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Homotopy-commutativity in spinor groups

By

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1. Introduction

For two subsets S and S' of a topological group G which contain the unit of G as its base points, we say S and S' homotopy-commute in G, when the commutator map c from $S \wedge S'$ to G which maps $(x, y) \in S \wedge S'$ to $xyx^{-1}y^{-1} \in G$ is null homotopic.

In [3], the first author showed the next theorem:

Theorem 1.1. *Let n, m be positive integers, and let $n + m \neq 4$ or 8. If n or m is even or if $\binom{n+m-2}{n-1} \equiv 0 \bmod 2$ then $SO(n)$ and $SO(m)$ do not homotopy-commute in $SO(n+m-1)$.*

In this paper, we describe the homotopy-commutativity of $Spin(n)$ and $Spin(m)$ in $Spin(n+m-1)$.

Definition 1.2. If $SO(n)$ and $SO(m)$ homotopy-commute in $SO(n+m-1)$, we say (n, m) is SO-irregular, and if not we say (n, m) is SO-regular. Also, If $Spin(n)$ and $Spin(m)$ homotopy-commute in $Spin(n+m-1)$, we say (n, m) is Spin-irregular, and if not we say (n, m) is Spin-regular.

Main theorems are the followings:

Theorem 1.3. *Assume neither $n-1$ nor $m-1$ is a power of 2 and $n+m \neq 4$ or 8. If n or m is even or if $\binom{n+m-1}{n-1} \equiv 0 \bmod 2$ then (n, m) is Spin-regular.*

For the case $n - 1$ is a power of 2, we give some results as following:

Theorem 1.4. *Set $n = 3$ and $m \equiv 1 \bmod 4$ then $(3, m)$ is Spin-irregular.*

Remark 1.5. Theorem 1.1 implies that if $m \not\equiv 1 \bmod 4$, $(3, m)$ is SO-regular.

Remark 1.6. In fact, since $Spin(5) \cong Sp(2)$ and $\pi_6(Sp(2)) \cong \pi_6(\mathbf{Sp}) \cong \widetilde{KSp}^{-7}(\mathrm{pt}) \cong 0$ where $\mathbf{Sp}$ is $\lim_{n\to\infty} Sp(n)$, the commutator map $c : Spin(3) \wedge Spin(3) \to Spin(5)$ is null homotopic and $(3, 3)$ is Spin-irregular. On the other

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hand, Theorem 1.1 implies (3, 3) is SO-regular. Therefore SO-regularity and Spin-regularity is not the same.

This paper is organized as follows: In §2 we give a sufficient condition for (n, m) to be Spin-regular, which is an existence of a map with an adequate property and show that, when $n+m$ is odd, (n, m) is Spin-regular. In §3 we introduce the maps $\phi_{i,j} : \Omega^i\mathbf{BO} \wedge \Omega^j\mathbf{BO} \to \Omega^{i+j}\mathbf{BO}$ to investigate $\widetilde{KO}^{-*}(Spin(n) \wedge Spin(m))$ and in §4 investigate its induced cohomology maps and prove Theorem 1.3 for the case both n and m are odd. In §5 we look into the case n and m are even and complete the proof of Theorem 1.3 and finally in §6 we give the proof of Theorem 1.4.

2. Lift of commutator map

Similarly to [3], consider the next fibrations:

$$Spin(n+m-1) \xrightarrow{i} \mathbf{Spin} \xrightarrow{p} \mathbf{Spin}/Spin(n+m-1),$$

$$SO(n+m-1) \xrightarrow{j} \mathbf{SO} \xrightarrow{q} \mathbf{SO}/SO(n+m-1),$$

where $\mathbf{SO}$ (resp. $\mathbf{Spin}$) is $\lim_{n\to\infty} SO(n)$ (resp. $\lim_{n\to\infty} Spin(n)$).

We refer to the cohomology rings of spaces which we use in this paper, that is,

$$\mathrm{H}^*(\Omega\,\mathbf{Spin}) = \mathbf{Z}/2\mathbf{Z}[\alpha_2, \alpha_4, \alpha_6, \dots]/(\alpha_{4k} - \alpha_{2k}^2),$$

$$\mathrm{H}^*(Spin(k)/Spin(k-1)) = \Delta(x_{k-l}, \dots, x_{k-1}),$$

$$\mathrm{H}^*(Spin(k)) = \Delta(x_3, x_5, x_6, x_7, x_9, \dots) \otimes \bigwedge(z).$$

In the last equality, the index i of x_i scans all integers neither of which is not a power of 2 and $3 \le i \le k-1$. Also, $\deg(\alpha_{2i}) = 2i$ and $\deg(x_i) = i$.

Further, it can be easily seen that $\mathrm{H}^*(\Omega\,\mathbf{Spin}/Spin(n+m-1)) = 0$ for $* \le n+m-3$ and $\mathrm{H}^{n+m-2}(\Omega\,\mathbf{Spin}/Spin(n+m-1)) = \mathbf{Z}/2\mathbf{Z}$ whose generator is written as α_{n+m-2}. When $n+m$ is even, $\Omega p^*(\alpha_{n+m-2}) = \alpha_{n+m-2} \in \mathrm{H}^*(\Omega\,\mathbf{Spin})$.

From above fibrations, we can deduce the following fibration sequences.

$$\cdots \longrightarrow \Omega\,\mathbf{Spin} \xrightarrow{\Omega p} \Omega(\mathbf{Spin}/Spin(n+m-1)) \xrightarrow{\delta_{Spin}}$$

$$Spin(n+m-1) \xrightarrow{i} \mathbf{Spin} \xrightarrow{p} \mathbf{Spin}/Spin(n+m-1),$$

$$\cdots \longrightarrow \Omega\mathbf{SO} \xrightarrow{\Omega q} \Omega(\mathbf{SO}/SO(n+m-1)) \xrightarrow{\delta_{SO}}$$

$$SO(n+m-1) \xrightarrow{j} \mathbf{SO} \xrightarrow{q} \mathbf{SO}/SO(n+m-1).$$

Let c_{SO} (resp. c_{Spin}) be the commutator map from $SO(n) \wedge SO(m)$ to $SO(n+m-1)$ (resp. from $Spin(n) \wedge Spin(m)$ to $Spin(n+m-1)$). Obviously we can see that $i \circ c_{Spin}$ and $j \circ c_{SO}$ are null homotopic. Thus there exists a lift of c_{SO} from $SO(n) \wedge SO(m)$ to $\Omega\mathbf{SO}/SO(n+m-1)$ and a lift of c_{Spin} from $Spin(n) \wedge Spin(m)$ to $\Omega\,\mathbf{Spin}/Spin(n+m-1)$.

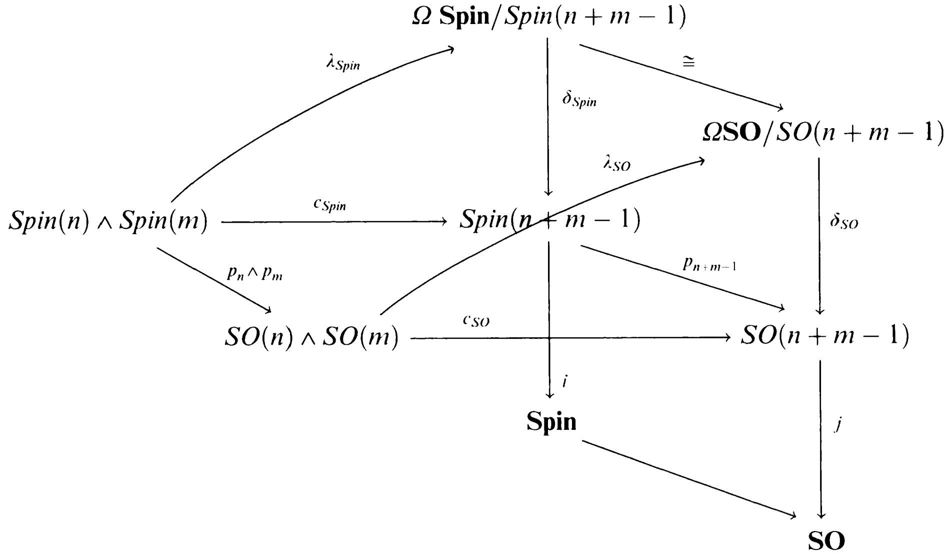
In [4], a lift of c_{SO} written as λ_{SO} was constructed and in [3], it is obtained that

$$\lambda_{SO}^{*}(\alpha_{n+m-2}) = x_{n-1} \otimes x_{m-1}. \tag{1}$$

Here set $\lambda_{Spin} = \lambda_{SO} \circ (p_n \wedge p_m)$.

Lemma 2.1. *λ_{Spin} is a lift of c_{Spin}, that is, $\delta_{Spin} \circ \lambda_{Spin} \simeq c_{Spin}$.*

Proof. See the diagram below.



Since $\delta_{SO} \circ \lambda_{SO} \simeq c_{SO}$ and $\delta_{SO} \simeq p_{n+m-1} \circ \delta_{Spin}$, it occurs that

$$\begin{aligned} p_{n+m-1} \circ \delta_{Spin} \circ \lambda_{Spin} &= \delta_{SO} \circ \lambda_{SO} \circ (p_n \wedge p_m) \\ &\simeq c_{SO} \circ (p_n \wedge p_m) \\ &= p_{n+m-1} \circ c_{Spin} \end{aligned} \tag{2}$$

Now consider the fibration $\mathbf{Z}/2\mathbf{Z} \to Spin(n+m-1) \to SO(n+m-1)$. Then for any CW complex X we have the exact sequence of base pointed homotopy sets:

$$[X, \mathbf{Z}/2\mathbf{Z}]_* \longrightarrow [X, Spin(n+m-1)]_* \xrightarrow{p_{n+m-1*}} [X, SO(n+m-1)]_*.$$

Thus p_{n+m-1_*} is injective and from (2) we can see

$$\delta_{Spin} \circ \lambda_{Spin} \simeq c_{Spin}.$$

In the rest of paper, c, λ, δ stands for c_{Spin}, λ_{Spin}, δ_{Spin} respectively.

Proposition 2.2. *Assume neither $n-1$ nor $m-1$ is a power of 2.*
1. *If $n+m$ is odd, c is not null homotopic.*
2. *Let $n+m$ is even. If for any continuous map x from $Spin(n) \wedge Spin(m)$ to $\Omega\,\mathbf{Spin}$, $x^*(\alpha_{n+m-2}) \neq x_{n-1} \otimes x_{m-1}$ in cohomology, then c is not null homotopic.*

Proof. If c is null homotopic, that is, $\delta \circ \lambda \simeq *$, then there exists a map $x : Spin(n) \wedge Spin(m) \to \Omega\,\mathbf{Spin}$ such that $\Omega p \circ x \simeq \lambda$.

From (1) we can see

$$\begin{aligned} x^*(\alpha_{n+m-2}) &= x^* \circ \Omega p^*(\alpha_{n+m-2}) \\ &= \lambda^*(\alpha_{n+m-2}) \\ &= (p_n \wedge p_m)^* \circ \lambda_{SO}^*(\alpha_{n+m-2}) \\ &= (p_n \wedge p_m)^*(x_{n-1} \otimes x_{m-1}) \\ &= x_{n-1} \otimes x_{m-1}, \end{aligned} \tag{3}$$

since neither $n-1$ nor $m-1$ is a power of 2. Thus the statement for the case $n+m$ is even is proved.

When $n+m$ is odd, it occurs that

$$\begin{aligned} \lambda^*(\alpha_{n+m-2}) &= x^* \circ \Omega p^*(\alpha_{n+m-2}) \\ &= x^*(0), \end{aligned}$$

since $H^*(\Omega\,\mathbf{Spin})$ is concentrated in even degrees. This contradicts to (3) and c is not null homotopic.

3. $\widetilde{KO}^{-*}(Spin(n) \wedge Spin(m))$

In this section we assume that both n and m are odd.

From Proposition 2.2 we should look into the homotopy set $[Spin(n) \wedge Spin(m), \Omega\,\mathbf{Spin}]$. By use of KO-theory we can say that,

$$[Spin(n) \wedge Spin(m), \Omega\,\mathbf{Spin}] \cong [Spin(n) \wedge Spin(m), \Omega_0\mathbf{SO}] \cong \widetilde{KO}^{-2}(Spin(n) \wedge Spin(m)),$$

since $\Omega^2\mathbf{BO} \cong \Omega\mathbf{SO}$.

Further more, the complex representation ring of $Spin(2k+1)$ is generated by real representations or symplectic representations. (See Proposition 6.19 in P. 290 of [8].) Thus Theorem 5.12. in [11] implies that, when n is odd, $KO^{-*}(Spin(n))$ is $KO^{-*}(\mathrm{pt})$ free. Therefore we have an decomposition of

$$\widetilde{KO}^{-*}(Spin(n) \wedge Spin(m)) \cong \widetilde{KO}^{-*}(Spin(n)) \otimes_{\widetilde{KO}^{-*}(\mathrm{pt})} \widetilde{KO}^{-*}(Spin(m)).$$

From now on, we identify $\widetilde{KO}^{-i}(X)$ with $[X, \Omega^i\mathbf{BO}]$.

Theorem 3.1. *There is a map* $\phi_{i,j} : \Omega^i\mathbf{BO} \wedge \Omega^j\mathbf{BO} \to \Omega^{i+j}\mathbf{BO}$ *such that for any CW-complexes* X, X' *and* $\alpha \in \widetilde{KO}^{-i}(X)$ *and* $\beta \in \widetilde{KO}^{-j}(X')$,

$$\alpha \hat{\otimes} \beta = \phi_{i,j} \circ (\alpha \wedge \beta) \qquad \text{in } \widetilde{KO}^{-(i+j)}(X \wedge X').$$

Proof. First we construct $\phi_{i,j}$. Let ξ_n be the universal vector bundle over $BO(n)$ and put $\eta_n = \xi_n - n$, $\eta_\infty = \lim_{n\to\infty} \eta_n$. And set $\phi_{0,0} : \mathbf{BO} \wedge \mathbf{BO} \to \mathbf{BO}$ as the classifying map of $\eta_\infty \hat{\otimes} \eta_\infty$. Let $\kappa_i : \Sigma^i\Omega^i\mathbf{BO} \to \mathbf{BO}$ be the map which satisfies

$$\mathrm{Ad}^i\kappa_i \simeq \mathrm{Id}_{\Omega^i\mathbf{BO}}.$$

Consider the composition of $\kappa_i \wedge \kappa_j$ and $\phi_{0,0}$:

$$\Sigma^i\Omega^i\mathbf{BO} \wedge \Sigma^j\Omega^j\mathbf{BO} \xrightarrow{\kappa_i \wedge \kappa_j} \mathbf{BO} \wedge \mathbf{BO} \xrightarrow{\phi_{0,0}} \mathbf{BO}.$$

We define $\phi_{i,j}$ as

$$\phi_{i,j} = \mathrm{Ad}^{i+j}(\phi_{0,0} \circ (\kappa_i \wedge \kappa_j)) : \Omega^i\mathbf{BO} \wedge \Omega^j\mathbf{BO} \to \Omega^{i+j}\mathbf{BO}.$$

Now, take $\alpha \in [X, \Omega^i\mathbf{BO}]$ and $\beta \in [X', \Omega^j\mathbf{BO}]$ and see the composition of $\alpha \wedge \beta$ and $\phi_{i,j}$:

$$\phi_{i,j} \circ (\alpha \wedge \beta) : X \wedge X' \to \Omega^i\mathbf{BO} \wedge \Omega^j\mathbf{BO} \to \Omega^{i+j}\mathbf{BO}.$$

Taking $\mathrm{Ad}^{-(i+j)}$ of the above composition, we obtain

$$\mathrm{Ad}^{-(i+j)}(\phi_{i,j} \circ (\alpha \wedge \beta)) = (\mathrm{Ad}^{-(i+j)}\phi_{i,j}) \circ (\Sigma^i\alpha \wedge \Sigma^j\beta)$$
$$: \Sigma^{i+j}(X \wedge X') \to \Sigma^{i+j}(\Omega^i\mathbf{BO} \wedge \Omega^j\mathbf{BO}) \to \mathbf{BO}.$$

From the definition of $\phi_{i,j}$, $\mathrm{Ad}^{-(i+j)}(\phi_{i,j} \circ (\alpha \wedge \beta))$ is the composition of following maps:

$$\Sigma^{i+j}(X \wedge X') \xrightarrow{\Sigma^i\alpha\wedge\Sigma^j\beta} \Sigma^{i+j}(\Omega^i\mathbf{BO} \wedge \Omega^j\mathbf{BO}) \xrightarrow{\kappa_i\wedge\kappa_j} \mathbf{BO} \wedge \mathbf{BO} \xrightarrow{\phi_{0,0}} \mathbf{BO}. \quad (4)$$

Lemma 3.2. *For any continuous map* $f : \Sigma^i X \to \mathbf{BO}$,

$$f \simeq \kappa_i(\Sigma^i\mathrm{Ad}^i f).$$

Proof. Consider the composition of $\mathrm{Ad}^i f$ and identity map of $\Omega^i\mathbf{BO}$.

$$X \xrightarrow{\mathrm{Ad}^i f} \Omega^i\mathbf{BO} \xrightarrow{\mathrm{Id}_{\Omega^i\mathbf{BO}}} \Omega^i\mathbf{BO}.$$

Taking Ad^{-i} of the above composition, we have

$$f = \mathrm{Ad}^{-i}(\mathrm{Id}_{\Omega^i\mathbf{BO}} \circ \mathrm{Ad}^i f) = \kappa_i \circ \Sigma^i\mathrm{Ad}^i f$$
$$: \Sigma^i X \xrightarrow{\Sigma^i\mathrm{Ad}^i f} \Sigma^i\Omega^i\mathbf{BO} \xrightarrow{\kappa_i} \mathbf{BO}.$$

By (4) and the above lemma, it follows that

$$\begin{aligned}\mathrm{Ad}^{-(i+j)}(\phi_{i,j}\circ(\alpha\wedge\beta)) &\simeq \phi_{0,0}\circ(\kappa_i\wedge\kappa_j)\circ(\Sigma^i\alpha\wedge\Sigma^j\beta)\\ &\simeq \phi_{0,0}\circ(\kappa_i\circ\Sigma^i\alpha)\wedge(\kappa_j\circ\Sigma^j\beta)\\ &\simeq \phi_{0,0}\circ(\mathrm{Ad}^{-i}\alpha\wedge\mathrm{Ad}^{-j}\beta).\end{aligned}$$

Since $f\in[X,\Omega^i\mathbf{BO}]$ corresponds to $(\mathrm{Ad}^{-i}f)^*(\eta_\infty)\in\widetilde{KO}^{-i}(X)$, the above equation says that $\phi_{i,j}\circ(\alpha\wedge\beta)$ corresponds to

$$(\mathrm{Ad}^{-i}\alpha\wedge\mathrm{Ad}^{-j}\beta)^*\phi_{0,0}^*(\eta_\infty)=\mathrm{Ad}^{-i}\alpha^*(\eta_\infty)\hat{\otimes}\,\mathrm{Ad}^{-j}\beta^*(\eta_\infty).$$

Therefore we obtain that

$$\alpha\hat{\otimes}\beta=\phi_{i,j}\circ(\alpha\wedge\beta)\qquad\text{in }\widetilde{KO}^{-(i+j)}(X\wedge X').$$

From the above theorem, we can deduce the next theorem.

Theorem 3.3. *Assume both n and m are odd. If, for all $(i,j)\in\mathbf{Z}/8\mathbf{Z}^2$ which satisfy $i+j=2$, $\phi_{i,j}^*(\alpha_{n+m-2})=\sum b_s\otimes c_t$ where $|b_s|=s$ and $|c_t|=t$ and $b_{n-1}\otimes c_{m-1}=0$ then $c:Spin(n)\wedge Spin(m)\to Spin(n+m-1)$ is not null homotopic.*

Proof. For any $\eta\in\widetilde{KO}^{-2}(Spin(n)\wedge Spin(m))$, there exist $\alpha_a\in\widetilde{KO}^{-i_a}(Spin(n))$ and $\beta_a\in\widetilde{KO}^{-j_a}(Spin(m))$ such that $\eta=\sum\alpha_a\hat{\otimes}\beta_a$ and $i_a+j_a=2$. Since α_{n+m-2} is primitive,

$$\eta^*(\alpha_{n+m-2})=\left(\sum\alpha_a\hat{\otimes}\beta_a\right)^*(\alpha_{n+m-2})=\sum(\alpha_a\hat{\otimes}\beta_a)^*(\alpha_{n+m-2})$$

and by Theorem 3.1,

$$(\alpha\hat{\otimes}\beta)^*(\alpha_{n+m-2})=(\alpha\wedge\beta)^*\circ\phi_{i,j}^*(\alpha_{n+m-2}).$$

If the hypothesis is satisfied, $\eta^*(\alpha_{n+m-2})$ can not be $x_{n-1}\otimes x_{m-1}$. Therefore from Proposition 2.2, c is not null homotopic.

4. The case n and m are odd

In this section we investigate the induced cohomology map of $\phi_{i,j}$ for $(i,j)\in(\mathbf{Z}/8\mathbf{Z})^2$, such that, $i+j=2$.

We start from the next lemma.

Lemma 4.1. *Assume $a\in\mathrm{H}^u(\Omega^{i+j}\mathbf{BO})$ is primitive and $\phi_{i,j}^*(a)=\sum_{s+t=u}b_s\otimes c_t$ where $|b_s|=s$ and $|c_t|=t$. Then b_s and c_t are primitive.*

Proof. Since for any $\alpha,\beta,\gamma\in\widetilde{KO}(X)$,

$$(p_1^*(\alpha)\oplus p_2^*(\beta))\otimes p_3^*(\gamma)=(p_1^*(\alpha)\otimes p_3^*(\gamma))\oplus(p_2^*(\beta)\otimes p_3^*(\gamma))$$

where $p_i:X\times X\times X\to X$ is the projection to i-th component, the next diagram commutes.

$$
\begin{array}{ccc}
\Omega^i\mathbf{BO}\times\Omega^i\mathbf{BO}\times\Omega^j\mathbf{BO} & \xrightarrow{\Omega^i\mu\times 1} & \Omega^i\mathbf{BO}\times\Omega^j\mathbf{BO} \\
\downarrow{\scriptstyle (1\times T\times 1)\circ(1\times 1\times\Delta)} & & \\
\Omega^i\mathbf{BO}\times\Omega^j\mathbf{BO}\times\Omega^i\mathbf{BO}\times\Omega^j\mathbf{BO} & & \downarrow{\scriptstyle \hat{\phi}_{i,j}} \\
\downarrow{\scriptstyle \hat{\phi}_{i,j}\times\hat{\phi}_{i,j}} & & \\
\Omega^{i+j}\mathbf{BO}\times\Omega^{i+j}\mathbf{BO} & \xrightarrow{\Omega^{i+j}\mu} & \Omega^{i+j}\mathbf{BO}
\end{array}
$$

Here T is the transposition map, Δ is the diagonal map and $\mu : \mathbf{BO}\times\mathbf{BO}\to\mathbf{BO}$ is the classifying map of $\eta_\infty\times\eta_\infty$ over $\mathbf{BO}\times\mathbf{BO}$. Further, $\hat{\phi}_{i,j}$ is the next composition:

$$\Omega^i\mathbf{BO}\times\Omega^j\mathbf{BO}\to\Omega^i\mathbf{BO}\wedge\Omega^j\mathbf{BO}\to\Omega^{i+j}\mathbf{BO}.$$

Let $a\in \mathrm{H}^n(\Omega^{i+j}\mathbf{BO})$ be a primitive element. Then we have

$$
\begin{aligned}
&(1\otimes\Delta^*)\circ(1\otimes T^*\otimes 1)\circ(\hat{\phi}^*_{i,j}\otimes\hat{\phi}^*_{i,j})\circ\Omega^{i+j}\mu^*(a)\\
&\quad=(1\otimes\Delta^*)\circ(1\otimes T^*\otimes 1)\circ(\hat{\phi}^*_{i,j}\otimes\hat{\phi}^*_{i,j})(a\otimes 1+1\otimes a)\\
&\quad=(1\otimes\Delta^*)\circ(1\otimes T^*\otimes 1)\Big(\sum b_s\otimes c_t\otimes 1\otimes 1+\sum 1\otimes 1\otimes b_s\otimes c_t\Big)\\
&\quad=(1\otimes\Delta^*)\Big(\sum b_s\otimes 1\otimes c_t\otimes 1+\sum 1\otimes b_s\otimes 1\otimes c_t\Big)\\
&\quad=\Big(\sum b_s\otimes 1\otimes c_t+\sum 1\otimes b_s\otimes c_t\Big)\\
&\quad=\Big(\sum(b_s\otimes 1+1\otimes b_s)\otimes c_t\Big).
\end{aligned}
$$

Also

$$
\begin{aligned}
(\Omega^i\mu^*\otimes 1)\circ\hat{\phi}^*_{i,j}(a)&=(\Omega^i\mu^*\otimes 1)\Big(\sum b_s\otimes c_t\Big)\\
&=\sum\Omega^i\mu^*(b_s)\otimes c_t.
\end{aligned}
$$

The above diagram says that these are the same. Therefore it occurs that $\Omega^i\mu^*(b_s)=b_s\otimes 1+1\otimes b_s$, that is, b_s is primitive. Similarly we can prove that c_t is primitive.

Theorem 4.2. *Let $i+j=2$ and n and m be odd. Assume $\phi_{i,j}(\alpha_{n+m-2})=\sum b_s\otimes c_t$ where $|b_s|=s$ and $|c_t|=t$. If $\binom{n+m-2}{n-1}\equiv 0 \bmod 2$, then $b_{n-1}\otimes c_{m-1}=0$.*

Proof. From assumption, (i,j) is $(1,1)$, $(2,0)$, $(3,7)$, $(4,6)$, $(5,5)$, $(6,4)$, $(7,3)$ or $(0,2)$. From the symmetricity, we shall look in to the cases $(i,j)=(1,1)$, $(2,0)$, $(3,7)$, $(4,6)$ and $(5,5)$.

For $\phi_{3,7}$, $\phi_{5,5}$, the proof is easy. From the assumption, $n-1$ and $m-1$ are even and by Lemma 4.1, b_{n-1} and c_{m-1} are primitive. On the other hand, it is known that all of the non-zero primitive elements of $\Omega^3\mathbf{BO}$, $\Omega^5\mathbf{BO}$ are in odd degrees. [7] Thus $b_{n-1}\otimes c_{m-1}=0$.

To start the proof for $\phi_{2,0}$, we investigate $\phi_{0,0}^*$.

Let $N=2^r$, $r\in\mathbf{N}$ and $\eta\in\widetilde{KO}(\mathbf{BO}(N)\wedge\mathbf{BO}(N))$ be the class of

$$\eta=(\xi_N-N)\hat{\otimes}(\xi_N-N).$$

We calculate the total Stiefel-Whitney class of η in $\mathrm{H}^*(\mathbf{B}(\mathbf{Z}/2\mathbf{Z})^N\wedge\mathbf{B}(\mathbf{Z}/2\mathbf{Z})^N)\supset\mathrm{H}^*(\mathbf{BO}\wedge\mathbf{BO})$. Let $t_1,\dots,t_N$ and $t'_1,\dots,t'_N$ be the generator of $\mathrm{H}^*(\mathbf{B}(\mathbf{Z}/2\mathbf{Z})^N\wedge\mathbf{B}(\mathbf{Z}/2\mathbf{Z})^N)$ where t_i corresponds to the first component and t'_i corresponds to the second. Then $w_k=\sigma_k(t_1,\dots,t_N)$ and $w'_k=\sigma_k(t'_1,\dots,t'_N)$ $(1\le k\le N)$ are the generators of $\mathrm{H}^*(\mathbf{BO}\wedge\mathbf{BO})$ where σ_k is k-th fundamental symmetric polynomial. (We put $w_0=1$.) Also we set $S'_l=\sum_{i=1}^N t_i'^l$.

Lemma 4.3. *The total Stiefel-Whitney class of η satisfies*

$$w(\eta)=1+\sum_{k=0}^{N-1}\sum_{l=0}^{k}\binom{N-k}{l}w_{N-k}\otimes S'_l \text{ modulo } (w_1\otimes 1, w_2\otimes 1,\dots,w_N\otimes 1)^2$$

in $\mathrm{H}^*(\mathbf{BO}(N)\wedge\mathbf{BO}(N))$ *for* $*<N$.

Proof. Since

$$\eta=\xi_N\hat{\otimes}\xi_N-\xi_N\hat{\otimes}N-N\hat{\otimes}\xi_N+N\hat{\otimes}N,$$

we can see that

$$w(\eta)=\prod_{1\le i\le N,1\le j\le N}(1+t_i+t'_j)\prod_{1\le i\le N}(1+t_i)^{-N}\prod_{1\le j\le N}(1+t'_j)^{-N}.$$

Here in the part of degrees less than N, $(1+t_i)^{-N}=(1+t_i^N)^{-1}=1$ and also $(1+t'_j)^{-N}=1$. Therefore modulo $\bigoplus_{i\ge N}\mathrm{H}^i(\mathbf{B}(\mathbf{Z}/2\mathbf{Z})^N\times\mathbf{B}(\mathbf{Z}/2\mathbf{Z})^N)$, we obtain that

$$\begin{aligned}w(\eta)&=\prod_{1\le i\le N,1\le j\le N}(t_i+1+t'_j)\\&=\prod_{j=1}^N\left(\sum_{k=0}^N w_k(1+t'_j)^{N-k}\right)\\&=\prod_{j=1}^N\left(1+\sum_{k=1}^N\sum_{l=0}^{N-k}\binom{N-k}{l}w_kt_j'^l\right).\end{aligned}$$

We proceed the calculation modulo $(w_1\otimes 1,w_2\otimes 1,\dots,w_N\otimes 1)^2$ and obtain

$$\begin{aligned}w(\eta)&\equiv 1+\sum_{k=1}^N\sum_{l=1}^{N-k}\binom{N-k}{l}w_kS'_l\\&\equiv 1+\sum_{1\le k,1\le l,k+l\le N}\binom{N-k}{l}w_kS'_l.\end{aligned}$$

Lemma 4.4. *Let* $k, l, r \in \mathbf{N}$. *If* $2^r > k + l$, *then* $\binom{2^r - k}{l} \equiv \binom{k+l-1}{l} \bmod 2$.

Proof. We set the binary expansion of $k-1$, l as

$$k - 1 = \sum_{0 \le i \le r-1} \varepsilon_i 2^i, \qquad l = \sum_{0 \le i \le r-1} \delta_i 2^i.$$

Then we have

$$\binom{2^r - k}{l} = \binom{(2^r - 1) - (k-1)}{l} \equiv \prod_{0 \le i \le r-1} \binom{1 - \varepsilon_i}{\delta_i}.$$

Therefore $\binom{2^r - k}{l} \equiv 0$ if and only if, for some i, $\binom{1-\varepsilon_i}{\delta_i} \equiv 0$, i.e., $\varepsilon_i = \delta_i = 1$.

Assume, for some i $(0 \le i \le r-1)$, $\varepsilon_i = \delta_i = 1$. Then let i_0 be the smallest such a number. Then i_0-th coefficient of the binary expansion of $k+l-1$ is 0, while $\delta_{i_0} = 1$. Thus we have $\binom{k+l-1}{l} \equiv 0$.

Vice versa if, for any i $(0 \le i \le r-1)$, not both ε_i and δ_i are 1, then

$$\binom{k+l-1}{l} \equiv \prod_{0 \le i \le r-1} \binom{\varepsilon_i + \delta_i}{\delta_i} \not\equiv 0.$$

Therefore $\binom{2^r - k}{l} \equiv \binom{k+l-1}{l} \bmod 2$.

Since $\phi_{0,0}$ is the classifying map of $\eta_\infty \hat{\otimes} \eta_\infty$, Lemma 4.3 implies that

$$\begin{aligned} \phi_{0,0}^*(w_i) &= \sum_{k+l=i} \binom{2^r - k}{l} w_k \otimes S_l' \\ &= \sum_{k+l=i} \binom{k+l-1}{l} w_k \otimes S_l' \text{ modulo } (w_1 \otimes 1, w_2 \otimes 1, w_3 \otimes 1, \ldots)^2 \end{aligned} \tag{5}$$

where r is sufficiently large.

Therefore

$$(\kappa_2 \wedge \mathrm{Id}_{\mathbf{BO}})^* \circ \phi_{0,0}^*(w_i) = \sum_{k+l=i, k:\text{even}} \binom{k+l-1}{l} \Sigma^2 a_{k-2} \otimes S_l',$$

since

$$\kappa_2^*(w_k) = \begin{cases} \Sigma^2 a_{k-2} & k : \text{even} \\ 0 & k : \text{odd} \end{cases}$$

and κ_2^* (decomposable element) $= 0$.

From definition, $\phi_{2,0} = \mathrm{Ad}^2(\kappa_2 \wedge \mathrm{Id} \circ \phi_{0,0})$ and then we have

$$\phi_{2,0}^*(a_{4i+2}) = \sum_{k+l=4i+2,\,k:\text{even}} \binom{k+l}{l} a_k \otimes S_l, \tag{6}$$

here we remark that $\binom{k+l+1}{l} = \binom{k+l}{l}$ when k and l are even. From (6), and since $a_{4k} = a_{2k}^2$, it occurs that

$$\phi_{2,0}^*(a_{2^p(4i+2)}) = \sum_{k+l=4i+2,\,k:\text{even}} \binom{k+l}{l} a_k^{2^p} \otimes S_l^{2^p},$$

Thus the coefficient of $b_{n-1} \otimes c_{m-1}$ in $\phi_{2,0}^*(a_{n+m-2})$ is 0 when $\binom{n+m-2}{n-1} = 0$ and the statement is true for $\phi_{2,0}$.

Second case is $\phi_{1,1}$. Consider the composition of following maps.

$$\Sigma\Omega\mathbf{BO} \wedge \Sigma\Omega\mathbf{BO} \xrightarrow{\kappa_1 \wedge \kappa_1} \mathbf{BO} \wedge \mathbf{BO} \xrightarrow{\phi_{0,0}} \mathbf{BO}.$$

From (5) and since κ_1^* (decomposable element) $= 0$ and

$$\kappa_1^*(w_k) = \Sigma x_{k-1}$$

$$\kappa_1^*(S_l) = \begin{cases} \Sigma x_{l-1} & k : \text{odd} \\ 0 & k : \text{even}, \end{cases}$$

the induced cohomology map of this composition can be obtained as

$$(\kappa_1 \wedge \kappa_1)^* \circ \phi_{0,0}^*(w_i) = (\kappa_1 \wedge \kappa_1)^* \left(\sum_{k+l=i} \binom{k+l-1}{l} S_l \otimes w_k' \right) \tag{7}$$

$$= \sum_{k+l=i,\,l:\text{odd}} \binom{k+l-1}{l} \Sigma x_{l-1} \otimes \Sigma x_{k-1}. \tag{8}$$

Here we remark that $\binom{k+l-1}{l} = 0$ when l is odd and k is even. Thus it occurs that

$$(\kappa_1 \wedge \kappa_1)^* \circ \phi_{0,0}^*(w_i) = \sum_{k+l=i,\,l:\text{odd},\,k:\text{odd}} \binom{k+l-1}{l} \Sigma x_{l-1} \otimes \Sigma x_{k-1}. \tag{9}$$

Similarly as the case of $\phi_{2,0}$, $\phi_{1,1} = \mathrm{Ad}^2(\kappa_1 \wedge \kappa_1 \circ \phi_{0,0})$ and from (9) we have

$$\phi_{1,1}^*(\alpha_{4i+2}) = \sum_{k+l=4(i+1),\,l:\text{odd},\,k:\text{odd}} \binom{k+l-1}{l} x_{l-1} \otimes x_{k-1}$$

$$= \sum_{k+l=4i+2,\,l:\text{even},\,k:\text{even}} \binom{k+l}{l} x_l \otimes x_k. \tag{10}$$

And also

$$\phi_{1,1}^*(\alpha_{2^p(4i+2)}) = \sum_{k+l=4i+2,\, l:\text{even},\, k:\text{even}} \binom{k+l}{l} x_l^{2^p} \otimes x_k^{2^p}. \tag{11}$$

Thus the coefficient of $b_{n-1} \otimes c_{m-1}$ in $\phi_{1,1}^*(a_{n+m-2})$ is also 0 when $\binom{n+m-2}{n-1} = 0$ and the statement is true for $\phi_{1,1}$.

The final case is $\phi_{4,6}$. Let $\xi_n^{\mathbf{R}}, \xi_n^{\mathbf{C}}$ and $\xi_n^{\mathbf{H}}$ be the universal bundle over $BO(n)$, $BU(n)$ and $BSp(n)$ respectively and put

$$\eta_n^{\mathbf{R}} = \xi_n^{\mathbf{R}} - n, \qquad \eta_n^{\mathbf{C}} = \xi_n^{\mathbf{C}} - n, \qquad \eta_n^{\mathbf{H}} = \xi_n^{\mathbf{H}} - n.$$

and

$$\eta_\infty^{\mathbf{R}} = \lim_{n\to\infty} \xi_n^{\mathbf{R}} - n, \qquad \eta_\infty^{\mathbf{C}} = \lim_{n\to\infty} \xi_n^{\mathbf{C}} - n, \qquad \eta_\infty^{\mathbf{H}} = \lim_{n\to\infty} \xi_n^{\mathbf{H}} - n.$$

Also set c be the classifying map to $(\eta_\infty^{\mathbf{R}})_{\mathbf{C}}$, complexification of $\eta_\infty^{\mathbf{R}}$, c' be the classifying map of $\eta_\infty^{\mathbf{H}}$ as a complex vector bundle and ψ be the classifying map of $\eta_\infty^{\mathbf{C}} \hat{\otimes} \eta_\infty^{\mathbf{C}}$ over $\mathbf{BU} \wedge \mathbf{BU}$.

We start from the next lemma.

Lemma 4.5. *The next diagram commutes.*

$$\begin{array}{ccc} \mathbf{BSp} \wedge \mathbf{BSp} & \xrightarrow{c' \wedge c'} & \mathbf{BU} \wedge \mathbf{BU} \\ \downarrow{\scriptstyle \phi_{4,4}} & & \downarrow{\scriptstyle \psi} \\ \mathbf{BO} & \xrightarrow{c} & \mathbf{BU} \end{array}$$

Proof. Consider the next composition:

$$\Sigma^4\mathbf{BSp} \wedge \Sigma^4\mathbf{BSp} \xrightarrow{\kappa_4 \wedge \kappa_4} \mathbf{BO} \wedge \mathbf{BO} \xrightarrow{\phi_{0,0}} \mathbf{BO} \xrightarrow{c} \mathbf{BU}.$$

Here in K-theory, $c^*(\eta_\infty^{\mathbf{C}}) = (\eta_\infty^{\mathbf{R}})_{\mathbf{C}}$ and $\phi_{0,0}^*((\eta_\infty^{\mathbf{R}})_{\mathbf{C}}) = (\eta_\infty^{\mathbf{R}})_{\mathbf{C}} \hat{\otimes} (\eta_\infty^{\mathbf{R}})_{\mathbf{C}}$. Also it is known that $\kappa_4^*((\eta_\infty^{\mathbf{R}})_{\mathbf{C}}) = (\zeta_{\mathbf{H}} - \mathbf{H}) \otimes_{\mathbf{C}} \eta_\infty^{\mathbf{H}}$ where $\zeta_{\mathbf{H}}$ is the $\mathbf{H}$ canonical line bundle over $\mathbf{H}P^1$. Therefore above composition pulls back $\eta_\infty^{\mathbf{C}}$ to $(\zeta_{\mathbf{H}} - \mathbf{H}) \hat{\otimes}_{\mathbf{C}} (\zeta_{\mathbf{H}} - \mathbf{H}) \hat{\otimes}_{\mathbf{C}} \eta_\infty^{\mathbf{H}} \hat{\otimes}_{\mathbf{C}} \eta_\infty^{\mathbf{H}}$.

On the other hand consider the next composition:

$$\Sigma^8\mathbf{BSp} \wedge \mathbf{BSp} \xrightarrow{\Sigma^8(c' \wedge c')} \Sigma^8\mathbf{BU} \wedge \mathbf{BU} \xrightarrow{\Sigma^8\psi} \Sigma^8\mathbf{BU} \xrightarrow{\kappa_8'} \mathbf{BU}.$$

Here κ_8' is defined as follows. From Bott Periodicity, we know that $\Omega^2\mathbf{BU} \simeq \mathbf{BU} \times \mathbf{Z}$. Thus there exists a map $\kappa_{2i}' : \Sigma^{2i}\mathbf{BU} \to \mathbf{BU}$ which satisfies $\mathrm{Ad}^{2i}\kappa_{2i}'$ is the inclusion map $\mathbf{BU} \to \Omega^{2i}\mathbf{BU}$. One can easily verify that

$$\kappa_2' \circ \Sigma^2\kappa_2' \circ \cdots \Sigma^{2i-2}\kappa_2' \simeq \kappa_{2i}'$$

and it is known that in K-theory $\kappa_2'^*(\eta_\infty^{\mathbf{C}}) = (\zeta_{\mathbf{C}} - \mathbf{C}) \hat{\otimes} \eta_\infty^{\mathbf{C}}$ where $\zeta_{\mathbf{C}}$ is the canonical line bundle over $\mathbf{C}P^1$. Therefore $\kappa_8'^* = (\zeta_{\mathbf{C}} - \mathbf{C})^4 \hat{\otimes} \eta_\infty^{\mathbf{C}}$. Now we can see that the above composition pulls back $\eta_\infty^{\mathbf{C}}$ to $(\zeta_{\mathbf{C}} - \mathbf{C})^4 \hat{\otimes} \eta_\infty^{\mathbf{H}} \hat{\otimes}_{\mathbf{C}} \eta_\infty^{\mathbf{H}}$.

Since $\tilde{K}^{-4}(\mathrm{pt}) = \mathbf{Z}$ and the second Chern class of $-(\zeta_{\mathbf{H}} - \mathbf{H})$ and $(\zeta_{\mathbf{C}} - \mathbf{C})^2$ coincide, we see that the above two compositions are homotopic each other.

Take the Ad^8 of two compositions and we obtain

$$c \circ \phi_{4,4} \simeq \psi \circ c'$$

Refer to the diagram of Lemma 4.5. We want to calculate $\phi_{4,4}(w_i)$. As we have done in the proof of Lemma 4.3, let $N = 2^r$, $r \in \mathbf{N}$ and $\theta \in \tilde{K}(BU(2N) \times BU(2N))$ be the class of $\theta = (\xi_{2N}^{\mathbf{C}} - 2N) \hat{\otimes} (\xi_{2N}^{\mathbf{C}} - 2N)$ where $\xi_{2N}^{\mathbf{C}}$ is the universal vector bundle over $BU(2N)$. Also let ψ_N be the classifying map of θ.

First, we calculate the total Chern class of θ in $\mathrm{H}^*(BT^{2N} \times BT^{2N}) \supset \mathrm{H}^*(BU(2N) \times BU(2N))$. Let $t_1, \ldots, t_{2N}, t_1', \ldots, t_{2N}' \in \mathrm{H}^*(BT^{2N} \times BT^{2N})$ be the generators as usual. Then in the part of degree less than $4N$,

$$\psi_N^*\left(1 + \sum_{i=1}^{\infty} c_i\right) = \prod_{1 \le i \le 2N, 1 \le j \le 2N} (1 + t_i + t_j').$$

Now we proceed the calculations of $(c' \wedge c')^* \psi_N^*(1 + \sum_{i=1}^{\infty} c_i)$ in $\mathrm{H}^*(BT^N \times BT^N) \supset \mathrm{H}^*(BSp(N) \times BSp(N))$. Let $s_1, \ldots, s_N, s_1', \ldots, s_N' \in \mathrm{H}^*(BT^N \times BT^N)$ be the generators. Then we can see

$$\begin{aligned}
&(c' \wedge c')^* \psi_N^*\left(1 + \sum_{i=1}^{\infty} c_i\right) \\
&= (c' \wedge c')^*\left(\prod_{1 \le i \le 2N, 1 \le j < 2N} (1 + t_i + t_j')\right) \\
&= \prod_{1 \le i \le N, 1 \le j \le N} (1 + s_i + s_j')(1 + s_i - s_j')(1 - s_i + s_j')(1 - s_i - s_j') \\
&= \prod_{1 \le i \le N, 1 \le j \le N} (1 + s_i + s_j')^4 \\
&= \left\{\prod_{1 \le i \le N, 1 \le j \le N} (1 + s_i^2 + s_j'^2)\right\}^2.
\end{aligned}$$

On the other hand, considering $\mathrm{H}^*(BSp(N)) \subset \mathrm{H}^*(\mathbf{BSp})$, in the part of degree less than $4N$,

$$\begin{aligned}
(c' \wedge c')^* \psi_N^*\left(1 + \sum_{i=1}^{\infty} c_i\right) &= \phi_{4,4}^* c^*\left(1 + \sum_{i=1}^{\infty} c_i\right) \\
&= \phi_{4,4}^*\left(1 + \sum_{i=1}^{\infty} w_i^2\right) \\
&= \phi_{4,4}^*\left(1 + \sum_{i=1}^{\infty} w_i\right)^2
\end{aligned}$$

Since $H^*(\mathbf{BSp} \wedge \mathbf{BSp})$ is a subalgebra of a polynomial algebra, the square of any element in $H^*(\mathbf{BSp} \wedge \mathbf{BSp})$ does not vanishes. Therefore

$$\phi_{4,4}^*\left(1+\sum_{i=1}^{\infty} w_i\right) = \prod_{1\le i\le N, 1\le j\le N} (1+s_i^2+s_j'^2).$$

in the part of degree less than $2N$.

We set $q_k' = \sigma_k(s_1'^2, \dots, s_N'^2)$ $(1 \le k \le N)$ which are the generators of $H^*(BSp(N))$ and $Q_l = \sum_{i=1}^N s_i^{2l}$ which is the primitive element of $H^*(BSp(N))$. Now we have in the part of degrees less than $2N$

$$\begin{aligned}\phi_{4,4}^*\left(1+\sum_{i=1}^{\infty} w_i\right) &= \prod_{1\le i\le N, 1\le j\le N} (1+s_i^2+s_j'^2)\\ &= \prod_{i=1}^{N}\left(\sum_{k=0}^{N}(1+s_i^2)^k q_{N-k}'\right)\\ &= \prod_{i=1}^{N}\left(1+\sum_{k=0}^{N-1}\sum_{l=0}^{k}\binom{k}{l} s_i^{2l} q_{N-k}'\right)\end{aligned}$$

Now we proceed the calculations modulo $(q_1', \dots, q_N')^2$.

$$\begin{aligned}\phi_{4,4}^*\left(1+\sum_{i=1}^{\infty} w_i\right) &\equiv 1+\sum_{k=0}^{N-1}\sum_{l=1}^{k}\binom{k}{l} Q_l q_{N-k}'\\ &\equiv 1+\sum_{k=1}^{N}\sum_{l=1}^{N-k}\binom{N-k}{l} Q_l q_k'\\ &\equiv 1+\sum_{i=1}^{N}\sum_{1\le k, 1\le l, k+l=i}\binom{N-k}{l} Q_l q_k'\end{aligned}$$

This leads us to the next lemma.

Lemma 4.6. *Modulo* $(1\otimes q_1, 1\otimes q_2, 1\otimes q_3, \dots)^2$,

$$\phi_{4,4}^*(w_i) = \begin{cases} \sum_{1\le k, 1\le l, k+l=j} \binom{k+l-1}{l} Q_l \otimes q_k & i = 4j \\ 0 & i \not\equiv 0 \bmod 4 \end{cases}$$

where $H^*(\mathbf{BSp}) = \mathbf{Z}/2\mathbf{Z}[q_1, q_2, q_3, \dots]$ *and* $Q_l \in H^*(\mathbf{BSp})$ *is the primitive element of degree* $4l$.

Let $\kappa' : \Sigma^2\Omega^6\mathbf{BO} \to \Omega^4\mathbf{BO}$ be the map which satisfies $\mathrm{Ad}^2(\kappa') = \mathrm{Id}_{\Omega^6\mathbf{BO}}$. Then it can be easily verified that $\mathrm{Ad}^2(\phi_{4,4} \circ \mathrm{Id}_{\Omega^4\mathbf{BO}} \wedge \kappa') = \phi_{4,6}$. Since

$$\kappa'^*(q_l) = \Sigma^2 b_{4l-2},$$

where $H^*(\Omega^2\mathbf{BSp}) = \bigwedge(b_2, b_4, b_6, \dots)$ and b_{4i-2} is primitive, it occurs that

$$(\mathrm{Id}_{\Omega^4\mathbf{BO}} \wedge \kappa')^* \phi_{4,4}^*(w_{4i}) = \sum_{1 \le k, 1 \le l, k+l=i} \binom{k+l-1}{l} Q_l \otimes \Sigma^2 b_{4k-2}$$

and

$$\phi_{4,6}^*(a_{4i-2}) = \sum_{1 \le k, 1 \le l, k+l=j} \binom{k+l-1}{l} Q_l \otimes b_{4k-2}.$$

Remark that $\binom{k+l-1}{l} = \binom{4k+4l-4}{4l} = \binom{4k+4l-2}{4l}$ and

$$\phi_{4,6}^*(a_{2^p(4i-2)}) = \sum_{1 \le k, 1 \le l, k+l=j} \binom{4k+4l-2}{4l} Q_l^{2^p} \otimes b_{4k-2}{}^{2^p}.$$

Therefore the statement is also true for $\phi_{4,6}$. Q.E.D. (Theorem 4.2)

From Theorem 3.3 and Theorem 4.2, the next theorem follows.

Theorem 4.7. *Assume neither $n-1$ nor $m-1$ is a power of 2 and both n and m are odd. If $\binom{n+m-2}{n-1} \equiv 0 \bmod 2$, (n, m) is Spin-regular.*

5. The case n and m are even

In this section we use integral cohomology. Consider the next diagram.

$$\begin{array}{ccccc}
S^{n-1} & \xrightarrow{\tilde{i}_n} & Spin(n) & \xrightarrow{\pi_n} & S^{n-1} \\
\downarrow{\scriptstyle p'_n} & & \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle \cong} \\
\mathrm{RP}^{n-1} & \xrightarrow{i_n} & SO(n) & \xrightarrow{\pi'_n} & S^{n-1}
\end{array}$$

Here π_n, π'_n is the map obtained from $Spin(n) \to Spin(n)/Spin(n-1) = S^{n-1}$ and $SO(n) \to SO(n)/SO(n-1) = S^{n-1}$ respectively. Also i_n is the inclusion map defined as follows. Let $l \in \mathrm{RP}^{n-1}$ be a line and let $e \in l$ be a unit vector. Then $i_n(l) = i'_n(l_0) i'_n(l)$ where $i'_n(l)(v) = v - 2(v, e)e$ and l_0 is the base point of RP^{n-1}. We set $p'_n : S^{n-1} \to \mathrm{RP}^{n-1}$ be the usual covering map then there is a map $\tilde{i}_n$ which makes diagram commutative. Moreover, when $n = 4$, π_n has a section $\varepsilon : S^{n-1} \to Spin(n)$, that is, $\pi_n \circ \varepsilon = \mathrm{Id}$.

We set c_{n-1} as the generator of $H^*(S^{n-1}; \mathbf{Z})$ and take $\delta \in H^*(Spin(n) \wedge Spin(m); \mathbf{Z})$ as $\delta = (\pi_n \wedge \pi_m)^*(c_{n-1} \otimes c_{m-1})$.

Lemma 5.1. *If n and m are even and neither n nor m is 4,*

$$H^{n+m-2}(Spin(n) \wedge Spin(m); \mathbf{Z}) = \langle \delta \rangle \oplus \mathrm{Ker}(\tilde{i}_n \wedge \tilde{i}_m)^*.$$

Proof. Since n is even, $i_n^*\pi_n'^*(c_{n-1})$ is the generator of $\mathrm{H}^{n-1}(\mathrm{RP}^{n-1}; \mathbf{Z}) \cong \mathbf{Z}$. Therefore

$$\tilde{i}_n^*\pi_n^*(c_{n-1}) = p_n'^* i_n^* \pi_n'^*(c_{n-1}) = 2c_{n-1}, \tag{12}$$

that is, $\tilde{i}_n \wedge \tilde{i}_m^*(\delta) = 4c_{n-1} \otimes c_{m-1}$.

Because $p_n'^* : \mathrm{H}^{n-1}(\mathrm{RP}^{n-1}; \mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}^{n-1}(S^{n-1}; \mathbf{Z}/2\mathbf{Z})$ is a 0-map and $\tilde{i}_n^* \circ p_n^* = p_n'^* \circ i_n^*$, we have $\tilde{i}_n^* \circ p_n^* = 0$ in mod 2 cohomology. Further, since, when $n \neq 4$, $p_n^* : \mathrm{H}^{n-1}(SO(n); \mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}^{n-1}(Spin(n); \mathbf{Z}/2\mathbf{Z})$ is epic, this implies that $\tilde{i}_n^* : \mathrm{H}^{n-1}(Spin(n); \mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}^{n-1}(S^{n-1}; \mathbf{Z}/2\mathbf{Z})$ is also a 0-map. Therefore $\mathrm{Im}\, \tilde{i}_n^* \subset \langle 2c_{n-1} \rangle$ in integral cohomology.

Now we obtain that $\mathrm{Im}(\tilde{i}_n \wedge \tilde{i}_m)^* = \langle 4c_{n-1} \otimes c_{m-1} \rangle = \langle (\tilde{i}_n \wedge \tilde{i}_m)^*(\delta) \rangle$ and from the freeness of $\mathrm{H}^{n+m-2}(S^{n+m-2}; \mathbf{Z})$ the statement follows.

Lemma 5.2. *If $n = 4$ and m are even and $m \neq 4$,*

$$\mathrm{H}^{n+m-2}(Spin(n) \wedge Spin(m); \mathbf{Z}) = \langle \delta \rangle \oplus \mathrm{Ker}(\varepsilon \wedge \tilde{i}_m)^*.$$

Proof. From (12) and $\varepsilon^*\pi_4^*(c_3) = c_3$,

$$(\varepsilon \wedge \tilde{i}_m)^*(\delta) = 2c_{n-1} \otimes c_{m-1}.$$

As seen in the proof of previous lemma, $\mathrm{Im}\, \tilde{i}_m^* \subset \langle 2c_{m-1} \rangle$ in integral cohomology and since ε is a section, $\mathrm{Im}\, \varepsilon^* = \langle c_3 \rangle$.

Now it follows that $\mathrm{Im}(\varepsilon \wedge \tilde{i}_m)^* = \langle 2c_3 \otimes c_{m-1} \rangle = \langle (\varepsilon \wedge \tilde{i}_m)^*(\delta) \rangle$ and from the freeness of $\mathrm{H}^{n+m-2}(S^{n+m-2}; \mathbf{Z})$ the statement follows.

Theorem 5.3. *Assume neither $n - 1$ nor $m - 1$ is a power of 2, both n and m are even, $n + m \equiv 0$ mod 4 and $n + m \geq 16$. Then (n, m) is Spin-regular.*

Proof. We use Proposition 2.2. Let $x : Spin(n) \wedge Spin(m) \to \Omega\, \mathbf{Spin}$ satisfies $x^*(\alpha_{n+m-2}) = x_{n-1} \otimes x_{m-1}$ in mod 2 cohomology. Then there exists $\eta \in \widetilde{KO}(\Sigma^2 Spin(n) \wedge Spin(m))$ which satisfies

$$w_{n+m}(\eta) = \Sigma^2 x_{n-1} \otimes x_{m-1}. \tag{13}$$

Here, since Pontrjagin square acts trivially in $\mathrm{H}^*(\Sigma^2 Spin(n) \wedge Spin(m); \mathbf{Z})$, by the second formula of Wu [12],

$$\rho_4(P_{(n+m)/4}(\eta)) = w'_{n+m}(\eta), \tag{14}$$

where w'_{n+m} is the image of w_{n+m} under the coefficient monomorphism $\mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/4\mathbf{Z}$ and ρ_4 is the map of mod 4 reduction.

When neither n nor m is 4, from (13), (14) and Lemma 5.1, we can see that

$$P_{(n+m)/4}(\eta) = \Sigma^2((4k + 2)\delta + \alpha),$$

where $\alpha \in \mathrm{Ker}(\tilde{i}_n \wedge \tilde{i}_m)^*$ and we obtain

$$P_{(n+m)/4}(\Sigma^2(\tilde{i}_n \wedge \tilde{i}_m)^*(\eta)) = (16k + 8)c_{n+m}.$$

When $n = 4$ and $m \neq 4$, (13), (14) and Lemma 5.2 imply that

$$P_{(n+m)/4}(\eta) = \Sigma^2((4k+2)\delta + \beta),$$

where $\beta \in \mathrm{Ker}(\varepsilon \wedge \tilde{i}_m)^*$ and we have

$$P_{(n+m)/4}(\Sigma^2(\varepsilon \wedge \tilde{i}_m)^*(\eta)) = (8k+4)c_{n+m}.$$

But for the generator η_0 of $\widetilde{KO}(S^{n+m})$, $P_{(n+m)/4}(\eta_0)$ is divisible by $\left(\frac{n+m}{2} - 1\right)!$. [1] When $n + m \geq 16$ this is a contradiction and the statement follows.

Theorem 5.4. *Assume neither $n-1$ nor $m-1$ is a power of 2, both n and m are even. If $n+m = 12$ or $n+m \equiv 2 \bmod 4$. Then (n, m) is Spin-regular.*

Proof. We use Proposition 2.2. Let $x : Spin(n) \wedge Spin(m) \to \Omega\,\mathbf{Spin}$ be the arbitrary continuous map.

When $n+m \equiv 2$ mod 4, that is, $n+m-2$ is divisible by 4, $x^*(\alpha_{n+m-2}) = x^*(\alpha_{(n+m-2)/2})^2$ in mod 2 cohomology. Thus $x^*(\alpha_{n+m-2})$ can be written in the form $\sum \alpha \otimes \beta$ where α and β are decomposable. Therefore $x^*(\alpha_{n+m-2}) \neq x_{n-1} \otimes x_{m-1}$.

Now let $n+m = 12$ and $n \leq m$. When $n \neq 4$, $x^*(\alpha_6) = x_3 \otimes x_3$ or 0 and when $n = 4$, $x^*(\alpha_6) = z \otimes x_3$, $x_3 \otimes x_3$ or 0. We can see

$$\mathrm{Sq}^2 x^*(\alpha_6) = x^*(\mathrm{Sq}^2\alpha_6) = x^*(\alpha_8) = x^*(\alpha_2)^4 = 0$$

while

$$\mathrm{Sq}^2 x_3 \otimes x_3 = x_5 \otimes x_3 + x_3 \otimes x_5,$$
$$\mathrm{Sq}^2 z \otimes x_3 = z \otimes x_5.$$

So $x^*(\alpha_6) = 0$ and we have

$$x^*(\alpha_{10}) = x^*(\mathrm{Sq}^4\alpha_6) = \mathrm{Sq}^4 x^*(\alpha_6) = 0.$$

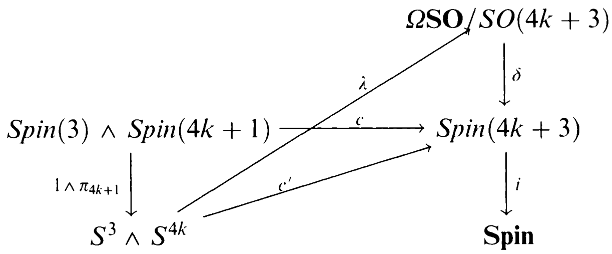
From Proposition 2.2, Theorems 4.7, 5.3, 5.4, we finally obtain Theorem 1.3.

6. $(3, 4k+1)$ is Spin-irregular

In this section we shall give the proof of Theorem 1.4 which requires that $(3, 4k+1)$ is Spin-irregular.

Since there are embeddings $Spin(3) \to Spin(4k+3)$, $Spin(4k+1) \to Spin(4k+3)$ where any element of $Spin(3)$ and any element of $Spin(4k) \subset Spin(4k+1)$ exactly commute in $Spin(4k+3)$. Let $A \in Spin(3)$, $B \in Spin(4k+1)$, $C \in Spin(4k) \subset Spin(4k+1)$. Then $A(BC)A^{-1}(BC)^{-1} = ABCA^{-1}C^{-1}B^{-1} = ABA^{-1}B^{-1}$ and the commutator of A and B is invariant under the right translation of $Spin(4k)$ on B.

Therefore there exists a map $c' : Spin(3) \wedge (Spin(4k+1)/Spin(4k)) \to Spin(4k+3)$ such that $c' \circ (1 \wedge \pi_{4k+1}) \simeq c$. See the diagram below. Remark that $Spin(3) \cong S^3$ and $Spin(4k+1)/Spin(4k) \cong S^{4k}$.



In the above diagram $\Omega\mathbf{SO}/SO(4k+3) \to Spin(4k+3) \to \mathbf{Spin}$ is a fibration and $i \circ c'$ is null homotopic. So there exists a map $\lambda : S^{4k+3} \to \Omega\mathbf{SO}/SO(4k+3)$, such that $\delta \circ \lambda \simeq c'$.

Since $\pi_{4k+4}(\mathbf{SO}/SO(4k+3)) \cong 0$ ([10]), $\pi_{4k+3}(\Omega\mathbf{SO}/SO(4k+3)) \cong 0$ and λ is null homotopic.

Thus $c \simeq \delta \circ \lambda \circ (1 \wedge \pi_{4n+1}) \simeq *$ and Theorem 1.4 is proved.

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