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Regularity bounds for projective subschemes of codimension two

By

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Abstract

New regularity bounds for subschemes of codimension two are established. They rest on the Serre correspondence to reflexive sheaves, a study of cohomological annihilators and the strong restriction property. Examples are described where the bounds are sharp.

1. Introduction

Let $\boldsymbol{P}^n$ be the projective n-space over an algebraically closed field of characteristic zero and let $X \subset \boldsymbol{P}^n$ be a (locally) Cohen-Macaulay subscheme of codimension two. The purpose of this paper is to find integers m_j, $1 \leq j \leq n-2$, such that the vanishing statement $H^j(\mathscr{I}_X(t))=0$ holds if $t \geq m_j$. We refer to such results as regularity bounds, since they yield estimates for the Castelnuovo-Mumford regularity of X. Now assume that X is a generically complete intersection. Then we can use the well-known Serre correspondence to reflexive sheaves of rank two. For this let e be the largest integer t such that $\omega_X(-t)$ has a global section ε which generates $\omega_X(-t)$ outside a subscheme of codimension ≥ 3 (in $\boldsymbol{P}^n$). The section ε gives rise to a reflexive sheaf $\mathscr{E}$ on $\boldsymbol{P}^n$ of rank two. In the correspondence between X and $\mathscr{E}$, the degree d of X and the smallest degree s of a hypersurface containing X occur naturally. There is one more integer we want to take into account. Since X is locally Cohen-Macaulay there is an integer k such that $\mathfrak{m}^k \cdot H^j_*(\mathscr{I}_X)=0$ for all j with $1 \leq j \leq n-2$. In that case X is called k-Buchsbaum. Similarly, it is defined when $\mathscr{E}$ is called k-Buchsbaum.

In this paper we establish regularity bounds for X depending on d, e, s, k, n. We do this in two steps. First we generalize results of Migliore and Miró-Roig [15] obtained for integral curves, to arbitrary codimension two subschemes of $\boldsymbol{P}^n$. This is done in Section 4. It requires regularity bounds for k-Buchsbaum sheaves on $\boldsymbol{P}^n$. Here we proceed by induction on n using the restriction to a

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general hyperplane. The difficulty is that the restriction of a k-Buchsbaum sheaf is not k-Buchsbaum any more in general. This problem is dealt with in Section 2. Since our sheaves are typically not locally free we have to extend the methods of [20]. Combining the results of Section 2 with results of Hartshorne and Sauer on stable and unstable sheaves on a plane, respectively, we prove the needed regularity bounds for sheaves in Section 3.

In a second step we show that some of the bounds of Section 4 can be improved if we consider integral subschemes of sufficiently large degree. The key is the consideration of the strong restriction property. We say that X has the restriction property if the general hyperplane section of X is not contained in a hypersurface of degree $s-1$. Sufficient conditions which ensure the restriction property have been obtained by Laudal, Gruson and Peskine, Mezzetti and Strano. However, if X has dimension ≥ 2 it might happen that X has the restriction property but its general hyperplane section does not have it. We say that X has the strong restriction property if X and all its consecutive (proper) general hyperplane sections have the restriction property. It is the purpose of Section 5 to present sufficient conditions for X to have the strong restriction property. Note that some of the results of this section hold true for subschemes of arbitrary codimension.

In the final section we show how the results of Section 4 can be improved if one knows that the subscheme X has the strong retriction property. Then we describe infinite series of surfaces and curves for which our improved regularity bounds are best possible.

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2. Cohomological annihilators and regularity

The aim of this section is to relate regularity bounds of a module to those of its restriction to a general subspace (cf. Proposition 2.2), by taking cohomological annihilators into account. Since we want to apply these results in the next section to sheaves which are not necessarily locally free, we cannot use [20] directly. Thus we show how the methods of [20] can be extended in order to achieve the required generalization.

Let $R=\oplus_{n\geq 0}R_n$ denote a graded Noetherian ring such that $R=R_0[R_1]$ and $K:=R_0$ is a field. Put $\mathfrak{m}=\oplus_{n>0}R_n$ the irrelevant maximal ideal of R. Let M denote a finitely generated graded R-module of Krull dimension δ. We fix the basic notation of [19]. In particular, a homogeneous element $x\in R$ is called M-filter regular, provided $0:_M x$ is an R-module of finite length. A system of (homogeneous) elements $\underline{x}=\{x_1,\cdots,x_r\}$ is called an M-filter regular sequence whenever

$$(x_1,\cdots,x_{i-1})M:x_i/(x_1,\cdots,x_{i-1})M \quad (i=1,\cdots,r)$$

is an R-module of finite length. For an arbitrary graded R-module N, let $e(N)$ denote

$$e(N):=\sup\{j\in\mathbf{Z}\,|\,N_j\neq 0\}$$

(where N_j denotes the j-th graded piece of the graded R-module N). Thus $e(\{0\})=-\infty$.

Furthermore we recall a definition, see [21], p. 266. For an integer $s\geq 0$ put

$$r_s(M):=\max\{i+e(H^i_{\mathfrak{m}}(M))\,|\,i\geq s\}.$$

Then $\operatorname{reg} M:=r_0(M)=r_{\operatorname{depth} M}(M)$ is called the Castelnuovo-Mumford regularity of M.

For a system of elements $\underline{x}=\{x_1,\cdots,x_r\}$ of R and an integer $0\leq i\leq r$, let $\underline{x}_i=\{x_1,\cdots,x_i\}$. Note that $\underline{x}_0$ is the empty set.

Lemma 2.1. *Let $\underline{l}=\{l_1,\cdots,l_r\}\subset R_1$ be an M-filter regular sequence. Suppose there is an integer $\mu\geq 0$ such that $l_u H^i_{\mathfrak{m}}(M)=0$, where $1\leq r\leq\delta-i$. Then there are the following bounds:*

(a) $e(H^i_{\mathfrak{m}}(M))\leq\mu-1+\max\{e(H^i_{\mathfrak{m}}(M/\underline{l}M)),\ r_{i+1}(M)-i+1\}$;

(b) $e(H^i_{\mathfrak{m}}(M))\leq\mu-1+e(H^i_{\mathfrak{m}}(M/\underline{l}M))$ *if* $H^i_{\mathfrak{m}}(M)=0$, *for all j with $i<j<i+r$.*

Proof. According to [20], Lemma 3.3 it holds

$$e(H^i_{\mathfrak{m}}(M))\leq\mu-1+e(H^i_{\mathfrak{m}}(M)/\underline{l}H^i_{\mathfrak{m}}(M)).$$

Hence [20], Theorem 2.3(a) yields

$$e(H^i_{\mathfrak{m}}(M))\leq\mu-1+\max\{e(H^i_{\mathfrak{m}}(M/\underline{l}M)),\ e(H^{i+1}_{\mathfrak{m}}(M/\underline{l}_jM))+2\,|\,0\leq j\leq r-2\}.$$

Using $r_i(M/\underline{l}_jM)\leq r_i(M)$ claim (a) follows.

In case (b) an easy induction shows that the assumption implies

$$H^{i+1}_{\mathfrak{m}}(M/\underline{l}_jM)=0\quad\text{if } 0\leq j\leq r-2.$$

Therefore [20], Theorem 2.3(a) reads as

$$e(H^i_{\mathfrak{m}}(M)/\underline{l}H^i_{\mathfrak{m}}(M))\leq e(H^i_{\mathfrak{m}}(M/\underline{l}M))$$

proving claim (b).

In case $r=1$ Lemma 2.1(b) has also been observed in [8].

Now we are ready for the main result of this section.

Proposition 2.2. *Let $\underline{l}=\{l_1,\cdots,l_r\}\subset R_1$ be an M-filter regular sequence. If $r\geq 0$ suppose there are integers $\mu_i,\cdots,\mu_{i+r-1}$, where $0\leq i\leq\delta-r$, such that*

$$\underline{l}^{\mu_j}_{r+i-j}H^j_{\mathfrak{m}}(M)=0 \text{ for all integers } j \text{ with } i\leq j<i+r.$$

Then it holds:

$$r_i(M)\leq r_i(M/\underline{l}M)+c_{i,r}$$

where

$$c_{i,r}=\max\{0,\ \mu_i+\cdots+\mu_{i+r-1}-1\}.$$

Proof. We induct on $r\geq 0$. The case $r=0$ being trivial, we assume $r>0$. Now we use descending induction on i. If $i=\delta-r$ then $r_i(M/\underline{l}M)=i+e(H^i_{\mathfrak{m}}(M/\underline{l}M))$ and [20], Proposition 4.3. shows the claim.

Let $0\leq i<\delta-r$. Then we know by the induction hypothesis

$$r_{i+1}(M)\leq r_{i+1}(M/\underline{l}_{r-1}M)+c_{i+1,r-1}.$$

Since $r_{i+1}(M/\underline{l}_{r-1}M)\leq r_i(M/\underline{l}M)$ we obtain

$$(*)\qquad r_{i+1}(M)\leq r_i(M/\underline{l}M)+c_{i+1,r-1}.$$

By definition it holds $i+e(H^i_{\mathfrak{m}}(M/\underline{l}M))\leq r_i(M/\underline{l}M)$. Therefore the Lemma above implies

$$i+e(H^i_{\mathfrak{m}}(M))\leq r_i(M/\underline{l}M)+c_{i+1,r-1}+\mu_i$$

proving $i+e(H^i_{\mathfrak{m}}(M)\leq r_i(M/\underline{l}M))+c_{i,r}$, if $\mu_{i+1}+\cdots+\mu_{i+r-1}>0$. Otherwise it follows $H^{i+1}_{\mathfrak{m}}(M)=\cdots=H^{i+r-1}_{\mathfrak{m}}(M)=0$ and Lemma 2.1(b) provides

$$i+e(H^i_{\mathfrak{m}}(M))\leq i+e(H^i_{\mathfrak{m}}(M/\underline{l}M))+\mu_i-1\leq i+e(H^i_{\mathfrak{m}}(M/\underline{l}M))+c_{i,r}.$$

Hence we have shown in any case that

$$i+e(H^i_{\mathfrak{m}}(M))\leq r_i(M/\underline{l}M)+c_{i,r}.$$

Using the estimation $(*)$ it follows

$$r_i(M)=\max\{i+e(H^i_{\mathfrak{m}}(M)),\ r_{i+1}(M)\}\leq r_i(M/\underline{l}M)+c_{i,r}$$

We will measure the "size" of the cohomology modules of M by the integers

$$\lambda_i(M)=\min\{\lambda\in N|\mathfrak{m}^\lambda H^i_{\mathfrak{m}}(M)=0\}.$$

Note that $\lambda_i(M)$ is finite if and only if $H^i_{\mathfrak{m}}(M)$ has finite length.

Corollary 2.3. *Suppose $H^j_{\mathfrak{m}}(M)$ has finite length for all $j\leq\delta-2$. Let $\underline{l}=\{l_1,\cdots,l_r\}\subset R_1$ be an M-filter regular sequence. Then it holds for all $i<\delta-r$*

$$r_i(M)\leq r_i(M/\underline{l}M)+d_{i,r}$$

where

$$d_{i,r}=\max\{0,\lambda_i(M)+\cdots+\lambda_{i+r-1}(M)-1\}.$$

Remark 2.4. In the corollary above we do not assume that $H^{\delta-1}_{\mathfrak{m}}(M)$ has finite length. Indeed, in our application this module is typically not of finite length. Thus the methods of [17] do not apply.

3. Reflexive sheaves of rank two

In this section we will generalize the vanishing theorems of [15] for reflexive sheaves of rank 2 on $\boldsymbol{P}^3$ to results for such sheaves on $\boldsymbol{P}^n$. First we consider stable sheaves and then unstable ones.

From now on R will be the polynomial ring $K[x_0,\cdots,x_n]$, where K is an algebraically closed field of characteristic zero and $n\geq 2$. Let $\mathscr{E}$ be a coherent sheaf on $\boldsymbol{P}^n=Proj(R)$. We denote by $H^j_*(\mathscr{E})$ the graded R-module $\oplus_{t\in\boldsymbol{Z}}H^j(\boldsymbol{P}^n,\mathscr{E}(t))$. For short we write $H^j(\mathscr{E}(t))=H^j(\boldsymbol{P}^n,\mathscr{E}(t))$. The sheaf $\mathscr{E}$ is called normalized if its first Chern class c_1 equals either 0 or -1.

Consider the module $E=H^0_*(\mathscr{E})$. The isomorphisms $H^{j+1}_{\mathfrak{m}}(E)\cong H^j_*(\mathscr{E})$, $j\geq 1$, allow us to apply the results of the previous section to the sheaf $\mathscr{E}$.

For an integer $s\geq 1$ we define

$$r_s(\mathscr{E})=\max\{j+1+e(H^j_*(\mathscr{E}))|j\geq s\}.$$

Then it holds $r_s(\mathscr{E})=r_{s+1}(E)$ for $s\geq 1$. Note also that $\operatorname{reg}\mathscr{E}=r_1(\mathscr{E})$ is the Castelnuovo-Mumford regularity of $\mathscr{E}$ in the sense of Mumford [18].

Definition 3.1. A reflexive sheaf $\mathscr{E}$ of rank 2 on $\boldsymbol{P}^n$ is said to be *k-Buchsbaum* if $\mathfrak{m}^kH^j_*(\mathscr{E})=0$ for all j with $1\leq j\leq n-2$.

Note that the module $E=H^0_*(\mathscr{E})$ is in general *not* k-Buchsbaum in the sense of [17] or [20], if the sheaf $\mathscr{E}$ is k-Buchsbaum. The reason is that we do not require a knowledge on the annihilator of $H^{n-1}_*(\mathscr{E})$ in the definition above. Instead, the definition is made up in such a way that $\mathscr{E}$ is k-Buchsbaum if and only if the zero scheme of a general section of $\mathscr{E}(t)$ is k-Buchsbaum for sufficiently large t (cf. Section 4).

The following preparatory result follows immediately from [6], Theorem 7.4.

Lemma 3.2. *Let $\mathscr{E}$ be a normalized stable reflexive sheaf of rank 2 on $\boldsymbol{P}^2$. Then it holds*

$$H^1(\mathscr{E}(t))=0 \quad \textit{if } t\geq c_2-c_1-2.$$

Now we are ready to generalize [15], Theorem 1.4.

Theorem 3.3. *Let $\mathscr{E}$ be a normalized stable reflexive sheaf of rank 2 on $\boldsymbol{P}^n$, $n\geq 3$. Suppose $\mathscr{E}$ is k-Buchsbaum. Then it holds for all integers j, with $1\leq j\leq n-2$:*

$$H^j(\mathscr{E}(t))=0 \quad \textit{if } t\geq c_2-c_1-2-j+(n-1-j)k,$$

unless $j=1$ and the restriction of $\mathscr{E}$ to a general linear subspace of dimension 3 is the null correlation bundle. In this case we have

$$H^1(\mathscr{E}(t))=0 \quad \textit{if } t\geq c_2-c_1-2+(n-2)k,$$

namely the previous bound increased by 1.

Proof. If $k=0$, then $H_*^j(\mathscr{E})=0$ for all j, where $1\le j\le n-2$. Thus we may assume $k\ge 1$. Let $L_{n-2}\subset L_{n-3}\subset\cdots\subset L_0=\boldsymbol{P}^n$ be a flag of general linear subspaces such that L_i has dimension $n-i$.

First, let us assume that the restriction $\mathscr{E}$ to L_{n-2} is a stable vector bundle. Hence $H^0(\mathscr{E}_{L_{n-2}})=0$ and $\mathscr{E}^*_{L_{n-2}}\cong\mathscr{E}_{L_{n-2}}(-c_1)$ imply by Serre duality $H^2(\mathscr{E}_{L_{n-2}}(t))=0$ if $t\ge -3-c_1$. According to Schwarzenberger (cf. [6], Lemma 3.2) it holds $4c_2>c_1^2$. Therefore Lemma 3.2 shows

$$(*)\qquad r_1(\mathscr{E}_{L_{n-2}})\le c_2-c_1-1.$$

Corollary 2.3 provides for j with $1\le j\le n-2$

$$\begin{aligned} j+1+e(H_*^j(\mathscr{E}))&\le r_j(\mathscr{E}_{L_{n-j-1}})+(n-j-1)k-1\\ &\le r_1(\mathscr{E}_{L_{n-2}})+(n-i-1)k-1 \text{ by induction on } j\ge 1\\ &\le c_2-c_1-2+(n-j-1)k\end{aligned}$$

where the last estimation is due to $(*)$. Our claim follows in this case.

Second, let us assume that $\mathscr{E}_{L_{n-2}}$ is not stable. Then, due to results of Barth and Hartshorne the restriction $\mathscr{E}_{L_{n-3}}$ is isomorphic to the null correlation bundle. Thus we have $r_2(\mathscr{E}_{L_{n-3}})=0$ and $c_1(\mathscr{E}_{L_{n-3}})=0$, $c_2(\mathscr{E}_{L_{n-3}})=1$. Arguing as in the first case we obtain, if $j\ge 2$,

$$\begin{aligned} j+1+e(H_*^j(\mathscr{E}))&\le r_2(\mathscr{E}_{L_{n-3}})+(n-1-j)k-1\\ &=c_2-c_1-2+(n-j-1)k.\end{aligned}$$

If $j=1$ we have:

$$1+1+e(H_*^1(\mathscr{E}))\le r_1(\mathscr{E}_{L_{n-2}})+(n-1-j)k-1,$$

and the conclusion follows, being $r_1(\mathscr{E}_{L_{n-2}})=1$.

Our next aim is to derive a vanishing result for unstable sheaves. A normalized rank 2 reflexive sheaf $\mathscr{E}$ on $\boldsymbol{P}^n$ is said to be *unstable of order r* if $\mathscr{E}$is not stable and r is the largest integer for which $H^0(\mathscr{E}^*(-r))\ne 0$.

We need the following result:

Lemma 3.4. *Let $\mathscr{E}$ be a rank 2 reflexive sheaf on $\boldsymbol{P}^n$, $n\ge 2$, unstable of order r. Then*:

(a) *if $L\subset\boldsymbol{P}^n$ is a general hyperplane, then $\mathscr{E}_L$ is a reflexive sheaf on L, unstable of order r;*
(b) *if $\mathscr{E}$ is normalized then $c_2+r^2+c_1r\ge 0$;*
(c) *is $\mathscr{E}$ is normalized and $n=2$, then*

$$H^1(\mathscr{E}(t))=0 \text{ if } t\ge c_2+r^2+(1+c_1)r-1.$$

Proof. All claims are in [27]: (a) is Proposition 1.1, (b) follows from formula (3) on p. 635 and (c) follows from Proposition 2.1 and Remark 2.3.1.

The following vanishing result generalizes [15], Theorem 1.5.

Theorem 3.5. *Let $\mathscr{E}$ be a normalized reflexive sheaf of rank 2 on $\boldsymbol{P}^n$ which is unstable of order r. Suppose $\mathscr{E}$ is k-Buchsbaum. Then it holds for all integers j with* $1 \leq j \leq n-2$:

$$H^j(\mathscr{E}(t))=0 \text{ if } t \geq c_2+(c_1+1)r+r^2-1-j+(n-1-j)k.$$

Proof. Consider a flag of linear spaces as in the proof of Theorem 3.3. Lemma 3.4(a) and Serre duality imply

$$H^2(\mathscr{E}_{L_{n-2}}(t))=0 \text{ if } t \geq r-2.$$

By Lemma 3.4(b) and (c) it follows:

$$r_1(\mathscr{E}_{L_{n-2}}) \leq \max\{c_2+r^2+(1+c_1)r, r\}=c_2+r^2+(1+c_1)r.$$

Then the same reasoning as in the proof of Theorem 3.3 provides the claim.

4. Subschemes of codimension two

In this section we will apply the vanishing results for reflexive sheaves in order to obtain Castelnuovo bounds for subschemes of codimension 2 which are related to a reflexive sheaf of rank 2 by the Serre correspondence.

Throughout this section we will assume that the subscheme $X \subset \boldsymbol{P}^n$ is a generically complete intersection of codimension 2. Note that every integral subscheme is a generically complete intersection.

Under the assumption above, for sufficiently small integers t the twist $\omega_X(-t)$ of the dualizing sheaf of X has a global section ε which generates $\omega_X(-t)$ outside a subscheme of $\boldsymbol{P}^n$ having codimension ≥ 3. Thus there is an integer $e=e(X)$ which is the maximum over these integers t and then we choose a corresponding section ε of $\omega_X(-e)$. We may think of ε as an element of $\operatorname{Ext}^1_{\boldsymbol{P}^n}(\mathscr{J}_X(e+n+1), \mathcal{O}_{\boldsymbol{P}^n})$ which leads to an extension

$$0 \rightarrow \mathcal{O}_{\boldsymbol{P}^n} \rightarrow \mathscr{E} \rightarrow \mathscr{J}_X(e+n+1) \rightarrow 0.$$

The assumptions on X and ε imply that $\mathscr{E}$ is a reflexive sheaf (cf. [7], Theorem 4.1 and [24], Theorem 2.2). For the Chern classes of $\mathscr{E}$ it holds according to [24], Proposition 2.4: $c_1(\mathscr{E})=e+n+1$ and $c_2(\mathscr{E})=\deg X$. For hort will denote the degree of X by d and put

$$s=\min\{t \in \boldsymbol{Z} \mid H^0(\mathscr{J}_X(t)) \neq 0\}.$$

The subscheme X is said to be *k-Buchsbaum* if

$$\mathfrak{m}^k H^j_*(\mathcal{J}_X)=0 \text{ for all } j \text{ with } 1 \leq j \leq n-2.$$

Using the exact sequence above it follows that X is k-Buchsbaum if and only if $\mathscr{E}$ has this property.

After these preparations we are ready to generalize [15], Theorem 3.1 to subschemes of arbitrary dimension ≥ 1. Regularity bounds using different invariants and/or assumptions can be found, for example, in [1], [8], [20].

Theorem 4.1. *Suppose X is k-Buchsbaum and $e<2s-(n+1)$. If we put $p=\left\lceil \frac{e+n+1}{2} \right\rceil$ and assume $p>0$, then it holds for all j with $1 \leq j \leq n-2$*

$$H^j(\mathcal{J}_X(t))=0 \text{ if } t \geq d-(e+n)p+p^2-2-j+(n-1-j)k,$$

unless $j=1$ and the restriction of X to a general linear space of dimension 3 is in the biliaison class of the union of two skew lines. In this case the above bound must be increased by 1.

Proof. As before consider the extension

$$0 \to \mathcal{O}_{\boldsymbol{P}^n} \to \mathscr{E} \to \mathcal{J}_X(e+n+1) \to 0.$$

Put $\mathscr{F}=\mathscr{E}(-p)$. Then $c_1(\mathscr{F})=c_1(\mathscr{E})-2p=e+n+1-2\left\lceil \frac{e+n+1}{2} \right\rceil \in \{-1,0\}$, i.e., $\mathscr{F}$ is a normalized reflexive sheaf. The assumption on e implies $e+n+1-p<s$. Therefore the exact sequence above yields that the sheaf $\mathscr{E}$ is k-Buchsbaum and

$$0=H^0(\mathcal{J}_X(e+n+1-p))=H^0(\mathscr{E}(-p))=H^0(\mathscr{F}),$$

thus $\mathscr{F}$ is stable. Since $c_2(\mathscr{F})=c_2(\mathscr{E})-c_1(\mathscr{E})p+p^2=d-(e+n+1)p+p^2$, and $H^j(\mathcal{J}_X(t+e+n+1-p))=H^j(\mathscr{F}(t))$ for $1 \leq j \leq n-2$, the claim follows from Theorem 3.3. Note that the curves in $\boldsymbol{P}^3$ which are zero-loci of sections of the null correlation bundle form the whole biliaison class of two skew lines.

Remark 4.2. If X is an integral subscheme, then the assumption $p>0$ is automatically satisfied. In fact, recall that the index of speciality of an arbitrary subscheme X of codimension 2 is

$$e'=e(H^{n-2}_*(\mathcal{O}_X)).$$

Hence Serre duality implies for a generically complete intersection X that $e' \geq e$. If X is integral, then equality holds and therefore $e+n+1$ is positive (cf. [9]).

Now we want to extend [15], Theorem 3.3. The result complements the previous one.

Theorem 4.3. *Suppose X is k-Buchsbaum and $e \geq 2s-(n+1)$. If we put*

$p=\left\lceil\frac{e+n+1}{2}\right\rceil$, *then it holds for all j with* $1\leq j\leq n-2$

$$H^j(\mathscr{J}_X(t))=0 \text{ if } t\geq d-(e+n+2-s)s+e+n-j+(n-1-j)k.$$

Proof. Again we consider the extension

$$0\to\mathcal{O}_{\boldsymbol{P}^n}\to\mathscr{E}\to\mathscr{J}_X(e+n+1)\to 0$$

and the sheaf $\mathscr{F}=\mathscr{E}(-p)$. Then $\mathscr{F}$ is a normalized k-Buchsbaum sheaf of rank 2. Moreover, it holds

$$H^0(\mathscr{F}^*(t+e+n+1-2p))=H^0(\mathscr{F}(t))=H^0(\mathscr{J}_X(t-p+e+n+1)).$$

Hence the assumption on e implies $H^0(\mathscr{F}^*(s-1-p))=0$ and $\mathrm{H}^0(\mathscr{F}^*(s-p))\neq 0$. Therefore $\mathscr{F}$ is unstable of order $p-s$ and Theorem 3.5 provides the claim.

We wish to point out that bounds for the vanishing of $H^j(\mathscr{J}_X(t))$ are also proved in [1]; these bounds, which are better than ours, are given under a stronger assumption on X.

5. Restriction results

In this section we study the restriction maps in low degrees. We define what we mean by "restriction property" and "strong restriction property" and establish sufficient conditions which ensure that a subscheme has these properties. In particular we first give sufficient conditions for a curve in $\boldsymbol{P}^3$ to have the restriction property, which are complementary to the known ones (cf. Lemma 5.5), and then we show that the strong restriction property for an equidimensional scheme X can be checked using a "general curve section" of X (cf. Theorem 5.11).

Let $X\subset\boldsymbol{P}^n$ be a subscheme of arbitrary codimension. We denote by $s(X)$ the integer

$$s(X)=\min\{t\in\boldsymbol{Z}\mid H^0(\mathscr{J}_X(t))\neq 0\}.$$

Suppose that the dimension of X is at least one. Then we say that X *has the restriction property* if it holds for a general hyperplane H that

$$s(X)=s(X\cap H).$$

A general hyperplane H gives rise to an exact sequence

$$0\to\mathscr{J}_X(-1)\to\mathscr{J}_X\to\mathscr{J}_{X\cap H}\to 0.$$

Taking global sections we get natural restriction maps

$$H^0(\mathscr{J}_X(t))\to H^0(\mathscr{J}_{X\cap H}(t)).$$

It follows that X has the restriction property if and only if these maps are surjective for all $t<s(X)$. In general the surjectivity for some integer t means that the hypersurfaces of degree t passing through $X \cap H$ can be lifted to hypersurfaces of degree t containing X.

The first sufficient conditions for the restriction property were given for integral curves in P^3 by Laudal. His lemma has been improved by Gruson and Peskine (cf. [5]). The proof in the "borderline cases" has been given by Strano (cf. [29]). There are other generalizations of Laudal's lemma due to Mezzetti (cf. [12] and [13]). For example, an integral surface $X \subset P^4$ of degree $d>s^2-s+2$ has the restriction property.

However, if $\dim X \geq 2$ and we take more than one hyperplane section of X new phenomena occur. Consider, for example, the quintic elliptic scroll $X \subset P^4$. It fits into an exact sequence

$$0 \to (\mathcal{O}_{P^4})^5 \to \Omega^2_{P^4}(3) \to \mathcal{J}_X(3) \to 0,$$

which shows that X does have the restriction property but its general hyperplane section does not have it. Thus we give the following definition which will be used later:

Definition 5.1. Let $X \subseteq P^n$ be a closed subscheme of dimension $m \geq 1$. We say that X has the *strong restriction property* if X and all its consecutive general hyperplane sections $X \cap H_1 \cap \cdots \cap H_i$, $(i<\dim X)$ have the restriction property. This is equivalent to say that $s(X)=s(X \cap L)$, where L is a general linear subspace of codimension m.

In order to investigate the restriction maps we will use the so-called Socle Lemma of Huneke and Ulrich. We need some notation. Let M be a finitely generated graded R-module. Then the socle of M, $\operatorname{soc}(M)$, is defined to be the annihilator $\operatorname{Ann}_M \mathfrak{m}$. It is always a finite-dimensional K-vector space. It can be trivial, but if M has finite length it is certainly not. Indeed, in this case $e(M)$ is finite and $[M]_{e(M)} \subset \operatorname{soc}(M)$.

We denote the initial degree of a module M by

$$a(M)=\min\{t \in Z \mid [M]_t \neq 0\}.$$

Then we can write $s(X)=a(I_X)$ where $I_X \subset R$ is the homogeneous ideal of X.

Now we can state the Socle Lemma (cf. [11], Corollary 3.11).

Lemma 5.2. *Let $l \in R$ be a general linear form and consider the exact sequence induced by multiplication by l:*

$$0 \to \ker \to M(-1) \xrightarrow{l} M \to \operatorname{coker} \to 0.$$

Then we have: If $\ker \neq 0$ *then* $a(\ker) > a(\operatorname{soc}(\operatorname{coker}))$.

The Socle Lemma may be viewed as the crucial step of a technique which is due to Strano and often called Strano's method. The following result is essentially due to Strano (cf. [28], Teorema 4) and follows quickly from the Socle Lemma (cf. [11]).

Corollary 5.3. *Let $C \subset \boldsymbol{P}^n$ be a curve and H a general hyperplane. If the restriction map $H^0(\mathscr{J}_C(t)) \to H^0(\mathscr{J}_{C \cap H}(t))$ is not surjective, then $[Tor^S_{n-2}(I_{C\cap H}, K)]_i \neq 0$ for some $i \leq t+n-1$, where S is the homogeneous coordinate ring of H and $I_{C\cap H} \subset S$ the homogeneous ideal of $C \cap H$.*

From this one can deduce the following form of Laudal's Lemma (cf. [29], Proposition 1 for details):

Lemma 5.4. *Let $C \subset \boldsymbol{P}^3$ be an integral curve of degree d. If the general hyperplane section of C is contained in a plane curve of degree b, then C is contained in a surface of degree b provided $d > b^2+1$ or $d = b^2+1$, $b \geq 4$ and C is not the zero-locus of a section of the null correlation bundle twisted by b.*

We want now to complement this statement by considering curves of degree 10.

Lemma 5.5. *Let $C \subset \boldsymbol{P}^3$ be an integral curve of degree 10 and genus $g \geq 8$. If the general hyperplane section of C is contained in a plane curve of degree 3, then C is contained in a surface of degree 3, unless C is the zero-locus of a section of the null correlation bundle twisted by 3.*

Proof. The proof shares some similarities and should be compared with the proof of [29], Proposition 1. Thus we will focus on the differences and refer to Strano's paper for more details of the remaining part.

Let $H \subset \boldsymbol{P}^3$ be a general hyperplane defined by the linear form $l \in R$. Put $Z = C \cap H$ and $S = R/lR$.

Suppose C is not contained in a cubic surface. Then Lemma 5.4 implies $s(Z) = 3$. Thus Z is contained in an irreducible cubic curve G because Z has the uniform position property. Moreover, since C has not the restriction property, the homogeneous ideal I_Z of Z must have a syzygy of degree ≤ 5 (cf. Corollary 5.3). Since G is irreducible this implies that I_Z has at least two minimal generators of degree 4. It follows that the Hilbert function of Z is 1 3 6 9 10 .... Now we use [2], Theorem 2.1 to conclude that I_Z has no minimal generator of degree ≥ 5. Altogether we get that Z has the following minimal free resolution

$$0 \to S(-6) \oplus S(-5) \to S^2(-4) \oplus S(-3) \to I_Z \to 0.$$

Moreover, we have $H^1(\mathscr{J}_Z(4)) = 0$ and thus $H^2(\mathscr{J}_C(3)) = 0$. Hence the Riemann-Roch theorem provides

$$h^0(\mathscr{J}_C(4)) = g - 6 + h^1(\mathscr{J}_C(4)).$$

According to our assumption $g \geq 8$ we see that C is contained in two quartic surfaces F_1, F_2. We use them in order to link C to a curve C' of degree 6. Then Z is linked by $F_1 \cap F_2 \cap H$ to $Z' = C' \cap H$. It follows that Z' has a free resolution as follows

$$0 \to S(-5) \oplus S^2(-4) \to S^2(-4) \oplus S(-3) \oplus S(-2) \to I_{Z'} \to 0.$$

This resolution is not minimal. In fact, since a quadric and a cubic can have at most one syzygy of degree 4 we conclude that $I_{Z'}$ has at most one minimal generator of degree 4. Thus there are at most two possibilities for the minimal free resolution of Z'.

First, let us assume that

$$0 \to S(-5) \to S(-3) \oplus S(-2) \to I_{Z'} \to 0$$

is the minimal resolution of Z'. Then Corollary 5.3 shows that the quadric containing Z' can be lifted to a quadric Q containing C'. Now we can follow Strano's argument and link the curve C' by the complete intersection, say $Q \cap F_1$, to a curve C'' which must have genus -1. Thus we obtain $g=11$ which implies that C is the zero-locus of a section of the null correlation bundle as claimed.

Second, let us assume that the minimal resolution of Z' is

$$0 \to S(-5) \oplus S(-4) \to S(-4) \oplus S(-3) \oplus S(-2) \to I_{Z'} \to 0.$$

We are seeking for a contradiction. Indeed, the fact that the quadric and cubic minimal generators of $I_{Z'}$ have a syzygy of degree 4 shows that they have a linear form h as common divisor. The ideal generated by h and the quartic minimal generator of $I_{Z'}$ define a subscheme of Z' of degree 4 which is contained in the line defined by h. Now we claim that C' contains a plane curve $D'$of degree 4. To show this we shall first prove that C' is contained in at least one cubic surface. By liaison C' has genus $g'=g-8 \geq 0$. Moreover $h^1(\mathscr{I}_{Z'}(3))=0$, whence $h^2(\mathscr{I}_{C'}(t)=0$ for $t \geq 2$. It follows:

$$\begin{aligned} h^0(\mathscr{I}_{C'}(3)) &= h^0(\mathcal{O}_{\mathbf{P}^3}(3)) - h^0(\mathcal{O}_{C'}(3)) + h^1(\mathscr{I}_{C'}(3)) \\ &\geq 20 - [\chi(\mathcal{O}_{C'}(3)) + h^1(\mathcal{O}_{C'}(3))] \\ &= 20 - (18 - g' + 1) \geq 1 \end{aligned}$$

Hence there exists at least one cubic surface F containing C'. Now, if H is a general hyperplane we have $F \cap H \supset C' \cap H \supset \Delta$, where Δ is a 0-dimensional subscheme of Z' of degree 4 on a line h. Then $F \cap H$ contains h, which implies that F contains a plane π (where $\pi \cap H = h$). Let D' be the curve obtained from $\pi \cap C'$ by removing the 0-dimensional components; since $H \cap D' = \Delta$, D' is the curve of degree 4 contained in C' we were seeking. It follows that D' is linked by $F_1 \cap F_2$ to a curve D which contains C. Since $\omega_{D'} \cong \mathcal{O}_{D'}$ (1) we obtain the exact sequence (cf., for example, [22], Lemma 3.5)

$$0 \to \mathscr{I}_{F_1 \cap F_2} \to \mathscr{I}_D \to \mathcal{O}_{D'}(-3) \to 0.$$

It follows immediately that D is contained in a cubic surface. Thus the same applies to C, which contradicts our assumption in the beginning of the proof.

Remark 5.6. It is not possible to skip the assumption on the genus in the last statement. The reason is that there is a sub-canonical curve of degree 10 and genus 6 which is not contained in a cubic surface, but whose general hyperplane section is contained in a plane cubic. This curve is the general hyperplane section of an abelian surface which is the zero-locus of a general section of $\mathscr{E}_{HM}(3)$ (cf. [10]), where $\mathscr{E}_{HM}$ denotes the Horrocks-Mumford bundle.

The preceding result sconcerning the restriction property can be generalized as follows:

Proposition 5.7. *Let $C \subset \boldsymbol{P}^3$ be an integral curve of degree d and genus g. Let $Z = C \cap H$ be its general hyperplane section. If C has degree $d \geq b^2+1$, then the restriction map $H^0(\mathscr{J}_C(t)) \to H^0(\mathscr{J}_Z t))$ is surjective for all $t \leq b$ unless $d = b^2+1$ and either*

C is the zero-locus of a section of the null correlation bundle twisted by b, or

$b=3$ and $g<8$, or

$b=2$.

Proof. If $H^0(\mathscr{J}_C(b))=0$, then Laudal's lemma 5.4 and Lemma 5.5 respectively imply $H^0(\mathscr{J}_Z(b))=0$ and we are done. Otherwise C is contained in an irreducible surface F of degree $\leq b$. Now we argue by contradiction. If the map $H^0(\mathscr{J}_C(t)) \to H^0(\mathscr{J}_Z(t))$ is not surjective for some $t \leq b$ then there is a plane curve G of degree t passing through Z such that G does not contain $F \cap H$. It follows that $G \cap F \cap H$ is a complete intersection of degree $\leq b \cdot t \leq b^2$ which contains Z. But this contradicts $\deg Z = \deg C \geq b^2+1$.

The preceding proposition gives sufficient conditions for a curve of codimension two to have the restriction property. The next lemma will be useful for concluding the strong restriction property of higher-dimensional schemes.

Lemma 5.8. *Let $X \subset \boldsymbol{P}^n$ be an equidimensional subscheme of dimension ≥ 2. Let $H, H' \subset \boldsymbol{P}^n$ be two general hyperplanes. Put $Y = X \cap H$ and $Z = Y \cap H'$. Let b be an integer. Then it holds: if the restriction map $H^0(\mathscr{J}_X(b)) \to H^0(\mathscr{J}_Y(b))$ is not surjective, then there exists $t \leq b$ such that the restriction map $H^0(\mathscr{J}_Y(t)) \to H^0(\mathscr{J}_Z(t))$ is not surjective.*

Proof. Consider the following two exact sequences

$$\begin{tikzcd}[column sep=small]
H^0_*(\mathscr{J}_X) \arrow[r] & H^0_*(\mathscr{J}_Y) \arrow[rr] \arrow[dr] & & H^1_*(\mathscr{J}_X)(-1) \arrow[r] & H^1_*(\mathscr{J}_X) \arrow[rr] \arrow[dr] & & H^1_*(\mathscr{J}_Y) \\
& & \ker \arrow[ur] \arrow[dr] & & & \operatorname{coker} \arrow[ur] \arrow[dr] & \\
& 0 \arrow[ur] & & 0 & 0 \arrow[ur] & & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^0_*(\mathscr{I}_Y) \arrow[r] & H^0_*(\mathscr{I}_Z) \arrow[rr] \arrow[dr] & & H^1_*(\mathscr{I}_Y)(-1) \arrow[r] & H^1_*(\mathscr{I}_Y) \arrow[rr] \arrow[dr] & & H^1_*(\mathscr{I}_Z) \\
& & \ker' \arrow[ur] \arrow[dr] & & & \operatorname{coker}' \arrow[ur] \arrow[dr] & \\
& 0 \arrow[ur] & & 0 & 0 \arrow[ur] & & 0
\end{tikzcd}$$

Our assumption means $a(\ker) \leq b$. Since X is equidimensional the module $H^1_*(\mathscr{I}_X)$ is finitely generated due to [22], Lemma 2.12. Therefore the Socle Lemma 5.2 yields the existence of a non-zero element $y \in \operatorname{coker}$ of degree $u < b$ such that $\mathfrak{m} \cdot y = 0$. Viewing y as an element of $H^1_*(\mathscr{I}_Y)$ this means in particular that y is annihilated by the linear form defining H'. Therefore we see that $y \in [\ker']_{u+1}$ and our assertion is proved because $t := u+1 \leq b$.

Let $X \subset \boldsymbol{P}^n$ be a subscheme of dimension $m \geq 1$. We call the intersection of X with a general linear space of codimension $m-1$ the *general curve section* of X. If $m=1$ it is understood that the general curve section of X coincides with X.

Corollary 5.9. *Let $X \subset \boldsymbol{P}^n$ be an equidimensional subscheme of dimension $m \geq 1$, and let C be its general curve section. Let Z be a general hyperplane section of C and assume that $s(Z)=s$ and that the restriction map*

$$H^0(\mathscr{I}_C(s)) \to H^0(\mathscr{I}_Z(s))$$

is surjective. Then X has the strong restriction property.

Proof. Put $X_0 := X$ and consider consecutive general hyperplane sections $X_1, \cdots, X_{m-1} = C, X_m = Z$ of X. Since by assumption the restriction maps $H^0(\mathscr{I}_C(t)) \to H^0(\mathscr{I}_Z(t))$ are surjective for all $t \leq s$, a repeated application of Lemma 5.8 implies that the restriction maps $H^0(\mathscr{I}_{X_i}(t)) \to H^0(\mathscr{I}_{X_{i+1}}(t))$ are surjective for $i=0, \cdots, m-1$ and $t \leq s$. In particular $s(X_i) = s(X_{i+1})$ for $i=0, \cdots, m-1$, which is our claim.

Remark 5.10. Corollary 5.9 is false if one assumes only that the general curve section has the restriction property: take for example X to be the Veronese surface in $\boldsymbol{P}^4$, or the Palatini scroll in $\boldsymbol{P}^5$ (see Lemma 6.10 below for details).

Let $X \subset \boldsymbol{P}^n$ be a subscheme of dimension $m \geq 1$. Then we can write its Hilbert polynomial as

$$p_X(t) = d \cdot \binom{m+t}{t} + h_{m-1} \cdot \binom{m-1+t}{t} + \cdots + h_0 \cdot \binom{t}{t}$$

with integers $h_0, \cdots, h_{m-1}$. Write $h_{m-1} = 1-d-g$. Then we call g the *sectional genus* of X, because it is the arithmetic genus of its general curve section. Using this notation we can state the main result of this section.

Theorem 5.11. *Let $X \subset \boldsymbol{P}^n, n \geq 3$ be an integral subscheme of codimension two and let C be its general curve section. Then X has the strong restriction property if one of the following conditions is satisfied*:

(a) $d > (s(X)-1)^2+1$;

(b) *C is not the zero-scheme of a section of $\mathcal{N}(s(X)-1)$, where $\mathcal{N}$ is the null correlation bundle on $\boldsymbol{P}^3$ and either*

  (b1) $d = (s(X)-1)^2+1$, $s(X) \geq 5$; *or*

  (b2) $d = 10$, $s(X) = 4$ *and the sectional genus of X is at least* 8.

Proof. According to Bertini's Theorem the general curve section C of X is integral, too. Hence the conclusion follows by Proposition 5.7 and Corollary 5.9.

Remark 5.12. Recall that a subscheme $X \subset \boldsymbol{P}^n$ of dimension m is said to be arithmetically Buchsbaum if $X \cap L$ is 1-Buchsbaum for any linear subspace L of dimension $> n-m$ which intersects X properly. It follows that an arithmetically Buchsbaum subscheme is 1-Buchsbaum. Note that the converse is, in general, only true for curves.

If a space curve C is the zero-scheme of a section of $\mathcal{N}(s-1)$ where $s = s(X)$, then C is arithmetically Buchsbaum and has sectional genus s^3-5s^2+8s-5. This information can be used in order to check if condition (b) of Theorem 5.11 is satisfied.

We conclude this section by showing that our Lemma 5.8 also allows to generalize a classical result of Roth [26], who considered surfaces of codimension two. A weaker version of our result was proved in [14].

Proposition 5.13. *Let $X \subset \boldsymbol{P}^n$ be an equidimensional subscheme of dimension ≥ 2. Let Y and Z be consecutive general hyperplane sections of X as in Lemma 5.8. Put $\sigma = a(H^0_*(\mathcal{J}_Y))$ and assume $h^0(\mathcal{J}_Z(\sigma)) = 1$. Then it holds $H^0(\mathcal{J}_X(\sigma)) \neq 0$.*

Proof. Consider the exact sequence

$$H^0(\mathcal{J}_Y(\sigma-1)) \to H^0(\mathcal{J}_Y(\sigma)) \to H^0(\mathcal{J}_Z(\sigma)).$$

Since $h^0(\mathcal{J}_Z(\sigma)) = 1$ we get by the definition of σ that $h^0(\mathcal{J}_Y(\sigma)) = 1$. If we had $a(H^0_*(\mathcal{J}_Z)) < \sigma$ it would follow $h^0(\mathcal{J}_Z(\sigma)) > 1$. Thus we have $a(H^0_*(\mathcal{J}_Z)) = \sigma$.

Now assume $H^0(\mathcal{J}_X(\sigma)) = 0$. Then the restriction map $H^0(\mathcal{J}_X(\sigma)) \to H^0(\mathcal{J}_Y(\sigma))$ is not surjective, hence Lemma 5.8 says that there is an integer $t \leq \sigma$ such that the map $H^0(\mathcal{J}_Y(t)) \to H^0(\mathcal{J}_Z(t))$ is not surjective. Since $H^0(\mathcal{J}_Z(j)) = 0$ if $j < \sigma$, we must have $t = \sigma$. Using $a(H^0_*(\mathcal{J}_Y)) = \sigma$ we obtain the following exact sequence

$$0 \to H^0(\mathcal{J}_Y(\sigma)) \to H^0(\mathcal{J}_Z(\sigma)) \to M \to 0$$

where M is a non-trivial K-vector space. But this contradicts the fact $h^0(\mathcal{J}_Y(\sigma)) = h^0(\mathcal{J}_Z(\sigma)) = 1$ and we are done.

6. Subschemes of codimension two with the strong restriction property

In this section we consider k-Buchsbaum subschemes which are a generically complete intersection of codimension 2. To these subschemes, in particular to the integral ones, Theorem 4.1 applies. We show that the bounds of 4.1 can be improved if the schemes have the strong restriction property. Then we describe two smooth surfaces which give rise to infinite series of examples where the new bound is sharp.

We use the notation of Section 4. Recall that e' denotes the index of speciality of X and that $e \leq e'$ is the largest integer such that $\omega_X(-e)$ has a section which gives rise to a reflexive sheaf. Recall also that $e=e'$ if X is integral.

Our first main result is the following.

Theorem 6.1. *Let $X \subseteq \boldsymbol{P}^n$ be an integral k-Buchsbaum subscheme of codimension 2, with sectional genus g. Assume that $s \geq p := \left\lceil \frac{e+n+1}{2} \right\rceil$. Suppose also that one of the following conditions is verified:*

(a) $d > (s-1)^2 + 1$

(b) $d = (s-1)^2 + 1$, $s \geq 5$ *and* $g \neq s^3 - 5s^2 + 8s - 5$

(c) $d = 10$, $s = 4$ *and* $g \geq 8$, $g \neq 11$.

Then it holds, for all j with $1 \leq j \leq n-2$, *that*:

$$H^j(\mathscr{I}_X(t)) = 0 \text{ if } t \geq d - (e+n)p + p^2 - 1 - j + (n-1-j)k - q^2$$

if $e+n+1$ *is even where* $q = \min\{s-p, p\}$, *and*

$$H^j(\mathscr{I}_X(t)) = 0 \text{ if } t \geq d - (e+n)p + p^2 - 2 - j + (n-1-j)k - q^2 + q$$

if $e+n+1$ *is odd where* $q = \min\{s-p+1, p\}$.

This theorem will be a consequence of a more general result. Before we state it we want to describe some examples where the bounds of Theorem 6.1 are attained.

Example 6.2. There is a smooth rational surface $X \subset \boldsymbol{P}^4$ of degree 10 and sectional genus 9 (cf. [4], example B1.15) which fits into an exact sequence

$$0 \to (\Omega^3_{\boldsymbol{P}^4}(-1))^2 \xrightarrow{\varphi} \mathcal{O}_{\boldsymbol{P}^4}(-4) \oplus (\Omega^1_{\boldsymbol{P}^4}(-3))^2 \to \mathscr{I}_X \to 0.$$

Thus we obtain that X is 1-Buchsbaum with $s=4$, $e=e'=-1$. Then we can apply 6.1 and get $e(H^1_*(\mathscr{I}_X)) \leq 3$ and $e(H^2_*(\mathscr{I}_X)) \leq 1$. Both estimates are the best possible because $h^1(\mathscr{I}_X(3)) = 2$ and $h^2(\mathscr{I}_X(1)) = 2$. Note that X is not arithmetically Buchsbaum according to [23].

Example 6.3. There is a smooth elliptic surface $X \subset \boldsymbol{P}^4$ of degree 10 and sectional genus 10 (cf. [4], Example B7.5) which admits an exact sequence

$$0 \to (\mathcal{O}_{\boldsymbol{P}^4}(-5))^2 \oplus \Omega^3_{\boldsymbol{P}^4}(-1) \xrightarrow{\varphi} (\mathcal{O}_{\boldsymbol{P}^4}(-4))^3 \oplus \Omega^1_{\boldsymbol{P}^4}(-3) \to \mathscr{I}_X \to 0.$$

This sequence implies that X is 1-Buchsbaum, but not arithmetically Buchsbaum (cf. [23]) and $s=4$, $e'=e=0$, $e(H^1_*(\mathscr{I}_X))=3$, $e(H^2_*(\mathscr{I}_X))=1$. Hence the estimates provided by Theorem 6.1 are optimal.

Remark 6.4. Observe, that the bounds in Theorem 6.1 are optimal for the general hyperplane sections of the surfaces described in the two examples above. These curves are 2-Buchsbaum but not 1-Buchsbaum since otherwise the surfaces were arithmetically Buchsbaum. Observe, that one could also use Theorem 6.1 to conclude that these curves are not 1-Buchsbaum. In fact, if they had this property they would violate the bound in Theorem 6.1.

The following result will imply Theorem 6.1.

Theorem 6.5. *Let $X \subseteq \boldsymbol{P}^n$ be a k-Buchsbaum subscheme of codimension 2 and generically complete intersection. Assume that $s \geq p := \left\lceil \frac{e+n+1}{2} \right\rceil$ and that X has the strong restriction property. Then it holds, for all j with $1 \leq j \leq n-2$, that:*

$$H^j(\mathscr{I}_X(t))=0 \text{ if } t \geq d-(e+n)p+p^2-1-j+(n-1-j)k-q^2$$

if $e+n+1$ is even and $p \geq 0$, where $q=\min\{s-p,p\}$;

$$H^j(\mathscr{I}_X(t))=0 \text{ if } t \geq d-(e+n)p+p^2-2-j+(n-1-j)k-q^2+q$$

if $e+n+1$ is odd and $p \geq 1$, where $q=\min\{s-p+1,p\}$.

Proof. The proof is very similar to the one of Theorem 4.1. Thus we will concentrate on the differences which are caused by using the restriction property of X. We distinguish two cases.

First, let us assume that $e+n+1=2p$. According to the assumption on X there is a global section $\varepsilon \in H^0(\omega_X(-e))$ which provides an extension

$$0 \to \mathcal{O}_{\boldsymbol{P}^n}(-p) \to \mathscr{F} \to \mathscr{I}_X(p) \to 0$$

where $\mathscr{F}$ is a reflexive sheaf with first Chern class $c_1=0$. Let $L \subset \boldsymbol{P}^n$ be a general plane and put $Z=X \cap L$. Then a restriction of the extension above gives the exact sequence

$$0 \to \mathcal{O}_L(-p) \to \mathscr{F}_L \to \mathscr{I}_Z(p) \to 0.$$

Let $q=\min\{j \in \boldsymbol{Z} \mid H^0(\mathscr{F}_L(j)) \neq 0\}$. Since X has the strong restriction property we have

$$H^0(\mathscr{I}_Z(s-1))=0 \text{ and } H^0(\mathscr{I}_Z(s)) \neq 0.$$

Therefore the last sequence implies

$$q=\min\{s-p,p\} \geq 0.$$

Let $c_2 = c_2(\mathscr{F})$. According to [6], Theorem 7.4(a), it holds

$$H^1(\mathscr{F}_L(j)) = 0 \text{ if } j \geq c_2 - q^2 - 1.$$

Serre's duality provides

$$H^2(\mathscr{F}_L(j)) \cong H^0(\mathscr{F}_L^*(-3-j)) \cong H^0(\mathscr{F}_L(-3-j))$$

whence $H^2(\mathscr{F}_L(j)) = 0$ if $j \geq -q-2$. It follows that $\operatorname{reg}(\mathscr{F}_L) \leq \max\{-q, c_2 - q^2\}$. By the above bounds we have

$$H^0(\mathscr{F}_L(q-1)) = H^2(\mathscr{F}_L(q-1))$$

whence $\chi(\mathscr{F}_L(q-1)) = -h^1(\mathscr{F}_L(q-1)) \leq 0$. By the Riemann-Roch formula (cf. [6], p. 242) we get $c_2 \geq q^2 + q$, and thus, because of $q \geq 0$,

$$\operatorname{reg}(\mathscr{F}_L) \leq \max\{-q, c_2 - q^2\} = c_2 - q^2.$$

Now Corollary 2.3 provides, as in the proof of Theorem 3.3,

$$j + 1 + e(H^j(\mathscr{F})) \leq c_2 - q^2 - 1 + (n - j - 1)k.$$

Our claim follows because $H^j(\mathscr{J}_X(t)) \cong H^j(\mathscr{F}(t-p))$ if $1 \leq j \leq n-2$.

Second, we assume $e + n + 1 = 2p - 1$. Then there is an extension

$$0 \to \mathcal{O}_{\mathbf{P}^n}(-p) \to \mathscr{F} \to \mathscr{J}_X(p-1) \to 0$$

where $\mathscr{F}$ is a reflexive sheaf with first Chern class $c_1 = -1$. Define L and Z as before. Restricting the sequence to L we obtain

$$q := \min\{j \in \mathbf{Z} \mid H^0(\mathscr{F}_L(j)) \neq 0\} = \min\{s - p + 1, p\} \geq 1.$$

Hence [6], Theorem 7.4(b), yields

$$H^1(\mathscr{F}_L(j)) = 0 \text{ if } j \geq c_2 - q^2 + q - 1.$$

As above, using Serre's duality and Riemann-Roch we get $c_2 \geq q^2$, whence

$$\operatorname{reg}(\mathscr{F}_L) \leq \max\{-q+1, c_2 - q^2 + q\} = c_2 - q^2 + q$$

and we conclude as in the first case.

Theorem 6.1 is implied by Theorem 6.5, because its numerical assumptions ensure that the strong restriction property is verified (cf. Theorem 5.11) and moreover $p \geq 1$ (see Remark 4.2).

Remark 6.6. Let us compare the above theorems with Theorem 4.1. It turns out that the stronger assumptions of Theorems 6.5 and 6.1 allow us to improve the bounds of Theorem 4.1 by $q^2 - 1$ if $e + n + 1$ is even and $q^2 - q$ otherwise.

Next, we want to show that we can use the surfaces in Examples 6.2 and 6.3

in order to obtain infinitely many surfaces where the bounds of Theorem 6.5 are attained. For this we need:

Lemma 6.7. *Let $X \subset \boldsymbol{P}^n$ be a generically complete intersection such that $s \leq e+n+1$. Let $\mathscr{E}$ be the reflexive sheaf associated to a suitable section of $\omega_X(-e)$, i.e., $\mathscr{E}$ fits into an exact sequence*

$$(*) \qquad 0 \to \mathcal{O}_{\boldsymbol{P}^n} \to \mathscr{E} \to \mathscr{J}_X(e+n+1) \to 0.$$

Assume that $e = e'(X)$ (e.g. X integral). Let Y be a subscheme of codimension 2 which is the zero-locus of a global section of $\mathscr{E}(t)$ for some $t>0$. Then it holds

$$e(Y) = e'(Y) = e+2t$$

$$s(Y) = s+t.$$

Moreover, X is k-Buchsbaum if and only if Y is k-Buchsbaum and X has the strong restriction property if and only if Y does.

If t is sufficiently large then we may assume that Y is an integral scheme.

Proof. Since $c_1(\mathscr{E}(t)) = c_1(\mathscr{E}) + 2t = e+n+1+2t$, there is an exact sequence

$$(+) \qquad 0 \to \mathcal{O}_{\boldsymbol{P}^n} \to \mathscr{E}(t) \to \mathscr{J}_Y(e+n+1+2t) \to 0.$$

The sequence $(*)$ implies $e(H^{n-1}_*(\mathscr{E})) \leq -n-1$ and $e(H^n_*(\mathscr{E})) = -n-1$. Hence the sequence $(+)$ provides $e((H^{n-1}_*(\mathscr{J}_Y)) = e+2t$ proving the first claim.

Now let $L \subset \boldsymbol{P}^n$ be a general plane and let Z be the zero-dimensional scheme $X \cap L$. Since by assumption $s-e-n-1 \leq 0$ we obtain using the sequences $(*)$ and $(+)$

$$H^0(\mathscr{E}(s-e-n-1)) \neq 0 \text{ and } H^0(\mathscr{E}(s-e-n-2)) = 0$$

and thus $s(Y) = s+t$, establishing the second claim.

Similarly, we get that $s(Z) = s(Y \cap L) + t$ if either X or Y has the strong restriction property. The fourth claim follows.

Our two exact sequences also provide for j with $1 \leq j \leq n-2$:

$$H^j_*(\mathscr{J}_X) \cong H^j_*(\mathscr{J}_Y)(t),$$

thus the third assertion. The last one is a consequence of [25], Teorema 3.

Example 6.8. (i) We want to apply the last result to the surface X considered in Example 6.2. Let $\mathscr{E}$ be the reflexive sheaf associated to a general section of $\omega_X(-e)$ and let Y_t, $(t>0)$, be the zero-locus of a general section of $\mathscr{E}(t)$. X meets the assumptions of Lemma 6.7. The latter ensures that Theorem 6.5 applies to the surface Y_t and gives bounds which exceed the bounds for X by t. Since

$$H^j_*(\mathscr{J}_X) \cong H^j_*(\mathscr{J}_{Y_t})(t),$$

and the bounds are optimal for X the bounds for Y_t provided by Theorem 6.5 are also best possible for all $t>0$.

(ii) Repeating the construction of the surfaces Y_t as above but starting with the surface X described in Example 6.3 we get another infinite series of surfaces where all the bounds of Theorem 6.5 are attained.

Thus we have just seen that Lemma 6.7 can be used to construct infinitely many schemes where the bounds of Theorem 6.5 are optimal, provided we know one such example.

Moreover, we want to point out that the bounds in Theorems 6.1 and 6.5 improve the bounds which were known before in the case of curves. Indeed, in the case of integral curves the bounds in Section 4 agree with the ones shown in [15]. But these bounds are improved in this section and optimal for the hyperplane sections of the curves Y_t of the privious example (cf. Remark 6.4).

Finally, we want to point out that the assumption on the strong restriction property cannot be dropped in Theorem 6.5.

Remark 6.9. If $X \subseteq \boldsymbol{P}^3$ is a quintic elliptic curve, we have $e=0$, $s=3$ and hence the bound for $H^1_*(\mathscr{J}_X)$ given by Theorem 6.5 is 1, while $h^1(\mathscr{J}_X(1)) \neq 0$. This is due to the fact that X does not have the restriction property, because the general hyperplane section of X lies obviously on a conic, and an irreducible quadric cannot contain a curve of degree 5 and genus 1.

In order to get further examples related to Theorem 6.5 we need the following result.

Lemma 6.10. *Let $n \geq 4$ and $r \geq n-2$ be two integers, and let $\varphi: \mathcal{O}^{n-2}_{\boldsymbol{P}^n} \oplus \mathcal{O}_{\boldsymbol{P}^n}(n-r-2) \to \Omega_{\boldsymbol{P}^n}(2)$ be a general morphism. Then its degeneracy locus is a subscheme $X \subseteq \boldsymbol{P}^n$ whose ideal sheaf fits into the exact sequence*

$$(*) \qquad 0 \to \mathcal{O}^{n-2}_{\boldsymbol{P}^n} \oplus \mathcal{O}_{\boldsymbol{P}^n}(n-r-2) \xrightarrow{\varphi} \Omega_{\boldsymbol{P}^n}(2) \to \mathscr{J}_X(r+1) \to 0.$$

Moreover X has the following properties:

(a) *X is integral of codimension 2, and is smooth if and only if $n=4$, 5;*

(b) $\deg(X) = r^2 - (n-3)r + 1 + \frac{(n-2)(n-3)}{2}$, $s(X) = r+1$;

(c) *for $1 \leq j \leq n-2$ it holds: $h^j(\mathscr{J}_X(t)) = 1$ if $(j,t) = (1, r-1)$, and $h^j(\mathscr{J}_X(t)) = 0$ otherwise;*

(d) $e(X) = 2(r-n+1)$;

(e) *if $L \subseteq \boldsymbol{P}^n$ is a general linear space of codimension c, $1 \leq c \leq n-2$, then $s(X \cap L) = r$ and $h^0(\mathscr{J}_{X \cap L}(r)) = c$;*

(f) *if L is as in (e), then for $1 \leq j \leq n-2-c$ it holds: $h^j(\mathscr{J}_{X \cap L}(t)) = 1$ if $(j,t) = (1, r-1)$ and $h^j(\mathscr{J}_{X \cap L}(t)) = 0$ otherwise.*

In particular, X is arithmetically Buchsbaum and does not have the restriction property, *while all its subsequent general hyperplane sections have the strong restriction property.*

Remark 6.11. If $n=4, 5$ the above proposition gives two well known subschemes, namely the Veronese surface in $\boldsymbol{P}^4$ and the Palatini scroll in $\boldsymbol{P}^5$.

Proof. The major part of this lemma is "well-known" (see e.g. the introduction of [13]), but no explicit proof seems to be available. So for the sake of completeness we sketch a proof here.

First of all we observe that $X \neq \emptyset$ (cf., for example,[22], proposition 6.3). Once we know that X exists, we get the exact sequence $(*)$. It implies $\dim X = n-2$, (c) (hence (f)) and that X is arithmetically Buchsbaum, thus equidimensional.

Moreover from $(*)$ and [3], Example 2.1 we have that X is smooth if and only if $n=4,5$, and $\dim(Sing\, X) \leq n-4$ for $n \geq 6$; and since X is equidimensional of codimension 2 it follows easily that X is integral. This completes the proof of (a).

From (a) we get that $e(X) = e(H_*^{n-1}(\mathscr{I}_X))$, and a straightforward calculation using $(*)$ yields (d).

Using repeatedly the isomorphism $(\Omega_{\boldsymbol{P}^m})_H \cong \Omega_{\boldsymbol{P}^{m-1}} \oplus \mathcal{O}_{\boldsymbol{P}^{m-1}}(-1)$ where H is a hyperplane, it is easy to prove (e).

Finally if $Z := X \cap L$ where $\dim L = 2$, we have an exact sequence

$$(**) \qquad 0 \to \mathcal{O}_{\boldsymbol{P}^2}^{n-2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(n-r-2) \to \Omega_{\boldsymbol{P}^2}(2) \oplus \mathcal{O}_{\boldsymbol{P}^2}(1)^{n-2} \to \mathscr{I}_Z(r+1) \to 0.$$

Then $\dim Z = 0$, and hence

$$\deg X = \deg Z = \binom{r+3}{3} - \chi(\mathscr{I}_Z(r+1)).$$

Now $\chi(\mathscr{I}_Z(r+1))$ can be computed from $(**)$, and a straight forward calculation produces (b).

Remark 6.12. (i) Let $n \geq 4$, $r \geq n-2$ and let $X \subseteq \boldsymbol{P}^n$ be a subscheme as in Lemma 6.10. The bound for the vanishing of $H^1(\mathscr{I}_X(t))$ given in 6.5 is equal to $r-1$, while $H^1(\mathscr{I}_X(r-1)) \neq 0$. This is due to the fact that X does not have the strong restriction property. Hence for all n this assumption cannot be avoided in Theorem 6.5.

(ii) If X is as in (i), the general curve section Y of X is integral and has the restriction property. Then the bound of 6.5 for H_*^1 applies, and it is sharp for all these curves Y (to simplify the calculation of the bound recall that $e+n+1 = c_1(\mathscr{E})$, where $\mathscr{E}$ is the reflexive sheaf associated to X. Since c_1 does not change by restrictiong to a general hyperplane, the only terms in the bounds of 6.5 which change when restricting to a general hyperplane is $n-q^2$ if $e+n+1$ is even, and $n-q^2+q$, if $e+n+1$ is odd. Note also that q decreases by 1 when passing from X to a general hyperplane section, and remains the same by passing to further sections).

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