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The Grassmannian of $k((z))$: Picard group, equations and automorphisms

By

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Abstract

It is shown that the Determinant line bundle generates the Picard group of the infinite Grassmannian and that it is defined by the Plücker equations. An approach to its automorphism group is also offered.

1. Introduction

This paper aims at generalizing some geometric properties of Grassmannians of finite dimensional vector spaces to the case of Grassmannians of infinite dimensional spaces.

Recall that infinite Grassmannians are schemes (see [1,3,12] or section §2 below for precise statements). Therefore, it is natural to use the standard techniques of algebraic geometry in the study of standard geometric problems (global sections of bundles, Picard group, automorphisms). The scheme structure of infinite Grassmannians has shown to be very useful in some moduli problems ([12,13,14]). Although none of our results is unexpected, we think that the literature lacks of rigorous proofs of them. However, it must be said that studies of similar properties have been carried out by several authors but with different approaches; for instance, some of them consider a Hilbert space and endow its Grassmannian with a structure of infinite complex space, others consider it simply as a set because they only need few properties.

For sake of brevity, we state our results whilst we describe the organization of the paper.

In §2 it is proved, using an explicit construction of global sections of the determinant bundle, that the Plücker morphism is a closed immersion (see Theorem 5.3). The section §3 is devoted to show that the Picard group is isomorphic to $\mathbf{Z}$ and that the determinant bundle generates it. The next section, §4, is concerned with the automorphisms of the Grassmannian. The characterization of finite dimensional

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projective spaces embedded in the Grassmannian (Theorem 4.4) and the fact that automorphisms of a connected component do extend to the entire Grassmannian (Lemma 4.6) are the key results of this section. Using them, we will, firstly, recover some known results (Theorem 4.8), and, secondly, prove that the automorphisms of the Grassmannian are "essentially" induced by linear maps (see Theorem 4.9).

The last section finishes with the applications of the previous results in the case of the Grassmannian of $k((z))$. In particular, the interpretation of semi-infinite wedge products as sections of the determinant bundle is now easily seen (the notion of admissible basis is not needed, see [15,16]). The Theorem 5.3 deserves a special mention : the equations of the Plücker morphism are the set of all Plücker relations ; that is, our Grassmannian scheme thus coincides with the Universal Grassmann Manifold of Sato-Sato [17], but not with the Segal-Wilson Grassmannian Gr_0 (see § 2 of [16]), which is interpreted as the set of points of Gr $(k((z)))$ with finitely many non-zero coordinates. In a certain sense, Gr $(k((z)))$ unifies both Grassmannians.

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2. Grassmannians

Backgrounds. (For a detailed approach to the scheme structure of infinite Grassmannians, see [1,12].)

We shall fix a pair $(V, \mathcal{B})$ of a k-vector space and a family of subspaces such that :

1. $A, B\in\mathcal{B} \Longrightarrow A+B,\ A\cap B\in\mathcal{B}$,
2. $A, B\in\mathcal{B} \Longrightarrow \dim(A+B)/A\cap B<\infty$,
3. the topology is separated : $\cap_{A\in\mathcal{B}}A=(0)$,
4. V is complete : $V\to \varprojlim_{A\in\mathcal{B}} V/A$ is an isomorphism,
5. every finite dimensional subspace of V/B is a neighbourhood of (0) : $\varinjlim_{A\in\mathcal{B}} A \to V/B$ is surjective (for $B\in\mathcal{B}$).

The completion with respect to the linear topology induced by $\mathcal{B}$ will be denoted with $\widehat{\ }$. For $T\to S$, a morphism of k-schemes, and $U\subseteq V_S:=V\otimes_k\mathcal{O}_S$, a sub-$\mathcal{O}_S$-module, define $\hat{U}_T:=U\hat{\otimes}_{\mathcal{O}_S}\mathcal{O}_T$.

Observe that the set of affine schemes $F_A:=\mathrm{Hom}_k(L_A, A)$ (where $A\in\mathcal{B}$ and $L_A\oplus A\simeq V$) is an open covering of the k-scheme infinite Grassmannian of $(V, \mathcal{B})$:

$$S\rightsquigarrow \mathrm{Gr}^\bullet(V)(S)=\left\{\begin{array}{c}\text{sub-}\mathcal{O}_S\text{-modules } L\subseteq\hat{V}_S \text{ quasi-coherent}\\ \text{such that there exist } A\in\mathcal{B} \text{ with } L\oplus\hat{A}_S\simeq\hat{V}_S\end{array}\right\}$$

From now on we will fix a subspace $V^+\in\mathcal{B}$. This choice allows us to introduce the index function as well as the determinant line bundle. Let $\mathcal{B}_0$ be the set of subspaces $A\in\mathcal{B}$ such that $\dim A/(A\cap V^+)=\dim V^+/(A\cap V^+)$.

Recall that the index function $i(L):=\dim_k(L\cap V^+)-\dim_k(V/L+V^+)$ gives rise to the decomposition into connected components $\mathrm{Gr}^\bullet(V)=\coprod\mathrm{Gr}^n(V)$, where

$\mathrm{Gr}^n(V) :=i^{-1}(n)$. For simplicity's sake, $\mathrm{Gr}^0(V)$ will be denoted by $\mathrm{Gr}(V)$.

The determinant bundle is the determinant of the perfect complex of $\mathcal{O}_{\mathrm{Gr}(V)}$-modules $\mathcal{L}\to\pi^*(V/V^+)$, where π is $\mathrm{Gr}(V)\to\mathrm{Spec}(k)$ and $\mathcal{L}$ is the universal object of $\mathrm{Gr}(V)$([1,10]).

Plücker morphism. Assume that Ω is a subspace of $H^0(\mathrm{Gr}(V), \mathrm{Det}_V^*)$ such that the canonical homomorphism $\Omega\otimes_k\mathcal{O}_{\mathrm{Gr}}(V)\to\mathrm{Det}_V^*$ is surjective. The universal property of the projective space implies the existence of a morphism:

$$\mathrm{Gr}(V)\to\mathbf{P}\Omega^*$$

which is called the "Plücker morphism". (Here and henceforth $\mathbf{P}E^*$ will denote the scheme Proj $(S^\bullet E)$ where $S^\bullet E$ is the symmetric algebra of a vector space E).

Let us show how such a subspace Ω can be obtained. Fix $A\in\mathcal{B}_0$. Then, $det(\pi_A)$ is a global section of the dual of the determinant of the complex $\mathcal{C}_A^\bullet\equiv\mathcal{L}\xrightarrow{\pi_A}\pi^*(V/A)$ and thus gives a section $\Omega_A\in H^0(\mathrm{Gr}(V), \mathrm{Det}_V^*)$ via the canonical isomorphism:

$$\mathrm{Det}_V^*\simeq\mathrm{Det}(\mathcal{C}_A^\bullet)^*\otimes(\wedge A/A\cap V^+)\otimes(\wedge V^+/A\cap V^+)^*$$

(where $\wedge$ denotes the top exterior algebra). Then, define Ω as the subspace:

$$\sum_{A\in\mathcal{B}_0}<\Omega_A>\subseteq H^0(\mathrm{Gr}(V), \mathrm{Det}_V^*)$$

Note that Ω is precisely the image of the homomorphism:

$$\Psi:\bigoplus_{A\in\mathcal{B}_0}(\wedge A/A\cap V^+)\otimes(\wedge V^+/A\cap V^+)^*\to H^0(\mathrm{Gr}(V), \mathrm{Det}_V^*)$$

defined by twisting the A-component by $det(\pi_A)$. An explicit expression for Ω is given by the following:

Lemma 2.1. *There exists a natural factorization*:

$$\begin{array}{ccc}\bigoplus_{A\in\mathcal{B}_0}(\wedge A/A\cap V^+)\otimes(\wedge V^+/A\cap V^+)^* & \xrightarrow{\Psi} & H^0(\mathrm{Gr}(V), \mathrm{Det}_V^*)\\ \downarrow & \nearrow & \\ \varinjlim_{\substack{B\in\mathcal{B}\\ B\subseteq V^+}}(\bigoplus_{k=0}^{\dim(V^+/B)}\wedge^k V/V^+\otimes(\wedge^k V^+/B)^*) & & \end{array}$$

Proof. Let us fix $B\in\mathcal{B}$ such that $B\subseteq V^+$. Note that the morphism:

$$\bigoplus_{\substack{A\in\mathcal{B}_0\\ A\cap V^+=B}}(\wedge A/A\cap V^+)\otimes(\wedge V^+/A\cap V^+)^*\to H^0(\mathrm{Gr}(V), \mathrm{Det}_V^*)$$

factors through a surjection onto $\wedge^k V/V^+\otimes(\wedge^k V^+/B)^*$, where $k=\dim_k(V^+/B)$. Therefore, the linear map Ψ factors through a surjection onto:

$$\bigoplus_{B\subseteq V^+}(\wedge^k V/V^+\otimes(\wedge^k V^+/B)^*)$$

Fixing B again, observe further that the induced morphism :

$$\bigoplus_{\substack{B\subseteq B'\\ B'\subseteq V^+}} \wedge^k V/V^+\otimes(\wedge^k V^+/B')^* \to H^0(\mathrm{Gr}(V),\ \mathrm{Det}_V^*)$$

($k=\dim_k(V^+/B)$) factors through a surjection onto :

$$\bigoplus_{k=0}^{\dim(V^+/B)} \wedge^k V/V^+\otimes(\wedge^k V^+/B)^* \tag{2.2}$$

Since the set (2.2) is a direct system as B varies, the statement follows easily.

Theorem 2.3. *The Plücker morphism* $\mathrm{Gr}(V)\to \mathbf{P}\Omega^*$ *is a closed immersion.*

Proof. Once the previous lemma has been proved, one proceeds as in [7]. Let U_A the affine open subscheme of $\mathbf{P}\Omega^*$ where the A-coordinate has no zeroes. It is clear that $\mathfrak{p}(F_A)\subseteq U_A$, and that it is enough to see that $\mathfrak{p}|_{F_A}$ is a closed immersion.

For the sake of clarity, it will be assumed that $A=V^+$. Nevertheless, the general case presents no extra difficulty. By fixing sections of $V\to V/V^+$ and $\Omega\to\Omega/<\Omega_+>$ one has identifications $F_{V^+}\simeq\mathrm{Hom}(V/V^+,V^+)$ and $U_{V^+}\simeq\mathrm{Hom}(\Omega/<\Omega_+>,<\Omega_+>)$. The restriction of the Plücker morphism to F_{V^+} is now a morphism :

$$\mathrm{Hom}(V/V^+,V^+)\to\mathrm{Hom}(\Omega/<\Omega_+>,<\Omega_+>) \tag{2.4}$$

By Lemma 2.1 one has that :

$$\Omega\simeq\varinjlim_{\substack{B\subseteq V^+\\ B\in\mathcal{B}}}\left(\bigoplus_{k=0}^{\dim(V^+/B)}\wedge^k V/V^+\otimes(\wedge^k V^+/B)^*\right)$$

and that $<\Omega_+>$ corresponds to $B=V^+$. Recalling that $V^+=\varprojlim V^+/B$, it follows that (2.4) is the inverse limit of:

$$\mathrm{Hom}(V/V^+,V^+/B)\to\mathrm{Hom}\left(\bigoplus_{k=1}^{\dim(V^+/B)}\wedge^k V/V^+\otimes(\wedge^k V^+/B)^*,<\Omega_+>\right)$$

where $B\in\mathcal{B}$ is such that $B\subsetneq V^+$.

Observe that all these spaces are affine schemes, and it then suffices prove that given $B\in\mathcal{B}$ such that $B\subsetneq V^+$ the morphism :

$$\mathrm{Hom}(V/V^+,V^+/B)\to\prod_{k=1}^{\dim(V^+/B)}\mathrm{Hom}(\wedge^k V/V^+\otimes(\wedge^k V^+/B)^*,<\Omega_+>)$$

is a closed immersion. This, however, is trivial since it is the graph of a morphism.

Related Grassmannians.

The Grassmannian of the dual space. Let $(V,\mathcal{B},V^+)$ be as usual. For a given submodule $U\subseteq \hat{V}_S$ (S a k-scheme), we introduce the following notation:

$$U^*:=\mathrm{Hom}_{\mathcal{O}_S}(U,\mathcal{O}_S)$$

$$U^c:=\{f\in U^* \text{ continuous}\}$$

where the topology in U is given by $\{\hat{A}_S\cap U|A\in\mathcal{B}\}$ and $\mathcal{O}_S$ has the discrete topology. And define:

$$U^\circ:=\{f\in(\hat{V}_S)^*|\ f|_U\equiv 0\}$$

$$U^\diamond:=\{f\in(\hat{V}_S)^c|\ f|_U\equiv 0\}$$

A long but straightforward check shows that the Grassmannian of $(V^c,\mathcal{B}^\diamond)$ exists, where $\mathcal{B}^\diamond:=\{A^\diamond \text{ where } A\in\mathcal{B}\}$.

Nevertheless, there is a canonical isomorphism between the Grassmannian of $(V,\mathcal{B})$ and that of $(V^c,\mathcal{B}^\diamond)$, whose expression for rational points is $I(L)=L^\diamond$.

The case of a metric space. Assume now that there is a given irreducible and skewsymmetric form on V, T_2: $V\times V\to k$ such that V^+ is maximal totally isotropic ($(V^+)^\perp=V^+$).

Thus, iT_2: $T\xrightarrow{\sim} V^c$ turns out to be a bicontinuous isomorphism of vector spaces and it therefore induces an isomorphism $\mathrm{Gr}^\bullet(V^c)\xrightarrow{\sim}\mathrm{Gr}^\bullet(V)$.

The composition of the latter isomorphism and I is an involution, R, of both $\mathrm{Gr}^\bullet(V)$ and $\mathrm{Gr}(V)$, whose expression at the rational points is given by:

$$\begin{aligned} R:\ \mathrm{Gr}^\bullet(V)&\longrightarrow \mathrm{Gr}^\bullet(V)\\ L&\longrightarrow L^\perp \end{aligned} \tag{2.5}$$

3. Picard group of Gr(V)

Restriction of sections. For a pair of rational points M, $N\in\mathrm{Gr}^\bullet(V)$ such that $N\subset M$, let Grass (M/N) denote the standard Grassmannian as defined in [7] I. We know from [1] that the morphism:

$$\begin{aligned} j:\ \mathrm{Grass}(M/N)&\to \mathrm{Gr}^\bullet(V)\\ L&\to \pi^{-1}(L) \end{aligned}$$

(where π: $M\to M/N$) is a closed immersion and that the composite:

$$\mathrm{Grass}^k(M/N)\xrightarrow{j}\mathrm{Gr}(V)\xrightarrow{\mathfrak{p}}\mathbf{P}H^0(\mathrm{Gr}(V),\mathrm{Det}_V^*)^* \tag{3.1}$$

factors through the Plücker morphism of $\mathrm{Grass}^k(M/N)$.

Looking at restrictions, we note first that if $A\in\mathcal{B}$ satisfies $V/M+A=(0)$ and $N\cap A=(0)$, then $j^{-1}(F_A)=F_{M\cap A}$. It follows that the restriction of the section Ω_A is $\Omega_{M\cap A}$, and that the restriction homomorphism of global sections:

$$\Omega \to H^0(\mathrm{Grass}^k(M/N), \mathrm{Det}^*_{M/N}) \tag{3.2}$$

is surjective.

Picard group

Theorem 3.3. *Let k be an algebraically closed field. Assume that* dim $(\mathrm{Gr}(V)) \geq 1$. *Then, the Picard group of* $\mathrm{Gr}(V)$ *is isomorphic to* $\mathbf{Z}$ *and the line bundle* Det_V *is a generator.*

Proof. If $\dim(\mathrm{Gr}(V))=1$, then $\mathrm{Gr}(V)$ is the projective line, and we have finished.

For the general case, recall that the Picard group of $\mathrm{Gr}(V)$ is canonically isomorphic to the class group of Cartier divisors, because $\mathrm{Gr}(V)$ is integral.

Fix $A \in \mathcal{B}_0$ and assume $Z_A := \mathrm{Gr}(V) - F_A$ (the locus where Ω_A vanishes) to be irreducible. This then implies the exactness of the sequence:

$$\mathbf{Z} \to \mathrm{Pic}(\mathrm{Gr}(V)) \to \mathrm{Pic}(F_A)$$

$$1 \to \mathcal{O}_{\mathrm{Gr}(V)}(-Z_A) = \mathrm{Det}_V$$

Observe that $\mathrm{Pic}(F_A)=0$, since F_A is the spectrum of a factorial ring (see [2] VII §3. 5.). Finally, $\mathrm{Det}_V^{*\otimes n}$ cannot be trivial for $n>0$, because its space of global sections has dimension greater than 1 and $H^0(\mathrm{Gr}(V), \mathcal{O}_{\mathrm{Gr}(V)}) = k([1])$.

The proof is therefore reduced to proving the following claim: *let k be an algebraically closed field, and* $\dim(\mathrm{Gr}(V)) \geq 2$. *There exists a subspace $A \in \mathcal{B}_0$ of V such that Z_A is irreducible.*

If $\mathrm{Gr}(V)$ is of finite type (that is, V is finite dimensional), the statement is an easy consequence of the Bertini Theorem (see [8] II.8.18 and III.7.9.1), since Z_A is precisely a hyperplane section of the Plücker morphism. Moreover, the Bertini Theorem implies that Z_A is irreducible for A generic.

Assume now that V is not finite dimensional. One first proves that if Z_A is reducible then its restriction to a finite dimensional Grassmannian is also reducible. Since the restriction homomorphism of global sections is surjective (3.2), one concludes that this is not possible for generic A, and the result follows.

4. Automorphisms of Grassmannians

The linear group. Fix a pair $(V, \mathcal{B})$ as usual. Given a k-scheme S, $\mathrm{Aut}_{\mathcal{O}_S}(V_S)$ will denote the automorphism group of $\hat{V}_S$ as an $\mathcal{O}_S$-module.

Definition 4.1.

- *A sub-$\mathcal{O}_S$-module $A \subseteq \hat{V}_S$ belongs to $\mathcal{B}$ if there exists $B \in \mathcal{B}$ such that $\hat{B}_S \subseteq A$ and the quotient is free of finite type.*
- *An automorphism $g \in \mathrm{Aut}_{\mathcal{O}_S}(\hat{V}_S)$ is bicontinuous (w.r.t. $\mathcal{B}$) if there exists $A \in \mathcal{B}$ such that both $g(\hat{A}_S)$ and $g^{-1}(\hat{A}_S)$ belong to $\mathcal{B}$.*
- *The linear group,* $\mathrm{Gl}(V)$, *associated with $(V, \mathcal{B})$, is the contravariant functor*

over the category of k-schemes given by :

$$S \rightsquigarrow \mathrm{Gl}(V)(S)=\{g\in \mathrm{Aut}_{\mathcal{O}_S}(\hat{V}_S) \text{ such that } g \text{ is bicontinuous}\}$$

Theorem 4.2. *There exists a canonical action of* $\mathrm{Gl}(V)$ *on (the functor of points of)* $\mathrm{Gr}^\bullet(V)$:

$$\mathrm{Gl}(V)\times \mathrm{Gr}^\bullet(V) \xrightarrow{\mu} \mathrm{Gr}^\bullet(V)$$

$$(g,L) \longrightarrow g(L)$$

Moreover, this action preserves Det_V^* ; *that is,* $f^*\mathrm{Det}_V \simeq f_g^*\mathrm{Det}_V$, *where* $f: S\to \mathrm{Gr}(V)$, *g is an element of* $\mathrm{Gl}(V)(S)$, *and* $f_g: S\to \mathrm{Gr}(V)$ *is the transform of f under g.*

Proof. Fix $g\in \mathrm{Gl}(V)(S)$. Note that it suffices check that $g(L)$ is a point of the Grassmannian for arbitrary $L\in F_A(S)$.

From $L\oplus \hat{A}_S \simeq \hat{V}_S$ it follows that $g(L)\oplus g(\hat{A}_S)\simeq \hat{V}_S$. Let $B\in \mathcal{B}$ be such that $g(\hat{A}_S)/\hat{B}_S$ is free of finite type. It follows from [1] that $g(L)\cap \hat{B}_S=0$ and $\hat{V}_S/(g(L)+\hat{B}_S)$ is locally free of finite type, and hence $g(L)\in \mathrm{Gr}^\bullet(V)(S)$, as desired.

The second claim is a direct consequence of the properties of the determinant ([10]) and the exactness of the sequence of complexes (written vertically) :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & g(L)\oplus\hat{B}_S & \longrightarrow & g(L)\oplus g(\hat{V}_S^+) & \longrightarrow & g(\hat{V}_S^+)/\hat{B}_S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{V}_S & \longrightarrow & \hat{V}_S & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

(where $B\in\mathcal{B}$ is such that $g(\hat{V}_S^+)/\hat{B}_S/$ is free of finite type).

Projective spaces in $\mathrm{Gr}(V)$. For the sake of notation, let us denote simply by D_L the stalk of Det_V^* at a rational point $L\in \mathrm{Gr}(V)$. Further, $\mathrm{Gr}(V)$ will be thought of as a closed subscheme of $\mathbf{P}\Omega^*$ (Theorem 2.3).

Theorem 4.3. *Three rational points L_1, L_2, L_3 of* $\mathrm{Gr}(V)$ *lie in a line iff* $L_1\cap L_2\subseteq L_3\subseteq L_1+L_2$ *and both inclusions have codimension* 1.

If this is the case, the line is $\mathrm{Grass}^1((L_1+L_2)/(L_1\cap L_2))\subset \mathrm{Gr}(V)$ *and does not depend on the choice of L_1, L_2.*

Proof. Consider the following commutative diagram :

$$\begin{array}{ccccc} \Omega & \xrightarrow{p_3} & \Lambda_3 & \xrightarrow{\rho_3} & \oplus_{i=1}^3 D_{L_i} \\ \Vert & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi_{12}} \\ \Omega & \xrightarrow{p_2} & \Lambda_2 & \xrightarrow{\rho_2} & \oplus_{i=1}^2 D_{L_i} \end{array}$$

where $\Lambda_j := \wedge^k(A\cap L_1+\cdots+A\cap L_j)^*$. Note that p_3, p_2, ρ_2 and π are surjective. Fix $A\in\mathcal{B}$ such that such that $(V/(A+L_i))=(0)$. It then follows that $k=\dim(A\cap$

L_i) does not depend on i and that $\wedge^k(A\cap L_i)^* \xrightarrow{\sim} D_{L_i}$.

Observe that $\{L_1, L_2, L_3\}$ lie in a line iff:

$$\left\{\begin{array}{l}\text{hyperplanes of } \mathbf{P}\Omega^* \\ \text{containing } L_1,\ L_2,\ L_3\end{array}\right\} = \left\{\begin{array}{l}\text{hyperplanes of } \mathbf{P}\Omega^* \\ \text{containing } L_1,\ L_2\end{array}\right\}$$

which is equivalent to $\ker(\rho_3)=\ker(\rho_2\circ\pi)$. But from the very definition of Λ_3 one easily sees that they are equal iff:

$$\wedge^k(A\cap L_3)\subseteq \wedge^k(A\cap L_1)+\wedge^k(A\cap L_2)$$

which implies that $A\cap L_3$ is contained in $A\cap L_1+A\cap L_2$ and that it has codimension 1. Observe, however, that this argument also holds for every $B\in\mathcal{B}$ such that $A\subseteq B$, and hence one concludes that $L_3\subseteq L_1+L_2$ has codimension 1, as desired.

When computing the codimension of $L_3\subseteq L_1+L_2$, one replaces A by $B\in\mathcal{B}$ such that $B\cap L_i=(0)$ and Λ_j by $\wedge^k(V/\sum_{i\leq j} B\cap L_i))$ in the previous discussion.

The converse is straightforward.

Recall that $n+2$ points of a n-projective space define a reference in it iff there is no $n+1$ of them lying in a $(n-1)$-dimensional subspace. For a family $\{L_i\}_{i\in I}$ of subspaces, one defines $\mathcal{Q}\{L_i\}:=(\sum L_i)/(\cap L_i)$.

The previous Theorem is generalized to the following characterization :

Theorem 4.4.

- *Let $\{L_i\}_{1\leq i\leq n+2}$ be points of* Gr(V) *defining a n-dimensional reference such that* $\dim_k \mathcal{Q}\{L_i\}=n+1$. *It then holds that* $\mathrm{Grass}^k\, \mathcal{Q}\{L_i\}$ ($k=\dim L_i/(\cap_j L_j)$, *which does not depend on i, and its value is* 1 *or n*) *is a n-dimensional projective space contained in* Gr(V).
- *If $X=\mathbf{P}_n\subseteq$ Gr(V), then there exists a reference $\{L_i\}_{1\leq i\leq n+2}$ in X such that* $\dim_k \mathcal{Q}\{L_i\}=n+1$ *and* $X=\mathrm{Grass}^k\, \mathcal{Q}\{L_i\}\subseteq \mathrm{Gr}(V)$, *where $k=1$ or $k=n$. (Note that k does not depend on $\{L_i\}$ but only on X).*

Automorphisms of Gr(V). Henceforth, $\mathrm{Aut}_{k\text{-scheme}}(\mathrm{Gr}(V))$ will simply be denoted by Aut(Gr(V)) and similarly for $\mathrm{Gr}^\bullet(V)$. It is clear that the automorphisms of Gr(V) are restrictions of linear transformations of $\mathbf{P}H^0(\mathrm{Gr}(V),\mathrm{Det}_V^*)^*$ via the Plücker morphism; that is :

Lemma 4.5. *Let $X\subseteq$ Gr(V) be a finite dimensional projective space and let ϕ be an automorphism of* Gr(V). *Then, $\phi(X)$ is a finite dimensional projective space.*

Lemma 4.6. *Fix $\phi\in$* Aut (Gr(V)). *There then exists a unique $\bar\phi\in$* Aut ($\mathrm{Gr}^\bullet(V)$) *with the following properties :*

1. *it is an extension of ϕ ($\bar\phi|_{\mathrm{Gr}(V)}=\phi$),*
2. *$\bar\phi$ is an inclusion-preserving or inclusion-reversing automorphism.*

Proof. Let us first define $\bar{\phi}(L)$ for $L\in \mathrm{Gr}^k(V)(k>0)$. Choose $L'\in \mathrm{Gr}^{-1}(V)$ such that $L'\subset L$ and hence $\mathbf{P}(L/L')\subseteq \mathrm{Gr}(V)$ is a finite dimensional projective space. Theorem 4.4 implies that there exists a finite family of points $\{M_i\}$ of $\mathbf{P}(L/L')$, such that:

$$\mathbf{P}(L/L')=\mathbf{P}\mathcal{Q}\{M_i\}$$

Using this theorem again and Lemma 4.5, it follows that:

$$\phi(\mathbf{P}(L'/L))=\mathrm{Grass}^r\mathcal{Q}\{\phi(M_i)\}$$

where $r=1$ or $r=-1$ (here $\mathrm{Grass}^{-1}(E)$ for a finite dimensional space E denotes the grassmannian of 1-codimensional subspaces of E). By proving that r is locally constant and that it is invariant under restrictions $\mathbf{P}(L/L')\subset\mathbf{P}(L''/L')\subseteq \mathrm{Gr}(V)(L\subset L'')$, it follows that r depends only on ϕ.

One now checks that the following definition fulfills the requirements:

$$\bar{\phi}(L') := \begin{cases} \cup\,\phi(M_i) & \text{if } r=1 \\ \cap\,\phi(M_i) & \text{if } r=-1 \end{cases}$$

Lemma 4.7. *Let V^+ be finite dimensional. Then the following conditions are equivalent:*

1. *there exists $\phi\in \mathrm{Aut}(\mathrm{Gr}(V))$ with an inclusion-reversing extension,*
2. *V is finite dimensional and $\dim_k V=2\dim_k V^+$,*
3. *there is an irreducible skewsymmetric metric in V, such that V^+ is maximal totally isotropic.*

Proof. Conditions 2 and 3 are clearly equivalent. The third implies the first since R, the automorphism of $\mathrm{Gr}(V)$ constructed in (2.5), extends naturally to an inclusion-reversing automorphism of $\mathrm{Gr}^\bullet(V)$.

Let us prove that 1 implies 2. Let $\phi\in \mathrm{Aut}(\mathrm{Gr}(V))$ and let $\bar{\phi}\in \mathrm{Aut}(\mathrm{Gr}^\bullet(V))$ be its inclusion-reversing extension. Observe that:

$$\mathrm{Gr}(V)=\mathrm{Grass}^{-k}(V)$$

where $k=\dim_k(V^+)$. Since $\bar{\phi}$ reverses inclusions and leaves $\mathrm{Gr}(V)$ invariant, it follows that:

$$\bar{\phi}: \mathrm{Grass}^{-k-r}(V)\xrightarrow{\sim}\mathrm{Grass}^{-k+r}(V)\quad \forall r\in\mathbf{Z}$$

Observe that for $r=-k-1$, the scheme on the left hand side is a projective space. By Theorem 4.4, one has that $\dim_k V=2k$.

The lemmas above enable us to state the following Theorem that includes the classical results about the automorphism group:

Theorem 4.8. *The group $\mathrm{Aut}(\mathrm{Gr}(V))$ is canonically isomorphic to:*

- $\mathrm{PGl}(V)$ *if* $\dim_k V<\infty$ *and* $\dim_k V\neq 2\dim_k V^+$;

- $\mathbf{PGl}(V)\times\mathbf{Z}/2$ *if* $\dim_k V<\infty$ *and* $\dim_k V=2\dim_k V^+$ *(these two connected components correspond to collineations and correlations)*;
- $\mathbf{PGl}(V)$ *if* $\dim_k V<\infty$ *and* $\dim_k V^+<\infty$;
- $\mathbf{PGl}(V^c)$ *if* $\dim_k V<\infty$ *and* $\dim_k (V/V^+)<\infty$.

(For a group G with centre $Z(G)$, *define* $\mathbf{P}G:=G/Z(G)$).

Remark 1. The two first statements were proved by Chow in [4], while the latter two are algebraic versions of the results of Cowen ([5]) and Kaup ([9]), which were given for Hilbert spaces. A similar group isomorphism cannot be expected for arbitrary infinite Grassmannians (see §1 of [9] for the case of a Banach space).

From Lemma 4.6 one deduces that the study of automorphism of the Grassmannian can therefore be restricted to those with an inclusion-preserving extension.

Nevertheless, since the study of the automorphism group is rather complicated we shall add some extra structure to the pair $(V,\mathscr{B})$ consisting of a separated linear topology on V with a basis $\mathfrak{C}$ such that: 1) $\dim_k(A+B)/(A\cap B)<\infty$ for all $A, B\in\mathfrak{C}$; and, 2) $A\in\mathrm{Gr}^\bullet(V)$ for all $A\in\mathfrak{C}$. The completion will be denoted by.

We now define:

$$\mathrm{Gl}(V,\mathfrak{C}):=\left\{\begin{array}{c}\phi\in\mathrm{Gl}(V)(k)\text{ bicontinuous w.r.t. }\mathfrak{C}\\ \text{such that }\phi(\mathrm{Gr}(V))=\mathrm{Gr}(V)\end{array}\right\}$$

$$\mathscr{A}(\mathfrak{C}):=\left\{\begin{array}{c}\phi\in\mathrm{Aut}(\mathrm{Gr}(V)\text{ bicontinuous w.r.t. }\mathfrak{C}.\\ \text{with an inclusion-preserving extension}\end{array}\right\}$$

where bicontinuous w.r.t. $\mathfrak{C}$ means that both $\phi(A)$ and $\phi^{-1}(A)\in\mathfrak{C}$ contain an element of $\mathfrak{C}$ (for all $A\in\mathfrak{C}$).

Theorem 4.9. *There exist injective morphisms of groups:*

$$\mathbf{PGl}(V,\mathfrak{C})\to\mathscr{A}(\mathfrak{C})\to\mathbf{PGl}(\check{V},\mathfrak{C})$$

such that the composite maps a continuous automorphism of V *to that canonically induced on* $\check{V}$.

Proof. The first morphism is deduced from Theorem 4.2. Now, take $\phi\in\mathscr{A}(\mathfrak{C})$. Since $\bar{\phi}$ is inclusion-preserving, we have the following commutative diagram:

$$\begin{array}{ccccc}\mathrm{Gr}(V/L) & \to & \mathrm{Gr}(V/L') & \to & \mathrm{Gr}(V)\\ \downarrow & & \downarrow & & \downarrow\\ \mathrm{Gr}(V/\phi(L)) & \to & \mathrm{Gr}(V/\phi(L')) & \to & \mathrm{Gr}(V)\end{array}$$

where $L'\subseteq L$ and $L, L'\in\mathfrak{C}$. Theorem 4.8 implies that $T_L: V/L\xrightarrow{\sim}V/\phi(L)$ is a morphism of inverse systems as L varies in $\mathfrak{C}$. One therefore obtains an isomorphism $T_\phi:\check{V}\xrightarrow{\sim}\check{V}$, which is bicontinuous w.r.t. $\mathfrak{C}$.

It remains to show that if $T_\phi=\lambda\cdot Id(\lambda\in k^*)$, then $\phi=Id$. But this follows from a standard argument about the restriction to F_A and $j_L^{-1}(F_A)$, where $A\in\mathcal{B}_0$, $L\in\mathfrak{C}$ and j_L: $\mathrm{Gr}(V/L)\to\mathrm{Gr}(V)$.

Example 2. Assume that $\dim(V/V^+)$ is finite and let $\mathfrak{C}$ be the set of all rational points of $\mathrm{Gr}(V)$. It then holds that $\mathbf{PGl}(V)\tilde{\to}\mathbf{PGl}(V,\mathfrak{C})\tilde{\to}\mathbf{PGl}(\check{V},\mathfrak{C})$ and $\mathcal{A}(\mathfrak{C})\tilde{\to}\mathrm{Aut}(\mathrm{Gr}(V))$. Summing up $\mathbf{PGl}(V)\tilde{\to}\mathrm{Aut}(\mathrm{Gr}(V))$. This is the case of finite dimensional Grassmannians.

5. Applications to $\mathrm{Gr}(k((z)))$

Global sections on finite Grassmannians. Let us now assume that V is a finite dimensional k-vector space and that $\mathcal{B}$ is the set of subspaces of V. Choose a basis $\{e_1,\cdots,e_d\}$ of V such that $V^+=<e_1,\cdots,e_{d-r}>$, and let $\{e_1^*,\cdots,e_d^*\}$ be its dual basis. Let $\mathcal{S}$ now be the set of strictly increasing sequences of $d-r$ integers $S\equiv 0<s_1<\cdots<s_{d-r}\leq d$; and for $S\in\mathcal{S}$ define $A_S=<e_{s_1},\cdots,e_{s_{d-r}}>$.

Observe that $\{F_{A_S}|S\in\mathcal{S}\}$ is again a covering of $\mathrm{Gr}(V)$ and that the rational points of $\mathrm{Gr}(V)$ are precisely the r-dimensional subspaces.

Carrying out similar arguments as above, one sees how to interpret the section Ω_S in terms of exterior products. More precisely, one obtains a natural isomorphism:

$$H^0(\mathrm{Gr}(V),\mathrm{Det}_V^*)\to\Lambda^r V^*$$

$$\Omega_S\to e_{\bar{s}_1}^*\wedge\cdots\wedge e_{\bar{s}_r}^*$$

where $\{\bar{s}_1,\cdots,\bar{s}_r\}=\{1,\cdots,d\}-\{s_1,\cdots,s_{d-r}\}$.

Let π : $V\to V'$ be a surjective morphism between two k-vector spaces, and let $\{e_1,\cdots,e_d\}$ and $\{e'_{\bar{d}+1},\cdots,e'_d\}(\bar{d}:=\dim_k(\ker\pi))$ be bases of V and V' respectively, such that $\pi(e_i)=0$ for $i\leq\bar{d}$, and $\pi(e_i)=e'_i$ for $i>\bar{d}$.

The naturally induced morphism between their Grassmannians defined by $L\to\pi^{-1}(L)$ gives rise to a restriction homomorphism of global sections $\wedge^{r+d}V^*\to\wedge^r V'^*$, which consists of the inner contraction with $\wedge\ker\pi$:

$$e_{j_1}^*\wedge\cdots\wedge e_{j_{r+d}}^*\to\begin{cases} e'^*_{j_{1+d}}\wedge\cdots\wedge e'^*_{j_{r+d}} & \text{if } j_i=i \text{ for } i\leq\bar{d},\\ 0 & \text{otherwise.}\end{cases}\tag{5.1}$$

$(1\leq j_1<j_2<\cdots<j_{r+d}\leq d)$.

Global sections on $\mathrm{Gr}(k((z)))$. Let V^+ be $k[[z]]$ and let $\mathcal{B}$ be the family:

$$\left\{\begin{array}{l}\text{subspaces } A\subseteq V \text{ containing } z^n\cdot k[[z]] \text{ as a}\\ \text{subspace of finite codimension (for } n\in\mathbf{Z})\end{array}\right\}$$

We need some notation. Let us denote by $\mathcal{S}$ the set of Maya diagrams of virtual cardinal zero; that is, the set of strictly increasing sequences of integers $S=\{s_i\}_{i\geq 0}$

such that:

- there exists an integer i_0 such that $\{i_0, i_0+1, \cdots\} \subseteq S$,
- $\#(S \cap \mathbf{Z}_{<0}) = \#(\mathbf{Z}_{\geq 0} - S)$ (condition of virtual cardinal zero).

For the sake of clarity, $z^i \in V$ will be denoted by e_i. Let A_S be the z-adic completion of the subspace $<\{e_{s_i}\}_{i\geq 0}>$ for a given Maya diagram $S \in \mathcal{S}$. Observe that the subschemes $\{F_{A_S} | S \in \mathcal{S}\}$ are an open covering of $\mathrm{Gr}(V)$.

When building global sections ([1]), note that the canonical isomorphisms:

$$\mathrm{Det}^* \mathscr{C}^\bullet_{A_S} \otimes \wedge (A_S / A_S \cap A_{S'}) \otimes \wedge (A_{S'} / A_S \cap A_{S'})^* \xrightarrow{\sim} \mathrm{Det}^* \mathscr{C}^\bullet_{A_{S'}}$$

(where $\wedge$ denotes the highest exterior power and $S, S' \in \mathcal{S}$) induce isomorphisms $\phi_{SS'}: \mathrm{Det}(\mathscr{C}^\bullet_{A_S}) \xrightarrow{\sim} \mathrm{Det}(\mathscr{C}^\bullet_{A_{S'}})$.

Moreover, these isomorphisms can be chosen in a compatible way; that is $\phi_{SS''} = \phi_{S'S''} \circ \phi_{SS'}$. This compatibility is based on the face that there is a "good choice" for a basis of $\wedge(A_S / A_S \cap A_{S'}) \otimes \wedge(A_{S'} / A_S \cap A_{S'})^*$; namely, $e_J \otimes e_K^*$ where $e_J := e_{j_1} \wedge \cdots \wedge e_{j_n}$, $e_K^* := e_{k_1}^* \wedge \cdots \wedge e_{k_n}^*$, $e_i^*(e_j) = \delta_{ij}$, $J = S - (S \cap S')$ and $K = S' - (S \cap S')$.

Then, the canonical global section $det(\pi_{A_S}) \in H^0(\mathrm{Gr}(V), \mathrm{Det}^*(\mathscr{C}^\bullet_{A_S}))$ gives rise to a global section of $H^0(\mathrm{Gr}(V), \mathrm{Det}^*_V)$, Ω_S, via the isomorphism $\mathscr{C}^\bullet_{A_S} \xrightarrow{\sim} \mathscr{C}^\bullet_{V^+}$. It is easy to check that the Plücker morphism $\mathrm{Gr}(V) \xrightarrow{\mathfrak{p}} \mathbf{P}\Omega_*$ is well defined, that is, defined at every point (Ω being $<\{\Omega_S\}_{S \in \mathcal{S}}>$).

In this situation the restriction of $\Omega_S (S \in \mathcal{S})$ by the homomorphism (3.2) admits a nice description:

$$j^* \Omega_S = \begin{cases} \Omega_{\tilde{S}} \text{ where } \tilde{S} := S - S(N) & \text{if } S(M) \subseteq S \subseteq S(N) \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

where $S(M)$ and $S(N)$ are Maya diagrams (but not of virtual cardinal zero) such that: $M \in F_{A_{S(M)}}$, $N \in F_{A_{S(N)}}$ and $S(M) \subseteq S(N)$.

Equations of $\mathrm{Gr}(k(z)))$. Take $L_i := <\{e_j\}_{j<i}>$ for an integer i. Note that $L_i \in \mathrm{Gr}^\bullet(V)$. Denote by j_i the induced morphism $\mathrm{Grass}\,(L_i / L_{-i}) \to \mathrm{Gr}^\bullet(V)$ and let G_i be $\mathrm{Grass}(L_i / L_{-i}) \cap \mathrm{Gr}(V)$. Recall from (3.1) that the composite $G_i \to \mathrm{Gr}(V) \to \mathbf{P}\Omega^*$ factors through $\mathbf{P}(\wedge^i L_i / L_{-i})$. Therefore, the corresponding commutative diagram of structural sheaves is the following:

$$\begin{array}{ccc} (S^\bullet \Omega)^\sim & \xrightarrow{\mathfrak{p}^*} & \mathcal{O}_{\mathrm{Gr}(V)} \\ \downarrow & & \downarrow \\ (S^\bullet (\wedge^i L_i / L_{-i})^*)^\sim & \xrightarrow{\mathfrak{p}_i^*} & (\bigoplus_{d \geq 0} H^0(G_i, \mathrm{Det}^{* \otimes d}))^\sim \end{array}$$

where $\sim$ denotes the homogeneous localization and $S^\bullet$ the symmetric algebra. Let I be the kernel of $\mathfrak{p}^*$ and I_i that of $\mathfrak{p}_i^*$. Theorem 2.3 implies that these ideals are the equations defining $\mathrm{Gr}(k((z)))$ and G_i, respectively.

Denote by ι_i^* the restriction morphism $\Omega \to \wedge^i (L_i / L_{-i})^*$, which is known to be surjective by (3.2). Let us denote by $<X>$ the free k-module generated by a set X. Let $\mathcal{S}_i$ be the set of strictly increasing sequences $-i \leq s_0 < s_1 < \cdots < s_{i-1} \leq i-1$. Note that $\wedge^i(L_i/L_{-i})^* \simeq <\mathcal{S}_i>$, $\Omega \simeq <\mathcal{S}>$ and that ι_i^* is the morphism induced by the map (see (5.2)) :

$$\mathcal{S} \longrightarrow \mathcal{S}_i$$

$$\{s_i\}_{i\geq 0} \to \begin{cases} \{s_0,\cdots,s_{i-1}\} & \text{if } -i\leq s_0 \text{ and } s_j=j \text{ for all } j\geq i \\ 0 & \text{otherwise} \end{cases}$$

This morphism has a natural section ; namely :

$$\mathcal{S}_i \longrightarrow \mathcal{S}$$

$$\{s_0,\cdots,s_{i-1}\} \to \{s_0,\cdots,s_{i-1},i,i+1,\cdots\}$$

Let σ_i be the induced morphism $S^\bullet<\mathcal{S}_i> \to S^\bullet<\mathcal{S}>$.

Bearing in mind (5.1), one constructs surjections $\mathcal{S}_j \to \mathcal{S}_i$ (for $j\geq i$) and sections $\mathcal{S}_i \to \mathcal{S}_j$, which render the family $\{\mathcal{S}_i\}_{i\geq 1}$ an inverse system and a direct system, respectively. Moreover, one has :

$$\mathcal{S} \simeq \varinjlim_{i} \mathcal{S}_i \tilde{\to} \varprojlim_{i} \mathcal{S}_i$$

from which one has that $S^\bullet<\mathcal{S}> \tilde{\to} \varinjlim S^\bullet<\mathcal{S}_i>$; and hence :

$$I = \varinjlim(I\cap\sigma_i(S^\bullet<\mathcal{S}_i>)) \tilde{\to} \varinjlim(\iota_i^* I\cap S^\bullet<\mathcal{S}_i>)$$

for every submodule $I\subset S^\bullet<\mathcal{S}>$.

Since $S^\bullet(\wedge^i L_i/L_{-i}) \simeq S^\bullet<\mathcal{S}_i>$, one has that $I = \varinjlim(\iota_i^* I_i)$. The same argument also implies that I is generated by its degree 2 homogeneous component, since I_i does (this is a classical result). That is, the ideal I is generated by the union of the generators of I_i for all i.

We have thus proved the following :

Theorem 5.3. Gr(V) *is defined by the set of all (finite) Plücker equations.*

Automorphisms of Gr($k((z))$). In this case, one consider on $k((z))$ the skew-symmetric form defined by $T_2(z^i,z^j) := \delta_{i+j,1}$ when $i\geq j$. Then, it follows easily that Aut(Gr $k((z))$) modulo the subgroup of automorphisms with an inclusion-preserving extension is isomorphic to $\mathbf{Z}/2$ and that the automorphism R of 2.5 is a generator.

It is now clear that it is not possible to represent arbitrary elements of $\check{\text{Gl}}(k((z)))$ as matrices. For instance, if $\mathfrak{C}$ is the set of subspaces $\{z^n\cdot k[z^{-1}]\}_{n\in\mathbf{Z}}$, then $\check{V}=k[[z^{-1}, z]]$. So, there is an associated $\mathbf{Z}\times\mathbf{Z}$-matrix to every $g\in\text{Gl}(V)$, but one recovers g from the matrix iff it is continuous w.r.t. the z-adic topology. This fact implies further relations between Gl(V,$\mathfrak{C}$) and the group Gl_{res} of [16].

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