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Estimating Siegel modular forms of genus 2 using Jacobi forms

By

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Abstract

We give a new elementary proof of Igusa's theorem on the structure of Siegel modular forms of genus 2. The key point of the proof is the estimation of the dimension of Jacobi forms appearing in the Fourier-Jacobi development of Siegel modular forms. This proves not only Igusa's theorem, but also gives the canonical lifting from Jacobi forms to Siegel modular forms of genus 2.

1. Igusa's theorem

On the Siegel upper-half plane of genus $g \in \mathbf{N}$

$$\mathbf{H}_g :=\{Z={}^tZ \in M_g(\mathbf{C}) | \mathrm{Im}(Z)>0\},$$

the symplectic group

$$Sp_g(\mathbf{R}) :=\left\{M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_{2g}(\mathbf{R}) \middle| \begin{matrix} {}^tAD-{}^tCB=E_g, \\ {}^tAC={}^tCA,\ {}^tBD={}^tDB \end{matrix}\right\}$$

acts by

$$\mathbf{H}_g \ni Z \rightarrow M\langle Z\rangle :=(AZ+B)(CZ+D)^{-1} \in \mathbf{H}_g.$$

For $k \in \mathbf{Z}$, a function F is a Siegel modular form of genus g and weight k, if F satisfies the following conditions:

(S1) $F : \mathbf{H}_g \rightarrow \mathbf{C}$ is holomorphic.

(S2) $F(Z)=\det(CZ+D)^{-k}F(M\langle Z\rangle)$ for any $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_g(\mathbf{Z})$.

(S3) F has the Fourier expansion

$$F(Z)=\sum_{T={}^tT\geq 0,\text{even integral}} a(T)\exp(\pi\sqrt{-1}\sigma(TZ)),$$

where σ means the trace of the matrix.

We remark that the condition (S3) is induced by the conditions (S1) and (S2) if g

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≥ 2. Let $G(k,Sp_g(\mathbf{Z}))$ be the $\mathbf{C}$-vector space of all Siegel modular forms of genus g and weight k. The following properties about $G(k, Sp_g(\mathbf{Z}))$ are well-known.
(S4) $G(0,Sp_g(\mathbf{Z}))=\mathbf{C}$ and $G(k,Sp_g(\mathbf{Z}))=\{0\}$ if $k<0$.
About more details of Siegel modular forms, for instance, see Freitag [3].

Put $\mathrm{M}_k := G(k,Sp_1(\mathbf{Z}))$ and $\mathrm{M}_* := \oplus_k \mathrm{M}_k$ be the induced graded ring. It is well-known that

$$\mathrm{M}_* = \mathbf{C}[e_4, e_6],$$

where e_k is the Eisenstein series of weight k for $k=4$, 6. These two generators are algebraically independent.

Put $\mathbf{M}_k := G(k,Sp_2(\mathbf{Z}))$ and $\mathbf{M}_* := \oplus_k \mathbf{M}_k$ be the induced graded ring. Igusa [4,5] determined the structure of $\mathbf{M}_*$.

Theorem 1 (Igusa). *The structure of* $\mathbf{M}_*$ *is*

$$\mathbf{M}_* = \mathbf{C}[E_4, E_6, \Delta_{10}, \Delta_{12}] \oplus \Delta_{35} \mathbf{C}[E_4, E_6, \Delta_{10}, \Delta_{12}],$$

where E_k *is the Eisenstein Series of weight* k *for* $k=4,6$ *and* Δ_k *is the unique cusp forms of weight* k *for* $k=10$, 12, 35.

Now, we give another and more elementary proof of this theorem. For an element of $\mathbf{M}_k$, we consider the Fourier expansion with respect to the last variable. This gives an injection from $\mathbf{M}_k$ to the spaces of Jacobi forms. We can give the upper bound of the dimension of this image by a similar method in the book of Eichler-Zagier[1]. Surprisingly, this upper bound coincides with the exact dimension of $\mathbf{M}_k$. This proves not only Igusa's theorem, but also gives the canonical lifting from Jacobi forms to Siegel modular forms of genus 2.

2. Estimation

For $k \in \mathbf{Z}$ and $m \in \mathbf{N}_0 := \{0, 1, 2, \cdots\}$, a function φ is a Jacobi form of weight k and index m, if φ satisfies the following conditions:
(J1) $\varphi : \mathbf{H}_1 \times \mathbf{C} \to \mathbf{C}$ is holomorphic.
(J2) $\varphi(\tau,z)=(c\tau+d)^{-k}\mathbf{e}\left(\frac{-mcz^2}{c\tau+d}\right)\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)$ for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, where $\mathbf{e}(*) := \exp(2\pi\sqrt{-1}*)$.
(J3) $\varphi(\tau,z)=q^{mx^2}\zeta^{2mx}\varphi(\tau,z+x\tau+y)$ for any $x, y \in \mathbf{Z}$, where $q := \mathbf{e}(\tau)$, $\zeta := \mathbf{e}(z)$.
(J4) φ has the Fourier expansion

$$\varphi(\tau,z)=\sum_{4mn-l^2 \geq 0} a(n,l) q^n \zeta^l.$$

Let $\mathbf{J}_{k,m}$ be the $\mathbf{C}$-vector space of all Jacobi forms of weight k and index m.
In the book of Eichler-Zagier [1], they prove the following theorem.

Theorem 2 (Eichler-Zagier [1] Theorem 3.1 and 3.4). *We have the following* *relations between Jacobi forms and elliptic modular forms.*

(A) *For each* $\nu \in \mathbf{N}_0$ *and* $k \in \mathbf{N}$, *there exists a polynomial* P_ν^k *that induces the map* D_ν :

$$\varphi(\tau,z)=\sum_{n,l} a(n,l)q^n \zeta^l \in \mathbf{J}_{k,m}$$

$$\mapsto D_\nu \varphi(\tau) := \sum_{n,l} P_\nu^k(l,nm)a(n,l)q^n \in \mathrm{M}_{k+\nu}.$$

(B) *The maps* D_ν *induce injections*

$$\begin{cases} \Pi_{\nu=0}^m D_{2\nu} : \mathbf{J}_{k,m} \to \Pi_{\nu=0}^m \mathrm{M}_{k+2\nu} & (k \text{ even}) \\ \Pi_{\nu=1}^{m-1} D_{2\nu-1} : \mathbf{J}_{k,m} \to \Pi_{\nu=1}^{m-1} \mathrm{M}_{k+2\nu-1} & (k \text{ odd}, m \geq 2) \end{cases}.$$

If k *is odd,* $\mathbf{J}_{k,0}=\{0\}$ *and* $\mathbf{J}_{k,1}=\{0\}$.

For $r \in \mathbf{N}_0$, we put

$$\mathbf{J}_{k,m}^{(r)} := \left\{ \varphi(\tau,z)=\sum_{n,l} a(n,l)q^n \zeta^l \in \mathbf{J}_{k,m} \middle| a(n,l)=0 \text{ if } n<r \right\}$$

and

$$\mathrm{M}_k^{(r)} := \left\{ f(\tau)=\sum_n a(n)q^n \in \mathrm{M}_k \middle| a(n)=0 \text{ if } n<r \right\}.$$

We remark $\mathbf{J}_{k,m}=\mathbf{J}_{k,m}^{(0)} \supset \mathbf{J}_{k,m}^{(1)} \supset \mathbf{J}_{k,m}^{(2)} \supset \cdots$ and $\mathrm{M}_k=\mathrm{M}_k^{(0)} \supset \mathrm{M}_k^{(1)} \supset \mathrm{M}_k^{(2)} \supset \cdots$. The following lemma is easily induced from Theorem 2.

Lemma 3. *We have the following estimation,*

$$\dim \mathbf{J}_{k,m}^{(r)} \leq \begin{cases} \sum_{\nu=0}^m \dim \mathrm{M}_{k+2\nu}^{(r)} & (k \text{ even}) \\ \sum_{\nu=1}^{n-1} \dim \mathrm{M}_{k+2\nu-1}^{(r)} & (k \text{ odd}, m \geq 2). \\ 0 & (k \text{ odd}, m \leq 1) \end{cases}$$

The element $\Delta(\tau) := q\Pi_{n=1}^{\infty}(1-q^n)^{24} \in \mathrm{M}_{12}^{(1)}$ induces the relation $\dim \mathrm{M}_k^{(r)} = \dim \mathrm{M}_{k-12}^{(r-1)}$. Hence $\dim \mathrm{M}_k^{(r)} = \dim \mathrm{M}_{k-12r}$.

We define $P_m : \mathbf{M}_k \ni F \mapsto \varphi_m \in \mathbf{J}_{k,m}$ by the Fourier-Jacobi development

$$\mathbf{M}_k \ni F\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} = \sum_{m \geq 0} \varphi_m(\tau, z)p^m,$$

where $p := \mathbf{e}(\omega)$. Under this development, the translations with respect to

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & y \\ x & 1 & y & * \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} \in Sp_2(\mathbf{Z}) \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \atop x,\ y \in \mathbf{Z} \right)$$

give the translation formulas (J2) and (J3) of Jacobi forms. The modular invariance by the translation with respect to

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in Sp_2(\mathbf{Z})$$

gives the condition $F\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} = (-1)^k F\begin{pmatrix} \omega & z \\ z & \tau \end{pmatrix}$. These elements generate $Sp_2(\mathbf{Z})$. We put

$$\mathbf{M}_k^{(r)} := \left\{ F\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} = \sum_m \varphi_m(\tau, r) p^m \in \mathbf{M}_k \middle| \varphi_m(\tau, z) = 0 \text{ if } m < r \right\}.$$

We remark $\mathbf{M}_k = \mathbf{M}_k^{(0)} \supset \mathbf{M}_k^{(1)} \supset \mathbf{M}_k^{(2)} \supset \cdots$. Then we have the exact sequence

$$0 \longrightarrow \mathbf{M}_k^{(r+1)} \longrightarrow \mathbf{M}_k^{(r)} \xrightarrow{P_r} \mathbf{J}_{k,r}.$$

The condition $F\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} = (-1)^k F\begin{pmatrix} \omega & z \\ z & \tau \end{pmatrix}$ gives an information of the image of P_r. In terms of the Fourier expansion $\mathbf{M}_k \ni F\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} = \sum a(n,l,m) q^n \zeta^l p^m$, the condition $F\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} = (-1)^k F\begin{pmatrix} \omega & z \\ z & \tau \end{pmatrix}$ means $a(n,l,m) = (-1)^k a(m,l,n)$.

Now we assume that k is even. The condition $a(n,l,m) = a(m,l,n)$ implies that the image of P_r is included in $\mathbf{J}_{k,r}^{(r)}$. Hence we have the exact sequence

$$0 \longrightarrow \mathbf{M}_k^{(r+1)} \longrightarrow \mathbf{M}_k^{(r)} \xrightarrow{P_r} \mathbf{J}_{k,r}^{(r)}.$$

and the estimation

$$\dim \mathbf{M}_k^{(r)} - \dim \mathbf{M}_k^{(r+1)} \leq \dim \mathbf{J}_{k,r}^{(r)}.$$

Because

$$\dim \mathbf{J}_{k,r}^{(r)} \leq \sum_{\nu=0}^{r} \dim \mathrm{M}_{k+2\nu-12r},$$

we have $\mathbf{J}_{k,r}^{(r)} = \{0\}$ for sufficiently large r, and hence $\mathbf{M}_k^{(r)} = \mathbf{M}_k^{(r+1)} = \cdots = \{0\}$. Then we have

$$\dim \mathbf{M}_k \leq \sum_{r=0}^{\infty} \dim \mathbf{J}_{k,r}^{(r)}$$

$$\leq \sum_{r=0}^{\infty} \sum_{\nu=0}^{r} \dim \mathbf{M}_{k+2\nu-12r}.$$

The Poincare series of the right-hand side, namely,

$$\sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \sum_{\nu=0}^{r} (\dim \mathbf{M}_{k+2\nu-12r}) t^k,$$

equals

$$\sum_{r=0}^{\infty} \sum_{\nu=0}^{r} \frac{t^{12r-2\nu}}{(1-t^4)(1-t^6)} = \frac{1}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}. \tag{1}$$

Assume the existence of the generators E_4, E_6, Δ_{10}, Δ_{12} and their algebraic independence for a while. Then (1) shows that the graded ring of even weight is generated by these generators. We have also the short exact sequence

$$0 \longrightarrow M_k^{(r+1)} \longrightarrow \mathbf{M}_k^{(r)} \xrightarrow{P_r} \mathbf{J}_{k,r}^{(r)} \longrightarrow 0.$$

Now we assume that k is odd. The condition $a(n,l,m) = -a(m,l,n)$, especially $a(m,l,m)=0$ implies that the image of P_r is included in$\mathbf{J}_{k,r}^{(r+1)}$. Hence we have the exact sequence

$$0 \longrightarrow M_k^{(r+1)} \longrightarrow \mathbf{M}_k^{(r)} \xrightarrow{P_r} \mathbf{J}_{k,r}^{(r+1)}.$$

Similarly to the even case, we have

$$\dim \mathbf{M}_k \leq \sum_{r=0}^{\infty} \dim \mathbf{J}_{k,r}^{(r+1)}$$

$$\leq \sum_{r=2}^{\infty} \sum_{\nu=1}^{r-1} \dim \mathbf{M}_{k+2\nu-12r-13}.$$

The Poincare series of the right-hand side is

$$\sum_{r=2}^{\infty} \sum_{\nu=1}^{r-1} \frac{t^{12r-2\nu+13}}{(1-t^4)(1-t^6)} = \frac{t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

This shows that any modular form of odd weight is the product of Δ_{35} and a modular form of even weight, if we admit the existence of Δ_{35} due to Igusa. We have also the short exact sequence

$$0 \longrightarrow M_k^{(r+1)} \longrightarrow \mathbf{M}_k^{(r)} \xrightarrow{P_r} \mathbf{J}_{k,r}^{(r+1)} \longrightarrow 0.$$

We have thus given a new proof of Igusa's theorem, assuming the existence of the generators E_4, E_6, Δ_{10}, Δ_{12}, Δ_{35} and the algebraic independence of the even generators. We now briefly discuss their existence and algebraic independence. Using theta series, the even generators were constructed in Igusa [4] and Freitag [2], while Δ_{35} was constructed in Igusa [5]. The explicit relation $\Delta_{35}^2 \in \mathbf{C}[E_4, E_6, \Delta_{10}, \Delta_{12}]$ was given by Igusa [6]. We remark that the existence of E_4, E_6, Δ_{10}, Δ_{12} also follows from the following proposition (cf. Eichler-Zagier [1] Theorem 6.2), which uses Jacobi forms and is a consequence of Saito-Kurokawa lifting.

Proposition 4. *For any $\varphi(\tau,z)=\sum a(n,l)q^n\zeta^l\in\mathbf{J}_{k,1}$, there exists $F\in\mathbf{M}_k$ such that $P_1(F)=\varphi$ and that $P_0(F)=const.\times a(0,0)\times e_k$.*

Finally, we prove the algebraic independence of E_4, E_6, Δ_{10} and Δ_{12}. These generators have the forms

$$E_4\begin{pmatrix}\tau & z\\ z & \omega\end{pmatrix}=e_4(\tau)+240e_{4,1}(z,\tau)p+\cdots$$
$$E_6\begin{pmatrix}\tau & z\\ z & \omega\end{pmatrix}=e_6(\tau)-504e_{6,1}(z,\tau)p+\cdots$$
$$\Delta_{10}\begin{pmatrix}\tau & z\\ z & \omega\end{pmatrix}=0+\varphi_{10,1}(z,\tau)p+\cdots$$
$$=0+(0-4\pi^2\Delta(\tau)z^2+\cdots)p+\cdots \text{ (cusp form)}$$
$$\Delta_{12}\begin{pmatrix}\tau & z\\ z & \omega\end{pmatrix}=0+\varphi_{12,1}(z,\tau)p+\cdots$$
$$=0+(12\Delta(\tau)+*z^2+\cdots)p+\cdots \text{ (cusp form)},$$

where $e_{k,1}$ is the unique Jacobi form of weight $k=4$, 6 and index 1, and $\varphi_{k,1}$ is the unique Jacobi cusp form of weight $k=10$, 12 and index 1. Now we show their algebraic independence. Suppose

$$\sum_{4b_i+6c_i+10d_i+12e_i=k} a_iE_4^{b_i}E_6^{c_i}\Delta_{10}^{d_i}\Delta_{12}^{e_i}=0 \tag{2}$$

and $a_i\neq0$ for any i. Let $s:=\min\{d_i+e_i\}$ and $t:=\min\{d_i|d_i+e_i=s\}$. Then the contribution to the coefficient of p^sz^{2t} in (2) only comes from $e_i=s-t$ and $d_i=t$. Hence we have

$$\sum_{4b_i+6c_i+10t+12(s-t)=k} a_ie_4^{b_i}e_6^{c_i}(-4\pi^2\Delta)^t(12\Delta)^{s-t}=0.$$

That is $\sum_{4b_i+6c_i+10t+12(s-t)=k}a_ie_4^{b_i}e_6^{c_i}=0$. From the algebraic independence of e_4 and e_6, we have $a_i=0$ when $e_i=s-t$, $d_i=t$. This is a contradiction. Hence E_4, E_6, Δ_{10}, Δ_{12} are algebraically independent.

3. Lifting

Let $\varphi_{35,2}:=P_2(\Delta_{35})\in\mathbf{J}_{35,2}^{(3)}$. We note here that $\Delta_{35}\in\mathbf{M}_{35}^{(2)}$ is a base of $\mathbf{M}_{35}$, which is a one-demensional $\mathbf{C}$-vector space, and that any automorphic form of odd weight with respect to $Sp_2(\mathbf{Z})$ can be divided by Δ_{35}. As seen in the previous section, we have the following commutative diagram

$$\begin{array}{ccccccccc}
\text{(odd weight) } 0 & \longrightarrow & \mathbf{M}_{k+35}^{(r+3)} & \longrightarrow & \mathbf{M}_{k+35}^{(r+2)} & \xrightarrow{P_{r+2}} & \mathbf{J}_{k+35,r+2}^{(r+3)} & \longrightarrow & 0 \\
 & & \wr\uparrow \times\Delta_{35} & & \wr\uparrow \times\Delta_{35} & & \wr\uparrow \times\Delta_{35,2} & & \\
\text{(even weight) } 0 & \longrightarrow & \mathbf{M}_{k}^{(r+1)} & \longrightarrow & \mathbf{M}_{k}^{(r)} & \xrightarrow{P_{r}} & \mathbf{J}_{k,r}^{(r)} & \longrightarrow & 0
\end{array}$$

We shall give the splitting map of these exact sequences. From the previous section, we have the dimension formula

$$\dim \mathbf{J}_{k,r}^{(r)} = \sum_{\nu=0}^{r} \dim \mathbf{M}_{k+2\nu}^{(r)}$$

for any even weight k, hence the generating function is

$$\sum_{k\in 2\mathbf{Z}} \sum_{r=0}^{\infty} \dim \mathbf{J}_{k,r}^{(r)} t^k s^r = \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \sum_{\nu=0}^{r} \dim \mathbf{M}_{k+2\nu-12r} t^k s^r$$

$$= \sum_{r=0}^{\infty} \sum_{\nu=0}^{r} \frac{t^{12r-2\nu} s^r}{(1-t^4)(1-t^6)}$$

$$= \frac{1}{(1-t^4)(1-t^6)(1-t^{10}s)(1-t^{12}s)}.$$

Therefore we have the expression

$$\mathbf{J}_{\text{even}}^{(*)} := \bigoplus_{k\in 2\mathbf{Z}} \bigoplus_{r=0}^{\infty} \mathbf{J}_{k,r}^{(r)} = \mathbf{C}[e_4, e_6, \varphi_{10,1}, \varphi_{12,1}].$$

We remark that this graded ring structure also can be induced from the theory of weak Jacobi forms, given by Eichler-Zagier [1] Theorem 9.3. The correspondences $e_k \mapsto E_k$ for $k=4$, 6 and $\varphi_{k,1} \mapsto \Delta_k$ for $k=10$, 12 induce the ring homomorphism L : $\mathbf{J}_{\text{even}}^{(*)} \to \mathbf{M}_*$. For even weight k,define $L_{k,r}: \mathbf{J}_{k,r}^{(r)} \to \mathbf{M}_k^{(r)}$ be a restriction of L. This $L_{k,r}$ satisfies $P_r \circ L_{k,r} = id$ and splits the exact sequence. For any odd weight k, define $L_{k,r}: \mathbf{J}_{k,r}^{(r+1)} \to \mathbf{M}_k^{(r)}$ by $L_{k,r}(\varphi) := \Delta_{35} \cdot L_{k-35,r-2}\left(\frac{\varphi}{\varphi_{35,2}}\right)$. We note here that $\frac{\varphi}{\varphi_{35,2}} \in \mathbf{J}_{k-35,r-2}^{(r-2)}$ for any $\varphi \in \mathbf{J}_{k,r}^{(r+1)}$ in view of the above commutative diagram. Also this $L_{k,r}$ satisfies $P_r \circ L_{k,r} = id$ and splits the exact sequence. We have proved the following theorem.

Theorem 5. *We have the following result:*

(A) *Let k be an even integer.*

(a) *The sequence* $0 \longrightarrow M_k^{(r+1)} \longrightarrow \mathbf{M}_k^{(r)} \xrightarrow{P_r} \mathbf{J}_{k,r}^{(r)} \longrightarrow 0$ *is exact.*

(b) $\mathbf{J}_{\text{even}}^{(*)} := \bigoplus_{k\in 2\mathbf{Z}} \bigoplus_{r=0}^{\infty} \mathbf{J}_{k,r}^{(r)} = \mathbf{C}[e_4, e_6, \varphi_{10,1}, \varphi_{12,1}]$.

(c) *Define the ring homomorphism* $L: \mathbf{J}_{\text{even}}^{(*)} \to \mathbf{M}_*$ *by* $e_k \mapsto E_k$ *for* $k=4$, 6 *and*

$\varphi_{k,1} \mapsto \Delta_k$ for $k=10, 12$, and $L_{k,r}: \mathbf{J}_{k,r}^{(r)} \to \mathbf{M}_k^{(r)}$ be a restriction of L. Then $L_{k,r}$ splits (a), that is, $P_r \circ L_{k,r} = id$.

(B) *Let k be an odd integer.*

(a) *The sequence* $0 \longrightarrow M_k^{(r+1)} \longrightarrow \mathbf{M}_k^{(r)} \xrightarrow{P_r} \mathbf{J}_{k,r}^{(r+1)} \longrightarrow 0$ is exact.

(b) *Define $L_{k,r}: \mathbf{J}_{k,r}^{(r+1)} \to \mathbf{M}_k^{(r)}$ by $L_{k,r}(\varphi) := \Delta_{35} \cdot L_{k-35,r-2}\left(\frac{\varphi}{\varphi_{35,2}}\right)$. Then $L_{k,r}$ splits* (a), *that is, $P_r \circ L_{k,r} = id$.*

Remark. Theorem 5(A) is proved only by using the theory of Jacobi forms, and we do not use the theory of theta series, which was used in the proof of Igusa and the one of Freitag. But in the proof of (B), we use the existence of Δ_{35}, that was given by Igusa with the use of theta series. The problem to construct $L_{35,2}$ directly not using Δ_{35}, is still open.

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References

[1] M. Eichler and D. Zagier, The Theory of Jacobi Forms, Progress in Math. 55, Birkhäuser, Boston, 1985.
[2] E. Freitag, Zur Theorie der Modulformen zweiten Grades (German), Nachr. Akad. Wiss. Göttingen, **11** (1965), 151-157.
[3] E. Freitag, Siegelsche Modulfunktionen (German), GMW254, Springer Verlag, Berlin, 1983.
[4] J. Igusa, On Siegel modular forms of genus two, Amer. J. Math., **84** (1962), 175-200.
[5] J. Igusa, On Siegel modular forms of genus two (II), Amer. J. Math., **86** (1964), 392-412.
[6] J. Igusa, Modular Forms and Projective Invariants, Amer. J. Math., **89** (1967),817-855.