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A certain class of distribution-valued additive functionals I —for the case of Brownian motion

By

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1. Introduction

Let B_s be a one-dimensional Brownian motion and T be a distribution which belongs to the class $\mathscr{D}'^{1}_{L^2_{loc}}$. M. Fukushima has proposed a definition of the integral $\int_0^t T(B_s)ds$ via Ito's formula and showed that the integral is a continuous additive functional of zero energy ([3]).

T. Yamada [11] and M. Yor [13] studied concretely principal values of Brownian local time which are typical examples in the class of additive functionals of zero energy.

It is well known that there is a one-to-one correspondence between the class of positive continuous additive functionals of d-dimensional Brownian motion and the class of Revuz measures ([7], [8]).

R. Bass [1] showed that additive functionals $A(a,t,\omega)$ for d-dimensional Brownian motion are jointly continuous in a and t, $a.s.$ and represented $A(a,t,\omega)$ as d-dimensional analogue to the Ito and Mckean [5] that states that any additive functional A_t of one-dimensional Brownian motion can be represented as

$$A_t = \int L_t^y \mu(dy),$$

where L_t^y is the local time at y for the one-dimensional Brownian motion and μ is the measure corresponding to A_t.

T. Yamada showed that any continuous additive functional of zero energy has a representation via convolution-type transform of the local time in the case of one-dimensional Brownian motion and generalized a representation formula given by R. Bass in the case of multi-dimensional Brownian motion ([12]).

In this paper, we show that $A_T(a:t,\omega) = \int_0^t T(X_s - a)ds$ is a continuous additive functional for some $T \in H_p^\beta$, where X_s is d-dimensional Brownian motion and this additive functional has jointly continuous modification in a and t, $a.s.$ and has zero energy.

Our method is very simple. It is principally based on the Fourier transform theory in distribution sense. The concrete estimate of the characteristic function

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of d-dimensional Brownian motion plays an essential role in the proof of our main result.

The present paper is organized as follows. In section 2, we define distribution valued additive functionals and prepare some notation.

In section 3, we discuss the existence and (a, t)-joint continuity of $A_T(a : t, \omega)$ for d-dimensional Brownian motion.

In section 4, we discuss the energy of $A_T(a : t, \omega)$ in the sense of M. Fukushima [4].

In the forthcoming paper we will show that $A_T(a : t, \omega)$ for 1-dimensional stable process with index α is a continuous additive functional for some $T \in H_p^\beta$ and this additive functional has jointly continuous modification in a and t, *a.s.* and has zero energy. And we will show some representation theorems for $A_T(a : t, \omega)$ in that paper.

2. Definitions and preliminary results

Throughout the paper, we shall use the following notation.

$$\mathbf{R} = \text{the set of all real numbers.}$$

$$\mathbf{N} = \text{the set of all natural numbers.}$$

$$\mathbf{C} = \text{the set of all complex numbers.}$$

$$\mathbf{R}^d = \{x = (x_1, \ldots, x_d) : x_i \in \mathbf{R} \text{ for } 1 \le i \le d\}.$$

For $p \in \mathbf{C}$, $\bar{p}$ denotes the complex conjugate of p.

$$\mathscr{D} = \{\phi(x) : \phi \text{ is an infinitely differentiable function on } \mathbf{R}^d \text{ and has a compact support}\}.$$

$$\mathscr{D}' = \{T : T \text{ is a continuous linear functional on } \mathscr{D}\}.$$

$$\mathscr{S} = \{\phi(x) : \phi \text{ is an infinitely differentiable function and } (1 + |x|^2)^k D^\alpha \phi(x) \text{ is bounded on } \mathbf{R}^d \text{ for any } k \text{ and } \alpha\}.$$

$$\mathscr{S}' = \{T : T \text{ is a linear continuous functional on } \mathscr{S}\}.$$

Here we take the topology for these spaces in Schwartz's sense.

Let (X_s) be the standard Brownian motion on $\mathbf{R}^d$ or one-dimensional real valued stable process with index α $(0 < \alpha \le 2)$.

Lemma 2.1. *Let $T \in \mathscr{D}'$, $\phi \in \mathscr{D}$ and set $T * \phi(x) = \langle T_y, \phi(x - y)\rangle_y$. Then*

$$\langle A_T(t, \omega), \phi\rangle = \int_0^t T * \phi(X_s(\omega))ds$$

is well-defined and we have

$$A_T(t, \omega) \in \mathscr{D}'.$$

Proof. Since $\phi \in \mathscr{D}$, there exists a compact set K which includes $supp\{\phi\}$. Then there exists a compact set $L = L_t(\omega)$ such that

$$K + X_s(\omega) \subset L_t(\omega) \qquad (0 \leq s \leq t).$$

And there exist the positive numbers C_L and N such that $|T * \phi(x)| \leq C_L\|\phi\|_K^N$ for every $K + x \in L$. Here we denote $\|\phi\|_K^N = \sup_{x \in K, |p| \leq N} |D^p\phi(x)|$.

Then we have

$$|\langle A_T(t, \omega), \phi \rangle| \leq t \sup_{0 \leq s \leq t} |T * \phi(X_s)| \leq tC_L\|\phi\|_K^N.$$

This implies that $A_T(t, \omega)$ is an element of $\mathscr{D}'$.

Remark 2.2. In particular, in the case of $T = T_f = f \in L^1_{loc}$, we have

$$\langle A_T(t, \omega), \phi \rangle = \int_0^t f * \phi(X_s) ds.$$

Moreover, let μ be a Radon measure and we set

$$(\mu * \phi)(x) = \int \phi(x - y)\mu(dy).$$

Then we have

$$\langle A_\mu(t, \omega), \phi \rangle = \int_0^t (\mu * \phi)(X_s) ds.$$

We define τ_x and θ_t as following:

$$\tau_x : X_t(\tau_x\omega) = X_t(\omega) + x$$

and

$$\theta_t : X_s(\theta_t\omega) = X_{t+s}(\omega).$$

Clearly, we have

Lemma 2.3.

$$\langle A_T(t, \tau_x\omega), \phi \rangle = \langle A_T(t, \omega), \phi(\cdot + x) \rangle \tag{2.1}$$

$$\langle A_T(s + t, \omega), \phi \rangle = \langle A_T(s, \omega), \phi \rangle + \langle A_T(t, \theta_s\omega), \phi \rangle. \tag{2.2}$$

Lemma 2.4. *Let T be an element of $\mathscr{S}'$. Then $A_T(t, \omega)$ is also an element of $\mathscr{S}'$.*

Proof. $A_T(t, \omega)$ is a linear form clearly.

We note that $T \in \mathscr{S}'$ if and only if there exist constants M and $p \in \mathbf{N}$ such that

$$|\langle T, \phi \rangle| \leq M\|\phi\|_p \qquad \text{for any } \phi \in \mathscr{S}, \tag{2.3}$$

where $\|\phi\|_p = \sup_{|\alpha| \leq p, x \in \mathbf{R}^d} (1 + |x|^2)^p |D^\alpha\phi(x)|$.

For $\phi \in \mathscr{S}$, we have

$$|\langle A_T(t,\omega), \phi\rangle| \leq t \sup_{0\leq s\leq t} |T * \phi(X_s)|$$

$$= t \sup_{0\leq s\leq t} |\langle T_y, \phi(X_s - y)\rangle_y|.$$

Then by (2.3) we get

$$|\langle A_T(t,\omega), \phi\rangle| \leq 2^p t M \|\phi\|_p \sup_{0\leq s\leq t} (1 + |X_s(\omega)|^2)$$

Therefore $A_T(t,\omega)$ is a continous linear form.

We denote the Fourier transform of $\phi(a)$ by $\hat{\phi}(\lambda)$:

$$\hat{\phi}(\lambda) = \int \phi(a) e^{i\lambda\cdot a}\, da,$$

and the Fourier inverse transform of $\psi(\lambda)$ by $\mathscr{F}^{-1}(\psi)(a)$:

$$\mathscr{F}^{-1}(\psi)(a) = \frac{1}{(2\pi)^d} \int \psi(\lambda) e^{-i\lambda\cdot a}\, d\lambda,$$

where $x \cdot y$ ($x \in \mathbf{R}^d$, $y \in \mathbf{R}^d$) denotes the inner product.

Let $T \in \mathscr{S}'$. We denote the Fourier transform of T by $\hat{T}$:

$$\langle \hat{T}, \phi\rangle = \langle T, \hat{\phi}\rangle \qquad \text{for any } \phi \in \mathscr{S}.$$

Definition 2.5. We say that T is an element of H_p^β ($1 \leq p \leq \infty, -\infty < \beta < \infty$) if and only if T is an element of $\mathscr{S}'$ and the Fourier transform of T has a version as a function $\hat{T}(\lambda)$ on $\mathbf{R}^d$ such that

$$\hat{T}(\lambda)(1 + |\lambda|^2)^{\beta/2} \in L^p.$$

Then we set

$$\|T\|_{H_p^\beta} = \|\hat{T}(\lambda)(1 + |\lambda|^2)^{\beta/2}\|_{L^p}.$$

We note $\mathscr{F}^{-1}(T)(\lambda) = (2\pi)^{-d}\hat{T}(-\lambda)$ for $T \in H_p^\beta$.

Lemma 2.6. *Let T be an element of H_p^β. Then $A_T(t,\omega)$ is also an element of H_p^β.*

Proof. For $\phi \in \mathscr{S}$, we have

$$\langle \hat{A}_T(t,\omega), \phi\rangle = \langle A_T(t,\omega), \hat{\phi}\rangle$$

$$= \int_0^t \left\langle T_\lambda, \int e^{-i\lambda\cdot x} e^{iX_s\cdot x} \phi(x) dx \right\rangle_\lambda ds$$

$$= \int_0^t \langle T_\lambda, (2\pi)^d \mathscr{F}^{-1}(e^{iX_s\cdot}\phi(\cdot))(\lambda)\rangle_\lambda\, ds$$

$$= (2\pi)^d \int_0^t \langle \mathscr{F}^{-1}(T_\lambda), e^{iX_s\cdot\lambda}\phi(\lambda)\rangle_\lambda \, ds$$

$$= \int_0^t \langle \hat{T}(-\lambda)e^{iX_s\cdot\lambda}, \phi(\lambda)\rangle_\lambda \, ds$$

$$= \left\langle \int_0^t \hat{T}(-\lambda)e^{iX_s\cdot\lambda} ds, \phi(\lambda)\right\rangle_\lambda .$$

Thus we get

$$\hat{A}_T(t,\omega) = \int_0^t \hat{T}(-\lambda)e^{i\lambda\cdot X_s} \, ds.$$

By $T \in H_p^\beta$, we have

$$\int_0^t e^{i\lambda\cdot X_s} ds \hat{T}(-\lambda)(1+|\lambda|^2)^{\beta/2} \in L^p.$$

We state the following lemma, which will play an important role in the next section. In fact, we will prove the boundedness of certain integrals by this lemma, which will appear in theorem 3.1 and 3.4.

Lemma 2.7. *We set*

$$J = \int \frac{d\mu}{(1+|\mu|^2)^p (1+|\mu+\lambda|^2)^q}.$$

Let $2p+2q > d$ *and* $p \geq q > 0$ or $p > 0 \geq q$.

(1) *If* $2p < d$ *and* $2q < d$, *then*

$$J \asymp \frac{1}{(1+|\lambda|^2)^{p+q-(d/2)}}. \tag{2.4}$$

(2) *If* $2p = d$, *then*

$$J \asymp \frac{1+\log^+|\lambda|}{(1+|\lambda|^2)^q}, \tag{2.5}$$

where $\log^+|x| = \max(\log|x|, 0)$.

(3) *If* $2p > d$, *then*

$$J \asymp \frac{1}{(1+|\lambda|^2)^q}. \tag{2.6}$$

Here we denote that "$f \asymp g$" means $k \leq f/g \leq K$ for some positive constants k and K, where $f, g \not\equiv 0$.

We will show the proof of this lemma in appendix.

Now let ρ_ε be the molifier:

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right).$$

where

$$\rho(x) = \begin{cases} C_d \exp\left(-\dfrac{1}{1-|x|^2}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

and C_d is the constant which satisfies that $\int \rho(x)dx = 1$.

In the remainder of this paper, we denote

$$A_T^{\varepsilon}(t,\omega) = \langle A_T(t,\omega), \rho_{\varepsilon} \rangle$$

and

$$A_T^{\varepsilon}(a : t,\omega) = A_T^{\varepsilon}(t, \tau_{-a}\omega).$$

We note that

$$\langle A_T^{\varepsilon}(t,\omega), \phi \rangle = \langle A_T(t,\omega), \rho_{\varepsilon} * \phi \rangle.$$

Here we emphasize $A_T^{\varepsilon}(a : t,\omega)$ is a function of a. We recall that $\rho_{\varepsilon} \to \delta_0$ as $\varepsilon \to 0$ and $\hat{\rho}_{\varepsilon}$ uniformly converges to one in wider sense tending ε to zero and $\|\hat{\rho}_{\varepsilon}\|_{\infty} \leq 1$.

In general, A_T^{ε} approximates the distribution A_T. However, under some conditions we will prove that the limit is a function, which realizes the local time associated to the distribution T, and study the (a,t)-joint continuity of $A_T(a : t,\omega)$.

3. Convergence and continuity theorems

In this section we write

$$\Gamma_N = \{(\lambda_1, \lambda_2) : |\lambda_1| \leq N, |\lambda_2| \leq N\} \qquad \text{for any } N > 0.$$

Let P_x be the probability measure of the d-dimensional standard Brownian motion $B_t(\omega)$ starting from x and we denote $p(t,x)$ the transition probability density function. We notice that the characteristic function of B_s is

$$E_x[e^{i\lambda B_s}] = \exp\left\{-\frac{|\lambda|^2}{2}s + i\lambda x\right\}.$$

Theorem 3.1. *Let $1 < p \leq \infty$ and q satisfy $1/p + 1/q = 1$. Suppose that $\beta > (d-2q)/q$ in the case where $d > 2q$ and that $\beta > (d-2q)/2q$ in the case where $d \leq 2q$.*

Then we have for $T \in H_p^{\beta}$ that

$$\lim_{\varepsilon \to 0} A_T^{\varepsilon}(a : t,\omega) = A_T(a : t,\omega) \qquad \text{in } L^2(dP_x),$$

holds.

Proof. Without loss of generality, we can assume that the Brownian motion starts from zero.

$$I = E_0[(A_T^{\varepsilon}(a : t, \omega))^2]$$

$$= 2E_0\left[\int_0^t ds \int_s^t du T * \rho_\varepsilon(B_u - a) T * \rho_\varepsilon(B_s - a)\right]$$

$$= 2\int dy_1 \int dy_2 \int_0^t ds \int_s^t du T * \rho_\varepsilon(y_1) T * \rho_\varepsilon(y_2) p(s, y_1 + a) p(u - s, y_2 - y_1).$$

By Parseval's equality with respect to dy_2 and dy_1, we have

$$I = \frac{2}{(2\pi)^{2d}} \int d\lambda_1 \int d\lambda_2 \int_0^t ds \int_s^t du \overline{\widehat{T * \rho_\varepsilon}(\lambda_1) \widehat{T * \rho_\varepsilon}(\lambda_2)}$$

$$\times e^{-(|\lambda_1 + \lambda_2|^2/2)s - (|\lambda_2|^2/2)(u-s) - i(\lambda_1 + \lambda_2)\cdot a}.$$

Note that $\widehat{T * \rho_\varepsilon}(\lambda) = \hat{T}(\lambda)\hat{\rho}_\varepsilon(\lambda)$ holds. Then

$$I = \frac{2}{(2\pi)^{2d}} \int d\lambda_1 \int d\lambda_2 \overline{\hat{T}(\lambda_1)\hat{\rho}_\varepsilon(\lambda_1)\hat{T}(\lambda_2)\hat{\rho}_\varepsilon(\lambda_2)} e^{-i(\lambda_1 + \lambda_2)\cdot a}$$

$$\times \int_0^t ds \int_s^t du e^{-(|\lambda_1 + \lambda_2|^2/2)s - (|\lambda_2|^2/2)(u-s)}.$$

By $|e^{ia}| = 1$ we get

$$|I| \le 2\left(\sup_{|\lambda| \le N} |\hat{\rho}_\varepsilon(\lambda)|\right)^2 \int\int_{\Gamma_N} d\lambda_1 d\lambda_2 |\hat{T}(\lambda_1)\hat{T}(\lambda_2)| \int_0^t ds \int_0^t du e^{-(|\lambda_1 + \lambda_2|^2/2)s - (|\lambda_2|^2/2)u}$$

$$+ 2(\|\hat{\rho}_\varepsilon(\lambda)\|_\infty)^2 \int\int_{\Gamma_N^c} d\lambda_1 d\lambda_2 |\hat{T}(\lambda_1)\hat{T}(\lambda_2)| \int_0^t ds \int_0^t du e^{-(|\lambda_1 + \lambda_2|^2/2)s - (|\lambda_2|^2/2)u}.$$

Now we set

$$J = \int_0^t e^{-ks} ds = \frac{1}{k}(1 - e^{-kt}),$$

where we suppose $Re(k) \ge 0$. If $|k| > 1$, then $1 + |k| \ge 2$. Using $|1 - e^{-kt}| \le 2$, we have

$$|J| \le \frac{2}{|k|} \le \frac{4}{1 + |k|}.$$

If $|k| \le 1$, we have

$$|J| \le t \le \frac{2t}{1 + |k|}.$$

Thus we get

$$\left|\int_0^t e^{-ks} ds\right| \le \frac{C}{1 + |k|} \qquad (k \in \mathbf{C}, Re(k) \ge 0), \tag{3.1}$$

where $C = \max(4, 2t)$. We obtain

$$|I| \leq 2\left(\sup_{|\lambda|\leq N} |\hat{\rho}_\varepsilon(\lambda)|\right)^2 t^2 \int\int_{\Gamma_N} d\lambda_1 d\lambda_2 |\hat{T}(\lambda_1)\hat{T}(\lambda_2)|$$

$$+ 2C^2(\|\hat{\rho}_\varepsilon(\lambda)\|_\infty)^2 \int\int_{\Gamma_N^c} d\lambda_1 d\lambda_2 |\hat{T}(\lambda_1)\hat{T}(\lambda_2)|(1+|\lambda_1+\lambda_2|^2)^{-1}(1+|\lambda_2|^2)^{-1}$$

$$= 2t^2\left(\sup_{|\lambda|\leq N} |\hat{\rho}_\varepsilon(\lambda)|\right)^2 I_1 + 2C^2(\|\hat{\rho}_\varepsilon(\lambda)\|_\infty)^2 I_2(\Gamma_N^c), \qquad \text{say.}$$

By Hölder's inequality we get

$$\text{(3.2)} \qquad I_1 \leq (\|T\|_{H_p^\beta})^2 \left(\int_{|\lambda|\leq N} d\lambda (1+|\lambda|^2)^{-q(\beta/2)}\right)^{2/q}.$$

Therefore I_1 is finite for any β.

Now we estimate $I_2(\Gamma_N^c)$. First we consider $I_2(\mathbf{R}^d \times \mathbf{R}^d) = I_2$. We apply Hölder's inequality to I_2.

$$\text{(3.3)} \quad I_2 = \int d\lambda_1 \int d\lambda_2 |\hat{T}(\lambda_1)\hat{T}(\lambda_2)|(1+|\lambda_1+\lambda_2|^2)^{-1}(1+|\lambda_2|^2)^{-1}$$

$$\leq (\|T\|_{H_p^\beta})^2$$

$$\times \left(\int d\lambda_1 \int d\lambda_2 (1+|\lambda_1|^2)^{-q\beta/2}(1+|\lambda_2|^2)^{-q-(q\beta/2)}(1+|\lambda_1+\lambda_2|^2)^{-q}\right)^{1/q}$$

$$\leq (\|T\|_{H_p^\beta})^2$$

$$\times \left(\int d\mu_1 \int d\mu_2 (1+|\mu_1-\mu_2|^2)^{-q\beta/2}(1+|\mu_2|^2)^{-q-(q\beta/2)}(1+|\mu_1|^2)^{-q}\right)^{1/q}.$$

We apply (2.6) to I_2. If β satisfies

$$\beta > \frac{d-2q}{2q} \qquad \text{in the case where } d \leq 2q$$

and

$$\beta > \frac{d-2q}{q} \qquad \text{in the case where } d > 2q,$$

then I_2 is finite, therefore we can make $I_2(\Gamma_N^c)$ small for sufficiently large N.

Now we note that $\sup_{|\lambda|\leq N} |\hat{\rho}_\varepsilon - \hat{\rho}_{\varepsilon'}| \to 0$ as $\varepsilon, \varepsilon' \to 0$ and $\|\rho_\varepsilon - \rho_{\varepsilon'}\|_\infty \leq 2$. Since

$$A_T^\varepsilon(a; t, \omega) - A_T^{\varepsilon'}(a : t, \omega) = \int_0^t T * (\rho_\varepsilon - \rho_{\varepsilon'})(B_s - a) ds,$$

$A_T^\varepsilon(a:t,\omega)$ is a Cauchy sequence in $L^2(dP_x)$ and $A_T^\varepsilon(a:t,\omega)$ converges $A_T(a:t,\omega)$ in $L^2(dP_x)$.

If $p=2$ then we can improve Theorem 3.1 as follows:

Theorem 3.2. *Suppose that $\beta \geq -1$ in the case where $d=1$ and that $\beta > (d-4)/2$ in the case where $d \geq 2$.*
For $T \in H_2^\beta$, tending ε to zero,

$$A_T^\varepsilon(a:t,\omega) \to A_T(a:t,\omega) \qquad \text{in } L^2(dP_x).$$

Proof. For the proof, it is sufficient to show that (3.3) is finite.
We set

$$\Lambda_1 = \left\{(\lambda_1,\lambda_2) : \frac{|\lambda_1|}{2} < |\lambda_1+\lambda_2|\right\}$$

and

$$\Lambda_2 = \left\{(\lambda_1,\lambda_2) : \frac{|\lambda_1|}{2} \geq |\lambda_1+\lambda_2|\right\}.$$

Then

$$\begin{aligned} I_2 &= \iint_{\Lambda_1} d\lambda_1 d\lambda_2 |\hat{T}(\lambda_1)\hat{T}(\lambda_2)|(1+|\lambda_1+\lambda_2|^2)^{-1}(1+|\lambda_2|^2)^{-1} \\ &\quad + \iint_{\Lambda_2} d\lambda_1 d\lambda_2 |\hat{T}(\lambda_1)\hat{T}(\lambda_2)|(1+|\lambda_1+\lambda_2|^2)^{-1}(1+|\lambda_2|^2)^{-1} \\ &= J_{\Lambda_1} + J_{\Lambda_2}, \qquad \text{say.} \end{aligned}$$

First, we estimate J_{Λ_1}. We have

$$\begin{aligned} J_{\Lambda_1} &\leq C_1\left(\int |\hat{T}(\lambda)|(1+|\lambda|^2)^{-1}d\lambda\right)^2 \\ &\leq C_1(\|T\|_{H_2^\beta})^2 \int (1+|\lambda|^2)^{-2-\beta} d\lambda. \end{aligned} \tag{3.4}$$

Next, we estimate J_{Λ_2}.

$$\begin{aligned} J_{\Lambda_2} &= \iint_{\Lambda_2} d\lambda_1 d\lambda_2 |\hat{T}(\lambda_1)|(1+|\lambda_1|^2)^{\beta/2}|\hat{T}(\lambda_2)|(1+|\lambda_2|^2)^{\beta/2} \\ &\quad \times (1+|\lambda_1|^2)^{-\beta/2}(1+|\lambda_2|^2)^{-\beta/2}(1+|\lambda_1+\lambda_2|^2)^{-1}(1+|\lambda_2|^2)^{-1} \\ &= \iint_{\Lambda_2} d\lambda_1 d\lambda_2 |\hat{T}(\lambda_1)|(1+|\lambda_1|^2)^{\beta/2}|\hat{T}(\lambda_2)|(1+|\lambda_2|^2)^{\beta/2} L, \qquad \text{say.} \end{aligned}$$

Let (λ_1,λ_2) belong to Λ_2. Then we get $|\lambda_1+\lambda_2| \leq |\lambda_2|$ and $2|\lambda_1+\lambda_2| \leq |\lambda_1|$. If $\beta \geq 0$, then we have

$$L \leq C_2(1+|\lambda_1+\lambda_2|^2)^{-2-\beta} \qquad \text{for some positive constant } C_2.$$

If $-1 \leq \beta < 0$, then using $|\lambda_2| \leq |\lambda_1+\lambda_2| + |\lambda_1| \leq 3/2|\lambda_1|$ and $|\lambda_1| \leq |\lambda_1+\lambda_2| + |\lambda_2| \leq 2|\lambda_2|$, we have

$$\begin{aligned} L &\leq C_3(1+|\lambda_1|^2)^{-1-\beta}(1+|\lambda_1+\lambda_2|^2)^{-1} \\ &\leq C_4(1+|\lambda_1+\lambda_2|^2)^{-2-\beta}. \end{aligned}$$

Hence, if $\beta \geq -1$ we get

$$L \leq C_5(1+|\lambda_1+\lambda_2|^2)^{-2-\beta},$$

where $C_5 = \max(C_2, C_4)$.

Thus, by change of variables ($\lambda_1+\lambda_2 = \mu$ and $\lambda_1 = \lambda$)

$$J_{\Lambda_2} \leq C_5 \int\int d\lambda d\mu |\hat{T}(\lambda)|(1+|\lambda|^2)^{\beta/2}|\hat{T}(\mu-\lambda)|(1+|\mu-\lambda|^2)^{\beta/2}(1+|\mu|^2)^{-2-\beta}.$$

Using Schwarz's inequality for the integration with respect to λ, we get

$$J_{\Lambda_2} \leq C_5(\|T\|_{H_2^\beta})^2 \int d\mu(1+|\mu|^2)^{-2-\beta}. \tag{3.5}$$

Thus we obtain a sufficient condition for the finiteness of I_2,

$$\beta \geq -1 \qquad \text{and} \qquad \beta > \frac{d-4}{2}$$

by (3.4) and (3.5).

If $p = 1$ we have the following theorem.

Theorem 3.3. *For $T \in H_1^\beta$, tending ε to zero,*

$$A_T^\varepsilon(a:t,\omega) \to A_T(a:t,\omega) \qquad \text{in } L^2(dP_x),$$

where we take $\beta \geq -1$.

Proof. For the proof of this result, it is sufficient to show (3.3) is finite. By Hölder's inequality we have

$$I_2 \leq (\|T\|_{H_1^\beta})^2 \|(1+|\lambda_1|^2)^{-\beta/2}(1+|\lambda_2|^2)^{-1-(\beta/2)}(1+|\lambda_1+\lambda_2|^2)^{-1}\|_\infty.$$

If $\beta \geq 0$, then clearly I_2 is finite. We consider the case of $\beta < 0$. We set

$$L = (1+|\lambda_1|^2)^{-\beta/2}(1+|\lambda_2|^2)^{-1-(\beta/2)}(1+|\lambda_1+\lambda_2|^2)^{-1}.$$

We consider Λ_1 and Λ_2 which are appeared in Theorem 3.2. First, we consider the case of (λ_1, λ_2) belongs to Λ_1. That is, $|\lambda_1| \leq 2|\lambda_1+\lambda_2|$. Thus we have

$$L \leq C_6(1+|\lambda_1|^2)^{-(\beta/2)-1}(1+|\lambda_2|^2)^{-1-(\beta/2)}. \tag{3.6}$$

Second, we consider the case of (λ_1, λ_2) belongs to Λ_2. Recall that $|\lambda_1| \le 2|\lambda_2|$. Thus we have

$$L \le C_7(1 + |\lambda_1|^2)^{-\beta-1}(1 + |\lambda_1 + \lambda_2|^2)^{-1}. \tag{3.7}$$

Therefore using (3.6) and (3.7), for the finiteness of I_2 we take $\beta \ge -1$.

Since the convergence of A_T^ε is in L^2, we can take a subsequence $\{A_T^{\varepsilon'}\}$ to converge almost surely. Thus we take the limit $A_T(a : t, \omega)$ as the almost everywhere convergence of $A_T^\varepsilon(a : t, \omega)$.

Next we discuss the (a, t)-joint continuity of $A_T(a : t, \omega)$.

Theorem 3.4. *Let $T \in H_p^\beta$ $(1 < p \le \infty)$, where we take β as Theorem 3.1. Suppose that $\delta = \min(1, (q\beta - d + 2q)/2q)$ in the case where $d > 2q$ and that $\delta = \min(1, (2q\beta - d + 2q)/2q)$ in the case where $d \le 2q$. Then $A_T(a : t, \omega)$ has (a, t)-jointly continuous modification, which is locally Hölder-continuous with exponent γ, where $0 < \gamma < \delta$.*

Proof. We will estimate

$$E_x[(A_T^\varepsilon(a : t, \omega) - A_T^\varepsilon(b : s, \omega))^{2n}]$$

and then we apply Kolmogorov–Čentsov theorem to get the joint continuity.

Without loss of generality, we suppose that $t > s$ and Brownian motion starts from zero and $b = 0$.

We set

$$\begin{aligned} &E_0[(A_T^\varepsilon(a : t, \omega) - A_T^\varepsilon(0 : s, \omega))^{2n}] \\ &\le 2^{2n}|E_0[(A_T^\varepsilon(a : t, \omega) - A_T^\varepsilon(0 : t, \omega))^{2n}]| \\ &\quad + 2^{2n}|E_0[(A_T^\varepsilon(0 : t, \omega) - A_T^\varepsilon(0 : s, \omega))^{2n}]| \\ &= 2^{2n}|I_a| + 2^{2n}|I_t|, \qquad \text{say.} \end{aligned}$$

First we estimate I_a.

$$\begin{aligned} I_a = (2n)! &\int dy_1 \dots \int dy_{2n} \\ &\times (T * \rho_\varepsilon(y_1 - a) - T * \rho_\varepsilon(y_1)) \dots (T * \rho_\varepsilon(y_{2n} - a) - T * \rho_\varepsilon(y_{2n})) \\ &\times \int_0^t du_1 \int_{u_1}^t du_2 \dots \int_{u_{2n-1}}^t du_{2n} p(u_{2n} - u_{2n-1}, y_{2n} - y_{2n-1}) \dots p(u_1, y_1). \end{aligned}$$

Then we set

$$F_n = \prod_{i=1}^n (T * \rho_\varepsilon(y_i - a) - T * \rho_\varepsilon(y_i))$$

and

$$P_n = \prod_{i=1}^{n} p(u_i - u_{i-1}, y_i - y_{i-1}) \qquad (\text{setting } y_0 = u_0 = 0).$$

We have

$$I_a = (2n)! \int dy_1 \dots \int dy_{2n} \int_0^t du_1 \int_{u_1}^t du_2 \dots \int_{u_{2n-1}}^t du_{2n} F_{2n-1} P_{2n-1}$$
$$\times (T * \rho_\varepsilon(y_{2n} - a) - T * \rho_\varepsilon(y_{2n}))(p(u_{2n} - u_{2n-1}, y_{2n} - y_{2n-1})).$$

By Parseval's equality with respect to dy_{2n}, we have

$$I_a = \frac{(2n)!}{(2\pi)^d} \int dy_1 \dots \int dy_{2n-1} \int d\lambda_{2n} \int_0^t du_1 \int_{u_1}^t du_2 \dots \int_{u_{2n-1}}^t du_{2n} F_{2n-1} P_{2n-1}$$
$$\times \overline{\widehat{T * \rho_\varepsilon}(\lambda_{2n})} e^{-(|\lambda_{2n}|^2/2)(u_{2n} - u_{2n-1}) + i\lambda_{2n} \cdot y_{2n-1}} \overline{(e^{i\lambda_{2n} \cdot a} - 1)}$$
$$= \frac{(2n)!}{(2\pi)^d} \int dy_1 \dots \int dy_{2n-1} \int d\lambda_{2n} \int_0^t du_1 \int_{u_1}^t du_2 \dots \int_{u_{2n-1}}^t du_{2n} F_{2n-2} P_{2n-2}$$
$$\times (T * \rho_\varepsilon(y_{2n-1} - a) - T * \rho_\varepsilon(y_{2n}))(p(u_{2n-1} - u_{2n-2}, y_{2n-1} - y_{2n-2})$$
$$\times \overline{\widehat{T\rho_\varepsilon}(\lambda_{2n})} e^{-(|\lambda_{2n}|^2/2)(u_{2n} - u_{2n-1}) + i\lambda_{2n} \cdot y_{2n-1}} \overline{(e^{i\lambda_{2n} \cdot a} - 1)}$$

By Parseval's equality with respect to dy_{2n-1}, we have

$$I_a = \frac{(2n)!}{(2\pi)^{2d}} \int dy_1 \dots \int dy_{2n-2} \int d\lambda_{2n-1} \int d\lambda_{2n} \int_0^t du_1 \int_{u_1}^t du_2 \dots \int_{u_{2n-1}}^t du_{2n}$$
$$\times F_{2n-2} P_{2n-2} \overline{\widehat{T * \rho_\varepsilon}(\lambda_{2n}) \widehat{T * \rho_\varepsilon}(\lambda_{2n-1})}$$
$$\times e^{-(|\lambda_{2n}|^2/2)(u_{2n} - u_{2n-1}) - (|\lambda_{2n} + \lambda_{2n-1}|^2/2)(u_{2n-1} - u_{2n-2})}$$
$$\times e^{i(\lambda_{2n} + \lambda_{2n-1}) \cdot y_{2n-2}} \overline{(e^{i\lambda_{2n} \cdot a} - 1)(e^{i\lambda_{2n-1} \cdot a} - 1)}.$$

Using Parseval's equality with respect to $dy_{2n-2} \dots dy_1$ in a similar way of above, we get

$$I_a = \frac{(2n)!}{(2\pi)^{2nd}} \int d\lambda_1 \dots \int d\lambda_{2n} \int_0^t du_1 \int_{u_1}^t du_2 \dots \int_{u_{2n-1}}^t du_{2n}$$
$$\times \overline{\hat{T}(\lambda_{2n}) \dots \hat{T}(\lambda_1) \hat{\rho}_\varepsilon(\lambda_{2n}) \dots \hat{\rho}_\varepsilon(\lambda_1)}$$
$$\times e^{-(|\lambda_{2n}|^2/2)(u_{2n} - u_{2n-1}) - (|\lambda_{2n} + \lambda_{2n-1}|^2/2)(u_{2n-1} - u_{2n-2}) - \dots - (|\lambda_{2n} + \dots + \lambda_1|^2/2) u_1}$$
$$\times \overline{(e^{i\lambda_{2n} \cdot a} - 1)(e^{i\lambda_{2n-1} \cdot a} - 1) \dots (e^{i\lambda_1 \cdot a} - 1)}.$$

By using Hölder's inequality, we get

$$|I_a| \le \frac{(2n)!}{(2\pi)^{2nd}} (\|T\|_{H_p^\beta})^{2n} (\|\hat{\rho}_\varepsilon\|_\infty)^{2n}$$
$$\times \Bigg(\int d\lambda_1 \dots \int d\lambda_{2n} (1+|\lambda_1|^2)^{-q\beta/2} \dots (1+|\lambda_{2n}|^2)^{-q\beta/2}$$
$$\times \Bigg(\int_0^t du_1 \int_{u_1}^t du_2 \dots \int_{u_{2n-1}}^t du_{2n}$$
$$\times e^{-(|\lambda_{2n}|^2/2)(u_{2n}-u_{2n-1})-(|\lambda_{2n}+\lambda_{2n-1}|^2/2)(u_{2n-1}-u_{2n-2})-\dots-(|\lambda_{2n}+\dots+\lambda_1|^2/2)u_1} \Bigg)^q$$
$$\times |e^{i\lambda_{2n}\cdot a}-1|^q |e^{i\lambda_{2n-1}\cdot a}-1|^q \dots |e^{i\lambda_1\cdot a}-1|^q \Bigg)^{1/q}.$$

We change the variables λ_i $(1 \le i \le 2n)$ to μ_j $(1 \le j \le 2n)$ as follows:

$$\mu_{2n} = \lambda_{2n}$$
$$\mu_{2n-1} = \lambda_{2n} + \lambda_{2n-1}$$
$$\dots$$
$$\mu_1 = \lambda_{2n} + \lambda_{2n-1} + \dots + \lambda_1.$$

Then we get

$$|I_a| \le \frac{(2n)!}{(2\pi)^{2nd}} (\|T\|_{H_p^\beta})^{2n} (\|\hat{\rho}_\varepsilon\|_\infty)^{2n}$$
$$\times \Bigg(\int d\mu_1 \dots \int d\mu_{2n} (1+|\mu_1-\mu_2|^2)^{-q\beta/2} \dots (1+|\mu_{2n-1}-\mu_{2n}|^2)^{-q\beta/2}$$
$$\times (1+|\mu_{2n}|^2)^{-q\beta/2}$$
$$\times \Bigg(\int_0^t du_1 \int_{u_1}^t du_2 \dots \int_{u_{2n-1}}^t du_{2n} e^{-(|\mu_{2n}|^2/2)(u_{2n}-u_{2n-1})-\dots-(|\mu_2|^2/2)(u_2-u_1)-(|\mu_1|^2/2)u_1} \Bigg)^q$$
$$\times |e^{i\mu_{2n}\cdot a}-1|^q |e^{i(\mu_{2n-1}-\mu_{2n})\cdot a}-1|^q \dots |e^{i(\mu_1-\mu_2)\cdot a}-1|^q \Bigg)^{1/q}$$
$$\le \frac{(2n)!}{(2\pi)^{2nd}} (\|T\|_{H_p^\beta})^{2n} (\|\hat{\rho}_\varepsilon\|_\infty)^{2n}$$
$$\times \Bigg(\int d\mu_1 \dots \int d\mu_{2n} (1+|\mu_1-\mu_2|^2)^{-q\beta/2} \dots (1+|\mu_{2n-1}-\mu_{2n}|^2)^{-q\beta/2}$$
$$\times (1+|\mu_{2n}|^2)^{-q\beta/2}$$
$$\times \Bigg(\int_0^t du_1 \int_0^t du_2 \dots \int_0^t du_{2n} e^{-(|\mu_{2n}|^2/2)u_{2n}-\dots-(|\mu_1|^2/2)u_1} \Bigg)^q$$
$$\times |e^{i\mu_{2n}\cdot a}-1|^q |e^{i(\mu_{2n-1}-\mu_{2n})\cdot a}-1|^q \dots |e^{i(\mu_1-\mu_1)\cdot a}-1|^q \Bigg)^{1/q}.$$

Now we notice that the inequality (3.1) and for any $1 \geq l_a > 0$

$$|e^{i\mu\cdot a} - 1| \leq K|a|^{l_a}(1 + |\mu|^2)^{l_a/2} \qquad \text{for some positive constant } K > 0.$$

Then we apply these inequalities to I_a:

$$|I_a| \leq \frac{(2n)!K_1}{(2\pi)^{2nd}}(\|T\|_{H_p^\beta})^{2n}(\|\hat{\rho}_\varepsilon\|_\infty)^{2n}|a|^{2nl_a}$$

$$\times\left(\int d\mu_1 \dots \int d\mu_{2n}(1 + |\mu_1 - \mu_2|^2)^{-(q\beta/2)+(ql_a/2)} \dots (1 + |\mu_{2n-1} - \mu_{2n}|^2)^{-(q\beta/2)+(ql_a/2)}\right.$$

$$\left.\times (1 + |\mu_1|^2)^{-q} \dots (1 + |\mu_{2n-1}|^2)^{-q}(1 + |\mu_{2n}|^2)^{-q(1-(l_a/2))-(q\beta/2)}\right)^{1/q},$$

where $K_1 = (CK)^{2n}$.

We first estimate of this integral. We set

$$|I_a^{2n}| = \int d\mu_{2n}(1 + |\mu_{2n-1} - \mu_{2n}|^2)^{-(q\beta/2)+(ql_a/2)}(1 + |\mu_{2n}|^2)^{-q(1-(l_a/2))-(q\beta/2)}.$$

Now we apply (2.6) to this integral. If β satisfies

$$2\left(q\left(1 - \frac{l_a}{2}\right) + \frac{q\beta}{2}\right) + 2\frac{q\beta}{2} - 2\frac{ql_a}{2} > d \qquad \text{and} \qquad 2\left(q\left(1 - \frac{l_a}{2}\right) + \frac{q\beta}{2}\right) > d,$$

then we get

$$|I_a^{2n}| \asymp (1 + |\mu_{2n-1}|^2)^{-(q\beta/2)+(ql_a/2)}.$$

Therefore, by induction, we reach the integral

$$\int d\mu_1(1 + |\mu_1|^2)^{-q(1-l_a/2)-(q\beta/2)}.$$

For the finiteness of this integral, we set the following condition:

$$2\left(q\left(1 - \frac{l_a}{2}\right) + \frac{q\beta}{2}\right) > d.$$

Thus we obtain the condition

$$\beta > \max\left(\frac{d - 2q + l_a}{q}, \frac{d - 2(q - l_a)}{2q}\right) \tag{3.8}$$

and

$$|I_a| \leq C_8|a|^{2nl_a}(\|T\|_{H_p^\beta})^{2n}(\|\hat{\rho}_\varepsilon\|_\infty)^{2n}, \tag{3.9}$$

where C_8 is a positive constant and only depends on n and d.

Next we estimate I_t in a similar way of I_a. But we notice that for any $l_t > 0$, $Re(k) > 0$ and fixed $t > 0$, there exists a positive constant K_t such that

$$\left|\int_0^s e^{-ku}\,du\right| \le K_t\left(\frac{s^{l_t}}{1+|k|}\right)^{1/(l_t+1)} \qquad \text{for } s \in [0,t].$$

Because it is easy to see that

$$s^{-l_t/(l_t+1)}(1+|k|)^{1/(l_t+1)}\left|\int_0^s e^{-ku}\,du\right|$$

is a bounded function on $(s,|k|) \in [0,t] \times [0,\infty)$. Then we have

$$\begin{aligned}|I_t| \le{}& \frac{(2n)!K_2}{(2\pi)^{2nd}}|t-s|^{2n(l_t/(l_t+1))}(\|T\|_p^\beta)^{2n}(\|\hat{\rho}_\varepsilon\|_\infty)^{2n}\\ &\times\Bigg(\int d\mu_1\ldots\int d\mu_{2n}(1+|\mu_1-\mu_2|^2)^{-q\beta/2}\ldots(1+|\mu_{2n-1}-\mu_{2n}|^2)^{-q\beta/2}\\ &\times(1+|\mu_1|^2)^{-q/(l_t+1)}\ldots(1+|\mu_{2n-1}|^2)^{-q/(l_t+1)}(1+|\mu_{2n}|^2)^{-(q\beta/2)-(q/(l_t+1))}\Bigg)^{1/q},\end{aligned}$$

where $K_2 = K_t^{2n}$.

We apply (2.6) to the integral with respect to $d\mu_1\ldots d\mu_n$ of I_t. Then we obtain the condition

$$\beta > \max\left(\frac{d-\dfrac{2q}{l_t+1}}{q}, \frac{d-\dfrac{2q}{l_t+1}}{2q}\right) \tag{3.10}$$

for the finiteness of this integral and

$$|I_t| \le C_t|t-s|^{2n(l_t/(l_t+1))}(\|T\|_{H_p^\beta})^{2n}(\|\hat{\rho}_\varepsilon\|_\infty)^{2n}, \tag{3.11}$$

where C_t is a positive constant and only depends on n, t and d.

Therefore by (3.8) and (3.10) we make l_a and l_t satisfy the following equalities:

$$d-\frac{2q}{l_t+1} = d-2q(1-l_a) \text{ and } d-\frac{2q}{l_t+1} = d-2q+ql_a.$$

Since l_a and l_t are positive, if β satisfies the condition in Theorem 3.1, then we obtain

$$\begin{aligned}&|E_0[(A_T^\varepsilon(a:t,\omega)-A_T^\varepsilon(0:s,\omega))^{2n}]|\\ &\qquad\le C_{BM}(|a|^{2n\delta}+|t-s|^{2n\delta})(\|T\|_{H_p^\beta})^{2n}(\|\hat{\rho}_\varepsilon\|_\infty)^{2n}\end{aligned} \tag{3.12}$$

where we take δ as follows and $C_{BM} = \max(C_8, C_t)$.

If $d > 2q$, then for $\beta > (d-2q)/q$ we take δ as $(q\beta-d+2q)/2q \ge \delta$ by (3.8) or (3.10) and if $d \le 2q$, then for $\beta > (d-2q)/2q$ we take δ as $(2q\beta-d+2q)/2q \ge \delta$ by (3.8) or (3.10).

Thus tending ε to zero, we get (a,t)-joint continuity of $A_T(a:t,\omega)$ by Kolmogorov–Čentsov theorem.

But we cannot still get the result corresponding to Theorem 3.2 and Theorem 3.3.

By Theorem 3.4, we can take the (a,t)-jointly continuous modification of $A_T(a:t,\omega)$.

Now we discuss the existence and (a,t)-joint continuity of $A_T(a:t,\omega)$ in the cases of $p=\infty$ and $p=2$ for $d=1$.

Example 3.5. We set $d=1$.

If $p=\infty$, we take $T\in H_\infty^\beta$, where $\beta>-1/2$. $A_T(a:t,\omega)$ has (a,t)-jointly continuous modification which is locally Hölder continuous with exponent $0<\gamma<\min(1,(2\beta+1)/2)$.

If $p=2$, we take $T\in H_2^\beta$, where $\beta>-3/4$. $A_T(a:t,\omega)$ has (a,t)-jointly continuous modification which is locally Hölder continuous with exponent $0<\gamma<\min(1,(4\beta+3)/4)$.

Let $T=\delta_0$. Then T belongs to $H_\infty^0\cap H_2^{-1/2-\varepsilon}$, where $\varepsilon>0$. $A_T(a:t,\omega)$ is the Brownian local time. $A_T(a:t,\omega)$ has (a,t)-jointly continuous modification which is locally Hölder continuous with exponent $0<\gamma<1/2$ in the case of $T=\delta_0\in H_\infty^0$ and exponent $0<\gamma<1/4$ in the case of $T=\delta_0\in H_2^{-1/2-\varepsilon}$, where $\varepsilon>0$.

Therefore we conclude that the exponent is $1/2-\varepsilon$, which agrees to the result in [6].

Let $T=v.p.\dfrac{1}{x}$. Then T belongs to $H_\infty^0\cap H_2^{-1/2-\varepsilon}$, where $\varepsilon>0$. Thus $A_T(a:t,\omega)$ has (a,t)-jointly continuous modification which has the same exponent in the case of $T=\delta_0$.

4. Energy of $A_T(a:t,\omega)$

In this section we will discuss the energy of $A_T(a:t,\omega)$. First we define the energy of additive functionals in [4].

Definition 4.1. For any additive functional $A(a:t,\omega)$, we set

$$e(A)=\lim_{t\downarrow 0}\frac{1}{2t}E_m[(A(a:t,\omega))^2]$$

whenever the limit exists. We call $e(A)$ the energy of $A(a:t,\omega)$.

For the Brownian motion and stable processes, we take $m=dx$.

To discuss the energy of $A_T(a:t,\omega)$ we prepare the convergence theorem of $A_T^\varepsilon(a:t,\omega)$ in $L^2(dP_x\times dx)$.

Theorem 4.2. *Suppose that* $2<p\leq\infty$. *For* $T\in H_p^\beta$, *tending* ε *to zero,*

$$A_T^\varepsilon(a:t,\omega)\to A_T(a:t,\omega)\qquad in\ L^2(dP_x\times dx),$$

where we take $\beta>(d-2q)/2q$. *Here* q *is Hölder conjugate of* p.

Proof. We only show that the following integral is bounded. The detail of the proof is the same in Theorem 3.1.

$$
\begin{aligned}
(4.1) \qquad I &= E_{dx}[(A_T^{\varepsilon}(a:t,\omega))^2] \\
&= \int dx E_x[(A_T^{\varepsilon}(a:t,\omega))^2] \\
&= \int dx E_0[(A_T^{\varepsilon}(a+x:t,\omega))^2] \\
&= \int dx E_0\left[2\int_0^t ds \int_s^t du T * \rho_{\varepsilon}(B_s - a - x) T * \rho_{\varepsilon}(B_u - a - x)\right].
\end{aligned}
$$

By Parseval's equality with respect to x, we have

$$
\begin{aligned}
I &= 2(2\pi)^{-d}\int d\lambda E_0\left[\int_0^t ds \int_s^t du |\widehat{T * \rho_{\varepsilon}}(\lambda)|^2 e^{-i\lambda\cdot(B_s - B_u)}\right] \\
&\leq 2(2\pi)^{-d}\int d\lambda \int_0^t ds \int_0^t du |\hat{T}(\lambda)\hat{\rho}_{\varepsilon}(\lambda)|^2 e^{-(|\lambda|^2/2)u} \\
&\leq 2(2\pi)^{-d} t \int d\lambda \int_0^t du |\hat{T}(\lambda)\hat{\rho}_{\varepsilon}(\lambda)|^2 e^{-(|\lambda|^2/2)u}.
\end{aligned}
$$

Using $|\int_0^t e^{-ku} du| \leq t$ for any $Re(k) > 0$ and (3.1), for $N > 0$ we have

$$
\begin{aligned}
(4.2) \qquad I &\leq 2(2\pi)^{-d} t\left(t \sup_{|\lambda|\leq N} |\hat{\rho}_{\varepsilon}(\lambda)|^2 \int_{|\lambda|<N} d\lambda |\hat{T}(\lambda)|^2\right. \\
&\qquad \left. + C\|\hat{\rho}_{\varepsilon}\|_{\infty}^2 \int_{|\lambda|\geq N} d\lambda |\hat{T}(\lambda)|^2 \frac{1}{1+|\lambda|^2}\right) \\
&= 2(2\pi)^{-d} t\left(t \sup_{|\lambda|<N} |\hat{\rho}_{\varepsilon}(\lambda)|^2 I_1 + \|\hat{\rho}_{\varepsilon}\|_{\infty}^2 I_2(|\lambda| \geq N)\right), \qquad \text{say.}
\end{aligned}
$$

By Hölder's inequality we get

$$
I_1 \leq (\|T\|_{H_p^{\beta}})^2 \left(\int_{|\lambda|<N} d\lambda (1+|\lambda|^2)^{-q\beta}\right)^{1/q}.
$$

Therefore I_1 is finite for any β.

Now we estimate $I_2(|\lambda| \geq N)$. However it is enough to get the bound of $I_2(|\lambda| \geq 0) = I_2$. Applying Hölder's inequality to I_2, we have

$$
(4.3) \qquad I_2 \leq (\|T\|_{H_p^{\beta}})^2 \left(\int d\lambda (1+|\lambda|^2)^{-q-q\beta}\right)^{1/q}.
$$

For the finiteness of this integral we get $2q + 2q\beta > d$. Thus if β satisfies $\beta > (d-2q)/2q$, then we can easily see that tending ε to zero, $\{A_T^{\varepsilon}(a:t,\omega)\}$ is a Cauchy sequence in $L^2(dP_x \times dx)$ and $A_T^{\varepsilon}(a:t,\omega)$ converges $A_T(a:t,\omega)$ in $L^2(dP_x \times dx)$.

If $p=2$, then (4.3) is

$$I_2 \leq (\|T\|_{H_2^{\beta}})^2 \|(1+|\lambda|^2)^{-1-\beta}\|_{\infty}.$$

Thus we have

Corollary 4.3. *For $T \in H_2^{\beta}$, tending ε to zero,*

$$A_T^{\varepsilon}(a:t,\omega) \to A_T(a:t,\omega) \qquad \text{in } L^2(dP_x \times dx),$$

where we take $\beta \geq -1$.

These results guarantee the existence of $A_T(a:t,\omega)$ for $T \in H_p^{\beta}$ wider than Theorem 3.1 and 3.2. Especially, $A_T(a:t,\omega)$ exists for any $T \in H_2^{-1}$ for any dimensions according to Corollary 4.3. However, the limit $A_T(a:t,\omega)$ exists almost everywhere P_x not for all P_x. Then we denote the limit by $A_T^{dx}(a:t,\omega)$ in this sense.

Now we show that $A_T^{dx}(a:t,\omega)$ has 0-energy for the same β.

Theorem 4.4. *Suppose that $\beta > (d-2q)/2q$ in the case where $p>2$ and that $\beta \geq -1$ in the case where $p=2$.*
Then, for any $T \in H_p^{\beta}$, we have $e(A_T^{dx})=0$.

Proof. By (4.2) we have

$$E_{dx}[(A_T^{\varepsilon}(a:t,\omega)^2]$$
$$\leq 2(2\pi)^{-d}t\left(t \sup_{|\lambda|\leq N} |\hat{\rho}_{\varepsilon}(\lambda)|^2 \int_{|\lambda|\leq N} d\lambda |\hat{T}(\lambda)|^2 + C\|\hat{\rho}_{\varepsilon}\|_{\infty}^2 \int_{|\lambda|>N} d\lambda \frac{|\hat{T}(\lambda)|^2}{1+|\lambda|^2}\right).$$

If β satisfies the condition, for sufficiently large N we make

$$\int_{|\lambda|>N} d\lambda \frac{|\hat{T}(\lambda)|^2}{1+|\lambda|^2}$$

small. For such N we take t independent of ρ_{ε} as making

$$t\int_{|\lambda|\leq N} d\lambda |\hat{T}(\lambda)|^2$$

small. These estimations are uniform in ε. On the other hand we know

$$A_T^{\varepsilon}(a:t,\omega) \to A_T^{dx}(a:t,\omega) \qquad \text{in } L^2(dP_x \times dx),$$

by Theorem 4.2 and Corollary 4.3.

Therefore we get

$$\lim_{t\downarrow 0}\frac{1}{2t}E_{dx}[(A_T(a:t,\omega)^2]=0.$$

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Appendix. proof of Lemma 2.7

Proof. We prove the case of $p \geq q > 0$. We set

$$L = (1 + |\mu|^2)^{-p}(1 + |\lambda + \mu|^2)^{-q},$$

$$\Lambda_1 = \left\{\mu : |\mu| \leq \frac{|\lambda|}{2}\right\},$$

$$\Lambda_2 = \left\{\mu : |\mu + \lambda| \leq \frac{|\lambda|}{2}\right\}$$

$$\Lambda_3 = \{\mu : |\mu| \leq 2|\lambda|\} - \Lambda_1 - \Lambda_2,$$

and

$$\Lambda_4 = (\Lambda_1 \cup \Lambda_2 \cup \Lambda_3)^c.$$

We will consider each case. First, we consider μ which belongs to Λ_4. Since

$$|\mu| \leq |\mu + \lambda| + |\lambda| \leq |\mu + \lambda| + \frac{1}{2}|\mu|$$

and

$$|\mu + \lambda| \leq |\mu| + |\lambda| \leq \frac{3}{2}|\mu|,$$

we get $|\mu + \lambda| \asymp |\mu|$.

Second, we consider μ which belongs to Λ_3. Since

$$\frac{1}{2}|\lambda| \leq |\mu| \leq 2|\lambda|$$

and

$$\frac{1}{2}|\lambda| \leq |\mu + \lambda| \leq 3|\lambda|,$$

we get $|\lambda| \asymp |\mu + \lambda| \asymp |\mu|$.

Third, we consider μ which belongs to Λ_1. Using

$$\frac{1}{2}|\lambda| \leq |\mu + \lambda| \leq 3|\lambda|,$$

we get $|\lambda + \mu| \asymp |\lambda|$.

Last, we consider μ which belongs to Λ_2. Using

$$\frac{1}{2}|\lambda| \leq |\mu| \leq 2|\lambda|,$$

we get $|\lambda| \asymp |\mu|$.

We consider the order of J in Lemma 2.7. First we suppose that $|\lambda| \leq 1$ and set

$$J = \int_{|\mu| \leq 2} \frac{d\mu}{(1+|\mu|^2)^p (1+|\lambda+\mu|^2)^q} + \int_{|\mu|>2} \frac{d\mu}{(1+|\mu|^2)^p (1+|\lambda+\mu|^2)^q}$$
$$= J(|\mu| \leq 2) + J(|\mu| > 2), \qquad \text{say.}$$

If $|\mu| > 2 \geq 2|\lambda|$, then μ belongs to A_4. Thus we get

$$J(|\mu| > 2) \asymp \int_{|\mu|>2} \frac{d\mu}{(1+|\mu|^2)^{p+q}} \asymp 1.$$

If $|\mu| \leq 2$, then we have

$$1 \geq L \geq \frac{1}{(1+2^2)^p (1+(2+1)^2)^q} = \frac{1}{5^p 10^q}.$$

Thus we get $J(|\mu| \leq 2) \asymp 1$. Therefore for $|\lambda| \leq 1$, we get $J \asymp 1$.

Next we suppose that $|\lambda| > 1$. We set

$$J(A_i) = \int_{A_i} \frac{d\mu}{(1+|\mu|^2)^p (1+|\lambda+\mu|^2)^q} \qquad (i = 1, 2, 3, 4).$$

First, we consider $J(A_4)$.

$$J(A_4) = \int_{A_4} \frac{d\mu}{(1+|\mu|^2)^p (1+|\lambda+\mu|^2)^q}$$
$$\asymp \int_{A_4} \frac{d\mu}{(1+|\mu+\lambda|^2)^{p+q}}$$
$$\asymp \frac{|\lambda|^d}{|\lambda|^{2p+2q}}$$
$$\asymp \frac{1}{(1+|\lambda|^2)^{p+q-(d/2)}}.$$

Second, we consider $J(A_3)$.

$$J(A_3) = \int_{A_3} \frac{d\mu}{(1+|\mu|^2)^p (1+|\lambda+\mu|^2)^q}$$
$$\asymp \int_{A_3} \frac{d\mu}{(1+|\lambda|^2)^{p+q}}$$
$$\asymp \frac{|\lambda|^d}{(1+|\lambda|^2)^{p+q}}$$
$$\asymp \frac{1}{(1+|\lambda|^2)^{p+q-(d/2)}}.$$

Now we consider the order of

$$K = \int_{|x| \le a} \frac{dx}{(1+|x|^2)^p},$$

where $a > 0$.

For a fixed $k > 0$, we have

$$K = \int_{|x| \le k} \frac{dx}{(1+|x|^2)^p} + \int_{k<|x| \le a} \frac{dx}{(1+|x|^2)^p}$$
$$\asymp 1 + \int_{k<|x| \le a} \frac{dx}{|x|^{2p}}.$$

If $2p > d$, we have

$$K \asymp 1.$$

If $2p = d$, we have

$$K \asymp 1 + \log a - \log k \asymp 1 + \log^+ a.$$

If $2p < d$, we have

$$K \asymp 1 + (a^{d-2p} - k^{d-2p}) \asymp a^{d-2p} \asymp \frac{1}{(1+a^2)^{p-(d/2)}}.$$

Keeping this discussion in mind, we return to our original problem.

$$J(\Lambda_1) = \int_{\Lambda_1} \frac{d\mu}{(1+|\mu|^2)^p(1+|\lambda+\mu|^2)^q}$$
$$\asymp \frac{1}{(1+|\lambda|^2)^q} \int_{\Lambda_1} \frac{d\mu}{(1+|\mu|^2)^p}$$
$$\asymp \begin{cases} \dfrac{1}{(1+|\lambda|^2)^{p+q-(d/2)}}, & 2p < d \\ \dfrac{1+\log^+|\lambda|}{(1+|\lambda|^2)^q}, & 2p = d \\ \dfrac{1}{(1+|\lambda|^2)^q}, & 2p > d. \end{cases}$$

$$J(\Lambda_2) = \int_{\Lambda_2} \frac{d\mu}{(1+|\mu|^2)^p(1+|\lambda+\mu|^2)^q}$$
$$\asymp \frac{1}{(1+|\lambda|^2)^p} \int_{\Lambda_2} \frac{d\mu}{(1+|\mu+\lambda|^2)^q}$$

$$\asymp \frac{1}{(1+|\lambda|^2)^p} \int_{A_1} \frac{d\nu}{(1+|\nu|^2)^q}$$

$$\asymp \begin{cases} \dfrac{1}{(1+|\lambda|^2)^{p+q-(d/2)}}, & 2q < d \\ \dfrac{1+\log^+|\lambda|}{(1+|\lambda|^2)^p}, & 2q = d \\ \dfrac{1}{(1+|\lambda|^2)^p}, & 2q > d\,. \end{cases}$$

Therefore we get Lemma 2.7.

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