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On an invariant of plumbed homology 3-spheres

By

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Abstract

The main purpose of this paper is to give some invariance property of a homology cobordism invariant of plumbed homology 3-spheres under a kind of blowing up process for auxiliary 4-V-manifolds. By using this property, we prove a homology cobordism invariance of an integral lift of the Rohlin invariant constructed by W. Neumann [6] and L. Siebenmann [12] in the set of all homology 3-spheres bounding plumbed 4-V-manifolds with $b_2^+ + b_2^- \leq 2$ which are obtained by blowing down of smooth spin 4-manifolds.

1. Introduction

The Neumann-Siebenmann invariant ($\bar{\mu}$-invariant) [6], [12] is an invariant for plumbed homology 3-spheres which is an integral lift of Rohlin's μ-invariant. This invariant vanishes for several plumbed homology spheres which are known to bound acyclic 4-manifolds, but it is not known whether this invariant has a homology cobordism invariance or not. On the other hand, in a joint work with M. Furuta [2], we defined a homology cobordism invariant (w-invariant) which is an integral lift of μ-invariant by using the Seiberg-Witten monopole equation on closed 4-V-manifolds. For two non-negative integers k^+, k^-, let $\mathscr{S}^{\text{plumb}}(k^+, k^-)$ be the set of all homology 3-spheres bounding plumbed 4-V-manifolds with $b_2^\pm \leq k^\pm$ which are blowing down of smooth spin 4-manifolds. The main purpose of this paper[1] is to give an invariance property of w-invariant under a kind of blowing up process for auxiliary V-manifolds and to prove a homology cobordism invariance of $\bar{\mu}$-invariant in the set $\mathscr{S}^{\text{plumb}}(k^+, k^-)$ satisfying $k^+ + k^- \leq 2$. In fact, we show that $\bar{\mu}$-invariant is equal to minus w-invariant in the set $\mathscr{S}^{\text{plumb}}(k^+, k^-)$ for any k^+, k^-. Recently N. Saveliev defined an invariant (ν-invariant) of homology 3-spheres by using instanton Floer homology and proved that this invariant is equal to $\bar{\mu}$-invariant in the set of all Seifert fibered homology 3-spheres [10], [11]. As a corollary, we see that the above three invariants $\bar{\mu}$, ν, and $-w$ are equal and has a homology cobordism invariance in the set $\mathscr{S}^{\text{plumb}}(0, 1)$. This paper is organized as follows. In section 2, we review several definitions and basic facts concerning w-invariant. In section 3, we consider V-spin structures around the singularities of V-manifolds. Here we consider only cyclic quotient singularities for later discussions. In section 4, we define a kind of blowing up process of the singularities

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for V-manifolds. This process is a truncation of the Hirzebruch-Jung resolution process. In this section, we prove an invariance property of w-invariant under blowing up processes. In section 5, we apply these properties to V-plumbing which is an extension of the plumbing to the V-manifold category defined in [1]. Here we prove that w-invariant is equal to minus $\bar{\mu}$-invariant.

2. w-invariant

In this section, we review several definitions and basic facts concerning w-invariant [2]. Let (Σ, X, c) be a triple consisting of a homology 3-sphere Σ, a compact 4-V-manifold X with $\partial Y = \Sigma$, and a V-spin^c-structure c on X. We assume that the V-manifold X has only isolated singularities in its interior. For the definitions concerning V-manifolds see [9]. Let Y be a compact smooth spin 4-manifold with $\partial Y = -\Sigma$. We patch X and Y along Σ and get the closed 4-V-manifold $X \cup_\Sigma Y$. Since Σ is a homology 3-sphere, the V-spin^c-structure c on X and the spin structure on Y can be patched uniquely and define a V-spin^c-structure on $X \cup_\Sigma Y$, which we denote by c. Let $\mathscr{D}(X \cup_\Sigma Y)$ be the Dirac operator on $X \cup_\Sigma Y$ associated to the V-spin^c-structure c. Then w-invariant is defined as follows.

Definition 1.

$$w(\Sigma, X, c) := \frac{1}{2}\operatorname{ind}_{\mathbf{R}} \mathscr{D}(X \cup_\Sigma Y) + \frac{1}{8}\operatorname{sign} Y.$$

Here $\operatorname{ind}_{\mathbf{R}} D$ is the *real* V-index of an elliptic operator D over V-manifold defined as $\dim_{\mathbf{R}} \operatorname{Ker}_{\mathrm{V}}(D) - \dim_{\mathbf{R}} \operatorname{Coker}_{\mathrm{V}}(D)$, and sign Y is the signature of the intersection form on $\mathrm{H}^2(Y, \partial Y; \mathbf{R}) \cong \mathrm{H}^2(Y; \mathbf{R})$. Note that each term on the right hand side is an integer. By the excision property for the indices of the elliptic operators over V-manifolds and the Atiyah-Singer index theorem, the invariant $w(\Sigma, X, c)$ does not depend on Y and its spin structure [2]. Let $\mathscr{X}(k^+, k^-)$ be the set of all triples (Σ, X, c) consisting of homology 3-spheres Σ, spin 4-V-manifolds X with $\partial X = \Sigma$ satisfying $b_2^\pm(X) \leq k^\pm$ and $b_1(X) = 0$, and V-spin structures c on X. Furthermore, let $\mathscr{S}(k^+, k^-)$ be the set of all homology 3-spheres Σ such that $(\Sigma, X, c) \in \mathscr{X}(k^+, k^-)$ for some (X, c). In a joint work with M. Furuta [2], we proved the following property of w-invariant.

Theorem 1 ([2]). 1. *If c comes from a V-spin structure on X then $w(\Sigma, X, c) \equiv \mu(\Sigma) \pmod 2$.*

2. *Suppose $k^+ + k^- \leq 2$. Then $w(\Sigma, X, c)$ does not depend on the choice of (X, c) with $(\Sigma, X, c) \in \mathscr{X}(k^+, k^-)$ and the map:*

$$w(k^+, k^-) : \mathscr{S}(k^+, k^-) \ni \Sigma \mapsto w(\Sigma, X, c) \in \mathbf{Z}$$

is a homology cobordism invariant.

3. Cyclic quotient singularities

In this section, we consider V-spin structures around the isolated singularities of 4-V-manifolds. Here we consider only cyclic quotient singularities which are needed for later discussions. Let $C(\alpha,\beta)$ be the cyclic quotient singularity [5] defined by:

$$C(\alpha,\beta)=\frac{\mathbf{C}\times\mathbf{C}}{\sigma(\alpha,\beta)},$$

where $\sigma(\alpha,\beta)$ is the $\mathbf{Z}/\alpha$ action on $\mathbf{C}\times\mathbf{C}$ defined by

$$\zeta_\alpha^l\cdot(z,w):=(\zeta_\alpha^l z,\zeta_\alpha^{\beta l}w),$$

for $\zeta_\alpha^l\in\mathbf{Z}/\alpha$ with $\zeta_\alpha=e^{2\pi\sqrt{-1}/\alpha}$ and (α,β) coprime. Note that the link $\dfrac{S^4}{\sigma(\alpha,\beta)}\subset\dfrac{\mathbf{C}\times\mathbf{C}}{\sigma(\alpha,\beta)}$ of $C(\alpha,\beta)$ is isomorphic to the lens space $L(\alpha,\beta)$. Then we have the following proposition.

Proposition 1. *Suppose (α,β) are coprime integers.*

1. *If α is odd then $C(\alpha,\beta)$ has a unique V-spin structure.*
2. *If α is even then $C(\alpha,\beta)$ has two V-spin structures.*

Proof. Fix a V-complex structure on $C(\alpha,\beta)$ which is the quotient of the standard complex structure on $\mathbf{C}\times\mathbf{C}$, and fix the induced V-Riemannian metric on $C(\alpha,\beta)$. The set of all V-spin^c-structures on $C(\alpha,\beta)$ can be identified with the topological Picard group $\mathrm{Pic}_{\mathrm{V}}^t(C(\alpha,\beta))$ of all line V-bundles on $C(\alpha,\beta)$ see [4], and $C(\alpha,\beta)$ has a V-spin structure if and only if the canonical line V-bundle K on $C(\alpha,\beta)$ has a square root $K^{1/2}$. $\mathrm{Pic}_{\mathrm{V}}^t(C(\alpha,\beta))$ is isomorphic to $\mathbf{Z}/\alpha$ and is generated by L_0 corresponding to the standard $U(1)$-representation of $\mathbf{Z}/\alpha$. The canonical line V-bundle K is isomorphic to $L_0^{-(1+\beta)}$, and it has a square root if and only if $1+\beta\equiv 2m \pmod{\alpha}$ for some integer m. If α is odd, m has only one solution modulo α, and $K^{1/2}$ is determined uniquely. On the other hand, since α,β coprime, if α is even, there always exist a solution m, and $m+(\alpha/2)$ is also a solution modulo α, and hence $K^{1/2}$ has two possibility.

Next we consider the inclusion map $i: C(\alpha,\beta)\backslash\{0\}\hookrightarrow C(\alpha,\beta)$, we obtain the following restriction map from the set of all V-spin^c-structures on $C(\alpha,\beta)$ to the set of all spin^c-structures on $C(\alpha,\beta)\backslash\{0\}$.

$$i^*:\mathrm{Spin}_{\mathrm{V}}^c(C(\alpha,\beta))\to\mathrm{Spin}^c(C(\alpha,\beta)\backslash\{0\})$$

Proposition 2. *The restriction map* $i^*:\mathrm{Spin}_{\mathrm{V}}^c(C(\alpha,\beta))\to\mathrm{Spin}^c(C(\alpha,\beta)\backslash\{0\})$ *is bijective.*

Proof. We fix a V-complex structure on $C(\alpha,\beta)$. By using the canonical spin^c-V-bundle on $C(\alpha,\beta)$, we can identify $\mathrm{Pic}_{\mathrm{V}}^t(C(\alpha,\beta))$ and $\mathrm{Spin}_{\mathrm{V}}^c(C(\alpha,\beta))$. Then the assertion follows from the induced map corresponding to the restriction map:

$$i^* : \mathrm{Pic}^t_V(C(\alpha,\beta)) \cong \mathbf{Z}/\alpha \to \mathrm{Pic}^t(C(\alpha,\beta)\backslash\{0\}) \cong \mathrm{H}^2(C(\alpha,\beta)\backslash\{0\};\mathbf{Z}) \cong \mathbf{Z}/\alpha$$

which is clearly bijective.

Corollary 1. *The restriction map from the set of all V-spin structures on $C(\alpha,\beta)$ to the set of all spin structures on $C(\alpha,\beta)\backslash\{0\}$:*

$$i^* : \mathrm{Spin}_V(C(\alpha,\beta)) \to \mathrm{Spin}(C(\alpha,\beta)\backslash\{0\})$$

is bijective.

Proof. Any spin bundle is obtained as the tensor product of the dual of a half canonical line $K^{1/2}$ and the canonical spin^c-bundle. On the other hand, half canonicals on each side corresponds bijectively, since the restriction map i^* : $\mathrm{Pic}^t_V(C(\alpha,\beta)) \to \mathrm{Pic}^t(C(\alpha,\beta)\backslash\{0\})$ is bijective.

Then we have the next corollary.

Corollary 2. *Let X be a 4-V-manifold with cyclic quotient singularities $x_1,\dots,x_n$ in its interior. Then any spin structure on $X\backslash\{x_1,\dots,x_n\}$ can be extended uniquely to a V-spin structure on X.*

4. Blowing up

In this section, we use a truncation of the Hirzebruch-Jung resolution process [5] to define a kind of blowing up of 4-V-manifolds. Let x be a singular point of a 4-V-manifold X whose neighborhood V has the identification $(V,x) \cong (C(\alpha,\beta),0)$. First we consider a continued fraction expansion:

$$\frac{\alpha}{\beta} = m_1 - \frac{1}{\frac{\alpha_1}{\beta_1}}.$$

We put $\alpha_1 := \beta$, $\beta_1 := m_1\beta - \alpha$. Note that (α_1,β_1) are coprime integers. Let $\mathbf{C}_1$, $\mathbf{C}_2$ be two copies of the complex plane $\mathbf{C}$. Then we define the following line V-bundle U_1 over a Riemannian V-sphere $\mathbf{C}P^1 = \mathbf{C}_1 \cup \frac{\mathbf{C}_2}{\mathbf{Z}/\alpha_1}$:

$$U_1 := \mathbf{C}_1 \times \mathbf{C} \cup_{\varphi_1} \frac{\mathbf{C}_2 \times \mathbf{C}}{\sigma(\alpha_1,\beta_1)},$$

where the map φ_1 is given by:

$$\varphi_1 : \frac{(\mathbf{C}_2\backslash\{0\}) \times \mathbf{C}}{\sigma(\alpha_1,\beta_1)} \ni [z,w] \mapsto (z^{-\alpha_1}, z^{-\beta_1+m_1\alpha_1}w) \in (\mathbf{C}_1\backslash\{0\}) \times \mathbf{C}.$$

Let $E_1 \subset U_1$ be the zero V-section of U_1. We put $X_0 := X\backslash\{x\}$, and $V_0 := V\backslash\{x\}$. Then we have a diffeomorphism

$$\Psi_1 : U_1\backslash E_1 \cong C(\alpha,\beta)\backslash\{0\} \cong V_0$$

which is defined by the following commutative maps:

$$\begin{array}{ccccc}
U_1\backslash E_1 = & \mathbf{C}_1\times(\mathbf{C}\backslash\{0\}) & \cup_{\varphi_1} & \dfrac{\mathbf{C}_2\times(\mathbf{C}\backslash\{0\})}{\sigma(\alpha_1,\beta_1)} \\
 & (z,w) & & [z,w] \\
\Psi_1\Big\downarrow & \Big\downarrow & & \Big\downarrow \\
 & [w^{1/\alpha}, zw^{\beta/\alpha}] & & [w^{1/\alpha}z, w^{\beta/\alpha}] \\
C(\alpha,\beta)\backslash\{0\} = & \dfrac{(\mathbf{C}\backslash\{0\})\times\mathbf{C}}{\sigma(\alpha,\beta)} & \cup & \dfrac{\mathbf{C}\times(\mathbf{C}\backslash\{0\})}{\sigma(\alpha,\beta)}
\end{array}$$

Note that we take $w^{\beta/\alpha} = (w^{1/\alpha})^{\beta}$, and the above maps do not depend on the choice of a branch of $w^{1/\alpha}$. Then we define a blowing up $\tilde{X}$ of the V-manifold X by:

$$\tilde{X} := X_0 \cup_{\Psi_1} U_1.$$

Note that we have an isomorphism $\tilde{X}\backslash E_1 \cong X_0$. Then we have the following theorem.

Theorem 2. *Let X be a closed 4-V-manifold, and $\tilde{X}$ its blowing up at a cyclic quotient singularity in X. Suppose that $\tilde{X}$ admits a V-spin structure $\tilde{c}$. Then X admits a V-spin structure c whose restriction to X_0 is isomorphic to the restriction of $\tilde{c}$, and we have the following equality.*

$$\mathrm{ind}_{\mathbf{R}}\, \mathscr{D}(\tilde{X}) = \mathrm{ind}_{\mathbf{R}}\, \mathscr{D}(X).$$

Proof. Suppose $\tilde{X}$ is obtained by blowing up of a singularity $(V,x) \cong (C(\alpha,\beta),0)$ of X. Note that $\tilde{X}$ is defined by $\tilde{X} := X_0 \cup_{\Psi_1} U_1$. Now we have a V-spin structure $\tilde{c}$ on $\tilde{X}$. We denote by c_1 the restriction of $\tilde{c}$ to U_1. Then we have a spin structure c_{V_0} which is the pull back of $c_1|_{U_1\backslash E_1}$ by the diffeomorphism $\Psi_1 : U_1\backslash E_1 \cong V_0$. By Corollary 2, we can extend c_{V_0} uniquely to the V-spin structure c_V on V. Hence we have V-spin structures $c, -c_V \cup_{V_0} c_1$, and $-c_V \cup_{V_0} c_V$ on X, $-V \cup_{\Psi_1} U_1$, and $-V \cup V$, respectively. Now by the excision argument for the indices of the Dirac operators on V-manifolds,

$$\mathrm{ind}_{\mathbf{R}}\, \mathscr{D}(\tilde{X}) - \mathrm{ind}_{\mathbf{R}}\, \mathscr{D}(X) = \mathrm{ind}_{\mathbf{R}}\, \mathscr{D}(-V \cup_{\Psi_1} U_1) - \mathrm{ind}_{\mathbf{R}}\, \mathscr{D}(-V \cup V).$$

Since $-V \cup V$ admits an orientation reversing diffeomorphism, $\mathrm{ind}_{\mathbf{R}}\, \mathscr{D}(-V \cup V) = 0$. Note that the closed spin 4-V-manifold $-V \cup_{\Psi_1} U_1$ has the second Betti number 1. We see that $\mathrm{ind}_{\mathbf{R}}\, \mathscr{D}(-V \cup_{\Psi_1} U_1) = 0$, since the index of the Dirac operator on the closed spin 4-V-manifold with $b_2^+ \leq 2$ and $b_2^- \leq 2$ always vanish (see Corollary 1 in [2]).

Then we have the next corollary.

Corollary 3. *Let X be a 4-V-manifold with boundary Σ a homology 3-sphere, and $\tilde{X}$ be a blowing up of a cyclic quotient singularity in X. Suppose that $\tilde{X}$ admits* *a V-spin structure $\tilde{c}$. Then X admits a V-spin structure c whose restriction to X_0 is isomorphic to that of $\tilde{c}$, and we have the following equality.*

$$w(\Sigma, \tilde{X}, \tilde{c}) = w(\Sigma, X, c).$$

Proof. Take a spin 4-manifold Y with $\partial Y = -\Sigma$, and apply the above theorem to the closed spin 4-V-manifold $X \cup_\Sigma Y$. Then we have:

$$\frac{1}{2}\mathrm{ind}_{\mathbf{R}}\,\mathscr{D}(\tilde{X} \cup_\Sigma Y) + \frac{1}{8}\mathrm{sign}\,Y = \frac{1}{2}\mathrm{ind}_{\mathbf{R}}\,\mathscr{D}(X \cup_\Sigma Y) + \frac{1}{8}\mathrm{sign}\,Y.$$

5. V-Plumbing

In the paper concerning an explicit computation of w-invariants of plumbing type homology 3-spheres [1], we extended the notion of the plumbing process to the V-manifold category. For several definitions and basic facts concerning the usual plumbing process, see [7]. Here, we need more general version of this process. First we define a Seifert graph $\Gamma = (V, E, \omega)$ as follows.

Definition 2. *$\Gamma = (V, E, \omega)$ is a Seifert graph if and only if:*

1. *(V, E) is a connected tree graph consisting of a set of vertices V and a set of edges E.*
2. *Each vertex $k \in V$ is assigned an unnormalized Seifert invariant:*

$$\omega(k) = \{(\alpha_{k1}, \beta_{k1}), \ldots, (\alpha_{kn_k}, \beta_{kn_k})\} \qquad (k \in V),$$

where $(\alpha_{ki}, \beta_{ki})$ are coprime integers $(\alpha_{ki} \neq 0)$.

3. *If two vertices k and k' are connected by an edge $e \in E$, then there is a map $\omega(e)$ which assigns an index labeling a singular orbit in the Seifert invariant to each vertex k, k':*

$$\omega(e)(k) = ki, \qquad \omega(e)(k') = k'j,$$

and the pair $(ki, k'j)$ must satisfy the compatibility condition:

(a) $\alpha_{ki} = \alpha_{k'j}$, *and*

(b) $\beta_{ki}\beta_{k'j} \equiv 1 \pmod{\alpha_{ki}}$.

4. *If two edges e and e' have the same vertex k as the common boundary, then $\omega(e)(k) \neq \omega(e')(k)$.*

We denote edges e satisfying $\omega(e)(k) = ki$, $\omega(e)(k') = k'j$ simply by $e = (ki, k'j)$.

Remark. More general Seifert graphs can be defined as the usual plumbing graphs. Since we are only interested in integral homology 3-spheres, the above definition will be sufficient for our discussion.

A plumbed 4-V-manifold $P(\Gamma)$ is constructed from a Seifert graph Γ as follows. For each vertex $k \in V$, we construct a line V-bundle

$$L_k := (S^2\backslash\{n_k\text{-points}\}) \times \mathbf{C} \cup_{\{\varphi_{ki}\}} \bigcup_{i=1}^{n_k} \frac{\tilde{D}_{ki} \times \mathbf{C}}{\sigma(\alpha_{ki}, \beta_{ki})},$$

where $\tilde{D}_{ki} \cong D^2 \subset \mathbf{C}$, $\sigma(\alpha_{ki}, \beta_{ki})$ is the action of $\mathbf{Z}/\alpha_{ki}$ on $\tilde{D}_{ki} \times \mathbf{C}$ defined by $\zeta_{ki}^l \cdot (z, w) = (\zeta_{ki}^l z, \zeta_{ki}^{\beta_{ki} l} w)$, for $\zeta_{ki}^l \in \mathbf{Z}/\alpha_{ki}$ with $\zeta_{ki} = e^{2\pi\sqrt{-1}/\alpha_{ki}}$, and the map φ_{ki} is given by:

$$\varphi_{ki} : \frac{(\tilde{D}_{ki}\backslash\{0\}) \times \mathbf{C}}{\sigma(\alpha_{ki}, \beta_{ki})} \ni [z, w] \mapsto (z^{\alpha_{ki}}, z^{-\beta_{ki}} w) \in (D_{ki}\backslash\{0\}) \times \mathbf{C},$$

here we identified a neighborhood U_{ki} of x_{ki} in S^2 and a unit disk D_{ki} in $\mathbf{C}$. Let DL_k be the D^2-V-bundle associated to the line V-bundle L_k. If two vertices k, k' are connected by an edge $e \in E$, and the edge e is assigned a pair $\omega(e) = (ki, k'j)$ then we glue two disk V-bundles DL_k and $DL_{k'}$ as follows. We choose each trivialization over a (singular) disk around the singular point labeled by $ki, k'j$, $DL_k|_{D_{ki}} \cong \dfrac{\tilde{D}_{ki} \times D^2}{\sigma(\alpha_{ki}, \beta_{ki})}$ respectively. Note that we have specified the V-manifold structure by the map φ_{ki}'s and we use these identifications. Then we glue them up by the map:

$$\sigma_e : DL_k|_{D_{ki}} \cong \frac{\tilde{D}_{ki} \times D^2}{\sigma(\alpha_{ki}, \beta_{ki})} \ni [z, w] \mapsto [w, z] \in \frac{\tilde{D}_{k'j} \times D^2}{\sigma(\alpha_{k'j}, \beta_{k'j})} \cong DL_{k'}|_{D_{k'j}}.$$

The map σ_e is well-defined by the compatibility condition. The plumbed 4-V-manifold $P(\Gamma)$ has singularities of the form of the cone on the lens space. The V-manifold $P(\Gamma)$, which is a rational homology manifold, has a rational intersection pairing. We denote by I_Γ the intersection matrix of $P(\Gamma)$. If Γ is a tree graph then the (k, k')-entry of I_Γ is:

$$(I_\Gamma)_{k,k'} = \begin{cases} e_k & k = k' \\ 1/\alpha_{ki} & (ki, k'j) \in E \\ 0 & \text{otherwise}, \end{cases}$$

where $e_k := \sum_{i=1}^{n_k} \beta_{ki}/\alpha_{ki}$. Let us denote the boundary of the plumbing $P(\Gamma)$ by $\Sigma(\Gamma)$.

Let $b^+(\Gamma)$ (resp. $b^-(\Gamma)$) be the number of positive (resp. negative) eigenvalues of I_Γ. Note that if all α_{ki}'s are ± 1, then we can regard the Seifert graph Γ as a usual integrally weighted graph $\Gamma' = (V, E, m)$ by defining an integral weight $m : V \to \mathbf{Z}$ as $m_k = \dfrac{\beta_{k1}}{\alpha_{k1}} + \cdots + \dfrac{\beta_{kn_k}}{\alpha_{kn_k}}$. Correspondingly 4-V-manifold $P(\Gamma)$ obtained by plumbing according to a Seifert graph Γ can be regarded as a smooth 4-manifold obtained by plumbing according to the corresponding integrally weighted graph Γ'. We write a vertex $k \in V$ with the Seifert invariant $\{(\alpha_{k1}, \beta_{k1}), \dots, (\alpha_{kn_k}, \beta_{kn_k})\}$ as in Figure 1. If two vertices k and k' are connected by an edge $e = (ki, k'j)$ then we write as in Figure 2.

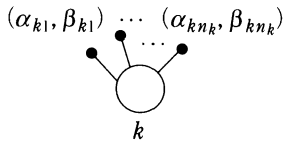


Fig. 1

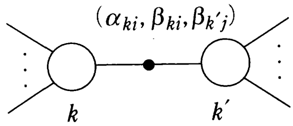
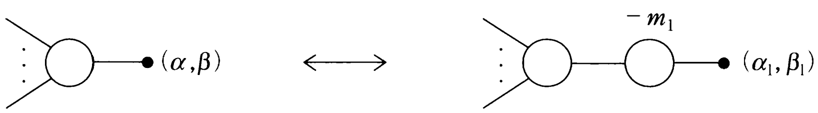
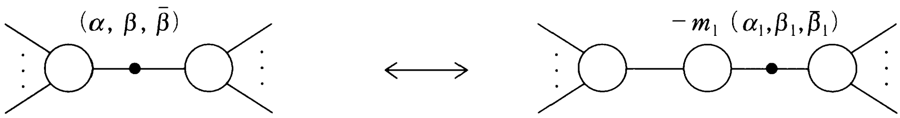


Fig. 2

Then the blowing up process introduced in section 4 applied to the V-plumbing gives the following operations of Seifert graphs. Here we omit the dots in the notation for the usual integral weights and usual plumbings. Note that these blowing up processes does not change the diffeomorphism type of the boundary of the V-plumbing.



$$(\alpha_1,\beta_1) = (\beta, m_1\beta - \alpha)$$



$$(\alpha_1,\beta_1,\bar\beta_1) = (\beta, m_1\beta - \alpha, \frac{\beta\bar\beta - 1}{\alpha})$$

6. The Neumann-Siebenmann invariant

Let Γ be an integrally weighted *tree* graph. Let $P(\Gamma)$ be a 4-manifold obtained by plumbing according to the graph Γ. Then the intersection matrix of I_Γ of $P(\Gamma)$ is:

$$(I_\Gamma)_{kk'} = \begin{cases} m_k & (k = k') \\ 1 & (k \neq k' \text{ and } (k, k') \in E) \\ 0 & (\text{otherwise}) \end{cases}$$

Let $\Sigma(\Gamma)$ be the boundary of $P(\Gamma)$. Then it is known that $\Sigma(\Gamma)$ is a homology 3-sphere if and only if:

(HS) $$\det I_\Gamma = \pm 1,$$

and $P(\Gamma)$ is spin if and only if:

(SP) all m_k's are even.

Suppose Γ satisfies the condition (HS). Then there exists a unique integral homology class $c \in \mathrm{H}_2(P(\Gamma); \mathbf{Z})$ (the spherical integral Wu class) satisfying:

1. $c \cdot x \equiv x \cdot x \pmod 2$ for any $x \in \mathrm{H}_2(P(\Gamma); \mathbf{Z})$,
2. $c \in \mathrm{H}_2(P(\Gamma); \mathbf{Z})$ is written by using the standard basis spheres $\{[E_i]\}$, $E_i \cong S^2$, for $\mathrm{H}_2(P(\Gamma); \mathbf{Z})$ as follows:

$$c = \sum_i \varepsilon_i [E_i], \qquad \varepsilon_i = 0, 1.$$

Then $\bar{\mu}$-invariant introduced by W. Neumann [6] and L. Siebenmann [12] is defined as follows.

Definition 3.

$$\bar{\mu}(\Sigma(\Gamma)) := \frac{1}{8}(\operatorname{sign} P(\Gamma) - c \cdot c)$$

Note that the right hand side is an integer and it is an integral lift of μ-invariant:

$$\bar{\mu}(\Sigma(\Gamma)) \equiv \mu(\Sigma(\Gamma)) \qquad (\text{mod } 2).$$

Let $\mathscr{S}^{\text{plumb}}(k^+, k^-)$ be the set of all plumbed homology 3-spheres $\Sigma(\Gamma)$ such that 1) Γ is an integrally weighted graph satisfying the conditions (HS) and (SP), 2) Γ is obtained by blowing up of a Seifert graph $\hat{\Gamma}$ satisfying $b^{\pm}(\hat{\Gamma}) \leq k^{\pm}$.

Note that the set of all Seifert fibered homology 3-spheres $\Sigma(\alpha_1, \dots, \alpha_n)$ with one of the α_i's is even is the class $\mathscr{S}^{\text{plumb}}(0, 1)$. Then we have the following theorem.

Theorem 3. *Suppose that* $k^+ + k^- \leq 2$. *Then the map:*

$$\bar{\mu}(k^+, k^-) : \mathscr{S}^{\text{plumb}}(k^+, k^-) \ni \Sigma(\Gamma) \mapsto \bar{\mu}(\Sigma(\Gamma)) \in \mathbf{Z}$$

is a homology cobordism invariant.

Proof. The spherical integral Wu class is $c = 0 \in \mathrm{H}_2(P(\Gamma); \mathbf{Z})$ for the spin 4-manifold $P(\Gamma)$. Since $P(\Gamma) \cup_{\Sigma(\Gamma)} -P(\Gamma)$ admits an orientation reversing diffeomorphism, we see that $\operatorname{ind}_{\mathbf{R}} \mathscr{D}(P(\Gamma) \cup_{\Sigma(\Gamma)} -P(\Gamma)) = 0$. Then by Corollary 3:

$$\begin{aligned}\bar{\mu}(\Sigma(\Gamma)) &= \frac{1}{8}\operatorname{sign} P(\Gamma) \\ &= -\left[\frac{1}{2}\operatorname{ind}_{\mathbf{R}} \mathscr{D}(P(\Gamma)\cup_{\Sigma(\Gamma)} - P(\Gamma)) + \frac{1}{8}\operatorname{sign}(-P(\Gamma))\right] \\ &= -w(\Sigma(\Gamma), P(\Gamma), c) \\ &= -w(\Sigma(\Gamma), P(\hat{\Gamma}), \hat{c}).\end{aligned}$$

Hence the assertion follows from Theorem 1-2 in section 2.

Recently, N. Saveliev defined an invariant ν of homology 3-spheres by using the instanton Floer homology, and he proved that $\bar{\mu}(\Sigma) = \nu(\Sigma)$ for any Seifert fibered homology 3-spheres [10], [11]. Thus we have the equality $\nu(\Sigma(\Gamma)) = \bar{\mu}(\Sigma(\Gamma)) = -w(\Sigma(\Gamma))$ for $\Sigma(\Gamma) \in \mathscr{S}^{\text{plumb}}(0,1)$. Therefore:

Corollary 4. *The map:* $\nu(0,1) : \mathscr{S}^{\text{plumb}}(0,1) \ni \Sigma(\Gamma) \mapsto \nu(\Sigma(\Gamma)) \in \mathbf{Z}$ *is a homology cobordism invariant.*

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