Geometric structure for $OpS^m_{1,1}$

By

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Abstract

With a symbol in $OpS^m_{1,1}$ or its kernel-distribution, one has a great difficulty to study the continuity or other properties, here one use the wavelets bases which come from the Beylkin-Coifman-Rokhlin (B-C-R) algorithm to study such operators. Each operator in $OpS^m_{1,1}$ corresponds to its wavelet coefficients; with this idea, one characterizes $OpS^m_{1,1}$ with a discrete space, and characterizes $OpS^0_{1,1}$ with a kernel-distribution space. As an application, theorem 1 of chapter 9 in tome II of [8] is a corollary of two theorems of this paper.

1. Introduction and continuous representations

There was a long period, when one studied an operator, one used only derivation and integration. Since 1950, the study of the operators split into two groups. One side, Kohn, Nireberg and Hörmander consider pseudo-differential operators with symbols systematically; another side, the Calderón-Zygmund (C-Z) school and wavelets group afterwards show a lot of interest for C-Z operators.

In differential equations, one has a good knowledge of an operator $\sigma(x, D)$ which is defined by its symbol $\sigma(x, \xi)$:

(1) $$\sigma(x, D)f(x) = (2\pi)^{-n} \int e^{ix\xi} \sigma(x, \xi) \hat{f}(\xi) d\xi.$$

In this paper, one considers that: $\sigma(x, D) \in OpS^m_{1,1}$, that is to say:

(2) $$|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| \leq C_{m,\alpha,\beta} |\xi|^{m+|\alpha|-|\beta|}, \quad \forall \alpha, \beta \in N^n.$$

According to the theory of L. Schwartz, a linear operator, which is continuous from $S(R^n)$ to $S'(R^n)$, corresponds to a kernel-distribution $K(x, y)$ given by:

(3) $$Tf(x) = \int K(x, y) f(y) dy.$$

In harmonic analysis, one considers generally the following C-Z operators.

Definition 1. For $0 < \gamma < 1$, $T \in F_\gamma$ if

(4) $T : L^2 \rightarrow L^2$ is continuous,

(5) $|K(x, y)| \leq \dfrac{C}{|x - y|^n}$,

(6) $|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \dfrac{C|x - x'|^\gamma}{|x - y|^{n+\gamma}}$, $\forall |x - x'| \leq \dfrac{1}{2}|x - y|$.

In particular, $T \in E_\gamma$ if $T \in F_\gamma$ and $T1 = T^*1 = 0$.

In fifties, Calderón found the following formal relation between the symbol and the kernel-distribution of an operator:

(7) $K(x, y) = (2\pi)^{-n} \int e^{i(x-y)\xi} \sigma(x, \xi) d\xi$.

(8) $\sigma(x, \xi) = \int K(x, y) e^{i(y-x)\xi} dy$.

$K(x, y)$ and $\sigma(x, \xi)$ are two continuous representations for an operator. If one applies only (7) or (8), one has a great difficulty to establish an internal relation between the symbol and the kernel-distribution. In this paper, one applies a discrete representation to characterize $OpS_{1,1}^m$ in wavelet coefficients and characterize $OpS_{1,1}^0$ in kernel-distributions spaces.

2. History of the discrete representation

After appearing of wavelets, one knows that some wavelet bases are unconditionals bases for most of functions spaces. Therefore, one hopes that the representation of a usual operator under such bases can reflect internal properties of operator. In fact, one has applied such a method, and has constructed a useful algebra of the C-Z operators where $T1 = T^*1 = 0$, cf [5], [6] and [7]; Meyer has established a very good relation between a sub-algebra of this algebra and a particular group of operators in $OpS_{1,1}^0$ cf [8]. One calls this method a standard representation of the operator, but one can not analyse a C-Z operator where $T1 \neq 0$ or $T^*1 \neq 0$. Afterwards, one has found, with the B-C-R algorithm, a characterization of all the C-Z operators, cf [1], [4] and [10], one calls this algorithm a non-standard representation for the operators.

One denotes first $\varphi(x)$, the father of Meyer wavelets, $\psi(x)$, the mother of Meyer wavelet; one denotes then $\Phi^{(0)}(x) = \varphi(x)$, $\Phi^{(1)}(x) = \psi(x)$. $\forall \varepsilon \in \{0, 1\}^n$, $\forall j \in Z$, $\forall k \in Z^n$, $\forall x \in R^n$, $\forall \xi \in R^n$, one denotes $\Phi_{j,k}^{(\varepsilon)}(x) = 2^{\frac{nj}{2}} \prod_{i=1}^{n} \Phi^{(\varepsilon_i)}(2^j x_i - k_i)$, $\hat{\Phi}^{(\varepsilon)}(\xi) = \prod_{i=1}^{n} \hat{\Phi}^{(\varepsilon_i)}(\xi_i)$. One denotes $\Gamma = \{\varepsilon, \varepsilon', j, k, l) : \varepsilon, \varepsilon' \in \{0, 1\}^n$ and $|\varepsilon| + |\varepsilon'| \neq 0, j \in Z, k, l \in Z^n\}$. Then one has that: $\{\Phi_{j,k}^{(\varepsilon)}(x) \Phi_{j,l}^{(\varepsilon')}(y)\}_{(\varepsilon, \varepsilon', j, k, l) \in \Gamma}$ is an orthonormal basis in $L^2(R^n \times R^n)$, cf [8]. As for an operator T, which is continous from Schwartz space $S(R^n)$ to $S'(R^n)$, one denotes:

(9) $$a^{(\varepsilon,\varepsilon')}_{j,k,l} = \langle T\Phi^{(\varepsilon')}_{j,l}(x), \Phi^{(\varepsilon)}_{j,k}(x) \rangle.$$

One calls $\{a^{(\varepsilon,\varepsilon')}_{j,k,l}\}_{(\varepsilon,\varepsilon',j,k,l)\in\Gamma}$ the representation of T under the basis $\{\Phi^{(\varepsilon)}_{j,k}(x)$ $\Phi^{(\varepsilon')}_{j,l}(y)\}_{(\varepsilon,\varepsilon',j,k,l)\in\Gamma}$ or the non-standard representation of T. This representation comes from the B-C-R algorithm, cf [1].

There exist the following relations among the non-standard representation, the kernel-distribution and the symbol:

(10) $$K(x,y) = \sum_\Gamma a^{(\varepsilon,\varepsilon')}_{j,k,l} \Phi^{(\varepsilon)}_{j,k}(x) \Phi^{(\varepsilon')}_{j,l}(y).$$

(11) $$\sigma(x,\xi) = \sum_\Gamma a^{(\varepsilon,\varepsilon')}_{j,k,l} \Phi^{(\varepsilon)}(2^j x - k)\hat{\Phi}^{(\varepsilon')}(2^{-j}\xi)e^{-i(x-2^{-j}l)\xi}.$$

Remark 1. If one considers symbolic operators in (2) with $|\alpha|+|\beta|$ finite, one use often Daubechies wavelets; in this paper, one considers only the case where $|\alpha|+|\beta|$ is infinite, one has to use Meyer's wavelets to give a characterization in wavelets.

3. Main results

The first, one introduces a definition:

Definition 2. (i) One calls that $T\in OpB^m$, if T satisfies the following condition:

(12) $$|a^{(\varepsilon,\varepsilon')}_{j,k,l}| \leq \frac{C_N 2^{jm}}{(1+|k-l|)^N}, \quad \forall N > 0, \; \forall(\varepsilon,\varepsilon',j,k,l)\in\Gamma.$$

(ii) One calls that $T\in OpA^m$, if $T\in OpB^m$, and T satisfies the following condition:

(13) $$\sum_l l^\alpha a^{(\varepsilon,0)}_{j,k,l} = 0, \quad \forall\alpha\in N^m, \; j\in Z, k\in Z^n, \varepsilon\in\{0,1\}^n\backslash\{0\}.$$

There exists a simple one-to-one relation between $OpS^m_{1,1}$ and OpA^m.

Theorem 1. *If* $n+m>0$, *then one has* $T\in OpS^m_{1,1} \Leftrightarrow T\in OpA^m$.

If $n+m>0$, and $T\in OpS^m_{1,1}$, (9) can be defined.

If $T\in OpA^m$, since T has a clear and simple structure, all of its properties can be studied easily, such as its continuity. One proves this theorem in the sections 5 and 6.

Then one introduces another theorem.

For all the cube $Q, \forall q>0, \forall f(x)\in C^q_0(Q)$, one denotes:

(14) $$\|f\|_{N^Q_q} = |Q|^{\frac{1}{2}}(\|f\|_\infty + |Q|^{\frac{|\alpha|}{n}}\|f\|_{\dot{C}^q}).$$

Definition 3. T is called an operator which has a weak continuity if there exists an integer q such that:

(15) $\qquad |\langle Tf,g\rangle| \le C_q\|f\|_{N_q^Q}\|g\|_{N_q^Q}, \ \forall$ cube $Q, \forall f(x)\in C_0^q(Q)$ and $g(x)\in C_0^q(Q)$.

Definition 4. (i) $T\in C_\infty$ if T satisfies the condition (15) and the following condition:

(16) $\qquad |\partial_x^\alpha\partial_y^\beta K(x,y)| \le \dfrac{C_{\alpha,\beta}}{|x-y|^{n+|\alpha|+|\beta|}}, \forall \alpha,\beta\in N^n.$

(ii) $T\in B_\infty$ if $T\in C_\infty$ and T satisfies the following condition:

(17) $\qquad\qquad\qquad Tx^\alpha = 0, \forall \alpha\in N^n.$

(iii) $T\in A_\infty$ if $T\in B_\infty$ and $T^*\in B_\infty$.

Then one can characterize the kernel-distribution space B_∞ with a space OpA^0, (which is defined in Definition 2). In fact, one has:

Theorem 2. $T\in OpA^0 \Leftrightarrow T\in B_\infty.$

One proves this theorem in section 4, here one introduces a corollary for Theorem 1 and Theorem 2 :

Corollary 1. $T\in A_\infty \Leftrightarrow T\in OpS_{1,1}^0$ and $T^*\in OpS_{1,1}^0.$

Meyer has given an elegant and sophistic proof in [8] for this famous theorem, but here, it is an immediate deduction of the Theorem 1 and the Theorem 2.

4. The proof of Theorem 2

In this section, first one gives two propositions of Meyer's wavelets, then one proves a lemma, finally one proves Theorem 2.

If $\varepsilon\ne 0$, let i_ε be the smallest index i, such that $\varepsilon_i\ne 0$ and $x_\varepsilon=(x_1,\cdots,x_{-1+i_\varepsilon},y_1,$

$x_{1+i_\varepsilon},\cdots,x_n)$, let $\Phi^{(\varepsilon,N)}(x)=\displaystyle\int_{-\infty}^{x_{i_\varepsilon}}\int_{-\infty}^{y_N}\cdots\int_{-\infty}^{y_2}\Phi^{(\varepsilon)}(x_\varepsilon)dy_1\cdots dy_N$, one introduces the first two proposition for the Meyer wavelets.

Proposition 1. (i) *If* $\varepsilon\ne 0$, *then* $\int x^\alpha\Phi^{(\varepsilon)}(x)dx = 0$, $\forall\alpha\in N^n$.
(ii) *If* $\varepsilon=0$, *then* $\int x^\alpha\Phi^{(0)}(x)dx = 0$, $\forall\alpha\in N^n$, *and* $\alpha\ne 0$.

One can find the proof of this Proposition in [8].

In [3], I. Daubechies has constructed the following wavelets: $\forall N>0$, there exists a wavelet $\Phi^{(0)}(x)$ such that $\Phi^{(0)}(x)\in C_0^N([-M_N,M_N]^n)$, and $\sum_k \Phi^{(0)}(x-k)=1$. But one can not use Daubechies wavelets to obtain a characterization for $OpS_{1,1}^m$ in

wavelets. Fortunately one can divide each Meyer's wavelet into a series of functions similar to Daubechies wavelets.

Proposition 2. *For every integer N, there exists a sequence of numbers $\{a_z^{(\varepsilon,N)}\}_{z\in Z^n}$ and a sequence of functions $\Phi^{(\varepsilon,N,z)}(x)\in C_0^N([-M_N,M_N]^n)$ such that:*

$$(18) \qquad \Phi^{(\varepsilon)}(x)=\sum_{z\in Z^n} a_z^{(\varepsilon,N)}\Phi^{(\varepsilon,N,z)}(x-z),$$

where $a_z^{(\varepsilon,N)}$ satisfies the following condition:

$$(19) \qquad |a_z^{(\varepsilon,N)}|\le \frac{C_N}{(1+|z|)^N}.$$

And if $\varepsilon\ne 0$, one has also:

$$(20) \qquad \int_{-\infty}^{+\infty} x^\alpha\Phi^{(\varepsilon,N,z)}(x)dx=0,\ \forall\alpha\in N^n\ and\ 0\le|\alpha|\le N.$$

Proof. (i) If $\varepsilon=0$, owing to that $\Phi^{(0)}(x)\in S(R^n)$, it is evident for the above proposition.

(ii) If $\varepsilon\ne 0$, one proves first $\Phi^{(\varepsilon,N)}(x)\in S(R^n)$. In fact, one has:

$$\Phi^{(\varepsilon,N)}(x)=C_{\varepsilon,N}\int_{-\infty}^{+\infty}\frac{\hat{\Phi}^{(1)}(\xi_{i_\varepsilon})}{\xi_{i_\varepsilon}^N}^{-1+i_\varepsilon}\prod_{i=1}^N \hat{\Phi}^{(\varepsilon i)}(\xi_i)\prod_{i=1+i_\varepsilon}^N \hat{\Phi}^{(\varepsilon i)}(\xi_i)e^{ix\xi}d\xi.$$

Then one has: $\forall\alpha,\beta\in N^m$,

$$x^\alpha\partial_x^\beta\Phi^{(\varepsilon,N)}(x)$$

$$=C_{\varepsilon,N,\alpha,\beta}\int_{-\infty}^{+\infty}\partial_{\xi_{i_\varepsilon}}^{\alpha_{i_\varepsilon}}\left(\frac{\hat{\Phi}^{(1)}(\xi_{i_\varepsilon})}{\xi_{i_\varepsilon}^{N-\beta_{i_\varepsilon}}}\right)^{-1+i_\varepsilon}\prod_{i=1}^N \partial_{\xi_i}^{\alpha_i}(\xi_i^{\beta_i}\hat{\Phi}^{(\varepsilon i)}(\xi_i))\prod_{i=1+i_\varepsilon}^N \partial_{\xi_i}^{\alpha_i}(\xi_i^{\beta_i}\hat{\Phi}^{(\varepsilon i)}(\xi_i))e^{ix\xi}d\xi.$$

According to the property of Meyer's wavelets (cf [8]), $\partial_{\xi_{i_\varepsilon}}^{\alpha_{i_\varepsilon}}\left(\dfrac{\hat{\Phi}^{(1)}(\xi_{i_\varepsilon})}{\xi_{i_\varepsilon}^{N-\beta_{i_\varepsilon}}}\right)^{-1+i_\varepsilon}$, $\prod_{i=1}^{...}$ $\partial_{\xi_i}^{\alpha_i}(\xi_i^{\beta_i}\hat{\Phi}^{(\varepsilon i)}(\xi_i))$ and $\prod_{i=1+i_\varepsilon}^N \partial_{\xi_i}^{\alpha_i}(\xi_i^{\beta_i}\hat{\Phi}^{(\varepsilon i)}(\xi_i))$ belong to $S(R^n)$. Hence $x^\alpha\partial_x^\beta\Phi^{(\varepsilon,N)}(x)\in L^\infty$, that is to say: $\Phi^{(\varepsilon,N)}(x)\in S(R^n)$. Therefore, there exists a sequence $a_z^{(\varepsilon,N)}$ which satisfies the condition (19) and $\tilde{\Phi}^{(\varepsilon,N,z)}(x)\in C_0^{2N}([-M_N,M_N]^n)$, such that $\Phi^{(\varepsilon,N)}(x)=\sum_{z\in Z^n} a_z^{(\varepsilon,N)}\tilde{\Phi}^{(\varepsilon,N,z)}(x-z)$. Let $\Phi^{(\varepsilon,N,z)}(x)=\partial_{i_\varepsilon}^N\tilde{\Phi}^{(\varepsilon,N,z)}(x)$, then one has that $a_z^{(\varepsilon,N)}$ and $\Phi^{(\varepsilon,N,z)}(x)$

satisfy all the conditions in Proposition 2.

Then one proves a lemma.

Lemma 1. $T \in OpB^0 \Leftrightarrow T \in C_\infty$.

Proof.   (i) First, one proves that: $T \in OpB^0$ implies that T satisfies the condition (15).   For all the cube Q, there exists an integer j_0 such that $2^{-nj_0} \leq |Q| < 2^{-n(j_0-1)}$. $\forall q > n$, $f(x) \in C_0^q(Q)$, $g(x) \in C_0^q(Q)$, one has:

$$|\langle Tf(x), g(x) \rangle| = |\sum_\Gamma a_{j,k,l}^{(\varepsilon,\varepsilon')} \langle \Phi_{j,k}^{(\varepsilon)}(x), g(x) \rangle \langle \Phi_{j,l}^{(\varepsilon')}(y), f(y) \rangle|$$

$$\leq |\sum_{j < j_0} a_{j,k,l}^{(\varepsilon,\varepsilon')} \langle \Phi_{j,k}^{(\varepsilon)}(x), g(x) \rangle \langle \Phi_{j,l}^{(\varepsilon')}(y), f(y) \rangle|$$

$$+ |\sum_{\substack{j \geq j_0 \\ \varepsilon \neq 0}} a_{j,k,l}^{(\varepsilon,\varepsilon')} \langle \Phi_{j,k}^{(\varepsilon)}(x), g(x) \rangle \langle \Phi_{j,l}^{(\varepsilon')}(y), f(y) \rangle|$$

$$+ |\sum_{\substack{j \geq j_0 \\ \varepsilon = 0}} a_{j,k,l}^{(\varepsilon,\varepsilon')} \langle \Phi_{j,k}^{(\varepsilon)}(x), g(x) \rangle \langle \Phi_{j,l}^{(\varepsilon')}(y), f(y) \rangle|$$

$$= I_1 + I_2 + I_3.$$

As for I_1, one has:

$$I_1 \leq C \sum_{j < j_0} |a_{j,k,l}^{(\varepsilon,\varepsilon')}| \| \langle \Phi_{j,k}^{(\varepsilon)}(x), g(x) \rangle| 2^{\frac{nj}{2}} \|f\|_\infty |Q|$$

$$\leq C_q \sum_{j < j_0} \sum_\varepsilon \sum_{\varepsilon'} \sum_k \sum_l \frac{|\langle \Phi_{j,k}^{(\varepsilon)}(x), g(x) \rangle|}{(1 + |k - l|)^{n+q}} 2^{\frac{nj}{2}} \|f\|_\infty |Q|$$

$$\leq C_q \sum_{j < j_0} 2^{nj} \langle \sum_\varepsilon \sum_k |\Phi^{(\varepsilon)}(2^j x - k)|, |g(x)| \rangle \|f\|_\infty |Q|$$

$$\leq C_q \sum_{j < j_0} 2^{nj} \|g\|_\infty |Q| \|f\|_\infty |Q|$$

$$\leq C_q |Q|^{\frac{1}{2}} \|f\|_\infty |Q|^{\frac{1}{2}} \|g\|_\infty$$

$$\leq C_q \|f\|_{N_q^Q} \|g\|_{N_q^Q}.$$

As for I_2, one has:

$$I_2 \leq \sum_{\substack{j \geq j_0 \\ \varepsilon \neq 0}} |a^{(\varepsilon,\varepsilon')}_{j,k,l}| \, |\langle \Phi^{(\varepsilon)}_{j,k}(x), g(x) \rangle \langle \Phi^{(\varepsilon')}_{j,l}(y), f(y) \rangle|$$

One use the integration by parts for x, one gets:

$$I_2 \leq \sum_{\substack{j \geq j_0 \\ \varepsilon \neq 0}} |a^{(\varepsilon,\varepsilon')}_{j,k,l}| 2^{-jq} |\langle \Phi^{(\varepsilon,q)}_{j,k}(x), \partial^q_{i_\varepsilon} g(x) \rangle \langle \Phi^{(\varepsilon')}_{j,l}(y), f(y) \rangle|$$

$$\leq C \sum_{\substack{j \geq j_0 \\ \varepsilon \neq 0}} |a^{(\varepsilon,\varepsilon')}_{j,k,l}| 2^{-j(q-\frac{n}{2})} |\langle \Phi^{(\varepsilon,q)}_{j,k}(x), \partial^q_{i_\varepsilon} g(x) \rangle| \|f\|_\infty |Q|$$

$$\leq C \sum_{\substack{j \geq j_0 \\ \varepsilon \neq 0}} |a^{(\varepsilon,\varepsilon')}_{j,k,l}| 2^{-j(q-n)} \langle |\Phi^{(\varepsilon,q)}(2^j x - k)|, |\partial^q_{i_\varepsilon} g(x)| \rangle \|f\|_\infty |Q|$$

$$\leq C_q \sum_{j \geq j_0} \sum_{\varepsilon \neq 0} 2^{-j(q-n)} \sum_l \frac{\sum_k \langle |\Phi^{(\varepsilon,q)}(2^j x - k)|, |\partial^q_{i_\varepsilon} g(x)| \rangle}{(1+|k-l|)^{n+q}} \|f\|_\infty |Q|$$

$$\leq C_q \sum_{j \geq j_0} \sum_{\varepsilon \neq 0} 2^{-j(q-n)} \|\partial^q_{i_\varepsilon} g\|_\infty |Q| \|f\|_\infty |Q|$$

$$\leq C_q |Q|^{\frac{1}{2}} \|f\|_\infty |Q|^{\frac{1}{2}+\frac{q}{n}} \|g\|_{\dot{C}^q}$$

$$\leq C_q \|f\|_{N^Q_q} \|g\|_{N^Q_q}.$$

As for I_3, making the same calculation as I_2, one gets:

$$I_3 \leq C_q \|f\|_{N^Q_q} \|g\|_{N^Q_q}.$$

So one has (15).

(ii) Then one proves (16). $\forall \alpha, \ \beta \in N^n$, one estimates $I_{\alpha,\beta} = |\partial^\alpha_x \partial^\beta_y K(x,y)|$. Let $N_{\alpha,\beta} = 2(n+1) + |\alpha| + |\beta|$, $\forall x, y \in R^n$, $x \neq y$, there exists an integer j_0 such that $2^{-j_0} \leq |x-y| < 2^{1-j_0}$. Hence, one has:

$$I_{\alpha,\beta} \leq \sum_\Gamma 2^{j(n+|\alpha|+|\beta|)} |a^{(\varepsilon,\varepsilon')}_{j,k,l}| \|(\partial^\alpha_x \Phi^{(\varepsilon)})(2^j x - k)\| |(\partial^\beta_y \Phi^{(\varepsilon')})(2^j y - l)|$$

$$\leq C_{\alpha,\beta} \sum_\Gamma \frac{2^{j(n+|\alpha|+|\beta|)}}{(1+|k-l|)^{N_{\alpha,\beta}}} \frac{1}{(1+|2^j x - k|)^{N_{\alpha,\beta}}} \frac{1}{(1+|2^j y - l|)^{N_{\alpha,\beta}}}$$

$$= \sum_{j < j_0} + \sum_{j \geq j_0}$$

$$= I_1 + I_2.$$

As for I_1, one has:

$$I_1 \leq C_{\alpha,\beta} \sum_{j < j_0} 2^{j(n+|\alpha|+|\beta|)} \sum_k \frac{1}{(1+|2^j x-k|)^{N_{\alpha,\beta}}} \sum_l \frac{1}{(1+|2^j y-l|)^{N_{\alpha,\beta}}}$$

$$\leq C_{\alpha,\beta} \sum_{j < j_0} 2^{j(n+|\alpha|+|\beta|)}$$

$$\leq \frac{C_{\alpha,\beta}}{|x-y|^{n+|\alpha|+|\beta|}} .$$

As for I_2, one applies the following inequality:

$$\frac{1}{(1+|k-l|)(1+|2^j x-k|)(1+|2^j y-l|)} \leq \frac{C}{|2^j x - 2^j y|} ,$$

then one has:

$$I_2 \leq C_{\alpha,\beta} \sum_{j \geq j_0} 2^{j(n+|\alpha|+|\beta|)} |2^j x - 2^j y|^{-N_{\alpha,\beta}+n+1}$$

$$\times \sum_k \frac{1}{(1+|2^j x-k|)^{n+1}} \sum_l \frac{1}{(1+|2^j y-l|)^{n+1}}$$

$$\leq C_{\alpha,\beta} \sum_{j \geq j_0} 2^{j(n+|\alpha|+|\beta|+2n+1-N_{\alpha,\beta})} |x-y|^{-N_{\alpha,\beta}+n+1}$$

$$\leq \frac{C_{\alpha,\beta}}{|x-y|^{n+|\alpha|+|\beta|}} .$$

(iii) Finally, one proves $T \in C_\infty$ implies $T \in OpB^0$. In fact, one has:

$$a_{j,k,l}^{(\varepsilon,\varepsilon')} = \iint K(x,y) \Phi_{j,l}^{(\varepsilon')}(y) \Phi_{j,k}^{(\varepsilon)}(x) dx dy$$

$$= \iint 2^{-j} K(2^{-j}(x+k), \ 2^{-j}(y+k+l-k)) \Phi^{(\varepsilon')}(y) \Phi^{(\varepsilon)}(x) dx dy$$

$$= \iint K_{j,k}(x, y + l - k)\Phi^{(\varepsilon')}(y)\Phi^{(\varepsilon)}(x)\,dxdy.$$

Each $K_{j,k}(x,y)$ defines an operator $T_{j,k}$, and $T_{j,k}$ satisfies the estimations in (15) and (16). For $N > N' + 3n + 1$, using Proposition 2, one has:

$$a_{j,k,l}^{(\varepsilon,\varepsilon')} = \sum_z \sum_{z'} a_z^{(\varepsilon,N)} a_{z'}^{(\varepsilon',N)} \iint K_{j,k}(x, y + l - k)\Phi^{(\varepsilon',N,z')}(y - z')\Phi^{(\varepsilon,N,z)}(x - z)\,dxdy.$$

If $|k - l| \le 64$ one has:

$$\left| a_{j,k,l}^{(\varepsilon,\varepsilon')} \right| \le C_N \sum_z \sum_{z'} \frac{1}{(1 + |z|)^N} \frac{1}{(1 + |z'|)^N} \le C_N.$$

If $|k - l| > 64$ and $\varepsilon \ne 0$, one has:

$$\left| a_{j,k,l}^{(\varepsilon,\varepsilon')} \right| \le \sum_z \sum_{|k-l+z-z'| \le 16} \left| a_z^{(\varepsilon,N)} \right| \left| a_{z'}^{(\varepsilon',N)} \right| \left| \iint K_{j,k}(x, y + l - k) \right.$$

$$\left. \times \Phi^{(\varepsilon',N,z')}(y - z')\Phi^{(\varepsilon,N,z)}(x - z)\,dxdy \right|$$

$$+ \sum_z \sum_{|k-l+z-z'| > 16} \left| a_z^{(\varepsilon,N)} \right| \left| a_{z'}^{(\varepsilon',N)} \right| \left| \iint K_{j,k}(x, y + l - k) \right.$$

$$\left. \times \Phi^{(\varepsilon',N,z')}(y - z')\Phi^{(\varepsilon,N,z)}(x - z)\,dxdy \right|$$

$$= I_1 + I_2.$$

As for I_1, using (15), one gets:

$$|I_1| \le C_N \sum_z \sum_{|k-l+z-z'| \le 16} \left| a_z^{(\varepsilon,N)} \right| \left| a_{z'}^{(\varepsilon',N)} \right|$$

$$\le C_N \sum_z \sum_{|k-l+z-z'| \le 16} \frac{1}{(1 + |z|)^N} \frac{1}{(1 + |z'|)^N}$$

$$\le \frac{C_{N'}}{(1 + |k - l|)^{N'}}$$

As for I_2, denoting $\Phi^{(\varepsilon,N,z)}(x) = \int_{-\infty}^{x_{i_\varepsilon}} \int_{-\infty}^{y_N} \cdots \int_{-\infty}^{y_2} \Phi^{(\varepsilon,N,z)}(x_\varepsilon)dy_1 \cdots dy_N$ and using the integration by parts, one gets:

$$|I_2| \leq \sum_z \sum_{|k-l+z-z'|>16} \left| a_z^{(\varepsilon,N)} \right| \left| a_{z'}^{(\varepsilon',N)} \right| \left| \iint \partial_{x_{i_\varepsilon}}^N K_{j,k}(x,y+l-k) \right.$$

$$\left. \times \Phi^{(\varepsilon',N,z')}(y-z')\Phi^{(\varepsilon,N,N,z)}(x-z)dxdy \right|.$$

Using (16), one gets:

$$|I_2| \leq C_N \sum_z \sum_{|k-l+z-z'|>16} \frac{1}{(1+|z|)^N} \frac{1}{(1+|z'|)^N} \iint \frac{|\Phi^{(\varepsilon',N,z')}(y-z')||\Phi^{(\varepsilon,N,N,z)}(x-z)|}{|x-y+k-l|^{n+N}}\,dxdy$$

$$\leq C_N \sum_z \sum_{|k-l+z-z'|>16} \frac{1}{(1+|z|)^N} \frac{1}{(1+|z'|)^N} \frac{1}{|z-z'+k-l|^{n+N}}$$

$$\leq C_N \frac{1}{(1+|k-l|)^{N'}} \sum_z \sum_{|k-l+z-z'|>16} \frac{1}{(1+|z|)^{n+1}} \frac{1}{(1+|z'|)^{n+1}}$$

$$\leq C_N \frac{1}{(1+|k-l|)^{N'}}.$$

If $|k-l| > 64$ and $\varepsilon = 0$, since that $\varepsilon' \neq 0$, by the same calculation as the case where $|k-l| > 64$ and $\varepsilon \neq 0$, one gets the same calculation.

Finally, one turns back to the proof of Theorem 2.
In fact, $\forall \alpha \in N^m$, one has:

$$Tx^\alpha = \int \sum_\Gamma a_{j,k,l}^{(\varepsilon,\varepsilon')} \Phi_{j,k}^{(\varepsilon)}(x) \Phi_{j,l}^{(\varepsilon')}(y) y^m dy$$

$$= \int \sum_\Gamma 2^{-j|\alpha|} a_{j,k,l}^{(\varepsilon,\varepsilon')} \Phi_{j,k}^{(\varepsilon)}(x) \Phi_{j,l}^{(\varepsilon')}(y)(2^j y - l + l)^m dy$$

$$= \int \sum_\Gamma 2^{-j|\alpha|} a_{j,k,l}^{(\varepsilon,\varepsilon')} \Phi_{j,k}^{(\varepsilon)}(x) \sum_\beta C_\alpha^\beta l^{\alpha-\beta} (2^j y - l)^\beta \Phi_{j,l}^{(\varepsilon')}(y) dy$$

According to the properties of the Meyer wavelets, one has:
(i) If $\varepsilon \neq 0$, then $\int y^\beta \Phi^{(\varepsilon)}(y)dy = 0, \forall \beta \in N^m$.

(ii) If $\alpha \neq 0$, then $\int y^\alpha \Phi^{(0)}(y)dy = 0$.

(ii) If $\alpha = 0$, then $\int y^\alpha \Phi^{(0)}(y)dy = 1$.

That is to say that one has formally the following equality:

$$Tx^\alpha = \sum_{\Gamma \cap \{\varepsilon' = 0\}} 2^{-j(|\alpha| + \frac{n}{2})} l^\alpha a_{j,k,l}^{(\varepsilon,\varepsilon')} \Phi_{j,k}^{(\varepsilon)}(x).$$

Since that $T \in C_\infty$, the above equality is true in the sense of distribution. Hence one has that:

$$Tx^\alpha = 0, \forall \alpha \in N^n \Leftrightarrow \sum_l l^\alpha a_{j,k,l}^{(\varepsilon,0)} = 0, \forall \alpha \in N^n.$$

This completes the proof of Theorem 2.

5. From symbol to non-standard representation

According to the definition of the symbol and the non-standard representation, one has:

$$a_{j,k,l}^{(\varepsilon,\varepsilon')} = \iint \sigma(x,\xi) \Phi^{(\varepsilon)}(2^j x - k) \hat{\Phi}^{(\varepsilon')}(2^{-j}\xi) e^{i(x - 2^{-j}l)\xi} dx d\xi$$

$$= \iint \Phi^{(\varepsilon)}(x) \sigma(2^{-j}(x + k), 2^j \xi) e^{i(x + k - l)\xi} \hat{\Phi}^{(\varepsilon')}(\xi) dx d\xi.$$

According to the estimation of (2), one has:

$$|a_{j,k,l}^{(\varepsilon,\varepsilon')}| \leq \iint |\Phi^{(\varepsilon)}(x)| 2^j \xi|^m |\hat{\Phi}^{(\varepsilon')}(\xi)| dx d\xi.$$

Since that $n + m > 0$, one has: $a_{j,k,l}^{(\varepsilon,\varepsilon')} \leq C_m 2^{jm}$. Furthermore, for $\alpha \in N^n$, one has:

$$(k - l)^\alpha a_{j,k,l}^{(\varepsilon,\varepsilon')}$$

$$= \iint (x + k - l - x)^\alpha \Phi^{(\varepsilon)}(x) \sigma(2^{-j}(x + k), 2^j \xi) e^{i(x - k - l)\xi} \hat{\Phi}^{(\varepsilon')}(\xi) dx d\xi$$

$$= \sum_{\alpha_1 + \bar{\alpha} = \alpha} C_\alpha^{\alpha_1} \iint x^{\alpha_1} \Phi^{(\varepsilon)}(x) \sigma(2^{-j}(x + k), 2^j \xi) \partial_\xi^{\bar{\alpha}} e^{i(x + k - l)\xi} \hat{\Phi}^{(\varepsilon')}(\xi) dx d\xi$$

$$= \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \alpha} C_\alpha^{\alpha_1, \alpha_2} 2^{j|\alpha_2|} \iint x^{\alpha_1} \Phi^{(\varepsilon)}(x) (\partial_\xi^{\alpha_2} \sigma)(2^{-j}(x + k), 2^j \xi)$$

$$\times e^{i(x + k - l)\xi} (\partial_\xi^{\alpha_3} \hat{\Phi}^{(\varepsilon')})(\xi) dx d\xi.$$

For $\varepsilon' \neq 0$, one applies (2), one gets:

$$|(k - l)^\alpha a_{j,k,l}^{(\varepsilon,\varepsilon')}|$$

$$\leq \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \alpha} C_\alpha^{\alpha_1, \alpha_2} 2^{j|\alpha_2|} \iint |x^{\alpha_1} \Phi^{(\varepsilon)}(x)| |2^j \xi|^{m - \alpha_2} |(\partial_\xi^{\alpha_3} \hat{\Phi}^{(\varepsilon')})(\xi)| dx d\xi$$

$$\leq C_m 2^{jm}.$$

For $\varepsilon'=0$, one has that $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_n)\neq 0$. Let $\Phi^{(\varepsilon,\alpha,s)}(x)=\displaystyle\int_{-\infty}^{x_{i_\varepsilon}}\int_{-\infty}^{y_s}\cdots\int_{-\infty}^{y_2}x_\varepsilon^\alpha$ $\Phi^{(\varepsilon)}(x_\varepsilon)dy_1\cdots dy_s$, then one has:

$$(k-l)^\alpha a_{j,k,l}^{(\varepsilon,0)}=$$

$$\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\s_1+s_2=s}}C_{\alpha,s}^{\alpha_1,\alpha_2,s_1}\,2^{j(|\alpha_2|-s_2)}\iint\Phi^{(\varepsilon,\alpha_1,s)}(x)(\partial_{x_{i_\varepsilon}}^{s_2}\partial_\xi^{\alpha_2}\sigma)(2^{-j}(x+k),\,2^j\xi)$$

$$\times\,\xi_{x_{i_\varepsilon}}^{s_2}e^{i(x+k-l)\xi}(\partial_\xi^{\alpha_3}\hat{\Phi}^{(\varepsilon')})(\xi)dxd\xi.$$

According to (2), one has:

$$\left|(k-l)^\alpha a_{j,k,l}^{(\varepsilon,0)}\right|$$

$$\leq\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\s_1+s_2=s}}C_{\alpha,s}^{\alpha_1,\alpha_2,s_1}\,2^{j(|\alpha_2|-s_2)}\iint|\Phi^{(\varepsilon,\alpha_1,s)}(x)|\,2^j\xi|^{m-|\alpha_2|+s_2}|\xi|^{s_1}$$

$$\times\,|(\partial_\xi^{\alpha_3}\hat{\Phi}^{(\varepsilon')})(\xi)|dxd\xi$$

$$\leq\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\s_1+s_2=s}}C_{\alpha,s}^{\alpha_1,\alpha_2,s_1}2^{jm}\iint|\Phi^{(\varepsilon,\alpha_1,s)}(x)|\,|\xi|^{m-|\alpha_2|+s}|(\partial_\xi^{\alpha_3}\hat{\Phi}^{(\varepsilon')})(\xi)|dxd\xi$$

$$\leq C_m 2^{jm}.$$

Then one proves (13). In fact, one has:

$$\sum_l l^\alpha a_{j,k,l}^{(\varepsilon,0)}$$

$$=\sum_l l^\alpha\iint\Phi^{(\varepsilon)}(x-k)e^{ix\xi}\sigma(2^{-j}x,2^j\xi)\hat{\Phi}^{(0)}(\xi)e^{-il\xi}dxd\xi$$

$$=\sum_l C_\alpha\iint\Phi^{(\varepsilon)}(x-k)e^{ix\xi}\sigma(2^{-j}x,2^j\xi)\hat{\Phi}^{(0)}(\xi)e^{-il\xi}dxd\xi.$$

One applies the integration by parts for ξ, one gets:

$$\sum_l l^\alpha a_{j,k,l}^{(\varepsilon,0)}$$

$$= \sum_{\substack{l \\ \alpha_1+\alpha_2+\alpha_3=\alpha}} C^{\alpha_1,\alpha_2}_\alpha \, 2^{j|\alpha_2|} \iint x^{\alpha_1} \Phi^{(\varepsilon)}(x-k) e^{ix\xi} (\partial^{\alpha_2}_\xi \sigma)(2^{-j}x, 2^j\xi)$$

$$\times \partial^{\alpha_3}_\xi \hat{\Phi}^{(0)}(\xi) e^{-il\xi} dx d\xi$$

$$= \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} C^{\alpha_1,\alpha_2}_\alpha \, 2^{j|\alpha_2|} \iint x^{\alpha_1} \Phi^{(\varepsilon)}(x-k) e^{ix\xi} (\partial^{\alpha_2}_\xi \sigma)(2^{-j}x, 2^j\xi)$$

$$\times \partial^{\alpha_3}_\xi \hat{\Phi}^{(0)}(\xi) \sum_l e^{-il\xi} dx d\xi.$$

Since that $:\sum_l e^{-il\xi} = \delta(\xi)$, one has: $\int \partial^{\alpha_3}_\xi \hat{\Phi}^{(0)}(\xi) \sum_l e^{-il\xi} d\xi = 0, \forall \alpha_3 \in N^n, \alpha_3 \neq 0$. Hence one gets:

$$\sum_l l^\alpha a^{(\varepsilon,0)}_{j,k,l} =$$

$$\sum_{\alpha_1+\alpha_2=\alpha} C^{\alpha_1,\alpha_2}_\alpha \, 2^{j|\alpha_2|} \iint x^{\alpha_1} \Phi^{(\varepsilon)}(x-k) e^{ix\xi} (\partial^{\alpha_2}_\xi \sigma)(2^{-j}x, 2^j\xi) \hat{\Phi}^{(0)}(\xi) \delta(\xi) dx d\xi.$$

Let $s \in Z, s > |m-|\alpha\|$ and $\Phi^{(\varepsilon,\alpha,s)}_k(x) = \int_{-\infty}^{x_{i_\varepsilon}} \int_{-\infty}^{y_s} \cdots \int_{-\infty}^{y_2} x^\alpha_\varepsilon \Phi^{(\varepsilon)}(x_\varepsilon - k) dy_1 \cdots dy_s$. Using the integration by parts for x, one gets:

$$\sum_l l^\alpha a^{(\varepsilon,0)}_{j,k,l}$$

$$= \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ s_1+s_2=s}} C^{\alpha_1,s_1}_{\alpha,s} \iint \Phi^{(\varepsilon,\alpha_1,s)}_k(x) \xi^{\alpha_1}_{i_\varepsilon} \, e^{ix\xi} 2^{j(|\alpha_2|-s_2)}$$

$$\times (\partial^s_{x_{i_\varepsilon}} \partial^{\alpha_2}_\xi \sigma)(2^{-j}x, 2^j\xi) \hat{\Phi}^{(0)}(\xi) \delta(\xi) dx d\xi.$$

One applies (2), one gets: $|\xi^{\alpha_1}_{i_\varepsilon} 2^{j(|\alpha_2|-s_2)} (\partial^s_{x_{i_\varepsilon}} \partial^{\alpha_2}_\xi \sigma)(2^{-j}x, 2^j\xi)| \leq |\xi|^{m+s-|\alpha_2|}$. Hence one has: $\int \xi^{\alpha_1}_{i_\varepsilon} e^{ix\xi} 2^{j(|\alpha_2|-s_2)} (\partial^s_{x_{i_\varepsilon}} \partial^{\alpha_2}_\xi \sigma)(2^{-j}x, 2^j\xi) \hat{\Phi}^{(0)}(\xi) \delta(\xi) d\xi = 0$. Hence one has got: $\sum_l l^\alpha a^{(\varepsilon,0)}_{j,k,l} = 0$.

6.   From non-standard representation to symbol

Denotes $\sigma_{\varepsilon,\varepsilon'}(x,\xi)=\sum\limits_{j,k,l}a_{j,k,l}^{(\varepsilon,\varepsilon')}\Phi^{(\varepsilon)}(2^jx-k)\hat{\Phi}^{(\varepsilon')}(2^{-j}\xi)e^{-i(x-2^{-j}l)\xi}$.   Let us prove that

each $\sigma_{\varepsilon,\varepsilon'}(x,\xi)$ satisfies the condition (2).   In fact, one has:

$$\partial_x^\alpha\partial_\xi^\beta\sigma_{\varepsilon,\varepsilon'}(x,\xi)$$

$$=\partial_x^\alpha\sum_{j,k,l}\sum_{\beta_1+\beta_2=\beta}2^{-j|\beta_2|}\,C_\beta^{\beta_1}\,a_{j,k,l}^{(\varepsilon,\varepsilon')}\Phi^{(\varepsilon)}(2^jx-k)(\partial_\xi^{\beta_2}\hat{\Phi}^{(\varepsilon')}(2^{-j}\xi)$$

$$\times(x-2^{-j}l)^{\beta_1}e^{-i(x-2^{-j}l)\xi}$$

$$=\sum_{j,k,l}\sum_{\substack{\beta_1+\beta_2=\beta\\\alpha_1+\alpha_2+\alpha_3=\alpha}}2^{j(|\alpha_2|-|\beta_2|)}\,C_{\alpha,\beta}^{\alpha_1,\alpha_2,\beta_1}\,a_{j,k,l}^{(\varepsilon,\varepsilon')}(\partial_x^{\alpha_2}\Phi^{(\varepsilon)})(2^jx-k)\xi^{\alpha_3}$$

$$\times(\partial_\xi^{\beta_2}\hat{\Phi}^{(\varepsilon')})(2^{-j}\xi)(x-2^{-j}l)^{\beta_1-\alpha_1}e^{-i(x-2^{-j}l)\xi}$$

$$=\sum_{j,k,l}\sum_{\substack{\beta_1+\beta_2=\beta\\\alpha_1+\alpha_2+\alpha_3=\alpha}}2^{j(|\alpha|-|\beta|)}\,C_{\alpha,\beta}^{\alpha_1,\alpha_2,\beta_1}\,a_{j,k,l}^{(\varepsilon,\varepsilon')}(2^jx-k+k-l)^{\beta_1-\alpha_1}$$

$$e^{-i(2^jx-k)2^{-j}\xi-i(k-l)2^{-j}\xi}(\partial_x^{\alpha_2}\Phi^{(\varepsilon)})(2^jx-k)(2^{-j}\xi)^{\alpha_3}(\partial_\xi^{\beta_2}\hat{\Phi}^{(\varepsilon')})(2^{-j}\xi)$$

$$=\sum_{j,k,l}\sum_{\substack{\beta_1+\beta_2=\beta\\\alpha_1+\alpha_2+\alpha_3=\alpha\\\beta_3+\beta_4=\beta_1-\alpha_1}}2^{j(|\alpha|-|\beta|)}\,C_{\alpha,\beta}^{\alpha_1,\alpha_2,\beta_1,\beta_3}\,a_{j,k,l}^{(\varepsilon,\varepsilon')}(x-l)^{\beta_3}e^{-i(k-l)2^{-j}\xi}(2^jx-k)^{\beta_4}$$

$$(\partial_x^{\alpha_2}\Phi^{(\varepsilon)})(2^jx-k)e^{-i(2^jx-k)2^{-j}\xi}(2^{-j}\xi)^{\alpha_3}(\partial_\xi^{\beta_2}\hat{\Phi}^{(\varepsilon')})(2^{-j}\xi).$$

Denotes $N_{\alpha,\beta}=|m|+|\alpha|+|\beta|+n$.   One considers two cases: (i) $\varepsilon'\neq0$, (ii) $\varepsilon'=0$.   For $\varepsilon'\neq0$, one has:

$$|\partial_x^\alpha\partial_\xi^\beta\sigma_{\varepsilon,\varepsilon'}(x,\xi)|$$

$$\leq\sum_{j,k}\sum_{|\alpha_2|+|\alpha_3|+|\beta_2|+|\beta_4|\leq4N_{\alpha,\beta}}2^{j(m+|\alpha|-|\beta|)}\,C_{\alpha,\beta}\,|(2^jx-k)^{\beta_4}(\partial_x^{\alpha_2}\Phi^{(\varepsilon)})(2^jx-k)|$$

$$\times|(2^{-j}\xi)^{\alpha_3}(\partial_\xi^{\beta_2}\hat{\Phi}^{(\varepsilon')}(2^{-j}\xi)|$$

$$\leq\sum_{j}\sum_{|\alpha_3|+|\beta_2|\leq4N_{\alpha,\beta}}2^{j(m+|\alpha|-|\beta|)}\,C_{\alpha,\beta}\,|(2^{-j}\xi)^{\alpha_3}(\partial_\xi^{\beta_2}\hat{\Phi}^{(\varepsilon')})(2^{-j}\xi)|.$$

Since that $\varepsilon' \neq 0$, then for each $\xi \neq 0$, there exists maximum a finite number of integer j such that $(\partial_\xi^{\beta_2}\hat{\Phi}^{(\varepsilon')})(2^{-j}\xi) \neq 0$. One applies then $|(2^{-j}\xi)^{\alpha_3}(\partial_\xi^{\beta_2}\hat{\Phi}^{(\varepsilon')})(2^{-j}\xi)| \leq C_{\alpha,\beta}$, one gets then: $|\partial_x^\alpha \partial_\xi^\beta \sigma_{\varepsilon,\varepsilon'}(x,\xi)| \leq C_{\alpha,\beta}|\xi|^{m+|\alpha|-|\beta|}$. For $\varepsilon' = 0$, denoting $F_{j,k}^{\varepsilon;\alpha}(\xi) = \sum_l (k-l)^\alpha a_{j,k,l}^{(\varepsilon,0)} e^{-i(k-l)\xi}$, then one has:

$$\partial_x^\alpha \partial_\xi^\beta \sigma_{\varepsilon,0}(x,\xi)$$

$$= \sum_{j,k} \sum_{\substack{\beta_1+\beta_2=\beta \\ \alpha_1+\alpha_2+\alpha_3=\alpha \\ \beta_3+\beta_4=\beta_1-\alpha_1}} 2^{j(|\alpha|-|\beta|)} C_{\alpha,\beta}^{\alpha_1,\alpha_2,\beta_1,\beta_3} F_{j,k}^{\varepsilon;\beta_2}(2^{-j}\xi)(2^j x-k)^{\beta_4}$$

$$(\partial_x^{\alpha_2}\Phi^{(\varepsilon)})(2^j x-k)e^{-i(2^j x-k)2^{-j}\xi}(2^{-j}\xi)^{\alpha_3}(\partial_\xi^{\beta_2}\hat{\Phi}^{(\varepsilon')})(2^{-j}\xi).$$

Let us prove first a property of $F_{j,k}^{\varepsilon;\alpha}(\xi)$.

Lemma 2. $\forall \alpha \in N^n, |\alpha| \leq N_{\alpha,\beta}, |F_{j,k}^{\varepsilon;\alpha}(\xi)| \leq C_{m,\alpha,\beta}2^{jm}|\xi|^{N_{\alpha,\beta}}.$

Proof. In fact, one has that:

$$\partial_\xi^{\alpha'} F_{j,k}^{\varepsilon;\alpha}(\xi) = (-i)^{\alpha'}\sum_l (k-l)^{\alpha+\alpha'} a_{j,k,l}^{(\varepsilon,0)} e^{-i(k-l)\xi}.$$

Since $|a_{j,k,l}^{(\varepsilon,0)}| \leq C_{m,\alpha,\beta}2^{jm}(1+|k-l|)^{-9N_{\alpha,\beta}}$, then for all $\alpha \in N^n$, $\alpha' \in N^n$ and $|\alpha|+|\alpha'| \leq 4N_{\alpha,\beta}$, one has:

$$(21) \qquad |\partial_\xi^{\alpha'} F_{j,k}^{\varepsilon;\alpha}(\xi)| \leq C_{m,\alpha,\beta}2^{jm}.$$

Furthermore, one has : $\sum_l l^\alpha a_{j,k,l}^{(\varepsilon,0)} = 0$, $\forall \alpha \in N^n$, $|\alpha| \leq N_{\alpha,\beta}$. Then one has: $\sum_l (k-l)^\alpha a_{j,k,l}^{(\varepsilon,0)} = 0$, $\forall \alpha \in N^n$, $|\alpha| \leq N_{\alpha,\beta}$. Hence, one has:

$$(22) \qquad \partial_\xi^{\alpha'} F_{j,k}^{\varepsilon;\alpha}(\xi)|_{\xi=0} = 0.$$

Using (21) and (22), one gets the conclusion.

One returns to the proof of Theorem 1. Using the above lemma, one has:

$$|\partial_x^\alpha \partial_\xi^\beta \sigma_{\varepsilon,0}(x,\xi)|$$

$$\leq \sum_{\substack{j,k}} \sum_{\substack{\beta_1+\beta_2=\beta\\ \alpha_1+\alpha_2+\alpha_3=\alpha\\ \beta_3+\beta_4=\beta_1-\alpha_1}} 2^{j(|\alpha|-|\beta|)} |C_{\alpha,\beta}| \|F_{j,k}^{\varepsilon,\beta_2}(2^{-j}\xi)\| |(2^j x-k)^{\beta_4}(\partial_x^{\alpha_2}\Phi^{(\varepsilon)})(2^j x-k)|$$

$$\times |(2^{-j}\xi)^{\alpha_3}(\partial_\xi^{\beta_2}\hat{\Phi}^{(0)})(2^{-j}\xi)|$$

$$\leq \sum_{j,k} \sum_{|\alpha_2|+|\alpha_3|+|\beta_2|+|\beta_4|\leq 4N_{\alpha,\beta}} 2^{j(m+|\alpha|-|\beta|)} |C_{\alpha,\beta}| \|(2^j x-k)^{\beta_4}(\partial_x^{\alpha_2}\Phi^{(\varepsilon)})(2^j x-k)|$$

$$\times |(2^{-j}\xi)^{N_{\alpha,\beta}}\| |(2^{-j}\xi)^{\alpha_3}(\partial_\xi^{\beta_2}\hat{\Phi}^{(0)})(2^{-j}\xi)|.$$

There exists a constant $C>0$ such that $\hat{\Phi}^{(0)}(\xi)=0$, for $|\xi|\geq C$. Then one has:

$$|\partial_x^\alpha \partial_\xi^\beta \sigma_{\varepsilon,0}(x,\xi)|$$

$$\leq C_{m,\alpha,\beta} \sum_{j\geq \log_2|\xi|-C} 2^{j(m+|\alpha|-|\beta|)} |2^{-j}\xi|^{N_{\alpha,\beta}}$$

$$\leq C_{m,\alpha,\beta} \sum_{j\geq \log_2|\xi|-C} 2^{j(m+|\alpha|-|\beta|-N_{\alpha,\beta})} |\xi|^{N_{\alpha,\beta}}$$

$$\leq C_{m,\alpha,\beta} |\xi|^{m+|\alpha|-|\beta|}$$

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