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Weak approximation, Brauer and R-equivalence in algebraic groups over arithmetical fields

By

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Abstract

We prove some new relations between weak approximation and some rational equivalence relations (Brauer and R-equivalence) in algebraic groups over arithmetical fields. By using weak approximation and local–global approach, we compute completely the group of Brauer equivalence classes of connected linear algebraic groups over number fields, and also completely compute the group of R-equivalence classes of connected linear algebraic groups G, which either are defined over a totally imaginary number field, or contains no anisotropic almost simple factors of exceptional type ${}^{3,6}\mathrm{D}_4$, nor E_6. We discuss some consequences derived from these, e.g., by giving some new criteria for weak approximation in algebraic groups over number fields, by indicating a new way to give examples of non stably rational algebraic groups over local fields and application to norm principle. Some related questions and relations with groups of Brauer and R-equivalence classes over arbitrary fields of characteristic 0 are also discussed.

Introduction

Let G be a linear algebraic group defined over a field k. There are two closely related questions in the arithmetic theory of algebraic groups over fields: the question of weak approximation and that of rationality of a given G. It is very difficult to study such questions for arbitrary groups over arbitrary fields. One should restrict to some class of groups and fields which are convenient in application.

Let X be a smooth algebraic variety defined over a field k of characteristic 0, $\bar{X} = X \times \bar{k}$, where $\bar{k}$ is an algebraic closure of k. Denote by $\mathscr{B}r X$ the usual Brauer group of X, $\mathrm{Br}(X)$ the cohomological Brauer group $\mathrm{H}^2_{et}(X, \mathbf{G}_m)$ of X, $\mathrm{Br}_1(X) = \mathrm{Ker}\,(\mathrm{Br}(X) \to \mathrm{Br}(\bar{X}))$, $\mathrm{Br}_0(X) = \mathrm{Im}\,(\mathrm{Br}(k) \to \mathrm{Br}(X))$. Following Manin, Colliot-Thélène and Sansuc (see [M1], [M2], [CTS1], [CTS2]), one defines the Brauer equivalence and R-equivalence as follows. First we construct a smooth compactification $\tilde{X}$ of X over k and we define a pairing

$$\tilde{X}(k) \times \mathrm{Br}(\tilde{X}) \to \mathrm{Br}(k), (x, b) \mapsto b(x),$$

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where $b \in H^2_{et}(\tilde{X}, \mathbf{G}_m)$ and $b(x)$ is the equivalence class of central simple algebras over k, which is considered as an element of $\mathrm{Br}(k)$.

Two points $x, y \in \tilde{X}(k)$ are said to be *Brauer equivalent* (*Br-equivalent*) if for any $b \in \mathrm{Br}(\tilde{X})$, we have $b(x) = b(y)$. The equivalence relation on $X(k)$ induced from the *Br*-equivalence relation on $\tilde{X}(k)$ is called *Brauer equivalence relation* and we denote by $X(k)/Br$ the set of Brauer equivalent classes of $X(k)$. It was shown in [CTS1], p. 212, that the above definition does not depend on the choice of smooth compactification $\tilde{X}$. Two points $x, y \in X(k)$ are called *R-equivalent* if there is a sequence of points $z_i \in \tilde{X}(k), x = z_1, y = z_n$, such that for each pair z_i, z_{i+1} there is a k-rational map $f : \mathrm{P}^1 \to X$, regular at 0 and 1, with $f(0) = z_i$, $f(1) = z_{i+1}, 1 \leq i \leq n-1$). We denote by $X(k)/R$ the set of R-equivalent classes of $X(k)$.

In the definition of Brauer equivalence above, one may also restrict to the subgroup $\mathrm{Br}_1 X$ of $\mathrm{Br} X$ to get a weaker equivalence relation. However, if $\bar{X}$ is rational over $\bar{k}$ (which is the main case we are interested in), it is known (cf. e.g. [CTS1], Lemme 16) that these two notions coincide. Moreover in [loc.cit], Prop. 16, it was shown that the Brauer equivalence is weaker than R-equivalence, i.e., two points of $X(k)$, being R-equivalent, are necessarily Br-equivalent. In [loc.cit], basic theory of Brauer equivalence on tori defined over a field k of characteristic 0 has been developed. In particular, in the arithmetic case, i.e., when k is a local or global field, formulae for computations of the *group* $T(k)/Br$ are given and it turns out to be a birational invariant of T. Though $T(k)/Br$ is "computable", the group itself and its computation is in general non-trivial.

In a subsequent paper [S], Sansuc developed Brauer theory of linear algebaic groups G over number fields, and applied it to obtain certain fundamental sequences connecting various arithmetic (obstruction to weak approximation), cohomological (Tate–Shafarevich group) and geometric invariants (the first Galois cohomology of the Picard group of a smooth compactification of G over an algebraic closure of k) for connected linear algebraic groups G over number fields k.

In this paper we continue the approach taken by Colliot-Thélène and Sansuc, to obtain certain connections between the above arithmetic, cohomological and birational (Brauer and R-) invariants of connected linear algebraic groups G over local and global fields of characteristic 0. As it was pointed out above, in general, the group of Brauer equivalence classes of G is non-trivial, even in the case of tori. Therefore it is natural to ask what kind of analogs in the case of arbitrary connected linear algebraic groups one can have.

In this paper we recall some useful facts from the Brauer theory in Section 1. In Section 2 we discuss a relation between the defect (obstruction) in weak approximation and the groups of Brauer and R-equivalence classes of tori over number fields, and in Section 3 we extend some results obtained here to the general case of connected linear algebraic groups over number fields. In particular regarding the group $G(k)/Br$, we computed it completely, which in fact gives apriori (or preliminary) information on the group $G(k)/R$. In Section 4 we present our main results and applications to obtain some new criteria for weak approximation, by recovering, extending, and giving some analogs to some classical results obtained by Colliot–Thélène and Sansuc, Harder and Sansuc. Until recently it was not known whether the group $G(k)/R$ is always finite for any connected linear algebraic group G defined over a number field k. In his paper in 1993 ([G2]), Gille gave a proof of this finiteness properly by using his norm principle and Kato–Saito's Hasse principle in higher dimension class field theory. However it was not known how one can compute the actual group $G(k)/R$, nor even suggested how it might look like. One of main results of this section (and of the paper) is Theorem 4.12, which allows us not only to have a new elementary approach to this finiteness result for all connected linear algebaric groups G (without using difficult higher dimensional Hasse principle), but also, by using the corresponding result for tori done in [CTS1], to compute completely the group $G(k)/R$ for those connected linear algebraic groups G, which either are defined over a totally imaginary number field k, or contain no anisotropic almost simple factors of exceptional type ${}^{3,6}\mathrm{D}_4$, nor E_6. In these last critical cases, it reduces the computation of $G(k)/R$ to a particular case of a well-known Platonov–Margulis conjecture about the normal structure of almost simple simply connected groups over number fields.

After the completion of this paper (cf. earlier versions of the paper: Preprint ICTP (September 1997), Duke University E-print alg-geom/9711015), there appeared the paper [G3] where Gille gave detailed proof of main results of [G1], [G2] with some refinements. It turned out that these refinements have one small overlap with our paper (being indicated below). Also, the exact sequence relating the defect of weak approximation and groups of R-equivalence classes [T4] was also discussed. In this paper we use some concepts and techniques developed in [CTS1], [CTS2], [G1], [G2] (with complete exposition in [G3]), [S]. In certain sense, this paper is a complement to these works.

Notation. For a reductive group H, we call *torus quotient* of H the factor group of H by its semisimple part $[H,H]$. Any connected linear algebraic group G over a field k of characteristic 0 is a semidirect product $G = LR_u(G)$, where L is a Levi (reductive) k-subgroup of G, and $R_u(G)$ is the unipotent radical of G. L is unique up to conjugacy by elements from $G(k)$ and by convention, we call the *semisimple part of* G the semisimple part of some fixed Levi k-subgroup of G. Let S be a finite set of valuations of a global field k, and G a connected linear algebraic group defined over k. Denote by $Cl_S(G(k))$ the closure of the group $G(k)$ in the product topology, where $G(k)$ is embedded diagonally into the direct product $\prod_{v \in S} G(k_v)$ and $G(k_v)$ has the v-adic topology, induced from that of k_v. We say that G has *weak approximation with respect to* S (or *in* S) if $Cl_S(G(k)) = \prod_{v \in S} G(k_v)$, and *has weak approximation over* k if it is so for any finite S. Let

$$\mathrm{A}(S,G) = \prod_{v \in S} G(k_v)/Cl_S(G(k)), \qquad \mathrm{A}(G) = \prod_{v} G(k_v)/Cl(G(k)),$$

the obstruction (or defect) of weak approximation in S and over k, respectively, where Cl denotes the closure in the product topology. Let $G(k)/R$ (resp. $G(k)/Br$) denote the group of R-equivalence (resp. Br-equivalence) classes of G

over k. Let $(\cdot)^{\sim}$ be the group $\mathrm{Hom}(\cdot, \mathbf{Q}/\mathbf{Z})$, and $(\cdot)^{\wedge}$ be the group $\mathrm{Hom}(\cdot, \mathbf{G}_m)$. $Ш(\cdot)$ denotes the Tate–Shafarevich group of $(\cdot)$. We denote also by $\mathrm{Br}_a X := \mathrm{Br}_1 X/\mathrm{Br}_0 X$, the *arithmetic Brauer group of* X. By $\mathrm{H}^i(k, G)$ we denote the Galois cohomology of G. Regarding the classification of absolutely almost simple algebraic groups we refer to [Ti].

1. Recall of some basic facts from Brauer theory of algebraic groups [CTS1], [S]

Let T be a torus defined over field k of characteristic 0 which is split over a Galois extension K/k with Galois group $\mathfrak{g}$. A k-torus N is called *induced* (or *quasi-split*), if its character module $\hat{N}$ has a $\mathbf{Z}$-basis, over which $Gal(\bar{k}/k)$ acts by permutations. A k-torus S is called $\mathfrak{g}$-*flasque* torus over k if $\mathrm{H}^{-1}(\mathfrak{h}, \hat{S}) = 0$ for all subgroup $\mathfrak{h} \subset \mathfrak{g}$. It is well-known ([CTS1], [V1]) that any torus T above has a $\mathfrak{g}$-flasque resolution, i.e., an extension

$$(1) \qquad 1 \to S \to N \to T \to 1$$

of T, where S is a $\mathfrak{g}$-flasque k-torus, and N is an induced k-torus. Denote by $\mathrm{Br}(k, K)$ the kernel of $\mathrm{H}^2(k, \mathbf{G}_m) \to \mathrm{H}^2(K, \mathbf{G}_m)$. The exact sequence (1) induces a homomorphism

$$(2) \qquad \mathrm{H}^1(k, \hat{S}) \to \mathrm{H}^2(k, \hat{T}),$$

which is injective, since N has trivial 1-cohomology.

One has a cup-product

$$T(k) \times \mathrm{H}^2(K/k, \hat{T}) \xrightarrow{\cup} \mathrm{Br}(k, K),$$

which defines, via (2), a pairing

$$\beta : T(k) \times \mathrm{H}^1(K/k, \hat{S}) \xrightarrow{\cup} \mathrm{Br}(k, K).$$

We have

1.1. Theorem ([CTS1]), Prop. 17 and Corol.). 1) *The map β defines the Brauer equivalence relation over $T(k)$, hence also a map*

$$\gamma : T(k)/Br \to \mathrm{Hom}(\mathrm{H}^1(K/k, \hat{S}), \mathrm{Br}(k, K)).$$

2) *We have the following anti-commutative diagram*

$$\begin{array}{ccc} T(k)/R & \xrightarrow{p} & T(k)/Br \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \gamma} \\ \mathrm{H}^1(k, S) & \xrightarrow{\omega} & \mathrm{Hom}(\mathrm{H}^1(K/k, \hat{S}), \mathrm{Br}(k, K)) \end{array}$$

Here δ is an isomorphism [CTS1, Theorem 2], and ω comes from the cup-product

$$\mathrm{H}^1(k, S) \times \mathrm{H}^1(k, \hat{S}) \xrightarrow{\cup} \mathrm{H}^2(k, \mathbf{G}_m).$$

3) *$T(k)/Br \simeq \mathrm{Im}(\omega)$ and $T(k)/Br$ is a birational invariant in the class of k-tori, stably equivalent to T.*

4) *If k is a $\mathfrak{p}$-adic local field, the Brauer equivalence on $T(k)$ coincides with R-equivalence on $T(k)$ and*

$$T(k)/Br \simeq \mathrm{H}^1(k, \hat{S})^{\sim}.$$

5) *If k is a number field, and μ is the composition map*

$$\mathrm{H}^1(k, \hat{S}) \xrightarrow{\Delta} \prod_v \mathrm{H}^1(k, \hat{S}) \xrightarrow{\lambda} \prod_v \mathrm{H}^1(k_v, \hat{S})$$

then $T(k)/Br \simeq [\mathrm{Im}(\lambda)/\mathrm{Im}(\mu)]^{\sim}$.

1.2. Theorem ([CTS1]), Prop. 19). *With above notation, let k be a number field. We have the following exact sequences:*

$$(R) \qquad 0 \to \text{Ш}(S) \to T(k)/R \xrightarrow{\rho_T} \prod_v T(k_v)/R \to \mathrm{A}(T) \to 0,$$

$$(V) \qquad 0 \to \mathrm{A}(T) \to \mathrm{H}^1(k, \hat{S})^{\sim} \to \text{Ш}(T) \to 0.$$

Slightly in different form, the exact sequence (V) is due to Voskresenskiĭ. The following result gives us the group structure on $G(k)/Br$, induced from that of $G(k)$. Denote by $\mathrm{Br}_e G$ the kernel of the homomorphism $\mathrm{Br}_1 G \to \mathrm{Br}\, k$, defined by specializing at the unit element $e \in G(k)$, which is isomorphic to $\mathrm{Br}_a(G)$ ([S], Lemme 6.9).

1.3. Proposition ([CTS1], p. 216, [S], Lem. 6.9(1)). *Let K be a field and G a connected linear algebraic group over K, assumed to be reductive if K is not perfect. Then the pairing*

$$G(K) \times \mathrm{Br}_e G \to \mathrm{Br}\, K$$

is biadditive. In particular, $G(K)/Br$ has a natural group structure induced from $G(K)$.

The following well-known fact (which is a direct consequence of the Hasse principle for Brauer group of global fields) was mentioned in [MT]:

1.4. Proposition ([MT]). *Let X be a smooth variety defined over a number field k. Then the restriction map*

$$X(k)/Br \to \prod_v X(k_v)/Br$$

is injective.

1.5. Remarks. 1) Notice that in Theorem 1.2, we have identified $\text{Ш}(S)$ with a subgroup of $T(k)/R$ via the isomorphism δ of Theorem 1.1, 2). The exact sequence (V), which is due to Voskresenskiĭ (see e.g. [V1], [S]), has been extended to the case of arbitrary connected linear algebraic groups over number fields by Sansuc [S].

2) We are interested in Brauer equivalence relation for connected linear algebraic groups, which are rational over algebraic closure $\bar{k}$ of k, hence their smooth compactifications are also rational and the Br-equivalence and Br_1-equivalence are the same.

Our objective is to study the analogs of the exact sequence (R) in the case of Brauer and R-equivalence over local and global fields, for tori in particular, and for connected linear algebraic groups in general.

2. A Brauer relative of exact sequence (R) for algebraic tori

Let S be a finite set of valuations of a number field k, T a k-torus, $T_S := \prod_{v \in S} T(k_v)$. Denote by $RT(L)$ (resp. $BT(L)$) the set of elements of $T(L)$ which are R- (resp. Br-) equivalent to 1 in $T(L)$, where L is a field extension of k. Let $RT_S = \prod_{v \in S} RT(k_v)$, $BT_S = \prod_{v \in S} BT(k_v)$. The following result was mentioned in [V2] (which is valid also for any field k with non-trivial v-adic valuations).

2.1. Proposition. *$RT_S \subset Cl_S(T(k))$ and is an open subgroup in T_S.*

From above one derives the following

2.2. Corollary.

$$A(S, T) \simeq \mathrm{Coker}\left(T(k)/R \to \prod_{v \in S} T(k_v)/R \right).$$

$$A(T) \simeq \mathrm{Coker}\left(T(k)/R \to \prod_{v} T(k_v)/R \right).$$

2.3. Corollary. $BT_S \subset Cl_S(T(k))$.

Proof. If v is a non-archimedean, then Theorem 1.1., (4) tells us that $BT(k_v) = RT(k_v)$. If v is archimedean, then it is well-known that T is rational over k_v, hence has trivial groups $T(k_v)/Br$ and $T(k_v)/R$, i.e., $BT(k_v) = RT(k_v)$.

In what follows we identify $T(k)$ with a subgroup of T_S via diagonal embedding.

2.4. Proposition. *We have*

1) $\mathrm{A}(S, T) \simeq \mathrm{Coker}(T(k)/Br \to \prod_{v \in S} T(k_v)/Br)$.

2) $\mathrm{A}(T) \simeq \mathrm{Coker}(T(k)/Br \to \prod_{v} T(k_v)/Br)$.

Proof. Notice that

$$\begin{aligned}\mathrm{Coker}(T(k)/Br \to T_S/BT_S) &= T_S/T(k)BT_S \\ &= T_S/Cl_S(T(k)),\end{aligned}$$

since BT_S contains RT_S so is also an open subgroup of T_S. So 1) and 2) follow by noticing that for almost all v

$$T(k_v)/Br = 1,$$

by [CTS1], p. 205.

We have the following close analog of an exact sequence in [CTS1] (see Theorem 1.2 above) in the case of R-equivalence of algebraic tori over number fields.

2.5. Proposition. 1) *With above notation we have the following exact sequence*

$$1 \to T(k)/Br \xrightarrow{\gamma_T} \prod_v T(k_v)/Br \to \mathrm{A}(T) \to 1.$$

2) *With notation of Theorem* 1.1, *if the restriction map* $\mathrm{H}^1(k, \hat{S}) \to \mathrm{H}^1(k_v, \hat{S})$ *is surjective for all* v, *then* $\mathrm{Ker}(\omega) = \text{Ш}(S)$.

Proof. 1) follows directly from Propositions 1.4 and 2.4.

2) First we show that under the given assumption,

$$\mathrm{Ker}(\omega) \subset \text{Ш}(S).$$

Consider the following commutative diagram[1]

$$\begin{array}{ccc} \mathrm{H}^1(k, S) & \xrightarrow{\omega} & \mathrm{Hom}(\mathrm{H}^1(k, \hat{S}), \mathrm{Br}(k)) \\ \big\downarrow q & & \\ \prod_v \mathrm{H}^1(k_v, S) & \overset{\omega'}{\simeq} & \prod_v \mathrm{Hom}(\mathrm{H}^1(k_v, \hat{S}), \mathrm{Br}(k_v)) \end{array}$$

where ω' is an isomorphism by Tate–Nakayama duality. We have, e.g. for $v \notin \infty$

$$\begin{aligned} \mathrm{H}^1(k_v, S) &\simeq \mathrm{H}^1(k_v, \hat{S})^{\sim} \\ &\simeq \mathrm{Hom}(\mathrm{H}^1(k_v, \hat{S}), \mathbf{Q}/\mathbf{Z}) \\ &= \mathrm{Hom}(\mathrm{H}^1(k_v, \hat{S}), \mathrm{Br}(k_v)) \end{aligned}$$

Therefore it suffices to show that $\omega'(q(\mathrm{Ker}(\omega))) = 0$. Denote $res_v : \mathrm{H}^1(k, \cdot) \to \mathrm{H}^1(k_v, \cdot)$ the restriction map of cohomology when passing to completion k_v. Since for $x \in \mathrm{H}^1(k, S), \omega(x)$ is the map

$$y \mapsto (x) \cup (y), \qquad y \in \mathrm{H}^1(k, \hat{S}),$$

the v-component of $\omega'(q(x))$ is given by

$$\omega'(q(x))_v : y_v \mapsto (res_v(x)) \cup (y_v), \qquad y_v \in \mathrm{H}^1(k_v, \hat{S}).$$

[1] This diagram is in fact commutative as soon as the equality $\mathrm{Ker}(\omega) = \text{Ш}(S)$ is established.

Since the cup-product is compatible with restriction maps, and each $y_v \in \mathrm{H}^1(k_v, \hat{S})$ has the form $res_v(y), y \in \mathrm{H}^1(k, \hat{S})$,

$$(res_v(x)) \cup (y_v) = (res_v(x)) \cup (res_v(y)) = res_v(x \cup y).$$

Therefore, if $x \in \mathrm{Ker}(\omega)$, then $res_v(x \cup y) = 0$ for all v, hence

$$\mathrm{Ker}(\omega) \subset \mathrm{Ker}(q) = \mathrm{III}(S).$$

Next we show that $\mathrm{III}(S) \subset \mathrm{Ker}(\omega)$. This is true in general without the condition on res_v above. We prove that there exists an exact sequence as follows

$$0 \to \mathrm{Ker}(\omega) \xrightarrow{\alpha} \mathrm{III}(S) \xrightarrow{\beta} T(k)/Br.$$

We define the map $\beta : \mathrm{III}(S) \xrightarrow{i} T(k)/R \xrightarrow{p} T(k)/Br$ to be the composite map, where i is the restriction of the isomorphism $\delta^{-1} : \mathrm{H}^1(k, S) \to T(k)/R$ (see Theorem 1.1) to $\mathrm{III}(S)$, and p is the projection. We show that β is a trivial homomorphism, and α is just the identity map.

a) $\mathrm{Ker}(\beta) \subset \mathrm{Ker}(\omega)$. We have

$$\mathrm{Ker}(\omega) = \{x \in \mathrm{H}^1(k, S) : (x) \cup (y) = 0, \forall y \in \mathrm{H}^1(k, \hat{S})\}.$$

We have an isomorphism $\delta : T(k)/R \simeq \mathrm{H}^1(k, S)$ ([CTS1, Théorème 2]), so

$$\begin{aligned} \mathrm{Ker}(\beta) &= \{x \in \mathrm{III}(S) \subset \mathrm{H}^1(k, S) : \delta^{-1}(x) \in BT/RT\} \\ &= \mathrm{III}(S) \cap \delta(BT/RT) \\ &= \{x \in \mathrm{III}(S) : (x) \cup (y) = 0, \forall y \in \mathrm{H}^1(k, \hat{S})\}, \end{aligned}$$

hence $\mathrm{Ker}(\beta) \subset \mathrm{Ker}(\omega)$.

b) $\mathrm{Im}(\beta) = \mathrm{Ker}(\gamma_T)(= 0)$. Consider the following commutative diagram

$$\begin{tikzcd} \mathrm{III}(S) \arrow[r, "i"] \arrow[dr, "\beta"'] & T(k)/R \arrow[r, "\rho_T"] \arrow[d, "p'"] & \prod_v T(k_v)/R \arrow[d, "q'", "\simeq"'] \\ & T(k)/Br \arrow[r, "\gamma_T"] & \prod_v T(k_v)/Br \end{tikzcd}$$

If $x \in \mathrm{Im}(\beta)$, then $x = p'(i(s))$, $s \in \mathrm{III}(S)$. Since $\rho_T \circ i = 0$ by Theorem 1.2, we have

$$\begin{aligned} q'(\rho_T(i(s))) &= \gamma_T(p'(i(s))) \\ &= \gamma_T(x) \\ &= 0, \end{aligned}$$

i.e. $x \in \mathrm{Ker}(\gamma_T)$.

Conversely, if $x \in \mathrm{Ker}(\gamma_T)$, $x = p'(t)$, since p' is surjective. Then $0 = \gamma_T(p'(t)) = q'(\rho_T(t))$, so $\rho_T(t) = 0$, since q' is an isomorphism (see Theorem 1.1 (4)). Hence $t \in \mathrm{Ker}(\rho_T) = \mathrm{Im}(i)$ since the upper row is exact by Theorem 1.2.

Since γ_T is injective (see Proposition 1.4), β is trivial and α is an isomorphism. Hence 2) is proved.

As a consequence of the above proposition, we have the following

2.6. Proposition. *We have the following exact sequence connecting the two groups of rational equivalence classes*

$$0 \to \text{Ш}(S) \to T(k)/R \to T(k)/Br \to 0.$$

In particular, the order of $T(k)/Br$ is equal to the index $n_T := [T(k)/R : \text{Ш}(S)]$.

Proof. We have (with above notation) the following commutative diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \text{Ш}(S) & \longrightarrow & T(k)/R & \xrightarrow{\rho_T} & \prod_v T(k_v)/R & \longrightarrow & \text{A}(T) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \lambda_T} & & \downarrow{\scriptstyle \lambda'_T} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle =} & & \\
 & & 1 & \longrightarrow & T(k)/Br & \xrightarrow{\gamma_T} & \prod_v T(k_v)/Br & \longrightarrow & \text{A}(T) & \longrightarrow & 1.
\end{array}$$

In this diagram, λ_T is induced from λ'_T and is just the quotient map. Indeed, we have the vertical isomorphism "$\simeq$" due to Theorem 1.1, 4), and it is clear that

$$\lambda'_T(\text{Ker}(\rho_T)) \subset \text{Ker}(\gamma_T).$$

Therefore it follows that

$$(T(k)/R)/\text{Ker}(\rho_T) \simeq T(k)/Br$$

and we are done.

3. Some reductive analogs

In this section we prove some analogs of results in Section 2 for the case of connected reductive groups G over number fields k. First we recall the following

3.1. Proposition ([T4]). *Let G be a connected linear algebraic groups defined over a number field k, S a finite set of valuations of k. For each $v \in S$ denote by RG_v the subgroup of $G(k_v)$ consisting of elements R-equivalent to 1, and by RG_S the direct product of RG_v for $v \in S$. Then $RG_S \subset Cl_S(G(k))$.*

3.2. Proposition ([T4]). *Let G, k, S be as above. Then we have the following canonical isomorphisms*

1) $\text{A}(S, G) \simeq \text{Coker}(G(k)/R \to \prod_{v \in S} G(k_v)/R)$.

2) $\text{A}(G) \simeq \text{Coker}(G(k)/R \to \prod_v G(k_v)/R)$.

We have the following analog in the case of Brauer equivalence relation.

3.3. Theorem. *Let G, k, S be as above. Let BG_v be the subgroup of $G(k_v)$ consisting of elements which are Br-equivalent to 1, and BG_S be the direct product of BG_v. Then*

$$BG_S \subset Cl_S(G(k)).$$

Proof. We follow the proof given in [T4]. We know by [CTS1] that for a torus T over $k, T(k_v)/R = T(k_v)/Br$, hence it follows that the Theorem holds for tori. Now we may assume that G is not a torus. Further we just follow the proof given in [T4], where R is replaced by Br everywhere.

3.4. Theorem. *With notation as above we have the following canonical isomorphisms*

1) $\mathrm{A}(S, G) \simeq \mathrm{Coker}(G(k)/Br \to \prod_{v \in S} G(k_v)/Br)$.

2) $\mathrm{A}(G) \simeq \mathrm{Coker}(G(k)/Br \to \prod_v G(k_v)/Br)$.

In other words, we have the following exact sequence

$$0 \to G(k)/Br \to \prod_v G(k_v)/Br \to \mathrm{A}(G) \to 0.$$

Proof. The same as in 2.4, by making use of Proposition 3.3.

We need the following technical result.

3.5. Proposition ([O]). *Let G be a connected reductive group defined over a field K. There exists a connected reductive K-group H with simply connected semisimple part and an induced K-torus Z such that the following sequence is exact.*

$$1 \to Z \to H \to G \to 1.$$

(Such H is called in the literature also a *z-extension* of G over K.)

The relation between the groups of Brauer equivalence classes of G and H is shown in the following statement, where we restrict ourselves only to the case of a field k of characteristic 0. In the second statement, the finiteness is not new, but we give a simple proof of this fact.

3.6. Proposition. *Let k be a field of characteristic* 0.

1) *If H is a z-extension of a connected reductive k-group G then there is a canonical isomorphism*

$$H(k)/Br \simeq G(k)/Br.$$

2) *If k is a local or number field, then for any connected linear algebraic group G, the group of Brauer equivalence classes $G(k)/Br$ is finite.*

3) *Let k be a field of characteristic* 0 *and let $k(t)$ be rational function field in variable t. Then the group of Brauer equivalence classes of a conected linear algebraic group G is stable under field extension $k \to k(t)$, i.e., $G(k)/Br \simeq G(k(t))/Br$.*

First we need the following (perhaps well-known to experts, but I do not know of any reference).

3.6.1. Lemma. *Let $X \xrightarrow{\pi} Y$ be a morphism of smooth varieties defined over a field k of characteristic* 0, *where $X(k) \neq \varnothing$, $Y(k) \neq \varnothing$, $\pi^* : \mathrm{Br}\, Y \to \mathrm{Br}\, X$ is the induced homomorphism.*

a. *The following diagram is commutative*

$$\begin{tikzcd} X(k) \times \mathrm{Br}_1 X \arrow[dr] \arrow[d, "\pi"'] & \\ Y(k) \times \mathrm{Br}_1 Y \arrow[r] \arrow[u, "\pi^*"', shift right=8] & \mathrm{Br}\, k \end{tikzcd}$$

i.e. the pairing

$$X(k) \times \mathrm{Br}_1 X \to \mathrm{Br}\, k$$

is functional in X, *i.e., given* $x \in X(k)$, $b \in \mathrm{Br}_1 X$, *then*

$$\pi^*(b)(x) = b(\pi(x)).$$

b. *Assume further that* X, Y *are quasi-projective and irreducible and* π *admits a k-section* $i : Y \to X$. *Then* π *and* i *extend to certain smooth compactifications* $\mathscr{X}$, $\mathscr{Y}$ *of* X, Y, *respectively, i.e. we have the following commutative diagram*

$$\begin{tikzcd} \mathscr{X} \arrow[r, shift left, "\pi'"] & \mathscr{Y} \arrow[l, shift left, "i'"] \\ X \arrow[u] \arrow[r, shift left, "\pi"] & Y \arrow[u] \arrow[l, shift left, "i"] \end{tikzcd}$$

where π' *has k-section* i' *extending* i.

Proof. a) Let $x \in X(k)$, $y = \pi(x) \in Y(k)$. Denote by $\mathcal{O}_{\bar{X},x}, \mathcal{O}_{\bar{Y},y}$ the local ring of $\bar{X}$ (resp. $\bar{Y}$) at x (resp. y). One has the following commutative diagram

$$\begin{tikzcd} \mathcal{O}_{\bar{Y},y} \arrow[r, "\pi^*"] \arrow[dr, "\pi''"'] & \mathcal{O}_{\bar{X},x} \arrow[d, "\pi'"] \\ & \bar{k} \end{tikzcd}$$

We denote

$$\mathfrak{g} := Gal(\bar{k}/k), \qquad (f_{s,t}) \in Z^2(\mathfrak{g}, \mathcal{O}^*_{\bar{Y},y})$$

the absolute Galois group of k and a 2-cocycle representative of $b \in \mathrm{Br}_1 Y$, respectively. Then by [S], Lemme 6.2, $b(y)$ is the class $[f_{s,t}(y)]$ in $\mathrm{Br}\, k$. This 2-cocycle gives rise to a 2-cocycle $[f_{s,t} \circ \pi] \in Z^2(\mathfrak{g}, \mathcal{O}^*_{\bar{X},x})$ which is nothing else than a representative of $a = \pi^*(b)$. Then $a(x)$ is just the class of

$$[(f_{s,t} \circ \pi)(x)] = [f_{s,t}(\pi(x))] = [f_{s,t}(y)] \in \mathrm{Br}\, k.$$

Therefore

$$\pi^*(b)(x) = b(\pi(x))$$

as required.

b) Denote by $\mathcal{X}$ (resp. $\mathcal{Y}$) the closure of X (resp. Y) in $\mathbf{P}^n$ (resp. $\mathbf{P}^m$) for some embedding $X \hookrightarrow \mathbf{A}^n \hookrightarrow \mathbf{P}^n$ (resp. $Y \hookrightarrow \mathbf{A}^m \hookrightarrow \mathbf{P}^m$). Since π is given by polynomial functions over k, it defines uniquely a k-morphism $\pi_1 : \mathcal{X} \to \mathcal{Y}$ extending π, since X, Y are dense in $\mathcal{X}, \mathcal{Y}$, respectively. In similar way we get extension i_1 of i. If $\mathcal{X}$ and $\mathcal{Y}$ are smooth we are done. Otherwise, assume first that $\mathcal{Y}$ is singular and Z denotes the singular locus of $\mathcal{Y}$. Let $Z' = \pi_1^{-1}(Z)$. Then by blowing-up $\mathcal{X}$ (resp. $\mathcal{Y}$) with center in Z' (resp. Z) we arrive at the following commutative diagram (see e.g. [Ha], Ch. II, Sec. 7, 7.12–7.16):

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "\pi'"] \arrow[d, "f"] & \mathcal{Y}' \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "\pi_1"] & \mathcal{Y} \end{tikzcd}$$

One checks that

$$i_1^{-1}(Z') = i_1^{-1}(\pi_1^{-1}(Z)) = Z.$$

Therefore in the above diagram one obtains also a k-morphism $i' : \mathcal{Y}' \to \mathcal{X}'$ making the diagram commute.

We show that π' has i' as its k-section. Let $\mathcal{Z}' = f^{-1}(Z')$, $\mathcal{Z} = g^{-1}(Z)$. We have the following commutative diagram

$$\begin{tikzcd} \mathcal{X}'\backslash\mathcal{Z}' \arrow[r, shift left, "\pi'"] \arrow[d, "f", "\simeq"'] & \mathcal{Y}'\backslash\mathcal{Z} \arrow[l, shift left, "i'"] \arrow[d, "g", "\simeq"'] \\ \mathcal{X}\backslash Z' \arrow[r, shift left, "\pi_1"] & \mathcal{Y}\backslash Z \arrow[l, shift left, "i_1"] \end{tikzcd}$$

Let $y \in \mathcal{Y}'\backslash\mathcal{Z}$ be an arbitrary element. Then we have

$$\begin{aligned} g(\pi'(i'(y))) &= \pi_1(f(i'(y))) \\ &= \pi_1(i_1(g(y))) = (\pi_1 \circ i_1)(g(y)) \\ &= g(y), \end{aligned}$$

hence we have

$$(\pi' \circ i')_{|\mathcal{Y}'\backslash\mathcal{Z}} = id.$$

Since $\mathcal{Y}'\backslash\mathcal{Z}$ is Zariski-dense in $\mathcal{Y}'$, it follows that $\pi' \circ i' = id$ as required.

By Hironaka [Hi], after a finite number of blow-ups, we may assume that $\mathcal{X}$ is singular and $\mathcal{Y}$ is smooth. This time we apply the same argument as above to get the following commutative diagram

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, shift right, "i'"'] \arrow[d, "g"] & \mathcal{X}' \arrow[l, shift right, "\pi'"'] \arrow[d, "f"] \\ \mathcal{Y} \arrow[r, shift right, "i_1"'] & \mathcal{X} \arrow[l, shift right, "\pi_1"'] \end{tikzcd}$$

where we blow-up the singular locus Z' of $\mathscr{X}$ (to get $\mathscr{X}'$) and its inverse image $i^{-1}(Z')$ (to get $\mathscr{Y}'$). Again as above we can show that i' is the cross-section of π'. By our construction, $\mathscr{X}'$ and $\mathscr{Y}'$ are smooth, projective and f and g define k-isomorphisms from some open subsets $U \subset \mathscr{X}$ and $V \subset \mathscr{Y}$ to X and Y, respectively. Thus we obtain smooth compactifications of X and Y with desired properties. The proof of the lemma is complete.

Proof of Proposition 3.6. 1) Let $\pi : H \to G$ be the projection. It induces an epimorphism

$$H(k) \to G(k),$$

hence also an epimorphism

$$\pi' : H(k)/Br \to G(k)/Br.$$

We show that π' is injective. Since Z is an induced k-torus, it is well-known that there is a k-section

$$i : G \to H, \qquad \pi \circ i = id_G.$$

By Lemma 3.6.1, b) We may choose k-compactifications $\mathscr{H}, \mathscr{G}$ of H, G, respectively with the following commutative diagram (notice that $\mathscr{H}, \mathscr{G}$ are $\bar{k}$-rational)

$$\begin{array}{ccccc} \mathscr{H}(k) & \times & \operatorname{Br}\mathscr{H} & \searrow & \\ {\scriptstyle i}\uparrow\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi^*}\uparrow\downarrow{\scriptstyle i^*} & & \operatorname{Br} k \\ \mathscr{G}(k) & \times & \operatorname{Br}\mathscr{G} & \nearrow & \end{array}$$

Since $\pi \circ i = id_G$, it follows that we have

$$(\pi \circ i)^* = i^* \circ \pi^* = id_{\operatorname{Br}\mathscr{G}}.$$

Therefore i^* is *surjective*. Now assume that $h \in H(k)$ such that

$$\pi(h) \cup \hat{g} = 0, \qquad \forall \hat{g} \in \operatorname{Br}\mathscr{G}.$$

Let $g = \pi(h)$. Then by Lemma 3.6.1, a) we have

$$i(g) \cup \hat{h} = \pi(i(g)) \cup i^*(\hat{h}) = 0, \qquad \forall \hat{h} \in \operatorname{Br}\mathscr{H},$$

since i^* is surjective. This implies that $i(g)$ is belong to the class $B\mathscr{H}(k)$ of all elements with trivial cup-product with $\operatorname{Br}\mathscr{H}$. Since

$$\pi(h) = g = \pi(i(g)),$$

one deduces that $h = i(g)z$, with $z \in Z(k)$. Since Z is k-rational, z is R-equivalent to the identity element e in $Z(k)$, it follows that h is R-equivalent to $i(g)$, hence also Br-equivalent to $i(g)$. Since $i(g) \in B\mathscr{H}(k)$, hence $h \in B\mathscr{H}(k)$, too, and we are done.

(In the case 2) we can prove our statement by using 2). Indeed, we have a birational equivalence

$$H \simeq G \times Z,$$

and this induces a bijection (see [CTS1], Section 7)

$$H(k)/Br \simeq (G(k)/Br \times Z(k)/Br) \simeq G(k)/Br.$$

since $H(k)/Br$ is finite by 2), π' is injective, hence also an isomorphism.)

2) We present two closely related proofs, one of which is a direct proof by using Theorem 3.4 above.

First proof. One reduces easily to the case where G is a connected reductive group. Take a z-extension H of G as above. We will show that $H(k)/Br$ is finite. Let $H = S\tilde{G}$, where $\tilde{G}$ is a simply connected semisimple group and S a central torus of H. We have the following exact sequence of k-groups

$$1 \to \tilde{G} \to H \to T \to 1$$

where T is a torus.

First we assume that k is a number field. Let $\mathscr{H}, \mathscr{T}$ be smooth compactifications of H, T, respectively, with a k-morphism $\pi' : \mathscr{H} \to \mathscr{T}$ extending the projection $H \to T$. We have the following commutative diagram, where all rows are exact by the main result of Sansuc [S]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{A}(H) & \longrightarrow & \mathrm{H}^1(k, \mathrm{Pic}\,\bar{\mathscr{H}})^{\sim} & \longrightarrow & \text{Ш}(H) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \mathrm{A}(T) & \longrightarrow & \mathrm{H}^1(k, \mathrm{Pic}\,\bar{\mathscr{T}})^{\sim} & \longrightarrow & \text{Ш}(T) & \longrightarrow & 0 \end{array}$$

where α, β, γ are induced by π.

By Lemma 3.8 (to be proved below) α is an isomorphism. We want to show that γ is injective (i.e. also a monomorphism of abelian groups, by [S]). We have the following commutative diagram

$$\begin{array}{ccccccc} T(k) & \xrightarrow{\delta} & \mathrm{H}^1(k, \tilde{G}) & \xrightarrow{i} & \mathrm{H}^1(k, H) & \xrightarrow{j} & \mathrm{H}^1(k, T) \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ \prod_v T(k_v) & \xrightarrow{\delta'} & \prod_v \mathrm{H}^1(k_v, \tilde{G}) & \xrightarrow{i'} & \prod_v \mathrm{H}^1(k_v, H) & \xrightarrow{j'} & \prod_v \mathrm{H}^1(k_v, T) \end{array}$$

where s is the diagonal embedding. The Hasse principle for simply connected groups says that p is a bijection. It is clear that j maps $\text{Ш}(H)$ into $\text{Ш}(T)$. hence it induces a map

$$\lambda : \text{Ш}(H) \to \text{Ш}(T).$$

Due to the functoriality of the commutative group Ш over number field ([S]), λ is also a homomorphism of commutative groups.

If $x \in \text{Ш}(H)$ such that $j(x) = 0$ in $\mathrm{H}^1(k, T)$ then $x = i(g)$, where $g \in \mathrm{H}^1(k, \tilde{G})$. Hence $q(i(g)) = i'(p(g))$ is the trivial element in $\prod_v \mathrm{H}^1(k_v, H)$, so $p(g) \in \mathrm{Im}\,(\delta')$. $p(g) = \delta'(t)$. Let $\prod_v T(k_v) = T_\infty \times T_f, \delta' = (\delta'_\infty, \delta'_f)$, where $T_\infty = \prod_{v \in \infty} T(k_v)$. $T_f = \prod_{v \notin \infty} T(k)$, $\delta'_\infty : T_\infty \to \prod_{v \in \infty} \mathrm{H}^1(k_v, \tilde{G})$, $\delta'_f : T_f \to \prod_{v \notin \infty} \mathrm{H}^1(k_v, \tilde{G}) = 0$. Let

$t = (t_\infty, t_f)$. Then $\delta'(t) = (\delta'_\infty(t_\infty), 0)$. Since $T(k)$ is dense in T_∞, we may choose a sequence $t_n \in T(k)$ such that $t_\infty = lim_n\, t_n$. Since $\prod_{v \in \infty} \mathrm{H}^1(k_v, \tilde{G})$ is finite, we conclude that for some N large enough, we have $\delta'_\infty(t) = \delta'_\infty(t_n)$ with $n > N$. Therefore $p(g) = p(\delta(t_n))$ or $g = \delta(t_n)$, since p is a bijection (Hasse principle). Therefore $x = 0$ and we have a monomorphism

$$\text{(3)} \qquad Ш(H) \hookrightarrow Ш(T)$$

as claimed.

From the above commutative diagram we conclude that β is injective which says that the functorial homomorphism

$$\text{(4)} \qquad \mathrm{H}^1(k, \mathrm{Pic}\, \bar{\mathscr{T}}) \to \mathrm{H}^1(k, \mathrm{Pic}\, \bar{\mathscr{H}})$$

is surjective. By Lemma 16 of [CTS1], the Brauer equivalence defined on $\mathscr{H}$ (resp. $\mathscr{T}$) coincides with the equivalence relation defined by the following pairing

$$\mathscr{H}(k) \times \mathrm{H}^1(k, \mathrm{Pic}\, \bar{\mathscr{H}}) \to \mathrm{Br}\, k$$

(resp.

$$\mathscr{T}(k) \times \mathrm{H}^1(k, \mathrm{Pic}\, \bar{\mathscr{T}}) \to \mathrm{Br}\, k.)$$

By Lemma 3.6.1 we have the following commutative diagram

$$\begin{tikzcd}
\mathscr{H}(k) \times \mathrm{H}^1(k, \mathrm{Pic}\, \bar{\mathscr{H}}) \arrow[d, "\pi"', shift right=8] \arrow[dr] & \\
\mathscr{T}(k) \times \mathrm{H}^1(k, \mathrm{Pic}\, \bar{\mathscr{T}}) \arrow[u, "\pi^*"', shift right=-4] \arrow[r] & \mathrm{Br}\, k
\end{tikzcd}$$

Assume that $h \in H(k) \subset \mathscr{H}(k)$ with $\pi(h) \cup \mathrm{H}^1(k, \mathrm{Pic}\, \bar{\mathscr{T}}) = 0$. Then we have

$$\pi^*(a)(h) = a(\pi(h)) = 0$$

for all $a \in \mathrm{H}^1(k, \mathrm{Pic}\, \bar{\mathscr{T}})$, and from surjective map in (4) we derive that $h \cup \mathrm{H}^1(k, \mathrm{Pic}\, \bar{\mathscr{H}}) = 0$. Thus we have proved that π induces an injective homomorphism

$$H(k)/Br \hookrightarrow T(k)/Br.$$

Therefore, the statement of 2) follows, if k is a number field, since if k is a field of finite type over **Q** and by [CTS1], Corollaire 1, p. 217, $T(k)/Br$ is finite. Therefore $H(k)/Br$, and *a fortiori* $G(k)/Br$, is also finite.

If k is a local field, we may assume that k is non-archimedean, since otherwise the rational equivalence relations considered are trivial. Then we may invoke the finiteness of the group $H(k)/R$ proved by Voskresenskiĭ [V2] to ensure the finiteness of $H(k)/Br$ due to the surjectivity $H(k)/R \to H(k)/Br$ ([CTS1], Prop. 16).

Second proof. By using Theorem 3.4 we see that the number field case is reduced to the case of local fields, since we know [CST1] that for almost all v, $G(k_v)/R$ (hence also $G(k_v)/Br$) is trivial. In the local field case we may use the finiteness of $G(k_v)/R$ proved by Voskresenskiĭ as above.

3) It is well-known that the natural homomorphism $\operatorname{Br} k \to \operatorname{Br} k(t)$ is injective. Therefore the induced homomorphism $i : G(k)/Br \to G(k(t))/Br$ is injective. To show that i is surjective we use the stability theorem due to Gille [G1, 3], which says that $G(k)/R \simeq G(k(t))/R$. Let $x \in G(k(t))$. Then there exists $y \in G(k)$ such that $x \equiv y(mod.\ RG(k(t)))$, hence $x \equiv y(mod.\ BG(k(t)))$, i.e., i is surjective.

3.6.2. Remark. It was proved in [G2], [G3], that if k is a number field then $G(k)/R$ is finite, hence $G(k)/Br$ is also. Here we did not use this finiteness result of Gille in the proof.

In the local non-archimedean field case we have the following result.

3.6.3. Proposition. *Let*

$$1 \to \tilde{G} \to H \xrightarrow{\pi} T \to 1$$

be an an exact sequence of connected linear algebraic groups defined over a non-archimedean local field k_v of characteristic 0 *where $\tilde{G}$ is semisimple, simply connected and T is a torus. Then we have*

$$H(k_v)/Br \simeq T(k_v)/Br.$$

Proof. We make use of Kneser's Theorem on the triviality of H^1 of simply connected groups. In fact, from the exact sequence of cohomology we see that π is surjective on k_v-points, thus gives a surjective map

$$\pi' : H(k_v)/Br \to T(k_v)/Br.$$

By the proof of Proposition 3.6 (see (4′) above) the natural homomorphism $H \to T$ induces the following injective map of commutative groups

$$H(k_v)/R \to T(k_v)/R,$$

hence from the surjectivity of π' above we conclude that $H(k_v)/Br \simeq T(k_v)/Br$.

Before we give the formulation of one of main results, we recall some definition and notation of Section 2.4.

Given a torus T defined over a number field k we denote by $V(T)$ a smooth compactification of T over k and by S the Neron–Severi k-torus of T, which is by definition the Cartier dual to the Picard group of $\bar{V}(T)(= V(T) \times \bar{k}), \hat{S} \simeq \operatorname{Pic} \bar{V}(T)$. The first Galois cohomology of S, which depends on T, does not depend on the chosen smooth compactification, and so is the Shafarevich–Tate group $\text{Ш}(S)$. By Proposition 2.5 we have the following exact sequence

$$1 \to T(k)/Br \xrightarrow{\gamma_T} \prod_v T(k_v)/Br \to \mathrm{A}(T) \to 1.$$

A connection between the above sequence for a connected linear algebraic group G and the one for torus quotient of a z-extension of reductive part of G is given by the following theorem.

3.7. Theorem. *Let G be a connected linear algebraic group defined over a number field k. Then we have the following commutative diagram, all rows of which are exact sequences*

$$
\begin{tikzcd}
1 \arrow[r] & G(k)/Br \arrow[r, "\gamma_G"] \arrow[d, "p"] & \prod_v G(k_v)/Br \arrow[r] \arrow[d, "q"] & \mathrm{A}(G) \arrow[r] \arrow[d, "r"] & 1 \\
1 \arrow[r] & T(k)/Br \arrow[r, "\gamma_T"] & \prod_v T(k_v)/Br \arrow[r] & \mathrm{A}(T) \arrow[r] & 1
\end{tikzcd}
\tag{5}
$$

where T is the torus quotient of any z-extension H of the reductive part of G, and all vertical maps are (functorial) isomorphisms (including local components ones). In particular, the image of $G(k)/Br$ via γ_G is a finite group of order n_T, where the notation is as above.

Proof. The exactness of the above sequences follows from Propositions 2.5, 1) and 3.4. By Proposition 3.6, 1), there is a canonical (functorial) isomoprhism $G(K)/Br \simeq H(K)/Br$ for any extension field K/k. Therefore it suffices to prove theorem 3.7 for H.

We have the following commutative diagram (see the notation above).

$$
\begin{tikzcd}
1 \arrow[r] & H(k)/Br \arrow[r, "\gamma_H"] \arrow[d, "p"] & \prod_v H(k_v)/Br \arrow[r] \arrow[d, "q"] & \mathrm{A}(H) \arrow[r] \arrow[d, "r"] & 1 \\
1 \arrow[r] & T(k)/Br \arrow[r, "\gamma_T"] & \prod_v T(k_v)/Br \arrow[r] & \mathrm{A}(T) \arrow[r] & 1
\end{tikzcd}
$$

where p, q, r are natural maps, induced from the projection $pr : H \to T$. By Proposition 3.6.3, p is injective and q and all its local components of are isomorphisms.

Next we need the following

3.8. Lemma. *With above notation, we have a canonical isomorphism of finite groups*

$$\mathrm{A}(G) \simeq \mathrm{A}(H) \overset{r}{\simeq} \mathrm{A}(T).$$

Proof. The first isomorphism is from [T4]. Consider the following commutative diagram

$$
\begin{tikzcd}
H(k) \arrow[r] \arrow[d, "\alpha"] & T(k) \arrow[r, "\delta"] \arrow[d, "\beta"] & \mathrm{H}^1(k, \tilde{G}) \arrow[d, "\gamma"] \\
H_S \arrow[r] & T_S \arrow[r, "\delta"] & \prod_{v \in S} \mathrm{H}^1(k_v, \tilde{G})
\end{tikzcd}
\tag{6}
$$

where we take S a sufficiently large finite set of valuations of k containing all the archimedean ones, such that

$$\mathrm{A}(H) = H_S/Cl_S(H(k)),$$

$$\mathrm{A}(T) = T_S/Cl_S(T(k)).$$

It is a general and well-known fact (see, e.g. [T4] for a discussion with references) that for any linear algebraic group P over k, $Cl_S(P(k))$ is an open subgroup of P_S. Also any connected linear algebraic group satisfies weak approximation with respect to the set ∞ of archimedean valuations.

Let $t_S \in T_S$, $t_S = (t_\infty, t_f)$, where t_∞ (resp. t_f) is the ∞-(resp. finite) component of t_S. Let $t_n \in T(k)$ such that $lim_n\ t_n = t_\infty$. Then for n large enough, the element (t_n, t_n) is very close to (t_∞, t_n) in the S-adic (product) topology, i.e., $(t_n^{-1}t_\infty, 1)$ is approaching 1 in T_S. Since $Cl_S(T(k))$ is open, there is N such that if $n > N$, then

$$(t_n^{-1}t_\infty, 1) \in Cl_S(T(k)),$$

i.e.,

$$(t_\infty, t_n) \in Cl_S(T(k)).$$

Since $t_S = (t_\infty, t_n)(1, t_n^{-1}t_f)$, it follows that

(7) *each coset of* $T_S/Cl_S(T(k))$ *has a representative from* $1 \times T_{S-\infty}$.

Since $\mathrm{H}^1(k_v, \tilde{G})$ is trivial for v non-archimedean by Kneser's Theorem [Kn2], it follows that

$$T_{S-\infty} = \pi(H_{S-\infty})$$

and from (7) we derive that the natural homomorphism

$$\mathrm{A}(S, H) = \mathrm{A}(H) \xrightarrow{\pi} \mathrm{A}(T) = \mathrm{A}(S, T)$$

is surjective.

Next we show that π is injective. Let $h_S \in H_S$ such that $\pi(h_S) \in Cl_S(T(k))$. Then

$$\pi(h_S) = lim_n\ t_n, \qquad t_n \in T(k),$$

hence from the commutative diagram (6) we derive

$$1 = \delta\pi(h_S) = lim_n\ \delta(t_n).$$

(Notice that here one endows $\mathrm{H}^1(k, \tilde{G}), \mathrm{H}^1(k_v, \tilde{G})$ with discrete topologies, and one checks readily that all maps in the diagram (6) are continuous.) Since $\prod_{v \in S} \mathrm{H}^1(k_v, \tilde{G})$ is finite, there is N_1 such that if $n > N_1$ then $\delta(t_n) = 1$. Let $h_n \in H(k)$, such that $\pi(h_n) = t_n$. Then $lim_n\ \pi(h_n^{-1}h_S) = 1$ since $Cl_S(H(k))$ is open in H_S, we deduce that

$$h_n^{-1}h_S \in \tilde{G}_S Cl_S(H(k))$$

for n large.

Since $\tilde{G}$ has weak approximation property (Kneser–Harder Theorem, see e.g. [S]), we have $\tilde{G}_S \subset Cl_S(H(k))$, hence $h_S \in Cl_S(H(k))$ as required. Hence $\mathrm{A}(H) \simeq \mathrm{A}(T)$ and the lemma is proved.

Continuation of the proof of Theorem 3.7. In the diagram (5) we know that r is an isomorphism by Lemma 3.8, q is an isomorphism and p is injective by Proposition 3.6.3. It follows that γ_H and γ_T have isomorphic images. Therefore

$$H(k)/Br \overset{p}{\simeq} T(k)/Br.$$

In particular, the order of $G(k)/Br$ is equal to n_T.

We have the following result which is an analog of [CTS1], Prop. 18, and [S], Th. 3.3. Let G be a connected linear algebraic group defined over number field k, H be a z-extension of a Levi subgroup of G, T be torus quotient of H, which is split over a finite extension K of k. Denote by S the Neron–Severi torus of T, V_0 the (finite) set of all valuations of k, such that their extensions of K have *non-cyclic* decomposition groups. Then for any finite set W of valuations of k we have the following formulas

3.9. Corollary (of the proof of Lemma 3.8). *There are canonical isomorphisms of finite groups*

$$\mathrm{A}(W, G) \simeq \mathrm{A}(W, T) \simeq \mathrm{Coker}\left(\mathrm{H}^1(k, S) \to \prod_{v \in W} \mathrm{H}^1(k_v, S)\right),$$

$$\mathrm{A}(G) \simeq \mathrm{A}(T) \simeq \mathrm{Coker}\left(\mathrm{H}^1(k, S) \to \prod_{W \cup V_0} \mathrm{H}^1(k_v, S)\right),$$

Proof. The proof of the first isomorphisms in these chain of isomorphisms follows directly from the proof of Lemma 3.8 above. The last ones related with S are deduced from Theorem 1.2 (well-known).

4. Some variations and applications to weak approximation

In this section we consider some applications of results obtained in previous sections, and also of those obtained in [T4]. We keep our notation as above. First we derive from Proposition 3.3 the following.

4.1. Proposition. *Let S be a finite set of valuations of k and G a connected linear algebraic group over k. If G has trivial group $G(k_v)/Br$ of Brauer equivalence classes for all $v \in S$, then G has weak approximation property in S.*

4.2. Conversely, assume that G does not have weak approximation property with respect to some v (necessarily non-archimedean). Then G has non-trivial group $G(k_v)/Br$ by Proposition 3.3, hence also non-trivial group $G(k_v)/R$. Therefore G is not stably rational *over* k_v, and a fortiori, over k.

4.2.1. Remarks. 1) Usually counter-examples to weak approximation over number fields k serve as examples of linear algebraic k-groups, which are non stably rational over k only. The statement 4.2 above shows that these examples are, in fact, stronger in the sense that they serve also as examples of non stably rational groups over bigger fields (say k_v). Of course, this remark can be also derived from the fact that the Brauer–Manin obstruction to weak approximation in connected algebraic groups is the only one ([S]) and the reader may consult a variety of examples in [S].

2) We mention the following one of the main results due to Sansuc [S], Corollaire 9.7, in the case G has no simple component of type E_8 (and also goes back to Voskresenskiĭ in torus case). Since the Hasse principle is also holds for E_8 by Chernousov, the following holds.

4.2.2. Theorem ([S], Cf. Thm 9.5.). *If G is a connected linear algebraic group, defined over a number field k, then we have the following exact sequence*

$$(V) \qquad 0 \to \mathrm{A}(G) \to \mathrm{H}^1(k, \operatorname{Pic} \bar{V}(G))^\sim \to \text{Ш}(G) \to 0.$$

In particular, if $\mathrm{H}^1(k, \operatorname{Pic} \bar{V}(G))$ is trivial, then G has weak approximation over k and satisfies Hasse principle for H^1.

We derive the following consequence of the proofs given in Section 3. In particular, to some extent, it explains what is behind the mysterious relation between the basic arithmetic and geometric invariants $\mathrm{A}(G)$, $\text{Ш}(G)$ and $\mathrm{H}^1(k, \operatorname{Pic} \bar{V}(G))$ of a connected linear algebraic group G defined over a number field k given by the above theorem.

4.2.3. Proposition. *Let G be a connected linear algebraic group over a number field k, H a z-extension of a k-Levi subgroup of G, T the torus quotient of H. Then there are canonical isomorphisms of finite commutative groups*

$$\mathrm{A}(G) \simeq \mathrm{A}(T), \qquad \mathrm{H}^1(k, \operatorname{Pic} \bar{V}(G))^\sim \simeq \mathrm{H}^1(k, \operatorname{Pic} \bar{V}(T))^\sim, \qquad \text{Ш}(G) \simeq \text{Ш}(T),$$

where $V(G), V(T)$ are some smooth compactifications of G, T over k, respectively.

Proof. Recall that we have canonical isomorphism $\mathrm{A}(G) \simeq \mathrm{A}(H)$ (see [T4]) and by Lemma 3.8 we have $\mathrm{A}(G) \simeq \mathrm{A}(T)$ (canonically). Let

$$1 \to Z \to H \to G \to 1$$

be the given z-extension of G. We show that the projection $\pi : H \to G$ induces an isomorphism of commutative groups

$$\text{Ш}(H) \simeq \text{Ш}(G).$$

Indeed, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}^1(k, H) & \xrightarrow{\pi'} & \mathrm{H}^1(k, G) & \longrightarrow & \mathrm{H}^2(k, Z) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_v \mathrm{H}^1(k_v, H) & \longrightarrow & \prod_v \mathrm{H}^1(k_v, G) & \longrightarrow & \prod_v \mathrm{H}^2(k_v, Z)
\end{array}$$

It is clear (by using the Hasse principle for Brauer group of global fields) that the induced map

$$\text{Ш}(H) \to \text{Ш}(G)$$

is onto. From the first row one sees that it has trivial kernel and by twisting argument one can see also that it is injective. By functoriality of Ш we have

$$\text{Ш}(H) \simeq \text{Ш}(G)$$

as isomorphism of commutative groups. (I became aware afterwards that the above isomorphism was mentioned earlier in [K], Lemma 4.3.2, b).)

Now by the functoriality of the exact sequence

$$0 \to \mathrm{A}(H) \to \mathrm{H}^1(k, \operatorname{Pic} \bar{V}(H))^{\sim} \to \text{Ш}(H) \to 0$$

in the argument H, we deduce (by diagram chasing) that π induces a canonical isomorphism of finite groups

$$\mathrm{H}^1(k, \operatorname{Pic} \bar{V}(H))^{\sim} \simeq \mathrm{H}^1(k, \operatorname{Pic} \bar{V}(G))^{\sim}.$$

We know from the proof of Proposition 3.6, 2) that there is canonical injective homomorphism of commutative groups

$$j : \text{Ш}(H) \hookrightarrow \text{Ш}(T).$$

We now show that j is surjective.

For a smooth k-variety X we denote by $\mathscr{B}r(X)$ the usual Brauer group (of equivalence classes of Azumaya algebras over X) of X,

$$\mathscr{B}r_a(X) = \operatorname{Ker}(\mathscr{B}r(X) \to \mathscr{B}r(\bar{X}))/\operatorname{Im}(\operatorname{Br} k \to \mathscr{B}r(X)),$$

$\mathscr{B}r^{\circ}(X)$ the subgroup of all elements of $\mathscr{B}r_a(X)$ which have trivial images via localization maps $\mathscr{B}r_a(X) \to \mathscr{B}r_a(X_{k_v})$.

By [S], Corollaire 6.11, we have the following exact sequence (by using the simply connectedness of $\tilde{G}$)

$$0 \to \mathscr{B}r T \to \mathscr{B}r H,$$

hence we also have the following exact sequence

$$0 \to \mathscr{B}r_a T \to \mathscr{B}r_a H,$$

and from this we derive the following monomorphism

$$\mathscr{B}r^{\circ}(T) \hookrightarrow \mathscr{B}r^{\circ}(H),$$

and by taking the Pontryagin dual we have a surjective homomorphism

$$(\mathscr{B}r^{\circ}(H))^{\sim} \to (\mathscr{B}r^{\circ}(T))^{\sim}.$$

By [S], Théorème 8.5 (in combining with Chernousov's result on Hasse principle for E_8), we have a functorial isomorphism of commutative groups

$$\text{III}(G) \simeq \mathscr{B}r^{\circ}(G)^{\sim}$$

or for arbitrary connected linear algebraic group G over k. From the above we see that it yields a surjective homomorphism

$$\text{III}(H) \to \text{III}(T)$$

as required. Thus we have canonical isomorphisms

$$\text{III}(G) \simeq \text{III}(H) \simeq \text{III}(T).$$

Now one can use the functoriality of the exact sequence (V) and canonical isomorphisms $\mathrm{A}(G) \simeq \mathrm{A}(H) \simeq \mathrm{A}(T)$ and $\text{III}(G) \simeq \text{III}(H) \simeq \text{III}(T)$ to get the isomorphism

$$\mathrm{H}^1(k, \operatorname{Pic} \bar{V}(G))^{\sim} \simeq \mathrm{H}^1(k, \operatorname{Pic} \bar{V}(H))^{\sim} \simeq \mathrm{H}^1(k, \operatorname{Pic} \bar{V}(T))^{\sim}.$$

Another (more functorial) way to see this canonical isomorphism is as follows. Instead of using the exact sequence (V), we use the following exact sequence for any connected linear algebraic group G defined over a number field k (see [S], Corollaire 8.14):

$$(S) \qquad 0 \to \mathrm{A}(G) \to \mathscr{B}r_{\omega}(G)^{\sim} \to \text{III}(G) \to 0,$$

where $\mathscr{B}r_{\omega}(G)$ is the subgroup of all elements of $\mathscr{B}r_a(G)$ which have almost all zero-images via localization maps

$$\mathscr{B}r_a(G) \to \mathscr{B}r_a(G_{k_v}).$$

Then one can check without difficulties that we have the following canonical isomorphisms

$$\mathscr{B}r_{\omega}(G) \simeq \mathscr{B}r_{\omega}(H) \simeq \mathscr{B}r_{\omega}(T)$$

and we may use (S) together with canonical isomorphism (see [S], Corollaire 9.4)

$$\mathscr{B}r_{\omega}(G) \simeq \mathscr{B}r_a(V(G)) \simeq \mathrm{H}^1(k, \operatorname{Pic} \bar{V}(G)).$$

Yet another way to prove our proposition is to use Kottwitz's theory [K] saying $\mathscr{B}r_a(G) \simeq \mathrm{H}^1(k, Z(\hat{G}))$, where $\hat{G}$ is the connected Langlands' dual of G and $Z(\hat{G})$ denotes the center of $\hat{G}$.

The proposition follows.

4.2.4. Remarks. One can give, along the proof given by Sansuc (which does not use the exactness proved for tori by Voskresenskiĭ), an alternative ("*short-cut*") proof of Theorem 4.2.2 (i.e. the exactness of the sequence (V)) by assuming only the exactness of this sequence for tori already proved by Voskresenskiĭ [V1].

STEP 1. We have [V1] the following exact sequence

$$0 \to \mathrm{A}(T) \to \mathrm{H}^1(k, \operatorname{Pic} \bar{V}(T))^{\sim} \to \text{III}(T) \to 0.$$

STEP 2. We have (see above)

$$\mathrm{A}(T) \simeq \mathrm{A}(H) \simeq \mathrm{A}(G), \qquad Ш(T) \simeq Ш(H) \simeq Ш(G).$$

STEP 3. Since $\bar{V}(T), \bar{V}(H), \bar{V}(G)$ are $\bar{k}$-rational, and $\mathrm{H}^3(k, \mathbf{G}_m) = 0$ (well-known), we derive from [M1], Lemme 3, and [V3] (that for almost all v, $\mathrm{H}^1(k_v, \mathrm{Pic}\, \bar{V}(G)) = 0$ and from the well-known fact that the torus dual to $Div_{\bar{V}(X)\backslash\bar{X}}\bar{V}(X)$ is an induced torus, by combining with the exact sequence

$$0 \to \mathscr{B}r_1 V(X) \to \mathscr{B}r_1 X \to \mathrm{H}^2(k, Div_{\bar{V}(X)\backslash\bar{X}})$$

the following

$$\mathrm{H}^1(k, \mathrm{Pic}\, \bar{V}(X)) \simeq \mathscr{B}r_a(V(X)) \simeq \mathscr{B}r_\omega(V(X)) \simeq \mathscr{B}r_\omega(X),$$

where X stands for T, H, G.

STEP 4. We have $\mathscr{B}r_\omega(T) \simeq \mathscr{B}r_\omega(H) \simeq \mathscr{B}r_\omega(G)$, as one can check easily (see above). Now the exact sequence (V) for G follows from these steps.

One may ask, by comparing with 4.2, if we have a similar situation assuming that $\mathrm{A}(G) = 0$, and $Ш(G) \neq 0$. However it is not true as the following classical example shows.

Example. Let $a, b \in \mathbf{Z}$ (the integers), and let $K = \mathbf{Q}(\sqrt{a}, \sqrt{b})$, be a bi-quadratic extension of $\mathbf{Q}$, where $\mathbf{Q}$ denotes the rational numbers. Denote by $T(a, b) = R^{(1)}_{K/\mathbf{Q}}(\mathbf{G}_m)$. Then (see [CTS1], Prop. 7, or [V1], p. 157) we have

$$\mathrm{H}^1(\mathbf{Q}, \mathrm{Pic}\, \bar{V}(T)) = \mathbf{Z}/2\mathbf{Z}.$$

If we choose a, b such that all the decomposition groups for K are cyclic then it is known (by Serre) that $\mathrm{A}(T) = 0$. For example, $a = 5, b = 29$ satisfy this condition. However, one checks that $T(5, 29)$ is rational over all completions of $\mathbf{Q}$, but $Ш(T(a, b)) = \mathbf{Z}/2\mathbf{Z}$.

In the next result we consider some applications to weak approximation in semisimple groups defined over number fields k.

4.3. Theorem. *Let G be a semisimple k-group such that G is of inner type over k_v for all $v \in S$. Then G has weak approximation over k with respect to S. In particular, if G is of inner type over k, it has weak approximation over k.*

The theorem follows from the following

4.4. Proposition. *If as a group over k_v, G is an inner type then G has trivial group $G(k_v)/R$.*

First we need the following result due to Gille [G1], Prop. 2.3.

4.4.1. Proposition ([G1]). *Let*

$$1 \to F \to G_1 \xrightarrow{\lambda} G_2 \to 1$$

be an isogeny of connected reductive groups, all defined over a field k of characteristic 0 *and $C_\lambda(k) = G_2(k)/\lambda(G_1(k))$. Then the following sequence of groups is exact*

$$G_1(k)/R \to G_2(k)/R \xrightarrow{\delta_R} C_\lambda(k)/R,$$

where the R-equivalence relation on $C_\lambda(k)$ is induced from that on $C_\lambda(k(t))$.

Note. In the case that $\mathrm{H}^1(k, G_1)$ is trivial, we identify $C_\lambda(k)$ with $\mathrm{H}^1(k, F)$ and also write the above exact sequence in the form

$$G_1(k)/R \to G_2(k)/R \to \mathrm{H}^1(k, F)/R \to 0.$$

Proof of Proposition 4.4. We distinguish two cases.

1) $v \in \infty$. It is well-known that any connected linear algebraic group G over $\mathbf{R}$ has trivial group of R-equivalences. Here is a short indication of proof. One reduces easily to proving that any semisimple element $s \in G(\mathbf{R})$ is R-equivalent to 1. But this follows from the fact that s belongs to some torus defined over $\mathbf{R}$, and any such torus is rational over $\mathbf{R}$.

2) *v is non-archimedean.* Let G_s be a k_v-split form of G, which is obtained from G by an inner twist. Denote by F the fundamental group of G, which is the same for G_s. One can check that simply connected groups have trivial groups of R-equivalence classes, we have by Proposition 4.4.1 ([G1], Prop. 2.3) the following exact sequences

$$0 \to G_s(k_v)/R \to \mathrm{H}^1(k_v, F)/R \to 0, \tag{8}$$

$$0 \to G(k_v)/R \to \mathrm{H}^1(k_v, F)/R \to 0. \tag{9}$$

Here we identify $\mathrm{H}^1(k_v, F)$ with the factor group $G_s(k_v)/\pi(\tilde{G}_s(k_v))$ (resp. $G(k_v)/\pi(\tilde{G}(k_v))$), where $\pi : \tilde{G}_s \to G_s$ (resp. $\pi : \tilde{G} \to G$) is the simply connected covering of G_s (resp. G) and take the factor group modulo the rational relation induced on $\mathrm{H}^1(k_v, F)$. We have also used the Kneser Theorem on the triviality of $\mathrm{H}^1(k_v, \tilde{G}_s)$ and $\mathrm{H}^1(k_v, \tilde{G})$. Since G_s is rational over k_v, the second group in (8) is trivial, therefore by (9) the group $G(k_v)/R$ is also trivial.

Thus by this proposition, and by Proposition 3.3, G has weak approximation with respect to S and Theorem 4.3 is proved.

4.5. Remarks. 1) One cannot simply drop the inner type assumption, since there are examples of semisimple quasi-split groups over number fields which do not have weak approximation property. First examples of such groups were given by Serre (see [Kn1] and [S] for more information).

2) In the case of number field, this result also extends the previously known (but more general) result by Harder, namely we derive from Theorem 4.3 the following.

4.5.1. Corollary (Harder [H1]). *If a semisimple group G defined over a number field k is split over k_v of all $v \in S$, then G has weak approximation in S.*

Now we apply our results to give new proofs (and also discuss some extension) of some results due to Harder and Sansuc. In the following theorem, the first result is due to Harder ([H2], Satz 2.2.3) and the second is due to Sansuc ([S], Cor. 5.4). The third, in the case that the given group is semisimple and split over a metacyclic extension of the given number field, is also due to Sansuc.

Let $\pi : \tilde{G} \to G$ be a central isogeny of semisimple groups, all defined over a field k, where $\tilde{G}$ is simply connected covering of G. π is called a *normal isogeny* (after Harder [H2]) if $\mu := \operatorname{Ker} \pi$ can be embedded into an induced k-torus M, such that M/μ is also an induced k-torus. One can show, for example, that adjoint groups have normal isogenies.

4.6. Theorem. *The following groups have weak approximation property over number fields.*

1) ([H2]) *Semisimple groups which are images of normal isogenies;*

2) ([S]) *Absolutely almost simple groups;*

3) *Inner forms of connected reductive groups which are split over a metacyclic extension of k_v for all non-archimedean v.*

Moreover, two connected reductive groups, which are inner form of each other have the same group of R-equivalence classes over local non-archimedean fields.[2]

Proof. 1) Let $\pi : \tilde{G} \to G$ be a normal isogeny defined over a number field k, and S any finite set of valuations of k. We show that for all $v \in (S - \infty)$, $G(K_v)/R$ is trivial. Indeed, let $\mu = \operatorname{Ker} \pi, M$ be an induced k-torus, such that M/μ is also an induced k-torus. As above (see Proposition 4.4.1), we have the following exact sequences

$$\tilde{G}(k_v) \to G(k_v) \to \mathrm{H}^1(k_v, \mu) \to 0,$$

$$\tilde{G}(k_v)/R \to G(k_v)/R \to \mathrm{H}^1(k_v, \mu)/R \to 0.$$

One can show easily that $\tilde{G}(k_v)/R$ is trivial. (Here is a short argument. One reduces to almost simple case. If G is isotropic, then it is well-known that $G(k_v)$ has no nontrivial normal subgroup, i.e. $G(k_v) = RG(k_v)$, since $RG(K_v)$ is a normal Zariski dense subgroup of $G(k_v)$. Otherwise, G is of inner type A_n by a result of Kneser, and in this case the result is well-known.)

Hence we have an isomorphism

$$G(k_v)/R \simeq \mathrm{H}^1(k_v, \mu)/R.$$

By considering similar exact sequences

$$1 \to \mu \to M \to M' \to 1,$$

$$M(k_v)/R \to M'(k_v)/R \to \mathrm{H}^1(k_v, \mu)/R \to 0,$$

where M, M' are induced tori and using that M, M' are rational, we get that

[2] This last fact was also mentioned independently by Gille in [G3].

$\mathrm{H}^1(k_v, \mu)/R$ is trivial, and so is $G(k_v)/R$. Therefore G has weak approximation over k by Proposition 3.3.

In the case G is an adjoint group, we can use a direct argument as follows. We may use the following (easy to show) fact: adjoint groups over local p-adic fields are rational. This was mentioned in the preprints [T1]–[T2]. One may also argue as follows. Let G_q be a quasi-split inner form of G defined over k, and F be its fundamental group. By using the same argument we have in the proof of Proposition 4.4, one concludes that $G(k_v)/R$ is trivial for all v. Therefore by Proposition 3.3, G has weak approximation over k.

2) Let G be an (absolutely) almost simple k-group. We want to show that for any finite set S of valuations of k, $G(k_v)/R$ is trivial for all $v \in S$. [In fact, one can show a stronger result in this case: see the paper by Chernousov and Platonov: The rationality problem of simple algebraic groups, C. R. Acad. Sci. Paris 322 (1996), 245–250, which have many results overlapped with results of [T2], where also other results were mentioned:

If v is a non-archimedean valuation, then G is rational over k_v if G is not of type A_n.]

Here we can use the following simple argument as follows. Let G_q be an almost simple quasi-split inner form of G. As in the case 1) we are reduced to proving the statement for quasi-split groups. Assuming that G is not split, then G is of type A_n, D_n, E_6, or ${}^{3,6}\mathrm{D}_4$. Let T be a maximal k-torus of G containing a maximal k_v-split torus of G. If G is not of trialitarian type, then we know by Tits [Ti] that T is split by a quadratic extension of k_v. The structure of tori split over a quadratic extensions are well-known: they are direct product of groups of type $\mathbf{G}_m$, $\mathrm{R}_{K/k_v}(\mathbf{G}_m)$, or $\mathrm{R}^{(1)}_{K/k_v}(\mathbf{G}_m)$ where K/k_v is a quadratic extension of k_v. In particular they are rational over k_v, and so is G by Bruhat decomposition (see, e.g., [BT]). In the trialitarian case one proves in the same way that maximal tori containing a maximal split torus are rational. Thus by Proposition 3.3, G has weak approximation in S for any S, thus also over k.

3) a) First we show that if S is a finite set of valuations of k and G is a connected reductive group which is split over metacyclic extension l_v of k_v for each non-archimedean $v \in S$ then G has weak approximation with respect to S. In fact we prove the following stronger result.

4.7. Proposition. *If G is a connected reductive group defined over a non-archimedean k_v and split over a metacyclic extension l_v of k_v then $G(k_v)/Br$ is trivial.*

Proof. It can be shown that there exists a maximal k_v-torus T of G which is split over l_v. (And in the case of number field k, one can show that there exists a maximal k-torus T such that T is l_v-split for $v \in S$.) Let H be a z-extension of G,

$$1 \to Z \to H \xrightarrow{\pi} G \to 1.$$

Let T_H be the maximal k_v-torus of G such that T_H is mapped onto T via π. Let $\tilde{G}$ be the simply connected covering of the semisimple part G' of G, and let $\tilde{T}$ be the maximal k_v-torus of $\tilde{G}$ which is mapped into T via the composite map

$$\tilde{G} \to G' \to G.$$

We have the following exact sequences of tori.

$$1 \to Z \to T_H \to T \to 1, \tag{10}$$

$$1 \to \tilde{T} \to T_H \to T_0 \to 1. \tag{11}$$

It is clear from (10), (11) that $\tilde{T}$ is also split over l_v for all $v \in S$. By [CTS1], Corollaire 3, p. 200, we have

$$\tilde{T}(k_v)/R = T(k_v)/R = \{1\}, \qquad \forall v \in S,$$

therefore $T_H(k_v)/R = \{1\}, \forall v \in S$.

Now we consider a maximal k_v-split torus T_1 of G. Then

$$Z_G(T_1) = T'H',$$

where T' is the connected center and H' is a semisimple k_v-group, anisotropic over k_v. If H' is trivial, i.e., G is quasi-split over k_v, then the torus T' is split over metacyclic extension l_v, so has trivial group of R-equivalence classes by [CTS1], Corollaire 3, p. 200, and so is G, since G and $Z_G(T_1)$ are birationally equivalent over k_v (using Bruhat decomposition). Therefore $G(k_v)/Br$ is trivial also. One may therefore assume that H' is non-trivial, and by replacing G by $Z_G(T_1)$, one may assume that G has semisimple part G' anisotropic over k_v.

By using a consequence of the Kneser's Theorem on the triviality of H^1 of simply connected groups over local non-archimedean fields [Kn2], we see that G' is necessarily a product of almost simple k_v-factors of type ${}^1\mathrm{A}$, which may be taken to be absolutely almost simple. So we have

$$\tilde{G} = H_1 \times \cdots \times H_t,$$

where H_i is simply connected of type ${}^1\mathrm{A}_{n_i-1}$ for all i.

We now recall the construction of the z-extension H of G. Let $G = G'P$, where P is a k_v-torus. Let $F = G' \cap P$, $F_1 = \{(f, f^{-1}) : f \in F\}$ and we have the following exact sequence

$$1 \to F_1 \to G' \times P \to G \to 1.$$

By taking the composite of two isogenies

$$\tilde{G} \times P \to G' \times P \to G,$$

we have an isogeny

$$1 \to F' \to \tilde{G} \times P \to G \to 1.$$

Thus one sees that since $\tilde{G}$ is of inner type (in fact the product of groups SL), the group F' can be embedded into a split torus Z defined over k_v. Then we take

$$F'_1 := \{(f, f^{-1}) : f \in F'\}$$

and take

$$H = (Z \times (\tilde{G} \times P))/F'_1.$$

From the very construction is follows from (10), (11) that T_H is split over l_v. Since $\tilde{T}$ and T_H are split over l_v, the same holds for T_0. Therefore $T_0(k_v)/R$ is trivial by [CTS1], p. 200. Since $T_0(k_v)/R = T_0(k_v)/Br$ ([CTS1], p. 217) and $H(k_v)/Br = T_0(k_v)/Br$ by Proposition 3.6.3 and the proof given there, we see that $H(k_v)/Br = G(k_v)/Br = \{1\}$. The proof of 4.7 is complete.

Now we see that $G(k_v)/Br = 1$ for all $v \in S$. Thus G has weak approximation for any given finite set S, which means that G has weak approximation over k.

3) b) Now we assume that G_1 is an inner form of a group G. First we prove the last statement in 3) of the theorem.

We need the following very useful fact, which is due to Ono in the case of tori.

4.8. Lemma (Sansuc [S], Lem. 1.10). *Let G be a connected reductive group defined over a field k. There exists a number n, induced k-tori T and T' such that we have the following central k-isogeny*

$$1 \to F \to \tilde{G}^n \times T' \to G^n \times T \to 1,$$

where $\tilde{G}$ is the simply connected covering of the semisimple part G' of G.

The finite covering of an algebraic group by a direct product of simply connected group with an induced torus (such as $\tilde{G}^n \times T' \to G^n \times T$ above) is called after Sansuc ([S], p. 14) a *special covering*. It is obvious that to prove our statement we may assume that the group G_1 itself has a special covering

$$1 \to F \to \tilde{G} \times T' \xrightarrow{\pi} G \to 1 \tag{12}$$

defined over k. Since the inner twist does not effect the center it is obvious that we have also a special covering

$$1 \to F \to \tilde{G}_1 \times T' \to G_1 \to 1, \tag{13}$$

where $\tilde{G}_1$ is the simply connected covering of the semisimple part of G_1. The exact sequences (12) and (13) induce the following exact sequences of groups of R-equivalences

$$(\tilde{G}(k_v) \times T'(k_v))/R \to G(k_v)/R \to \mathrm{H}^1(k_v, F)/R \to 0, \tag{14}$$

$$(\tilde{G}_1(k_v) \times T'(k_v))/R \to G_1(k_v)/R \to \mathrm{H}^1(k_v, F)/R \to 0, \tag{15}$$

(compare with (8) and (9)). Since the first groups in the exact sequences (14), (15) are trivial, we obtain

$$G(k_v)/R \simeq G_1(k_v)/R \simeq \mathrm{H}^1(k_v, F)/R.$$

Now assume that G_1 is an inner form of a group G satisfying 3) a) above. We will show that G_1 has weak approximation with respect to any finite set S of valuations v where G has metacyclic splitting field extension l_v/k_v. By Proposition 3.4, it suffices to show that $G_1(k_v)/Br$ is trivial for all $v \in S$.

Let $G = G'S$, where G' is the semisimple part of G, and S a central torus. Let $F = S \cap G'$, $\tilde{G}$ be the simply connected covering of G', $\pi_G : \tilde{G} \to G'$ the canonical isogeny. As in part a) we denote

$$F_1 = \{(f, f^{-1}) : f \in F\} \hookrightarrow S \times G',$$

so we have a central isogenies

$$1 \to F_1 \to S \times G' \xrightarrow{\alpha} G = SG' \to 1,$$

$$1 \to F_2 \to S \times \tilde{G} \xrightarrow{\beta} S \times G' \to 1,$$

where $F_2 = \{(x, 1) : x \in \mathrm{Ker}\,(\pi_G)\} \simeq \mathrm{Ker}\,\pi_G$. Denote by

$$\pi : S \times \tilde{G} \to SG'$$

the composite of isogenies α and $\beta, \pi : (s, \tilde{g}) \mapsto s\pi_G(\tilde{g})$. Then one checks that

$$\begin{aligned} \mu := \mathrm{Ker}\,\pi &= \{(\pi_G(\tilde{g}), \tilde{g}^{-1}) : \tilde{g} \in \pi_G^{-1}(F)\} \\ &\simeq \pi_G^{-1}(F) \\ &\hookrightarrow \tilde{F} := Cent(\tilde{G}) \end{aligned}$$

Since G_1 is an inner twist of G. $G_1 = SG_1'$, where G_1' is the semisimple part of G_1, and $S \cap G_1' = S \cap G = F$. We define

$$\begin{aligned} \mu_1 &:= \{(x, x^{-1}) : x \in \mu\}, \\ H &= (Z \times (S \times \tilde{G}))/\mu_1, \\ H_1 &= (Z \times (S \times \tilde{G}_1))/\mu_1. \end{aligned}$$

Then from the construction it follows that

$$H = \tilde{G}P, \qquad H_1 = \tilde{G}_1 P,$$

where P is the connected center of H and H_1, and also

$$\tilde{G} \cap P = \tilde{G}_1 \cap P. \tag{16}$$

Therefore we have

$$T_0 := H/\tilde{G} \simeq T_1 := H_1/\tilde{G}_1. \tag{17}$$

From the proof of the part a) above (Proposition 4.7) we see that $T_1(k_v)/Br$ is trivial since $T_0(k_v)/Br(= H(k_v)/Br = G(k_v)/Br)$ is trivial (see the proof of 4.7 of part a)). Therefore by Proposition 3.6.3 and relations (16), (17) $H_1(k_v)/Br$ (which is isomorphic to $T_1(k_v)/Br$) is also trivial, thus so is $G_1(k_v)/Br$. The proof of Theorem 4.6 is complete.

Now we consider some other analogs related with R-equivalence relations. We have the following extension of similar property of tori (see Theorem 1.1, (4)) over completions of a number field k.

4.9. Theorem. *Let G be a connected linear algebraic group defined over a local field k_v. Then the group of R-equivalence classes and the group of Brauer equivalence classes coincide:*

$$G(k_v)/R = G(k_v)/Br.$$

Proof. As above, we may assume that v is non-archimedean and G is reductive.

STEP 1. *Let G be a connected reductive k_v-group, G_q be its quasi-split inner form defined over k_v. Then we have*

$$G(k_v)/R \simeq G_q(k_v)/R. \tag{18}$$

This has been proved in Theorem 4.6, 3).

STEP 2. *Let G_q be a connected reductive quasi-split k_v-group. Then*

$$G_q(k_v)/R = G_q(k_v)/Br. \tag{19}$$

Proof. Take a maximal k_v-torus T of G_q containing a maximal k_v-split torus S of G_q. Then we have

$$T = Z_{G_q}(S),$$

and Bruhat decomposition for G_q shows that (see (CTS1], Section 7) we have

$$T(k_v)/R = G_q(k_v)/Br,$$

$$T(k_v)/Br = G_q(k_v)/Br.$$

Since for tori T we have

$$T(k_v)/R = T(k_v)/Br,$$

by Theorem 1.1, 4), hence $G_q(k_v)/R = G_q(k_v)/Br$.

STEP 3. *If G_q is a quasi-split inner form of a connected reductive k_v-group G, then*

$$G(k_v)/Br \simeq G_q(k_v)/Br. \tag{20}$$

Indeed, by Proposition 3.6 we may asume that the semisimple part $\tilde{G}$ (resp. $\tilde{G}_q$) of G (resp. G_q) is simply connected. From the proof of Theorem 4.6, 3) b (see

(16), (17)), it follows that we have the following canonical isomorphism

$$G_q/\tilde{G}_q \simeq G/\tilde{G} \simeq T,$$

where T is quotient torus of G (and T is defined on the same field as G). We know by Proposition 3.6.3 that

$$G_q(k_v)/Br \simeq T(k_v)/Br \simeq G(k_v)/Br,$$

hence (20) holds.

Now the theorem follows from the combination of (18), (19), (20).

4.10. Remarks. 1) In [CTS2], Remarque 2.8.17, it was shown that for a given smooth variety X over a local $\mathfrak{p}$-adic field and under some condition $(H^!_\lambda)$ on the universal torsor under some torus, the Brauer and R-equivalence are the same. Also, it is a very general method to obtain such kind of results (e.g. one may obtain similar results for tori over $\mathfrak{p}$-adic fields (see [CTS2], Section 2, for details).

2) All results above tell us that if a connected reductive group G over a number field k fails to have weak approximation over k, then for some valuation v (which is necessarily non-archimedean), and the quasi-split inner form G_q of G (which is necessarily non-split), we have $G_q(k_v)/Br \neq 1$.

3) Our assumption in Theorem 4.6 on the existence of metacyclic extension of k_v splitting G has local character, so it is weaker than that of Sansuc [S], Corollaire 5.4, p. 34.

The following *local-global* statement (or principle) would show that our result is equivalent to that of Sansuc:

A connected reductive group G defined over a number field k has a metacyclic splitting field if and only if it is so over all completions k_v of k.

4) Equally, it is natural (and important) to ask for which class $\mathcal{G}$ of finite groups the following holds. We say that a finite Galois extension k'/k is a $\mathcal{G}$-extension if $Gal(k'/k) \in \mathcal{G}$. We require that $\mathcal{G}$ be a kind of formation of groups, i.e., it is closed with respect to the operations of taking subgroups, factor groups and finite direct product. Then we ask *when* the following holds:

A connected reductive group over a number field k has a $\mathcal{G}$-splitting field if and only if it is so over all completions k_v.

Now we are able to formulate and prove a close analog of the exact sequence (Br) for groups of R-equivalence classes.

4.11. Theorem. *Let G be a connected linear algebraic group defined over a number field k. Let H be a z-extension of the reductive part of G, T be its torus quotient and S be the Neron–Severi torus of T. Then in the following exact sequence*

$$(R') \qquad 1 \to \operatorname{Ker} \rho_G \to G(k)/R \xrightarrow{\rho_G} \prod_v G(k_v)/R \to \mathrm{A}(G) \to 1$$

the subgroup $\mathrm{Ker}\,\rho_G$ *has finite index* $n_T = [T(k)/R : \text{III}(S)]$, *(or the same, Card$(T(k)/Br)$) in* $G(k)/R$. *Moreover the following sequence is exact*

$$1 \to \mathrm{Ker}\,\rho_G \to G(k)/R \to G(k)/Br \to 1,$$

and the image of $G(k)/R$ *in* $\prod_v G(k_v)/R$, *being isomorphic to* $G(k)/Br$, *is also isomorphic to* $T(k)/Br$.

Proof. The fact that the first sequence is exact follows from Proposition 3.2. From Theorem 3.7 we have the following commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Ker}\,\rho_G \arrow[r] \arrow[d, "\lambda_G"] & G(k)/R \arrow[r, "\rho_G"] \arrow[d, "\lambda'_G"] & \prod_v G(k_v)/R \arrow[r] \arrow[d, "\simeq"] & \mathrm{A}(G) \arrow[r] \arrow[d, "="] & 1 \\
& 1 \arrow[r] & G(k)/Br \arrow[r, "\gamma_G"] & \prod_v G(k_v)/Br \arrow[r] & \mathrm{A}(G) \arrow[r] & 1.
\end{tikzcd} \tag{21}$$

In the above diagram, the homomorphism λ_G is induced from λ'_G since we have the vertical isomorphism "$\simeq$" due to Theorem 4.9, and it is clear that

$$\lambda'_G(\mathrm{Ker}\,(\rho_G)) \subset \mathrm{Ker}\,(\gamma_G).$$

Therefore it follows that

$$(G(k)/R)/\mathrm{Ker}\,(\rho_G) \simeq G(k)/Br$$

and the image of $G(k)/R$ in the product $\prod_v G(k_v)/R$ is isomorphic to the group $G(k)/Br \simeq T(k)/Br$ and has order equal to $n_T = [T(k)/R : \text{III}(S)]$ by Proposition 2.6 and Theorem 3.7.

From Theorem 4.11 it follows that to determine the structure of $G(k)/R$ one needs to understand the structure of $\mathrm{Ker}\,\rho_G$, which is given in the following theorem. We also derive the following analog of the exact sequence (R) in Section 1 in the case the number field k is totally imaginary and also in many other cases, namely if the semisimple part of G contains no anisotropic factors of (exceptional, trialitarian) type D_4 nor E_6.

4.12. Theorem. 1) *Let G be a connected linear algebraic group defined over number field k. Then we have the following commutative diagram, where all rows and columns are exact sequences*

$$\begin{tikzcd}
& \tilde{G}(k)/R \arrow[r, "="] \arrow[d] & \tilde{G}(k)/R \arrow[d, "\pi_R"] & & \\
1 \arrow[r] & \mathrm{Ker}\,\rho_G \arrow[r] \arrow[d, "p"] & G(k)/R \arrow[r, "\lambda'_G"] \arrow[d, "q"] & G(k)/Br \arrow[r] \arrow[d, "r"', "\simeq"] & 1 \\
1 \arrow[r] & \text{III}(S) \arrow[r] \arrow[d] & T(k)/R \arrow[r, "\lambda'_T"] \arrow[d] & T(k)/Br \arrow[r] & 1 \\
& 1 & 1 & &
\end{tikzcd}$$

and $\tilde{G} \xrightarrow{\pi} G_s$ is the simply connected covering of the semisimple part G_s of G, T is the torus quotient of the reductive part of a z-extension H of G, S the Neron–Severi torus of T and r is an isomorphism.

2) *If the semisimple part of G contains no anisotropic almost simple factors of types D_4 (trialitarian) nor E_6 then $\tilde{G}(k)/R = 1$. In particular, p and q are isomorphisms, $G(k)/R \simeq T(k)/R$ and the following exact equence (R') holds for G.*

$$(R') \qquad 1 \to Ш(S) \to G(k)/R \xrightarrow{\rho_G} \prod_v G(k_v)/R \to A(G) \to 1.$$

In general, (R') holds for all connected linear algebraic groups G if and only if all simply connected almost simple groups have trivial group of R-equivalence classes.

3) *If k is totally imaginary number field, then p, q are also isomorphisms and the exact sequence (R') holds for G.*

First we need the following results.

4.13. Theorem ([CM], Thm. 4.3.). *Let G be an almost simple algebraic group of outer type ${}^2A_n$ defined over a field k, $G(k) = SU(\Phi, D)$, where Φ is the associated hermitian form with respect to an involution J of second kind over a division algebra D of center K. Let Σ_J (resp. Σ'_J) be the group of elements which are J-symmetric (resp. with J-symmetric reduced norm) of D. Then*

$$G(k)/R \simeq \Sigma'_J/\Sigma_J.$$

4.14. Theorem ([H3]). *Assume that k is totally imaginary number field. Then any simply connected semisimple k-group has trivial (Galois) 1-cohomology, and anisotropic almost simple k-groups are of type A_n.*

4.15. Proposition ([T4]). *Let H be a z-extension of a connected reductive group G, all defined over a field k. Then the natural projection $H \to G$ induces canonical isomorphism of abstract groups*

$$H(k)/R \simeq G(k)/R.$$

Proof of Theorem 4.12. 1) The assertion regarding the exactness of two rows and that r is an isomorphism in the above diagram follows from the commutative diagram (21), and the bijectivity of r follows from Theorem 3.7.

4.16. Lemma. *With above notation*, $\operatorname{Ker} q \subset \operatorname{Ker} \rho_G$.

Proof. Indeed, if $x \in \operatorname{Ker} q$ then $r(\lambda'_G(x)) = \lambda'_T(q(x)) = 0$, hence $\lambda'_G(x) = 0$ since r is an isomorphism. As we mentioned above, the rows in the diagram are exact, so $x \in \operatorname{Ker} \rho_G$.

The following is a well-known (and trivial) result from homological algebra.

4.17. Lemma. *In the above diagram, p is surjective if and only if q is surjective.*

4.18. Lemma. *Let T be a torus defined over a field k, S a finite set of discrete valuations of k. Then*

$$Cl_S(RT(k)) = RT_S.$$

In particular, if S consists of real valuations then $RT(k)$ is dense (in the S-adic topology) in $T(k)$.

Proof. Let

$$1 \to N \to P \xrightarrow{q} T \to 1$$

be a flasque resolution of T over k. Here P is an induced k-torus and N is flasque. By [CTS1], Théorème 2, in the above exact sequence we have

$$q(P(k)) = RT(k),$$

hence

$$Cl_S(RT(k)) = Cl_S(q(P(k))).$$

If $x \in q(Cl_S(P(k))), x = q(y)$, where $y = lim_n\, p_n, p_n \in P(k)$ then

$$\begin{aligned} x &= q(lim_n\, p_n) = lim_n\, q(p_n) \\ &\in Cl_S(q(P(k))) = Cl_S(RT(k)), \end{aligned}$$

hence $q(Cl_S(P(k))) \subset Cl_S(RT(k))$.

Since P has weak approximation property, $Cl_S(P(k)) = P_S$, and as mentioned above, $q(P_S) = RT_S$, hence

$$RT_S \subset Cl_S(RT(k)).$$

On the other hand, by Proposition 2.1, RT_S is an open subgroup of T_S containing $RT(k)$, hence the first assertion follows. The rest of the lemma follows from the previous one and also from the fact that any torus over the real numbers are rational.

4.19. Lemma. *With notation as in the theorem, q (hence also p) is surjective.*

Proof. We need only show that

$$T(k) = q(G(k))RT(k),$$

where we may assume that G has simply connected semisimple part $\tilde{G}$, and $q : G \to T = G/\tilde{G}$ is the projection.

By Lemma 4.18 we know that

$$T_\infty = Cl_\infty(RT(k)).$$

For $x \in T(k)$ we have

$$x = lim_n\, r_n, r_n \in RT(k)$$

(the limit is taken with repsect to the archimedean ∞-topology). We have the following commutative diagram similar to (6):

$$\begin{array}{ccccc} G(k) & \longrightarrow & T(k) & \xrightarrow{\delta} & \mathrm{H}^1(k,\tilde{G}) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ G_\infty & \longrightarrow & T_\infty & \xrightarrow{\delta_\infty} & \prod_{v\in\infty} \mathrm{H}^1(k_v,\tilde{G}) \end{array} \tag{22}$$

where all rows are exact and all arrows are continuous with respect to the topologies induced from G_∞ and T_∞. We have

$$\begin{aligned} \delta_\infty(\beta(x)) &= lim_n\, \delta_\infty(\beta(r_n)) \\ &= \delta_\infty(\beta(r_n)), \qquad \forall n > N_0, \end{aligned}$$

for some fixed N_0, since $\prod_{v\in\infty} \mathrm{H}^1(k_v,\tilde{G})$ is finite. Hence

$$\begin{aligned} \delta_\infty(\beta(x)) &= \gamma(\delta(x)) \\ &= \gamma(\delta(r_n)) \end{aligned}$$

for $n > N_0$. Since γ is an isomorphism (i.e. bijection) by Hasse principle, one concludes that

$$\delta(x) = \delta(r_n), \qquad \forall n > N_0.$$

One checks, by using the interpretation of the coboundary map δ (see [Se]) that

$$x = r_n q(g),$$

for some $g \in G(k)$. Thus

$$T(k) = RT(k)q(G(k)),$$

i.e., q (and p, by Lemma 4.17) is surjective.

With this lemma, the proof of the exactness of the first column in Theorem 4.12 is complete. Next we consider the exactness of the second column. We have the following general result.

4.20. Lemma. *Let k be a field of characteristic 0, G a connected reductive group with simply connected semisimple part $\tilde{G}$, $T = G/\tilde{G}$. Then we have the following exact sequence of groups*

$$\tilde{G}(k)/R \to G(k)/R \to T(k)/R.$$

Proof. It is obvious that if the lemma is true for some power $G^n = G \times \cdots \times G$, then it is also true for G, so by virtue of Lemma 4.8 we may assume that G has a special covering $\tilde{G} \times T'$, where T' is an induced k-torus, and we have the following exacts sequence of algebraic groups, all defined over k:

$$1 \to F \to \tilde{G} \times T' \to G \to 1,$$

where F is a finite central subgroup of $\tilde{G} \times T'$. From this we derive the following 3×3-commutative diagram

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & F \cap \tilde{G} \arrow[r] \arrow[d] & F \arrow[r, "u"] \arrow[d] & F' \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \tilde{G} \arrow[r] \arrow[d, "l"'] & \tilde{G} \times T' \arrow[r] \arrow[d, "\pi"] & T' \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \tilde{G} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

Since $\tilde{G}$ is simply connected, l is an isomorphism, hence $F \cap \tilde{G} = 1$, and u is also an isomorphism. From the diagram above we derive the following commutative diagram

$$\begin{tikzcd}
(\tilde{G}(k) \times T'(k))/R \arrow[r, "\pi_R"] \arrow[d, "p'"] & G(k)/R \arrow[r] \arrow[d, "q'"] & \mathrm{H}^1(k, F)/R \arrow[d, "r'", "\simeq"'] \\
T'(k)/R \arrow[r] & T(k)/R \arrow[r] & \mathrm{H}^1(k, F')/R
\end{tikzcd}$$

where all rows are exact (see Proposition 4.4.1), and r' is an isomorphism. Since T' is an induced k-torus, $T'(k)/R = 1$, and

$$(\tilde{G}(k) \times T'(k))/R \simeq \tilde{G}(k)/R.$$

If $x \in G(k)/R$ such that $q'(x) = 1$, then by chasing on this diagram we see that $x \in \operatorname{Im} \pi$, thus

$$\operatorname{Ker} q' = \operatorname{Im} (\tilde{G}(k)/R \to G(k)/R)$$

as required.

From this the exactness of the second column of the diagram in the theorem is proved, hence we have finished the proof of 1).

2) The "general" part of 2) follows directly from 1). Next we show that if the semisimple part of G does not contain anisotropic almost simple factors of exceptional types D_4 and E_6 then (R′) holds for G.

To see this, we reduce the proof to the following situation. Namely, one may assume that the group G above is connected reductive with anisotropic semisimple part. Indeed, it is clear that we may assume G to be reductive.

STEP 1. *If H is an almost simple simply connected group over k then $H(k)$ has no proper noncentral normal subgroups except possibly for the following types:*

- *anisotropic* A_n, D_4 *(exceptional)*, E_6*;*
- *isotropic exceptional types* ${}^2\mathrm{E}_6^{35}$, ${}^2\mathrm{E}_6^{29}$.

This is the result of many authors and the readers are refered to Chapter 9, Section 9.1 of [PR] for further information. (Quite recently Y. Segev and G. Seitz announced that the Platonov–Margulis conjecture is true for the case ${}^1\mathrm{A}_n$.)

STEP 2. *If H is as in Step 1, but H can be of anisotropic type A_n, then* $H(k)/R = 1$.

This follows from Step 1, the well-known fact that $RH(k)$ is an infinite normal subgroup of $H(k)$, Theorem 4.13 in combination with results of Wang (that the group $\mathrm{SK}_1(A) = 1$ for any central simple algebra A over number field k), and the result of Platonov–Yanchevskiĭ (that $\Sigma'_J/\Sigma_J = 1$ for number field k).

4.21. Lemma. *If H is simply connected either of isotropic type ${}^2\mathrm{E}_6^{35}$ or ${}^2\mathrm{E}_6^{29}$ then $H(k)/R = 1$.*

Proof. If S' is a maximal k-split torus of H, then it is well-known by [CTS1] that we have the following functorial isomorphism of abstract groups

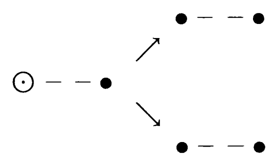
$$H(K)/R \simeq Z_H(S')(K)/R,$$

for any field extension K of k. Hence by Theorem 3.4, we have

$$\mathrm{A}(H) \simeq \mathrm{A}(Z_H(S')).$$

(One can show in general that this last isomorphism holds for any field k, see [T3].)

CASE ${}^2\mathrm{E}_6^{35}$. Let S' be a maximal k-split torus of H. The Tits index of H is as follows



One can check that the centralizer $Z := Z_H(S')$ of S' in H is

$$Z_H(S') = S'L,$$

where L is an almost simple simply connected k-group of type ${}^2\mathrm{A}_5$. Since $L(k)/R = 1$ by Step 2, from the result of 1) (namely from the commutative diagram in the theorem), we have the following exact sequence (R′) for Z:

$$1 \to \text{Ш}(S) \to Z(k)/R \to \prod_v Z(k_v)/R \to \mathrm{A}(Z) \to 1, \tag{23}$$

where S is the Neron–Severi torus of the torus Z/L since Z is the z-extension of itself. Since Z/L is a k-split torus, S has trivial cohomology, so $\text{Ш}(S)$ is trivial. As we notice earlier that

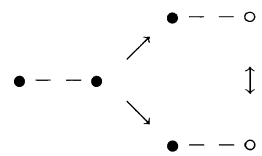
$$Z(k_v)/R \simeq H(k_v)/R$$

is trivial for all v since H is simply connected, hence from (23) it follows that

$$1 = Z(k)/R = H(k)/R$$

as required.

CASE ${}^2\mathrm{E}_6^{29}$. The Tits index of H is as follows



We have

$$Z := Z_H(S') = S'T_0L,$$

where L is a simply connected k-group of (classical) type D_4 (hence satisfies $L(k)/R = 1$), and T_0 is a one-dimensional k-torus. As above we have the exact sequence (R') for the group Z. In this case, the torus quotient

$$T = Z/L = S'T_0/(S'T_0 \cap L)$$

is a two-dimensional k-torus, which is rational over k by a classical result of Voskresenskiĭ [V1]. Therefore

$$T(k)/R = 1.$$

By [CTS1], Proposition 19(ii), we have the following exact sequence

$$0 \to \mathrm{III}(T)^{\sim} \to \mathrm{Br}_a X \to \prod_v \mathrm{Br}_a X_v \to T(k)/R^{\sim} \to \mathrm{III}(S)^{\sim} \to 0.$$

(Here X denotes a smooth compactification of T over k and $\hat{S} = \mathrm{Pic}(\bar{X})$.) In particular, $\mathrm{III}(S) = 0$. Further we argue as above to obtain that $Z(k)/R$ is trivial, hence so is $H(k)/R$.

So from 1), Steps 1, 2 and from Lemma 4.21 it follows that the exact sequence (R') holds for G except possibly the case the semisimple part of G contains anisotropic almost simple factors of exceptional types D_4 and/or E_6. Hence 2) is proved.

3) We claim that $H(k)/R = 1$ for any almost simple simply connected group H over k. If H is anisotropic then by Theorem 4.14, H is of type A_n and the claim follows from Steps 1, 2 in 2). Also from there, by combining with Lemma 4.21, we know that the claim holds for any isotropic group H. So over totally imaginary fields k, $H(k)/R$ is trivial for all simply connected semisimple k-groups H. To obtain (R') we may use the result of 1) and Theorem 4.11. We supply also a proof of this fact independent of 1) as follows.

By using the same argument as in the proof of Theorem 4.4 (or 4.6) and by using the Harder's result on the triviality of the Galois cohomology of simply connected groups (Theorem 4.14), we can show that

$$G(k)/R \simeq G_q(k)/R,$$

where G_q is a quasi-split inner form of G over k and we may assume also that G has simply connected reductive part and that G is reductive. Let T_q be a maximal k-torus of G_q containing a maximal k-split torus S of G_q. By the same argument as in the proof of 4.6, 4.7, it follows that G and G_q have isomorphic torus quotients T. It is also well-known that for simply connected quasi-split semi-simple groups G'_q, any maximal torus containing a maximal k-split torus is also quasi-split (i.e. induced) torus. Denote such a torus by T'_q. Then

$$1 \to T'_q \to T_q \to T \to 1$$

is an exact sequence of k-tori, and T'_q is cohomologically trivial. Since this is a z-extension, from Proposition 4.15 it follows that

$$T_q(k)/R \simeq T(k)/R,$$

hence

$$G(k)/R \simeq T(k)/R,$$

and this is true for any field extension of k. In particular,

$$G(k_v)/R \simeq T(k_v)/R$$

for all v, hence from (R) we deduce the exact sequence

$$(R') \qquad 1 \to \text{Ш}(S) \to G(k)/R \to \prod_v G(k_v)/R \to \mathrm{A}(G) \to 1,$$

where T is the torus quotient of any z-extension of G and S is its Neron–Severi torus.

The proof of Theorem 4.12 is therefore complete.

Remark. In our earlier preprint [T4], we propose another way to express an exact sequence connecting groups of R-equivalence classes, weak approximation obstruction $\mathrm{A}(G)$ and the Tate–Shafarevich group of some *finite Galois module*. It is clear that the above exact sequence (R$'$) is, in a sense, more true (or natural) analog of the initial exact sequence (R) for tori established by Colliot–Thélène and Sansuc.

We derive the following consequence describing a relation between $RG(k)$ and RG_S, extending the corresponding result for tori (Lemma 4.18) over number fields.

4.22. Theorem. *Let k be a number field, S a finite set of valuations of k, G a connected linear algebraic k-group. Then*

$$BG_S = RG_S = Cl_S(RG(k)) = Cl_S(BG(k)).$$

In other words, the groups $RG(k)$ and $BG(k)$ have weak approximation over k.

Proof. We claim that

$$RG_S = \pi(\tilde{G}_S)Cl_S(RG(k)),$$

where $\pi : \tilde{G} \to G_s$ is the universal covering of the semisimple part G_s of G.

We may assume that G is reductive. We first show that RG_S contains the set on right hanside. It is known and easy to see that RG_S is an open subgroup in $\prod_{v \in S} G(k_v)$, hence also closed. We know that for simply connected groups

$$\tilde{G}_S = R\tilde{G}_S = Cl_S(\tilde{G}(k)),$$

hence

$$\pi(\tilde{G}_S) = \pi(R\tilde{G}_S) \subset RG_S,$$

and

$$\pi(\tilde{G}_S)Cl_S(RG(k)) \subset RG_S.$$

To prove the other inclusion, first we assume that $\tilde{G}$ is the semisimple part of G. Let $T = G/\tilde{G}$. We have the following commutative diagram

$$\begin{array}{ccccc}
G(k) & \xrightarrow{p} & T(k) & \xrightarrow{\delta} & \mathrm{H}^1(k, \tilde{G}) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
G_S & \xrightarrow{p} & T_S & \xrightarrow{\delta} & \prod_{v \in S} \mathrm{H}^1(k_v, \tilde{G})
\end{array}$$

Let $x \in RG_S$. Then $p(x) \in RT_S = Cl_S(RT(k))$ by Lemma 4.18, so

$$p(x) = lim_n\, r_n, \qquad r_n \in RT(k).$$

Then

$$\delta(r_n) \to \delta(p(x)) = 1, \qquad n \to \infty,$$

hence

$$\delta(r_n) = 1, \qquad \forall n > N,$$

for some fixed N. Therefore $r_n \in p(G(k)), r_n = p(g_n), g_n \in G(k)$ for $n > N$. Thus

$$lim_n\, p(xg_n^{-1}) = 1.$$

Let

$$G = \tilde{G}T',$$

where T' is a k-torus. The natural isogeny

$$p' : \tilde{G} \times T' \to G$$

induces an open map

$$p' : \tilde{G}_S \times T'_S \to G_S.$$

In particular, $\tilde{G}_S T'_S$ is an open subgroup of G_S. Since $Cl_S(RT'(k)) = RT'_S$ by Lemma 4.18, and RT'_S is open in T'_S by Proposition 2.1, it follows that $Cl_S(RT'(k))$ is open in T'_S. Hence $\tilde{G}_S Cl_S(RT'(k))$ is an open subgroup of G_S, and so is $\tilde{G}_S Cl_S(RG(k))$. Let $V_n, n = 1, 2, \ldots$ be a nested system of open neighbourhoods of 1 in T_S such that $V_{n+1} \subset V_n$ for all n,

$$\bigcap_n V_n = \{1\},$$

and $p(xg_n^{-1}) \in V_n$, for all n. Then

$$xg_n^{-1} \in p^{-1}(V_n), \qquad \forall n.$$

Since T' is rational over k, $T'_S \subset Cl_S(RG(k))$ and since $\tilde{G}_S T'_S$ is an open subgroup of G_S, it follows that for all n we have

$$x \in p^{-1}(V_n) g_n \subset p^{-1}(V_n) \tilde{G}_S Cl_S(RG(k)).$$

Since

$$\begin{aligned}\bigcap_n p^{-1}(V_n) &= p^{-1}(\bigcap_n V_n)\\ &= p^{-1}(1)\\ &= \tilde{G}_S,\end{aligned}$$

so (V_n) form a nested system of open neighbourhoods of $\tilde{G}_S$. Therefore for some N,

$$p^{-1}(V_n) \subset \tilde{G}_S Cl_S(RG(k)), \qquad \forall n > N,$$

since $\tilde{G}_S Cl_S(RG(k))$ is an open subgroup of G_S. Thus

$$x \in \tilde{G}_S Cl_S(RG(k)) = Cl_S(\tilde{G}(k)) Cl_S(RG(k)).$$

since, by Kneser's and Harder's results, $\tilde{G}$ has weak approximation property over k.

In general case, let H be a z-extension of G,

$$1 \to Z \to H \xrightarrow{\pi} G \to 1,$$

where π induces the covering isogeny $\tilde{G} \to G_s \subset G$. By Proposition 4.15 the projection π induces surjections

$$RH_S \to RG_S, \qquad RH(k) \to RG(k),$$

hence

$$\begin{aligned}RG_S &= \pi(RH_S)\\ &= \pi(\tilde{G}_S) Cl_S(RH(k)))\\ &= \pi(\tilde{G}_S)\pi(Cl_S(RH(k)))\\ &\subset \pi(\tilde{G}_S) Cl_S(\pi(RH(k)))\\ &= \pi(\tilde{G}_S) Cl_S(RG(k)),\end{aligned}$$

and the claim is proved. Since $R\tilde{G}(k)$ is an infinite normal subgroup of $\tilde{G}(k)$, $Cl_S(R\tilde{G}(k))$ is an infinite normal subgroup of $Cl_S(\tilde{G}(k)) = \tilde{G}_S$. It is known that $\tilde{G}_S$ has no proper infinite normal subgroup (consequence of Kneser–Tits conjecture over local fields). Thus

$$Cl_S(R\tilde{G}(k)) = \tilde{G}_S,$$

hence

$$\tilde{G}_S \subset Cl_S(RH(k))$$

and $RG_S = Cl_S(RG(k))$. The assertion regarding BG_S follows from Theorem 4.9. Theorem 4.22 is proved.

We derive also the following finiteness result of the group of R-equivalence classes in a slightly different way than the one given in [G2]. Namely we do not use the Kato–Saito's Hasse principle for arithmetical varieties (compare [G2] and [G3]).

4.23. Theorem ([G2, G3]). *If k is a number field then $G(k)/R$ is finite.*

Proof. From well-known theorem of Margulis–Prasad it follows that if $\tilde{G}$ is a simply connected semisimple k-group then $\tilde{G}(k)/R$ is finite. From the commutative diagram in Theorem 4.12 and from the finiteness of $T(k)/R$ (see [CTS1], Corol. 2, p. 200) it follows that $G(k)/R$ is finite.

4.24. Corollary. *Let G be an adjoint semisimple group defined over a number field k and $\tilde{G}$ be its simply connected covering. If $\tilde{G}(k)/R = 1$ then $G(k)/R = 1$. In particular, it is so if G contains no anisotropic factors of exceptional types* D_4, E_6.

Proof. Let $F = \mathrm{Ker}(\tilde{G} \to G), \tilde{G}_q, G_q$ be quasi-split inner forms of $\tilde{G}, G$ respectively. Denote by $\tilde{T}_q, T_q$ their corresponding maximal k-torus containing maximally k-split torus, which are known to be induced tori. From the exact sequence

$$\tilde{G}(k)/R \to G(k)/R \to \mathrm{H}^1(k, F)/R,$$

and from the assumption it follows that $G(k)/R \to \mathrm{H}^1(k, F)/R$ is injective. By considering the corresponding sequence for tori

$$\tilde{T}_q(k)/R \to T_q(k)/R \to \mathrm{H}^1(k, F)/R \to 0,$$

(we use the fact that $\mathrm{H}^1(k, \tilde{T}_q) = 0$) and from the fact that these tori are induced, so we have trivially $\mathrm{H}^1(k, F)/R = 0$, thus $G(k)/R$ is trivial.

5. Remarks, problems and conjectures

5.1. From Theorem 4.12 it follows that over an arbitrary field k, the finiteness of groups $G(k)/R$ for connected reductive groups G depends only on the finiteness of tori and simply connected groups over k. It seems natural to state the following

Problem 1. *Study the finiteness of $G(k)/R$ and $G(k)/Br$ for connected linear algebraic groups G over finitely generated (over the prime field) k.*

Problem 2. *Same problem as above, but only for purely transcendental extensions of* $\mathbf{Q}, \mathbf{Q}_p, \mathbf{R}, \mathbf{C}$.

5.2. It is natural to make the following

Conjecture 1. *The exact sequence (R') holds for any connected linear algebraic group G over any number field k.*

Notice from above that this conjecture is equivalent to the following conjecture

Conjecture 2. *$G(k)/R$ is trivial for anisotropic almost simple simply connected group G of exceptional type* D_4 *or* E_6 *over a number field k.*

As we have seen from above, the last conjecture is a consequence of another stronger conjecture due to Platonov and Margulis. (See [PR, Chapter 9] for more information.)

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