3.4), are such that $M_p=2A_i+H$, $M_q=2A_j+H$, $M_r=2A_k+H$ and $|2A_i|=|2A_j|=|2A_k|$ is a pencil of genus 3 without base points or fixed components.

Consider then the canonical double plane of S . The branch curve B has degree 16. The curve B contains three lines b_1, b_2 and b_3 corresponding to the base points p, q , and r of $|M|$. The curves A_i, A_j and A_k , which are 1-connected, are contracted to points $\beta_1 \in b_1, \beta_2 \in b_2, \beta_3 \in b_3$. The points $\beta_1, \beta_2, \beta_3$ have to be aligned since $M - F = A_1 + A_2 + A_3$. Furthermore there is a point δ coming from the contraction of H , which is also 1-connected. From the above analysis it follows that $\delta = b_1 \cap b_2 \cap b_3$. Furthermore the line containing $\beta_1, \beta_2, \beta_3$ does not pass through δ for the usual reasons.

Let $B = B' + b_1 + b_2 + b_3$. The curve B' has degree 13 and it is easy to see, with the usual arguments, that its only relevant singularities are a 5-tuple point at δ and three [4,4]-points at $\beta_1, \beta_2, \beta_3$, having tangent lines b_1, b_2, b_3 , respectively.

This, in view of the example (1.4, iv), proves the assertion for $K^2=8$.

(6.6) **Remark.** The analysis we performed in the proofs of the above propositions (6.4) and (6.5) shows that the minimal regular surfaces with $p_g=3$, with canonical system having a fixed part, with no pencil of curves of genus 2 and non birational bicanonical map, are, all specializations in moduli of the same surfaces with canonical system without fixed part. Moreover our analysis also shows exactly in which way the specialization takes place. In conclusion the possible cases are as follows:

(i) $K^2=6$ and the surface is a specialization of the Du Val ancestor with $p_g=4$, $K^2=8$ described in example (1.4, iii), from which we keep the notation. The case in question corresponds to the fact that the branch curve G of the double plane described in (1.4, iii) acquires a 4-tuple point x as indicated in general in table (1.3). The fixed component for $|K|$ arises when x is collinear with the [5,5]-points α_1, α_2 of G and the image of the fixed part of $|K|$ is the line containing the three points x, α_1, α_2 ;

(ii) $K^2=7$ and again the surface is a specialization of the Du Val ancestor with $p_g=4$, $K^2=8$ described in example (1.4, iii). As indicated in table (1.3), the branch curve G of the double plane described in (1.4, iii) acquires here a [3,3]-point x . However there are two possible specializations which enable $|K|$ to acquire a fixed part:

(ii₁) as in case (i), x is collinear with the [5,5]-points α_1, α_2 of G and the image of the fixed part of $|K|$ is the line containing the three points x, α_1, α_2 ;

(ii₂) one of the two points α_1, α_2 , say α_1 , becomes infinitely near to γ along the line a_1 , and the fixed part of $|K|$ is contracted to γ .

The two cases above are actually different, inasmuch as in the former one the movable part $|M|$ of $|K|$ has $M^2=4$, whereas in the later one has $M^2=5$;

(iii) $K^2=8$ and the surface is a particular member of the family of Du Val ancestors with $p_g=3, K^2=8$ described in example (1.4, iv), from which we keep the notation. The particularity consists in the fact that the three [5,5]-points $\alpha_i, i=1, 2, 3$, of the branch curve B are collinear along a line which is the image of the fixed part of $|K|$;

(iv) the surface contains a pencil of curves of genus 3 with a base point, and it is therefore a specialization of the Du Val ancestor with $p_g=6, K^2=9$ described in the example (1.4, ii), from which we keep the notation. A fixed part of $|K|$ can be produced when more than one essential singularity of the branch curve B' on the cone Q_0 , acquired as described in table (1.3), happens to lie on one and the same line of the cone.

7. Appendix

In this appendix we prove various results that have been used throughout the paper.

(7.1) **Theorem.** *I) Let D be a 2-connected curve on a surface S and let x, y be two distinct multiple points of D . Then x and y are not separated by $|K+D|$ if and only if D decomposes as a sum of two curves A, B satisfying:*

(i) $A \cdot B = 2$;

(ii) x, y are non-singular points of A and $\mathcal{O}_A(x+y) \simeq \mathcal{O}_A(B)$.

Furthermore if x, y are not separated by $|K+D|$, the decomposition appearing above is such that $A \cap B = \{x, y\}$ or $A \leq B$.

II) Let D be a 2-connected curve with $p_a(D) \geq 2$ on a surface S and let x, y be two smooth points of D (possibly $x=y$). Then, if x, y are not separated by $|\omega_D|$ one of the following occurs:

(a) either D is an irreducible hyperelliptic curve and $|\mathcal{O}_D(x+y)|$ is the unique g_2^1 on D ;
 (b) or D is reducible, x, y belong to the same component Γ of D , which is an hyperelliptic curve, and D decomposes as a sum $\Gamma + F_1 + \dots + F_n$ satisfying:

(i) F_1, \dots, F_n are curves such that $\Gamma \cdot F_i = 2$, for every $i \in \{1, \dots, n\}$;

(ii) $\mathcal{O}_\Gamma(F_i) \simeq \mathcal{O}_\Gamma(x+y)$, for every $i \in \{1, \dots, n\}$;

(iii) $|\mathcal{O}_\Gamma(x+y)|$ is a g_2^1 on Γ ;

(iv) $F_i \cdot F_j = 0$, for $i \neq j$;

(v) $\mathcal{O}_{F_i}(F_k) \simeq \mathcal{O}_{F_i}$ for all $k < i$;

(vi) F_i is 1-connected for every $i \in \{1, \dots, n\}$;

(vii) if $k < i$, either $F_i \cap F_k = \emptyset$ or $F_i < F_k$;

(c) or $x \neq y$, and $x \in \Gamma, y \in \Delta$ where $\Gamma, \Delta \leq D$ are non-singular rational irreducible

curves. Furthermore either $D=\Gamma+\Delta$, or $D=\Gamma+\Delta+F_1+\dots+F_n$ where F_1, \dots, F_n are curves satisfying (iv) and (v) of (b) and such that $\Gamma \cdot F_i = \Delta \cdot F_i = 1$, for every $i \in \{1, \dots, n\}$.

Proof. Part I) of this theorem is theorem 3.2 of [M], whilst part II) except assertions (vi) and (vii) is theorem 4.2 of [M]. Assertion (vi) is a trivial consequence of the other assertions. In fact assertions (ii) and (iv) imply that $F_i \cdot (D - F_i) = 2$, for every $i \in \{1, \dots, n\}$. Using lemma (A.4) of [CFM] and 2-connectedness of D one has then assertion (vi). For assertion (vii) assume that $F_i \cap F_k \neq \emptyset$. Then we can write $F_i = \Delta_0 + \Delta_1$, $F_k = \Delta_0 + \Delta_2$, where Δ_0 is a curve and Δ_1, Δ_2 are effective divisors without common components. Suppose that $\Delta_1 \neq 0$. Since by assertion (v), $\mathcal{O}_{F_i}(F_k) \simeq \mathcal{O}_{F_i}$, $\Delta_1 \cdot (\Delta_0 + \Delta_2) = 0$. Now Δ_1 and Δ_2 have no common components and therefore necessarily $\Delta_1 \cdot \Delta_0 \leq 0$. This contradicts the 1-connectedness of F_i we just proved, showing that $\Delta_1 = 0$.

(7.2) **Lemma.** Let $D := C + Z$ be a curve with C an irreducible hyperelliptic curve, Z a (-2) -cycle such that $C \cdot Z = 2$ and $\mathcal{O}_C(Z) \simeq \eta$ where $|\eta|$ is the unique g_2^1 on C . Then:

- (i) there exists a line bundle \mathcal{L} on D such that $|\mathcal{L}|$ is a g_2^1 on D and such that $\mathcal{L}^{\otimes n} \simeq \omega_D$;
- (ii) let \mathcal{M} be an invertible sheaf on D such that $\mathcal{M}^{\otimes p} \sim \mathcal{L}^{\otimes p}$, $\mathcal{M}^{\otimes p} \not\sim \mathcal{L}^{\otimes p}$ and $\mathcal{M}_C \simeq \mathcal{L}_C$. If there exists $m \in \mathbb{N}$ with $\mathcal{M}^{\otimes m} \simeq \mathcal{L}^{\otimes m}$, then $C \cap Z = P + Q$, where P, Q are distinct points of Z such that $\mathcal{O}_C(P + Q) \simeq \eta$.

Proof. Remark first that it is possible to glue the trivial sheaf on Z and η on C in such a way that for the resulting sheaf \mathcal{L} one has $|\mathcal{L}| = g_2^1$. Let $2p_a(D) - 2 = 2n$. Notice also that D is 1-connected. Since $\mathcal{L}^{\otimes n} \sim \omega_D$ and $h^0(D, \mathcal{L}^{\otimes n}) \geq n + 1$, using Serre's duality, the Riemann-Roch theorem and lemma (A.1) of [CFM], one gets (i).

For (ii) remark that the hypothesis on \mathcal{M} imply that $\mathcal{L}^{\otimes m} \otimes \mathcal{M}^{-m}$ is a torsion element of $\ker(\text{Pic}^0(D) \rightarrow \text{Pic}^0(C))$. Therefore this kernel must be \mathbb{C}^* and so we have the assertion.

(7.3) **Proposition.** Let D be a 0-connected divisor such that $h^0(D, \mathcal{O}_D) = 3$. Then D decomposes as $D = D_1 + D_2 + D_3$, where D_1, D_2, D_3 are curves such that:

- (i) $h^0(D_i, \mathcal{O}_{D_i}) = 1$, for $i \in \{1, 2\}$;
- (ii) $\mathcal{O}_{D_1}(D_2 + D_3) \simeq \mathcal{O}_{D_1}$ and $\mathcal{O}_{D_2}(D_3) \simeq \mathcal{O}_{D_2}$;
- (iii) $D_1 \cdot D_2 = D_2 \cdot D_3 = D_1 \cdot D_3 = 0$.

Furthermore, if $\mathcal{O}_{D_1}(D_2) \simeq \mathcal{O}_{D_1}$, the image of the restriction map $r: H^0(D, \mathcal{O}_D) \rightarrow H^0(D_3, \mathcal{O}_{D_3})$ is 1-dimensional.

Proof. Proposition (2.4) of [M] states that a 0-connected curve D on a non-singular surface such that $h^0(D, \mathcal{O}_D) = n \geq 2$ decomposes as $D = D_1 + \dots + D_n$, where D_1, \dots, D_n are curves such that:

- (a) $h^0(D_i, \mathcal{O}_{D_i}) = 1$, for $i \in \{1, \dots, n-1\}$;
- (b) $\mathcal{O}_{D_i}(D_{i+1} + \dots + D_n) \simeq \mathcal{O}_{D_i}$ for all $i \in \{1, \dots, n-1\}$;

Assertions (i) and (ii) of the lemma are just assertions (a) and (b) above. For assertion (iii) notice that from 0-connectedness of D , we have $D_2 \cdot (D_1 + D_3) \geq 0$ and $D_3 \cdot (D_1 + D_2) \geq 0$. From $D_1 \cdot (D_2 + D_3) = 0$ and $D_2 \cdot D_3 = 0$ we have then $D_1 \cdot D_2 = D_1 \cdot D_3 = 0$.

For the last assertion, let us notice that, since $\mathcal{O}_{D_1}(D_2) \simeq \mathcal{O}_{D_1}$, we have the exact sequence $0 \rightarrow \mathcal{O}_{D_1} \rightarrow \mathcal{O}_{D_1+D_2} \rightarrow \mathcal{O}_{D_2} \rightarrow 0$, whence $h^0(D_1 + D_2, \mathcal{O}_{D_1+D_2}) = 2$. Moreover, since $\mathcal{O}_{D_1} \simeq \mathcal{O}_{D_1}(-D_3)$, $\mathcal{O}_{D_2} \simeq \mathcal{O}_{D_2}(-D_3)$, also $\mathcal{O}_{D_1+D_2} \simeq \mathcal{O}_{D_1+D_2}(-D_3)$. Since $\text{Ker } r \simeq H^0(D_1 + D_2, \mathcal{O}_{D_1+D_2}(-D_3))$, we have immediately the last assertion.

(7.4) Lemma. *Let C, D be curves on a surface S such that the intersection form of S restricted to the subspace generated by the irreducible components of C and D is negative definite. Then:*

- (i) *if $C^2 = D^2 = C \cdot D$, then $C = D$;*
- (ii) *if $C^2 = D^2 = -1$, C and D are 1-connected. Furthermore if $C \neq D$, then $C \cdot D = 0$ and either $C \cap D = \emptyset$ or $C \leq D$ or $D \leq C$;*
- (iii) *if $C^2 = -1, D^2 = -2$ then $-1 \leq C \cdot D \leq 1$ and if $C \cdot D = -1$ then either $D < C$ or $C < D$.*

Proof. For (i), notice first that, since the intersection form of S restricted to the subspace generated by the irreducible components of C and D is negative definite, $C^2 < 0$ and $D^2 < 0$. Hence $C \cdot D < 0$ and so C and D have common components. Write then $C = A + B, D = A + E$ where B and E are effective divisors without common components and A is a curve. Then, since $B \cdot E \geq 0$, the relation $0 = (C - D)^2 = B^2 - 2BE + E^2$ implies that $B^2 + E^2 \geq 0$, which yields $B^2 = E^2 = 0$, hence $B = E = 0$, proving (i).

The first part of (ii) is clear since any decomposition of $C = A + B$ with $A \cdot B \leq 0$ would have $A^2 \geq 0$ or $B^2 \geq 0$. For the second notice that, since the intersection form is negative definite, we get $(C + D)^2 < 0$ and $(C - D)^2 \leq 0$ and therefore, if $C \neq D$, one has $C \cdot D = 0$. Suppose now that $C \cdot D = 0$ and C and D have common components. Then we can write $C = A + B, D = A + E$ where B and E have no common components. Suppose both B and E are non-zero. Then we get $B^2 - 2B \cdot E = (C - D)^2 = C^2 + D^2 = -2$ yielding $B^2 = E^2 = -1$ and $B \cdot E = 0$. This leads to a contradiction, because $0 > (D + B)^2 = -2 + 2D \cdot B = -2 + 2A \cdot B \geq 0$, since C is 1-connected, and therefore $B \cdot A > 0$. Hence either B or E is zero, thus either $D < C$ or $C < D$.

The proof of (iii) is very similar and we omit it.

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