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Direct proof of the perfect block diagonalization of systems of pseudo-differential operators in the ultradifferentiable classes

Dedicated to Professor Kiyoshi ASANO on his 60th anniversary

By

Waichiro MATSUMOTO

Abstract

We give direct proofs on the perfect block diagonalization and on the transformation to Arnold-Petkov's normal form of matrices of pseudo-differential operators in the ultradifferentiable classes.

1. Introduction

Let K be a compact set in $\mathbf{R}^l$, R be a positive number, $\{M_n\}$ be a non-decreasing and logarithmically convex sequence of positive numbers and $B\{M_n\}_R(K)$ be $\{f(x) \in C^\infty(K) : \left|\left(\frac{\partial}{\partial x}\right)^\alpha f(x)\right| \leq CR^{|\alpha|}M_{|\alpha|}$ *on* K *for arbitrary* α *in* $\mathbf{Z}_+^l\}$, where $\mathbf{Z}_+ = \mathbf{N} \cup \{0\} = \{0,1,2,\cdots\}$, $|\alpha| = \alpha_1 + \cdots + \alpha_l$ for $\alpha = (\alpha_1,\cdots,\alpha_l) \in \mathbf{Z}_+^l$ and C is a positive constant depending on f but not on α. We call $B\{n!^s\}_R(K)$ for $s>1$ the Gevrey class of order s. In case of the Gevrey classes, we have $B\{n!^s\}_R(\mathrm{K}) \times \mathrm{B}\{n!^s\}_R(K) \subset B\{n!^s\}_R(K)$ (See Proposition 4.1.) On the other hand, in case of the real analytic class, $B\{n!\}_R(K) \times B\{n!\}_R(K) \not\subset B\{n!\}_R(K)$. For example, in case of $l=1$, let us take $K = \{|x| \leq 1\}$ and $f(x) = \frac{1}{2-x}$. We have $\max_{x\in K}\left|\left(\frac{\partial}{\partial x}\right)^n f\right| = n!$ and $\max_{x\in K}\left|\left(\frac{\partial}{\partial x}\right)^n (f \times f)\right| = (n+1)n!$. Thus, $f \in B\{n!\}_1(K)$ but $f \times f \notin B\{n!\}_1(K)$. This is a difficulty on the theory of pseudo-differential operators in the ultradifferentiable classes. L. Boutet de Monvel and P. Krée[3] introduced an elegant norm of formal symbols and overcame this difficulty. T. Nishitani[17] obtained the perfect factorization of full symbols in the ultradifferentiable classes using the same norm. However, never-the-less the all terms are obtained algebraically step by step (see H. Kumano-go[7], [8], V.I. Arnold [1], K. Kajitani[5] and V.M. Petkov[18]), his proof is a successive approximation and not a direct estimate of each term obtained algebraically.

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In order to treat the ultradifferentiable classes in a unified way without L. Boutet de Monvel and P. Krée's norm, a way is often used standing on the fact that $B\{n!\}_{R_1}(K)\times B\{n!\}_{R_2}(K)\subset B\{n!\}_R(K)$ for $R_1\neq R_2$ and $R=\max\{R_1, R_2\}$ because we encounter the products of knowns and knowns or knowns and unknowns for the results in L. Boutet de Monvel and P. Krée[3]. However, for the perfect factorization, we encounter the products of unknowns and unknowns. Thus, we need consider products of type $B\{M_n\}_R(K)\times B\{M_n\}_R(K)$. (See the proofs of Theorems 1 and 2.) The following fact is well-known that $B\{M_{n-k_0}\}_R(K)\subset B\{M_n\}_R(K)\subset B\{M_{n-k_0}\}_{R'}(K)$, $\forall R'>R$ for $\log M_n=o(n^2)$ ($\exists R'>R$ for $\log M_n=O(n^2)$, respectively) and a positive integer k_0. (See S. Manderbrojt[9] and W. Matsumoto[10].) An idea is to consider the product of functions in $B\{M_{n-k_0}\}_R(K)$ where $k_0=2$ for the space of functions and $k_0=3$ for the space of formal symbols. For example, we can show that $B\{M_{n-k_0}\}_R(K)\times B\{M_{n-k_0}\}_R(K)\subset B\{M_{n-k_0}\}_R(K)$ if $\{M_n/n!\}$ is logarithmically convex and $k_0\geq 2$ (Proposition 2.3). By this idea, we can show the results in [3] and in [17] estimating step by step the terms algebraically obtained. We give the results in the matrix form.

The advantages of the direct proof are the following:

1) For the perfect block diagonalization, which corresponds to the perfect factorization in T. Nisitani[17], each term of the unknown formal symbol has an ambiguity. We can settle it freely if we use the direct method.

2) We can also treat $C^\infty([T_l,T_2]; S\{M_n,L_n\}(O(t)))$ defined in Subsection 3.4.

Through Sections 2 and 3, we assume that $\{M_n/n!\}$ and $\{L_n/n!\}$ are logarithmically convex and non-decreasing. In Section 2, we offer some fundamental inequalities and the results on the operations of functions. In Section 3, we give the results on the operator product of formal symbols, for example, the perfect block diagonalization, the normal form of Arnold-Petkov and the final normal form. In Subsection 3.4, we also give the results on $C^\infty([T_1,T_2]; S\{M_n,L_n\}(O(t)))$. In Appendix, we reconsider the product and the division assuming the logarithmical convexity of $\{M_n/n!^s\}$ $(s>1)$, which the analytic class does not satisfy but every Gevrey class does.

Theorem 2 in Subsection 3.3 had already been used in W. Matsumoto[12] to obtain the main theorem, which is presented as Theorem 3 in Subsection 3.4 in this article. (The result in [12] is essential to obtain the results in W. Matsumoto and H. Yamahara[15], [16] and W. Matsumoto[13], [14].) The latter theorem in Subsection 3.4 will be applied in a forthcoming paper on the necessary condition for the Cauchy-Kowalevskaya theorem of Nagumo type on systems.

2. Fundamental inequalities and operations on functions

2.1. Fundamental inequalities (1). Let $\{M_n\}_{n=0}^\infty$ and $\{L_n\}_{n=0}^\infty$ be logarithmically convex and non-decreasing sequences of positive numbers. (We say that $\{M_n\}$ is logarithmically convex when $M_n^2\leq M_{n-1}M_{n+1}$.) When we consider functions and formal symbols of ultradifferentiable class, we can replace finite M_n's arbitrarily. Then, we can assume that $M_0=M_1=1$. It is convenient to set $M_n=1$ for negative n's. Thus, $(-3)!=(-2)!=(-1)!=1$ and, more generally, $j!=j_+!$ for j in $\mathbf{Z}$, where $j_+=$

$\max\{j, 1\}$. Through this paper, we assume the following:

Assumption. *$\{M_n/n!\}$ and $\{L_n/n!\}$ are logarithmically convex and non-decreasing.*

For the results in this paper, this assumption can be relaxed to a weaker one. However, when we further consider the composition of functions and the theorem of the implicit function, it seems difficult to verify the sufficiency of the weaker condition but it is easy to see that our Assumption is also sufficient for these. Further, the logarithmical convexity is easier to judge on the concrete examples. Thus, we assume the logarithmical convexity of $\{M_n/n!\}$ and $\{L_n/n!\}$. Of course, when Assumption is satisfied for $n\gg 1$, we can find an equivalent sequence which satisfies Assumption for all n's.

Let α and β be elements in $\mathbf{Z}_+^{1+l}$. We set $\alpha!=\alpha_0!\alpha_1!\cdots\alpha_l!$, $\alpha+\beta=(\alpha_0+\beta_0,\cdots,\alpha_l+\beta_l)$ and we denote $\beta\leq\alpha$ when $\beta_i\leq\alpha_i$ for $0\leq i\leq l$. We set $\binom{k}{j}=k!/j!(k-j)!$ for $0\leq j\leq k$ and $\binom{\alpha}{\alpha'}=\binom{\alpha_0}{\alpha'_0}\cdots\binom{\alpha_l}{\alpha'_l}$ for $\alpha'\leq\alpha$.

The following inequalities are easily seen but used again and again.

Lemma 2.1. (1) *For $0\leq h\leq i\leq j\leq k$ and $i+j\leq h+k$,*

$$i!j!\leq h!k!.$$

(2)

$$\sum_{\alpha\in\mathbf{Z}^l_+,\ |\alpha|=k}\frac{k!}{\alpha!}=l^k.$$

(3) *If $|\alpha|=k$,*

$$\sum_{|\alpha'|=j,\ \alpha'\leq\alpha}\binom{\alpha}{\alpha'}=\binom{k}{j}.$$

(4) *For $0\leq j_i\leq k_i$ $(i=1, 2)$ and $k_0\geq 0$,*

$$\binom{k_1}{j_1}\frac{(k_2-k_0)!}{(j_2-k_0)!(k_2-j_2-k_0)!}\leq\frac{(k_1+k_2-k_0)!}{(j_1+j_2-k_0)!(k_1+k_2-j_1-j_2-k_0)!}.$$

(5) *Let $\{N_n\}$ be logarithmically convex, that is $N_n^2\leq N_{n-1}N_{n+1}$, and non-decreasing. If p, q and k are non-negative, the following holds;*

$$N_{p-k}N_{q-k}\leq N_{p+q-k}.$$

(6) *Let $\{M_n/n!\}$ be logarithmically convex and non-decreasing. If p, q and k are non-negative, it holds that*

$$\frac{M_{p-k}M_{q-k}}{M_{p+q-k}}\leq\frac{(p-k)!(q-k)!}{(p+q-k)!}.$$

(7) *Let $\{N_n\}$ be logarithmically convex and non-decreasing. If $k\geq -1$, then $\{(N_{n-k})^{1/n}\}_n$ is non-decreasing on n. (The restriction $k\geq -1$ is not essential.*

When $N_n=1$ for $n\leq n_0$, we can relax it to $k\geq -n_0$ and this is always realized by replacing $\{N_n\}$ to a suitable equivalent one.)

(8) *Let a be a positive number. There exists a positive constant c_{a+1} such that*

$$\sum_{j=0}^{k}\binom{k}{j}^{-a}\leq c_{a+1}.$$

(9) *Let a be a positive number. There exists a positive constant c'_{a+1} such that*

$$\sum_{j=1}^{k-1}\left[\frac{j!(k-j)!}{(k-1)!}\right]^a\leq c'_{a+1}.$$

Proof. The assertions from (1) to (4) are well-known.

(5) As $\frac{N_n}{N_{n-1}}\leq\frac{N_{n+1}}{N_n}$, $N_pN_q\leq N_{p+q}$ is easily seen. When $p-k$ and $q-k$ are non-negative, it implies $N_{p-k}N_{q-k}\leq N_{p+q-2k}\leq N_{p+q-k}$. When $p-k<0$, $N_{p-k}=1$ and $N_{q-k}\leq N_{p+q-k}$. The case where $q-k<0$ is shown by the same way. When both of $p-k$ and $q-k$ are negative, $N_{p-k}N_{q-k}=1\leq N_{p+q-k}$.

(6) Setting $N_n=M_n/n!$, (5) means this.

(7) Let us set $a_n=\log N_n-\log N_{n-1}$. $\{a_n\}$ is non-decreasing and $\log N_{-k}=0$. Then,

$$\frac{1}{n}\log N_{n-k}-\frac{1}{n-1}\log N_{n-k-1}=\frac{1}{n}\sum_{j=1}^{n}a_{j-k}-\frac{1}{n-1}\sum_{j=1}^{n-1}a_{j-k}$$

$$=\frac{1}{n}a_{n-k}-\frac{1}{n(n-1)}\sum_{j=1}^{n-1}a_{j-k}\geq\frac{1}{n}(a_{n-k}-a_{n-k-1})\geq 0.$$

(8) We take $j_0\geq 1/a$. For $j_0\leq j\leq k/2$, it holds that

$$\frac{j!(k-j)!}{k!}\leq\frac{j_0!}{(k-j_0+1)^{j_0}}.$$

Thus, we obtain

$$\sum_{j=0}^{k}\binom{k}{j}^{-a}\leq 2j_0+\sum_{j=j_0}^{k-j_0}j_0!^a(k-j_0+1)^{-1}\leq 2j_0+j_0!^a=c_{a+1}.$$

(9) This is shown by the same way as (8). □

The following is the key lemma for the operations of functions.

Lemma 2.2. *Let k_0 be an integer greater than or equal to 2. There exists a positive constant $c[k_0]$, and the followings hold.*
(1) *For k in $\mathbf{Z}_+$,*

$$\sum_{j=0}^{k}\binom{k}{j}\frac{(j-k_0)!(k-j-k_0)!}{(k-k_0)!}\leq C[k_0].$$

(2) *For* α *in* $\mathbf{Z}_{+}^{l}$,

$$\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}\frac{M_{|\alpha'|-k_{\circ}}M_{|\alpha''|-k_{\circ}}}{M_{|\alpha|-k_{\circ}}}\leq c[k_{\circ}].$$

Proof. (1) Because the proof is same for each $k_{\circ}$, we give it for $k_{\circ}=2$. For $k\geq 6$,

$$\sum_{j=0}^{k}\binom{k}{j}\frac{(j-2)!(k-j-2)!}{(k-2)!}$$

$$\leq 2\sum_{j=0}^{[k/2]}\frac{k(k-1)}{j_{+}(j-1)_{+}(k-j)(k-j-1)}$$

$$=10\sum_{j=0}^{[k/2]}\frac{1}{j_{+}(j-1)_{+}}$$

$$=10\left(1+1+\sum_{j=2}^{[k/2]}\left(\frac{1}{j-1}-\frac{1}{j}\right)\right)$$

$$<30.$$

On the other hand, by the direct calculation, the left-hand side of (1) is majorized by 9 for $0\leq k\leq 5$. Thus, we can see that there exists a constant $c[2]$ which satisfies (1) and it is less than 30. (We can also show that $c[3]<84$ through the direct calculation up to $k=7$.)

(2)

$$\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}\frac{M_{|\alpha'|-k_{\circ}}M_{|\alpha''|-k_{\circ}}}{M_{|\alpha|-k_{\circ}}}$$

$$\leq\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}\frac{(|\alpha'|-k_{\circ})!(|\alpha''|-k_{\circ})!}{(|\alpha|-k_{\circ})!}$$

$$=\sum_{j=0}^{k}\left(\sum_{\alpha'+\alpha''=\alpha,\,|\alpha'|=j}\binom{\alpha}{\alpha'}\right)\frac{(|\alpha'|-k_{\circ})!(|\alpha''|-k_{\circ})!}{(|\alpha|-k_{\circ})!}$$

$$=\sum_{j=0}^{k}\binom{k}{j}\frac{(j-k_{\circ})!(k-j-k_{\circ})!}{(k-k_{\circ})!}$$

$$\leq c[k_{\circ}],$$

where we set $|\alpha'|=j$ and $|\alpha|=k$ and used Lemma 2.1 (3), (6) and Lemma 2.2 (1). □

2.2. Formal symbols. In this subsection, we give the definitions of formal symbols. From an arbitrary asymptotic expansion of a symbol of a pseudo-differential operator in an ultradifferentiable class, a true symbol in the same class can be constructed and the ambiguity is of class $S^{-\infty}$. (See L. Boutet de Monvel and P. Krée [3], L. Boutet de Monvel[2] and W. Matsumoto[11].) Therefore, in order to consider many problems on partial differential equations in a ultradifferentiable class, it is sufficient to consider asymptotic expansions, which we call here *formal symbols.* Let us set $a(t,x,\xi)^{(\beta)}_{(\alpha)}=D_t^{\alpha_0}D_{x_1}^{\alpha_1}\cdots D_{x_l}^{\alpha_l}\left(\frac{\partial}{\partial\xi}\right)^{\beta}a(t,x,\xi)$ for $\alpha\in\mathbf{Z}_+^{1+l}$ and $\beta\in\mathbf{Z}_+^l$, where $D_t=\frac{1}{\sqrt{-1}}\frac{\partial}{\partial t}$, etc.

Now, we define a formal symbol of class $\{M_n, L_n\}$ on a real domain. We say that a set O in $\mathbf{R}_t\times\mathbf{R}_x^l\times\mathbf{R}_\xi^l$ is conic when $(t,x,\xi)\in O$ implies $(t,x,\lambda\xi)\in O$ for arbitrary positive λ and that a subset Γ in O is conically compact in O when Γ is conic and $\Gamma\cap\{|\xi|=1\}$ is compact in $O\cap\{|\xi|=1\}$, where $|\xi|=\sqrt{\sum_{i=1}^l\xi_i^2}$.

Definition 1. (*Formal symbol of class* $\{M_n, L_n\}$, [12]). We say that the formal sum $a(t,x,\xi)=\sum_{i=0}^\infty a_i(t,x,\xi)$ is a formal symbol of class $\{M_n, L_n\}$ (=*f.s. of class* $\{M_n, L_n\}$) on O when there exists a real number $\varkappa$ such that

1) $a_i(t,x,\xi)$ belongs to $C^\infty(O)$ and positively homogeneous of degree $\varkappa-i$ on ξ, $(i\in\mathbf{Z}_+)$.
2) For arbitrary conically compact subset Γ in O, there are positive constants C, R and R' and we have

$$|a_{i(\alpha)}^{(\beta)}(t,x,\xi)|\leq CR'^iR^{|\alpha|+|\beta|}M_{i+|\alpha|-3}L_{i+|\beta|-3}(i-3)!^{-1}|\xi|^{\varkappa-i-|\beta|}\quad on\quad\Gamma,\tag{2.1}$$
$$(i\in\mathbf{Z}_+,\ \alpha\in\mathbf{Z}_+^{1+l},\ \beta\in\mathbf{Z}_+^l).$$

Next, we introduce a holomorphic formal symbol and a meromorphic one. We say that a set O in $\mathbf{C}_t\times\mathbf{C}_x^l\times\mathbf{C}_\xi^l$ is conic when $(t,x,\xi)\in O$ implies $(t,x,\lambda\xi)\in O$ for arbitrary positive λ and that a subset Γ in O is conically compact in O when Γ is conic and $\Gamma\cap\{\|\xi\|=1\}$ is compact in $O\cap\{\|\xi\|=1\}$, where $\|\xi\|=\sqrt{\sum_{i=1}^l|\mathrm{Re}\,\xi_i|^2+|\mathrm{Im}\,\xi_i|^2}$. We say that Σ is a subvariety of O if it is a zero set of a holomorphic function in O.

Definition 2. (*Meromorphic and holomorphic formal symbols,* [12]).

I. We say that the formal sum $a(t,x,\xi)=\sum_{i=0}^\infty a_i(t,x,\xi)$ is a meromorphic formal symbol (=*m.f.s.*) on O when there exist a conic subvariety Σ in O and a real number $\varkappa$ such that

1) $a_i(t,x,\xi)$ is meromorphic in O, holomorphic in $O\backslash\Sigma$ and positively homogeneous of degree $\varkappa-i$ on ξ, $(i\in\mathbf{Z}_+)$.
2) For arbitrary conically compact set Γ in $O\backslash\Sigma$, there are positive constants C, R and R' and we have

$$|a_{i(\alpha)}^{(\beta)}(t,x,\xi)| \leq CR'^{i}R^{|\alpha|+|\beta|}(i+|\alpha|-3)!(i+|\beta|-3)!(i-3)!^{-1}|\xi_1|^{\kappa-i} \quad on \quad \Gamma,$$
$$(2.2) \qquad (i\in\mathbf{Z}_+,\ \alpha\in\mathbf{Z}_+^{1+l},\ \beta\in\mathbf{Z}_+^{l}).$$

II. The formal sum $\sum_{i=0}^{\infty} a_i$ is called a holomorphic formal symbol ($=$*h.f.s.*) when it is a meromorphic formal symbol with $\Sigma=\emptyset$.

Remark 2.1. We use ξ_1 as a holomorphic scale of order in case of a complex domain and Σ includes $\{\xi_1=0\}$. Of course, ξ_1 can be replaced by another ξ_i and Σ includes $\{\xi_i=0\}$.

Remark 2.2. In (2.2), it is important that Σ is independent of i.

Remark 2.3. When $\{M_n\}$ and $\{L_n\}$ satisfy the differentiable condition, that is, $\log M_n$ and $\log L_n$ are $O(n^2)$, the definition is equivalent if we replace $M_{i+|\alpha|-3}L_{i+|\beta|-3}(i-3)!^{-1}$ in the right-hand side of (2.1) by $M_{i+|\alpha|}L_{i+|\beta|}i!^{-1}$ taking other R and R'. Further, always taking other R and R', when $\{M_n\}$ and $\{L_n\}$ satisfy the separativity condition, that is, $M_{p+q}\leq R_0^{p+q}M_pM_q$ for a positive R_0 and so on $\{L_n\}$ (essentially $M_n=n!^s$, $L_n=n!^{s'}$, $s, s'\geq 1$), we can replace it by $M_{|\alpha|}L_{|\beta|}M_iL_ii!^{-1}$, then, especially if $L_n=n!$, by $M_{|\alpha|}|\beta|!M_i$. Therefore, on the holomorphic and meromorphic formal symbols, we can replace $(i+|\alpha|-3)!(i+|\beta|-3)!(i-3)!^{-1}$ by $\alpha!\beta!i!$. (See S. Manderbrojt[9] and W. Matsumoto[10].) Thus, for a separative $\{M_n\}$ and $L_n=n!$, we can construct a true symbol of class $\{M_n\}$ from a formal symbol of class $\{M_n, n!\}$. (See L. Boutet de Monvel and P. Krée[3], L. Boutet de Monvel[2] and W. Matsumoto [11].)

The number κ is called the order of the formal symbol a and denoted by *ord a*. When $a_i=0$ for $0\leq i\leq i_0-1$ and $a_{i_0}\not\equiv 0$, $\kappa-i_0$ is called the true order of a and denoted by *true ord a*. The order of 0 is posed $-\infty$. We set $S^{\kappa}\{M_n,L_n\}(O)=\{$*the f.s.'s of class* $\{M_n,L_n\}$ *on* O *of order* $\kappa\}$, $S_M^{\kappa}(O)=\{$*the m.f.s.'s on* O *of order* $\kappa\}$, $S_H^{\kappa}(O)=\{$*the h.f.s.'s on* O *of order* $\kappa\}$ and $S\{M_n,L_n\}(O)=\cup_{\kappa\in\mathbf{R}}S^{\kappa}\{M_n,L_n\}(O)$, etc. As our consideration is common to every space of formal symbols of ultradifferentiable class, we simply represent it by S^{κ} and S. For the holomorphic and meromorphic formal symbols, we always regard (2.2) as a special case of (2.1) and replace $|\xi|$ below by $|\xi_1|$.

2.3. Product. We consider the product of functions in this subsection.

Proposition 2.3 (Product). *Let* $c[k_0]$ *be that in Lemma* 2.2 *for* $k_0\geq 2$.

(1) *If the followings are satisfied on a compact set* K *by positive constants* R *and* C_j $(j=1, 2)$

$$|f_j(x)_{(\alpha)}| \leq C_jR^{|\alpha|}M_{|\alpha|-k_0}$$

the product of f_1 *and* f_2 *satisfies*

$$|(f_1(x)f_2(x))_{(\alpha)}| \leq c[k_0]C_1C_2R^{|\alpha|}M_{|\alpha|-k_0},$$

where $\alpha\in\mathbf{Z}_+^l$.

(2) *If the followings are satisfied on a conically compact set* Γ *by positive constants* R, R', C_j, *real numbers* $\varkappa_j$ *and nonnegative integers* i_j ($j=1, 2$):

$$(2.3)\qquad |a_{j(\alpha)}^{(\beta)}(t,x,\xi)| \leq C_jR'^{i_j}R^{|\alpha|+|\beta|}M_{i_j+|\alpha|-3}L_{i_j+|\beta|-3}(i_j-3)!^{-1}|\xi|^{\varkappa_j-i_j-|\beta|}$$

the product of a_1 *and* a_2 *satisfies*

$$(2.4)\qquad |(a_1(t,x,\xi)a_2(t,x,\xi))_{(\alpha)}^{(\beta)}|$$
$$\leq c[3]^2C_1C_2R'^{i_1+i_2}R^{|\alpha|+|\beta|}M_{i_1+i_2+|\alpha|-3}L_{i_1+i_2+|\beta|-3}(i_1+i_2-3)!^{-1}|\xi|^{\varkappa_1+\varkappa_2-i_1-i_2-|\beta|},$$

where $\alpha\in\mathbf{Z}_+^{1+l}$ *and* $\beta\in\mathbf{Z}_+^l$. (*We can replace "3" by* $k_0>3$ *in* (2.3) *and* (2.4). *However, later on we use only the above form.*)

Proof. (1) Applying Lemma 2.2 (2), we can see the following:

$$|(f_1(x)f_2(x))_{(\alpha)}| \leq C_1C_2R^{|\alpha|}M_{|\alpha|-k_0}\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}\frac{M_{|\alpha'|-k_0}M_{|\alpha''|-k_0}}{M_{|\alpha|-k_0}}$$

$$\leq c[k_0]C_1C_2R^{|\alpha|}M_{|\alpha|-k_0}\quad .$$

(2) The proof is similar as that of (1). We group i's into two cases 1) $i\geq 5$ and 2) $i\leq 4$, and further 1) into *i*) $i_j\geq 3$ ($j=1, 2$), *ii*) $i_1<3$ and $i_2\geq 3$, *iii*) $i_1\geq 3$ and $i_2<3$ and 2) into *ii'*) $i_1<3$ and $i_2\geq 3$, *iii'*) $i_1\geq 3$ and $i_2<3$ and *iv*) $i_j<3$ ($j=1, 2$). In case of $i_1=i_2=0$, the proof is just same as that of (1). (See also the proof of Lemma 3.1 (1) in Section 3.) □

2.4. Division. Under our Assumption, $B\{M_n\}_R(K)$ is not closed on the division by non-vanishing element. In fact, taking $l=1$, $K=\{|x|\leq 1\}$ and $f(x)=2-x$, we have $1\leq|f(x)|\leq 3$ on K, $f_{(1)}(x)=1$ and $f_{(n)}(x)=0$ for $n\geq 2$. Therefor if we take $C_\varepsilon=\varepsilon^{-1}$, we have $|f_{(n)}(x)|\leq C_\varepsilon\varepsilon^n n!$ for $\varepsilon\leq 1/3$ *i.e.* $f(x)$ belongs to $B\{n!\}_\varepsilon$(K). However, as $\max_{x\in K}|(1/f(x))_{(n)}|=n!$ for arbitrary n, $1/f(x)$ does not belong to $B\{n!\}_R(K)$ for $R<1$. Further, even though $|f_{(n)}(x)|\leq C_\varepsilon\varepsilon^n(n-2)!$, $1/f(x)$ does not belong to $B\{(n-2)!\}_1(K)$. On the other hand, under our Assumption, the division by non-vanishing element in $B\{M_n\}_R(K)$ belongs to the class replaced R by another one. (See W. Rudin[19].) We give a proof of this result for the case of $B\{M_{n-k_0}\}_R(K)$ in this subsection.

As we see above, in the real analytic class, we cannot keep R by the division. Never-the-less, every Gevrey class $B\{n!^s\}_R(K)$ ($s>1$) is closed on the division. We prove this in Appendix.

Proposition 2.4 (Division). *Let* $c[k_0]$ *be that in Lemma* 2.2.
(1) *When* $f(x)$ *satisfies the following*

(2.5) $$|f(x)_{(\alpha)}| \le CR^{|\alpha|}M_{|\alpha|-k_\circ} \quad on\ K,$$

and

$$|f(x)| \ge c_m > 0 \quad on\ K,$$

it follows that

(2.6) $$|(1/f(x))_{(\alpha)}| \le (1/c_m)R(1)^{|\alpha|}M_{|\alpha|-k_\circ} \quad on\ K,$$

where $R(1)=c[k_\circ]CR/c_m$.

(2) *When* $a(t,x,\xi)$ *satisfies the following*

(2.7) $$|a^{(\beta)}_{(\alpha)}(t,x,\xi)| \le CR^{|\alpha|+|\beta|}M_{|\alpha|-3}L_{|\beta|-3}|\xi|^{\kappa-|\beta|} \quad on\ \Gamma,$$

and

$$|a(t,x,\xi)| \ge c_m|\xi|^{\kappa} \quad on\ \Gamma, \quad (c_m>0),$$

it follows that

(2.8) $$|(1/a(t,x,\xi))^{(\beta)}_{(\alpha)}| \le (1/c_m)R(2)^{|\alpha|+|\beta|}M_{|\alpha|-3}L_{|\beta|-3}|\xi|^{-\kappa-|\beta|} \quad on\ \Gamma,$$

where $R(2)=c[3]^2CR/c_m$. (*We can replace* "3" *by* $k_\circ>3$ *in* (2.7) *and* (2.8). *However, later on we use only the above form.*)

Proof. (1) We show this by the induction on $k=|\alpha|$. Let us set $g(x)=1/f(x)$.
1) *Case of* $k=0$. As $1=|f(x)g(x)|\ge c_m|g(x)|$, (2.6) holds for $k=0$.
2) *Case of* $k>1$. We assume (2.6) holds for arbitrary α'' with $|\alpha''|<k$ and consider the case of $|\alpha|=k$.

As

$$0=(f(x)g(x))_{(\alpha)}=\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}f(x)_{(\alpha')}g(x)_{(\alpha'')},$$

by Lemma 2.2 (2), it holds that

$$\begin{aligned}|f(x)||g(x)_{(\alpha)}| &\le \sum_{\alpha'+\alpha''=\alpha,\ |\alpha'|\ge 1}\binom{\alpha}{\alpha'}|f(x)_{(\alpha')}||g(x)_{(\alpha'')}| \\ &\le (C/c_m)(R/R(1))R(1)^{|\alpha|}M_{|\alpha|-k_\circ}\sum_{\alpha'+\alpha''=\alpha,\ |\alpha'|\ge 1}\binom{\alpha}{\alpha'}\frac{M_{|\alpha'|-k_\circ}M_{|\alpha''|-k_\circ}}{M_{|\alpha|-k_\circ}} \\ &\le R(1)^{|\alpha|}M_{|\alpha|-k_\circ}.\end{aligned}$$

This shows that (2.6) also holds in case of $|\alpha|=k$.

Thus (2.6) holds for arbitrary α.

(2) This is shown by the same way as the proof of (1). □

We denote the inverse matrix of a matrix F by $(F)^{-1}$.

Proposition 2.5 (Inverse matrix).
(1) *Let $F(x)$ be an* $\mathrm{N}\times\mathrm{N}$ *matrix whose entries satisfy* (2.5). *If* $|\det F|\geq c_m>0$, *there exist the inverse matrix* $(F(x))^{-1}=(g^{pq})_{1\leq p,\, q\leq \mathrm{N}}$ *and positive constants* C_1 *and* $c(1)$ *determined by* C, c_m, N *and* k_0 *such that* $R_1=c(1)R$ *and*

$$|g^{pq}(x)_{(\alpha)}| \leq C_1 R_1^{|\alpha|} M_{|\alpha|-k_0} \quad on\ K.$$

(2) *Let $F(t,x,\xi)$ be an* $\mathrm{N}\times\mathrm{N}$ *matrix whose entries satisfy* (2.1) *with* $i=0$.
If $|\det F|\geq c_m|\xi|^{\varkappa}$ $(c_m>0)$, *there exist the inverse matrix* $(F(t,x,\xi))^{-1}=(g^{pq}(t,x,\xi))_{1\leq p,\, q\leq \mathrm{N}}$ *and positive constants* C_2 *and* $c(2)$ *determined by* C, c_m *and* N *such that* $R_2=c(2)R$ *and*

$$|g^{pq(\beta)}_{(\alpha)}(t,x,\xi)| \leq C_2 R_2^{|\alpha|+|\beta|} M_{|\alpha|-3} L_{|\beta|-3} |\xi|^{-\varkappa-|\beta|} \quad on\ \Gamma, \tag{2.9}$$
$$(\alpha\in\mathbf{Z}_+^{1+l},\ \beta\in\mathbf{Z}_+^{l}).$$

Proof. The inverse matrix $(g^{pq})_{1\leq p,\, q\leq \mathrm{N}}$ of F is given by $g^{pq}=\Delta_{qp}/\det F$, where Δ_{qp} is the (q, p)-cofactor of F. Then, by Propositions 2.3 and 2.4, (1) is evident. (2) is also obtained by the same reason. □

3. Operations on the operator product

In this section, we give the results on the formal symbols by the operator product.

3.1. Fundamental inequalities (2). We define the operator product on S in Subsection 3.2. Corresponding to it, the following is the key lemma on the operations on the formal symbols..

Lemma 3.1. *Let $\bar{R}$ be a positive constant greater than or equal to* 2. *There exists a positive constant c_0 and the followings hold.*
(1)

$$\sum_{0\leq j\leq k,\ 0\leq q\leq p,\ i_1+i_2+r=i} \bar{\mathrm{R}}^{-r}\binom{k}{j}\binom{p}{q}\frac{(i-3)!}{(i_1-3)!(i_2-3)!r!}\cdot \tag{3.1}$$
$$\frac{(i_1+j-3)!(i_2+k-j+r-3)!(i_1+q+r-3)!(i_2+p-q-3)!}{(i+k-3)!(i+p-3)!}\leq c_0.$$

(2)

$$\sum_{0\leq j\leq k,\ 0\leq q\leq p,\ i_1+i_2+r=i,\ i_2<i} \bar{\mathrm{R}}^{1-i_1-r}\binom{k}{j}\binom{p}{q}\frac{(i-3)!}{(i_1-3)!(i_2-3)!r!}\cdot$$
$$\frac{(i_1+j-3)!(i_2+k-j+r-3)!(i_1+q+r-3)!(i_2+p-q-3)!}{(i+k-3)!(i+p-3)!}\leq c_0.$$

(3)

$$\sum_{\alpha'+\alpha''=\alpha,\ \beta'+\beta''=\beta,\ i_1+i_2+|\gamma|=i}(l\bar{R})^{-r}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\frac{(i-3)!}{(i_1-3)!(i_2-3)!\gamma!}\cdot$$

$$\frac{M_{i_1+|\alpha'|-3}M_{i_2+|\alpha''|+|\gamma|-3}L_{i_1+|\beta'|+|\gamma|-3}L_{i_2+|\beta''|-3}}{M_{i+|\alpha|-3}L_{i+|\beta|-3}}\leq c_0.$$

(4)

$$\sum_{\alpha'+\alpha''=\alpha,\ \beta'+\beta''=\beta,\ i_1+i_2+|\gamma|=i,\ i_2<i}l^{-r}\bar{R}^{l-i_1-r}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\frac{(i-3)!}{(i_1-3)!(i_2-3)!\gamma!}\cdot$$

$$\frac{M_{i_1+|\alpha'|-3}M_{i_2+|\alpha''|+|\gamma|-3}L_{i_1+|\beta'|+|\gamma|-3}L_{i_2+|\beta''|-3}}{M_{i+|\alpha|-3}L_{i+|\beta|-3}}\leq c_0.$$

Proof. (1) We group i's into two cases: 1) $i\geq 5$ and 2) $i\leq 4$.

Case 1) $i\geq 5$. We further group these i's into four cases: *i*) $i_j\geq 3$ $(j=1, 2)$, *ii*) $i_1\leq 2$ and $i_2\geq 3$, *iii*) $i_1\geq 3$ and $i_2\leq 2$ and *iv*) $i_j\leq 2$ $(j=1, 2)$.

i) $i_j\geq 3$ $(j=1, 2)$. When $i=5$, this case is empty. Taking $k_0=0$ in Lemma 2.1 (4), we have

$$\binom{k}{j}\binom{i-6}{i_1-3}\leq\binom{i+k-6}{i_1+j-3},$$

that is,

$$\binom{k}{j}\frac{(i_1+j-3)!(i_2+k-j+r-3)!}{(i+k-3)!}\leq\frac{(i_1-3)!(i_2+r-3)!}{(i-6)!(i+k-5)(i+k-4)(i+k-3)}.$$

By the same way, we also have

$$\binom{p}{q}\frac{(i_1+q+r-3)!(i_2+p-q-3)!}{(i+p-3)!}\leq\frac{(i_1+r-3)!(i_2-3)!}{(i-6)!(i+p-5)(i+p-4)(i+p-3)}.$$

Therefore, by the relations $i+k-3\geq i-3$, $i+k-4>k+1$, $i+p-5\geq p+1$ and $(i_1+r-3)!(i_2+r-3)!\leq(i-6)!r!$ (Lemma 2.1 (1)), we have

the left-hand side of (3.1)

$$\leq\sum\bar{R}^{-r}\frac{(i-5)(i-4)(i-3)}{(i+k-5)(i+k-4)(i+k-3)(i+p-5)(i+p-4)(i+p-3)}$$

$$\leq\sum_{0\leq j\leq k,\ 0\leq q\leq p,\ i_1+i_2+r=i}\bar{R}^{-r}\frac{1}{(i+k-4)(i+k-3)(i+p-5)}$$

$$\leq\frac{1}{(k+1)(p+1)(i-3)}\sum_{j=0}^{k}\sum_{q=0}^{p}\sum_{i_1=3}^{i-3}\sum_{r=0}^{i-i_1}2^{-r}\qquad<2.$$

ii) $i_1 \le 2$ and $i_2 \ge 3$. Taking $k_\circ = 0$ in Lemma 2.1 (4), we have

$$\binom{k}{j}\binom{i-3}{i_2+r-3} \le \binom{i+k-3}{i_1+j},$$

and

$$\binom{p}{q}\binom{i-3}{i_2-3} \le \binom{i+p-3}{i_1+q+r}.$$

By the relations $(i_1+r)!(i_2+r-3)! \le (i-3)!r!$ (Lemma 2.1 (1)), $(i_1-3)!=1$, $i_1! \le 2$ and $\sum_{j=0}^{\infty} 1/(j-1)_+(j)_+ = 3$, we have

the left-hand side of (3.1)

$$\le \sum \bar{R}^{-r} \frac{2}{(i_1+j-2)_+(i_1+j-1)_+(i_1+j)_+(i_1+q+r-2)_+(i_1+q+r-1)_+(i_1+q+r)_+}$$

$$\le 2\sum_{j=0}^{k} \frac{1}{(j-1)_+(j)_+} \sum_{q=0}^{p} \frac{1}{(q-1)_+(q)_+} \sum_{i_1=0}^{2} \sum_{r=0}^{i-i_1} 2^{-r}$$

$$< 108.$$

iii) $i_1 \ge 3$ and $i_2 \le 2$. In this case, (3.1) is provable by the same way as in Case *ii*).

iv) $i_j \le 2$ $(j=1, 2)$. As $i \ge 5$, $i_1+r=i-i_2 \ge 3$ and $i_2+r=i-i_1 \ge 3$, applying Lemma 2.1 (4) with $k_\circ=0$, we have

$$\binom{k}{j}\binom{i-3}{i_2+r-3} \le \binom{i+k-3}{i_2+k-j+r-3},$$

and

$$\binom{p}{q}\binom{i-3}{i_1+r-3} \le \binom{i+p-3}{i_1+q+r-3}.$$

By the relations $(i_1+r-3)!(i_2+r-3)! \le (i-3)!(r-3)!$ (Lemma 2.1 (1)), $(i_j-3)!=1$, $i_j! \le 2$ $(j=1, 2)$ and $\sum_{j=0}^{\infty} 1/(j-1)_+(j)_+ = 3$, we have

the left-hand side of (3.1)

$$\le \sum \bar{R}^{-r} \frac{4}{(i_1+j-2)_+(i_1+j-1)_+(i_1+j)_+(i_2+p-q-2)_+(i_2+p-q-1)_+(i_2+p-q)_+}$$

$$\le 4\sum_{j=0}^{k} \frac{1}{(j-1)_+(j)_+} \sum_{q=0}^{p} \frac{1}{(q-1)_+(q)_+} \sum_{i_1=0}^{2} \sum_{r=0}^{i-i_1} 2^{-r}$$

$$< 216.$$

Thus, we obtain (3.1) in Case 1).

*Case*2) $i\leq 4$. We group these i's into three cases: *ii'*) $i_1\leq 2$ and $i_2\geq 3$, *iii'*) $i_1\geq 3$ and $i_2\leq 2$ and *iv'*) $i_j\leq 2$ $(j=1, 2)$.

The calculations in the cases *ii'*) and *iii'*) are same as in the cases *ii*) and *iii*).

iv') $i_j\leq 2$ $(j=1, 2)$. As

$$\frac{(i_1+j-3)!(i_2+k-j+r-3)!}{(i+k-3)!}\leq\frac{(j-3)!(k-j-3)!}{(k-3)!}$$

and

$$\frac{(i_1+q+r-3)!(i_2+p-q-3)!}{(i+p-3)!}\leq\frac{(q-3)!(p-q-3)!}{(p-3)!}$$

by the relations $(k-h)_+/(k-j-h)_+\leq(k-h)_+/(k/2-h)_+\leq 2\{1+(h/(k-2h)_+)\}\leq 2(1+h)$ $(0\leq j\leq k/2, 0\leq h\leq 2)$, $(i-3)!=(i_j-3)!=1$ $(j=1, 2)$ and $\sum_{j=0}^{\infty}1/(j-1)_+(j)_+=3$, we have

the left-hand side of (3.1)

$$\leq\sum\bar{R}^{-r}\frac{(k-2)_+(k-1)_+(k)_+}{(j-2)_+(j-1)_+(j)_+(k-j-2)_+(k-j-1)_+(k-j)_+}\cdot$$

$$\frac{(p-2)_+(p-1)_+(p)_+}{(q-2)_+(q-1)_+(q)_+(p-q-2)_+(p-q-1)_+(p-q)_+}$$

$$\leq 4\,(2^3 3!)^2\sum_{j=0}^{k/2}\frac{1}{(j-1)_+(j)_+}\sum_{q=0}^{p/2}\frac{1}{(q-1)_+(q)_+}\sum_{i_1=0}^{2}\sum_{r=0}^{i-i_1}2^{-r}$$

$$<2^{11}3^5.$$

Thus, in Case 2), also (3.1) holds.

(2) By the inequality

$$\sum_{i_1+i_2+r=i,\ i_2<i}\bar{R}^{1-i_1-r}\leq\sum_{r=1}^{i}\bar{R}^{1-r}+\sum_{i_1=1}^{i}\sum_{r=0}^{i-i_1}\bar{R}^{-r}$$

$$<\sum_{i_1=0}^{i}\sum_{r=0}^{i-i_1}\bar{R}^{-r},$$

(2) is obtained by the same way as (1).

(3) and (4) are the immediate consequences of Lemma 3.1 (1) and (2) respectively, applying Lemma 2.1 (2), (3) and (6). (See also the proof of Lemma 2.2 (2)). □

3.2. Operator product and inverse. Corresponding to the asymptotic expansion of the symbol of the product of pseudo-differential operators, we introduce the operator product of formal symbols.

Definition 3. Let $a=\sum_{i=0}^{\infty} a_i$ and $b=\sum_{i=0}^{\infty} b_i$ be formal symbols. We set

$$(3.2)\quad a\circ b=\sum\nolimits_{i=0}^{\infty} c_i,\quad c_i(\mathrm{t,x},\xi)=\sum\nolimits_{i_1+i_2+|\gamma|=i}\frac{1}{\gamma!}a_{i_1}^{(\gamma)}(t,x,\xi)b_{i_2(\gamma)}(t,x,\xi)$$

and call it the operator product of a and b.

The following is the direct consequence of Lemma 3.1 (3).

Proposition 3.2 (Operator Product). *We assume that $R'\geq 2lR^2$ and that formal symbols $a^j=\sum_{i=0}^{\infty} a_i^j(t,x,\xi)$ satisfy the following ($j=1$, 2):*

$$|a_{i(\alpha)}^{j(\beta)}(t,x,\xi)|\leq C_jR'^iR^{|\alpha|+|\beta|}M_{i+|\alpha|-3}L_{i+|\beta|-3}(i-3)!^{-1}|\xi|^{\kappa_j-i-|\beta|}\quad on\ \Gamma,$$
$$(i\in\mathbf{Z}_+,\ \alpha\in\mathbf{Z}_+^{1+l},\ \beta\in\mathbf{Z}_+^l).$$

Then, the operator product $a\equiv a^1\circ a^2=\sum_{i=0}^{\infty} a_i(t,x,\xi)$ satisfies

$$|a_{i(\alpha)}^{(\beta)}(t,x,\xi)|\leq c_\circ C_1C_2R'^iR^{|\alpha|+|\beta|}M_{i+|\alpha|-3}L_{i+|\beta|-3}(i-3)!^{-1}|\xi|^{\kappa_1+\kappa_2-i-|\beta|}\quad on\ \Gamma,$$
$$(i\in\mathbf{Z}_+,\ \alpha\in\mathbf{Z}_+^{1+l},\ \beta\in\mathbf{Z}_+^l).$$

Now, let us consider the inverse formal symbol. For the inverse of a as the formal symbol by the operator product, we denote it by a^{-1} and for the inverse of a_0 as a function by $1/a_0$ or $(a_0)^{-1}$.

Proposition 3.3 (Inverse). (1) *We assume that a formal symbol $a=\sum_{i=0}^{\infty} a_i(t,x,\xi)$ satisfies the estimate* (2.1) *and*

$$|a_0(t,x,\xi)|\geq c_m|\xi|^\kappa\quad on\ \Gamma,\quad (c_m>0).$$

Then, the inverse $a^{-1}=\sum_{i=0}^{\infty} b_i(t,x,\xi)$ satisfies

$$|b_{i(\alpha)}^{(\beta)}|\leq C_3R_3'^iR_3^{|\alpha|+|\beta|}M_{i+|\alpha|-3}L_{i+|\beta|-3}|\xi|^{-\kappa-i-|\beta|}\quad on\ \Gamma,$$

where $R_3=R(2)=c[3]^2CR/c_m$, $R_3'=(c_\circ^2C/c_m)$ max $\{R', 2lR_3^2\}$ and $C_3=c_\circ/c_m$. ($c[3]$, $c_\circ$ and $R(2)$ are those in Lemma 2.2, 3.1 *and Proposition* 2.4, *respectively.*)

(2) *Let $\mathcal{N}(t,x,\xi)$ be an $\mathrm{N}\times\mathrm{N}$ matrix whose entries satisfy* (2.1). *If*

$$|\det\mathcal{N}_0(t,x,\xi)|\geq c_m|\xi|^\kappa\quad on\ \Gamma,\quad (c_m>0),$$

there exists the inverse $\mathcal{N}^{-1}(t,x,\xi)=(h^{pq})_{1\leq p,q\leq\mathrm{N}}$, $h^{pq}=\sum_{i=0}^{\infty} h_i^{pq}$ and it satisfies

$$(3.3)\quad \begin{aligned}&|h_{i\ (\alpha)}^{pq(\beta)}(t,x,\xi)|\leq C_2'R_2'^iR_2^{|\alpha|+|\beta|}M_{i+|\alpha|-3}L_{i+|\beta|-3}(i-3)!^{-1}|\xi|^{-\kappa-i-|\beta|}\quad on\ \Gamma,\\ &\qquad (i\in\mathbf{Z}_+,\ \alpha\in\mathbf{Z}_+^{1+l},\ \beta\in\mathbf{Z}_+^l),\end{aligned}$$

where C_2 and R_2 are those in Proposition 2.5 *and $R_2'=(\mathrm{N}c_\circ)^2CC_2$* max $\{R', 2lR_2^2\}$ *and $C_2'=\mathrm{N}c_\circ C_2$.*

As we need to make attention to the choice of R_j', we introduce the following norm of a matrix $A=(a^{pq}(t,x,\xi))_{1\le p,\,q\le N}$ with $\varkappa$ in $\mathbf{R}$, i in $\mathbf{Z}_+$, a positive number R and a conic set Γ

$$\|A\|_{\varkappa,i,R,\Gamma} = \max_{1\le p,\,q\le N} \max_{\alpha\in\mathbf{Z}_+^{1+l},\,\beta\in\mathbf{Z}_+^l} \max_{(t,x,\xi)\in\Gamma}$$

$$|a^{pq(\beta)}_{(\alpha)}(t,x,\xi)|\{R^{|\alpha|+|\beta|}M_{i+|\alpha|-3}L_{i+|\beta|-3}(i-3)!^{-1}|\xi|^{\varkappa-i-|\beta|}\}^{-1}.$$

Proof. As the proofs of (1) and (2) are similar, we give only the latter.
(*Step* 1) By Proposition 2.5, every entry of $(\mathcal{N}_0)^{-1}$ satisfies the estimate (2.9). As $(\mathcal{N}_0)^{-1}$ itself is a formal symbol, we take the product of $\mathcal{N}$ and $(\mathcal{N}_0)^{-1}$. Let us set $\mathcal{N}\circ(\mathcal{N}_0)^{-1}=F=\sum_{i=0}^{\infty}F_i$. F_0 becomes I_N. By Proposition 3.2, F_i satisfies

$$\|F_i\|_{0,i,c(2)R,\Gamma} \le C''R''^i,$$

where $C''=\mathrm{N}c_{\circ}CC_2$ and $R''=\max\{R',\ 2l(R_2)^2\}$.

(*Step* 2) We set $F^{-1}=G=\sum_{i=0}^{\infty}G_i$. G_0 is also I_N. We show the following estimate by the induction on i :

$$\|G_i\|_{0,i,c(2)\mathrm{R},\Gamma} \le R_2'^i, \tag{3.4}$$

where $R_2'=\mathrm{N}c_{\circ}C''R''$.

As $G_0=I_N$, (3.4) is satisfied for $i=0$.

Assuming (3.4) for $i_2<i$, we consider G_i. By the relation $G_i=-\sum_{i_1+i_2+|\gamma|=i,\ i_2<i}(1/\gamma!)F_{i_1}^{(\gamma)}G_{i_2(\gamma)}$ and Lemma 3.1 (4), we see

$$\|G_i\|_{0,i,c(2)R,\Gamma} \le \mathrm{N}c_{\circ}C''(R''/R_2')R_2'^i \le R_2'^i.$$

Thus, (3.4) holds for arbitrary i in $\mathbf{Z}_+$.

(*Step* 3) As $\mathcal{N}^{-1}=(\mathcal{N}_0)^{-1}\circ G$, by Proposition 3.2, we obtain (3.3). $\square$

3.3 Block Diagonalization and Arnold-Petkov's normal form. In this subsection, we consider the following matrix ;

$$P(t,x,D_t,\xi) = D_t - \mathcal{A}(t,x,\xi),\quad \mathcal{A} = \sum_{i=0}^{\infty}\mathcal{A}_i(t,x,\xi) \in M_{\mathrm{N}}(S^{\varkappa})\quad (\varkappa\in\mathbf{N}). \tag{3.5}$$

From now on, for simplicity, we assume that C and R in (2.1) are greater than or equal to one.

Theorem 1 (Perfect Block Diagonalization). *We assume that every entry of* $\mathcal{A}$ *satisfies* (2.1) *and that the eigenvalues* $\bigcup_{1\le k\le d}\{\lambda_{kj}(t,x,\xi)\}_{j=1}^{m_k}$ *of* $\mathcal{A}_0$ $(\sum_{k=1}^{d}m_k=\mathrm{N})$ *satisfies*

$$|\lambda_{kj}(t,x,\xi)-\lambda_{k'j'}(t,x,\xi)| \ge c|\xi|^{\varkappa}\quad on\ \Gamma\quad (c>0,\ k\ne k',\ 1\le j\le m_k,\ 1\le j'\le m_{k'}).$$

Then, for every point $(t_{\circ},x_{\circ},\xi_{\circ})$ *in* Γ, *there exist a conically compact neighborhood* Γ', $\mathcal{N}_{\circ}(t,x,\xi)=\sum_{i=0}^{\infty}\mathcal{N}_i$ *in* $GL(\mathrm{N};S^0(\Gamma'))$, $\mathcal{N}_i=(n_i^{pq})_{1\le p,\,q\le N}$ *and* $\mathcal{B}_k(t,x,\xi)=\sum_{i=0}^{\infty}\mathcal{B}_{ki}$ *in* $M_{m_k}(S^{\varkappa}(\Gamma'))$, $\mathcal{B}_{ki}=(b_{ki}^{pq})_{1\le p,\,q\le m_k}$ *such that*

$$\mathcal{N}_0^{-1}(t,x,\xi)\circ P(t,x,D_t,\xi)\circ\mathcal{N}_0(t,x,\xi)=\oplus_{1\le k\le d}P^k,$$
$$P^k(t,x,D_t,\xi)=I_{m_k}D_t-\mathcal{B}_k(t,x,\xi), \tag{3.6}$$

where $\mathcal{B}_{k0}$ has the eigenvalues $\{\lambda_{kj}(t,x,\xi)\}_{j=1}^{m_k}$. Further the following estimates hold :

$$|b_{ki(\alpha)}^{pq(\beta)}(t,x,\xi)|\le C_4R_4'^iR_4^{|\alpha|+|\beta|}M_{i+|\alpha|-3}L_{i+|\beta|-3}(i-3)!^{-1}|\xi|^{\varkappa-i-|\beta|}\quad on\ \Gamma',$$
$$|n_{i\ (\alpha)}^{pq(\beta)}(t,x,\xi)|\le C_4'R_4'^iR_4^{|\alpha|+|\beta|}M_{i+|\alpha|-3}L_{i+|\beta|-3}(i-3)!^{-1}|\xi|^{-i-|\beta|}\quad on\ \Gamma', \tag{3.7}$$
$$(i\in\mathbf{Z}_+,\ \alpha\in\mathbf{Z}_+^{1+l},\ \beta\in\mathbf{Z}_+^l),$$

where the constants C_4, C_4', R_4 and R_4' are determined only by P.

In case of meromorphic formal symbol, $\mathcal{N}_0$ and $\mathcal{B}_k$ belong to $GL(\mathrm{N};S_M^0(O))$ and $M_{m_k}(S_M^{\varkappa}(O))$, respectively.

Proof. Essentially the proof is similar to that of Proposition 3.3.

(*Step*1) The projection to the generalized eigenspace of $\{\lambda_{kj}(t,x,\xi)\}_{j=1}^{m_k}$ is given by

$$\mathcal{P}_k(t,x,\xi)=\frac{1}{2\pi\sqrt{-1}}\int_C(\tau I_{\mathrm{N}}-\mathcal{A}_0(t,x,\xi/|\xi|)^{-1}\mathrm{d}\tau, \tag{3.8}$$

where C is a simple closed path encircling only $\{\lambda_{kj}(t,x,\xi)\}_{j=1}^{m_k}$.
As $\min_{\tau\in C}|\det(\tau-\mathcal{A}_0(t,x,\xi/|\xi|)|\ge c_m>0$, by Proposition 2.5 (2), we have the estimate :

$$\|(\tau I_{\mathrm{N}}-\mathcal{A}_0(t,x,\xi/|\xi|)^{-1}\|_{0,0,c(2)R,\Gamma}\le C_2. \tag{3.9}$$

Let us take m_k linearly independent column vectors of $\mathcal{P}_k$ at (t_0,x_0,ξ_0) $(1\le k\le d)$. $\mathcal{N}_0(t,x,\xi)$ is constituted by them and satisfies the estimate (3.9) replacing C_2 by another $\hat{C}_2$. As *det* $\mathcal{N}_0$ does not vanish on a conically compact neighborhood Γ' of (t_0,x_0,ξ_0) in Γ, we have $|\det\mathcal{N}_0|\ge c_m'>0$ on Γ'. In case of meromorphic formal symbols, there exists a conic subvariety Σ' and $\det\mathcal{N}_0\neq 0$ on $\Gamma\backslash\Sigma'$. For $\Gamma'\subset\Gamma\backslash\Sigma'$, applying Proposition 2.5 (2) once again, we obtain

$$\|(\mathcal{N}_0(t,x,\xi))^{-1}\|_{0,0,c(2)c(2)'R,\Gamma'}\le\tilde{C}_2. \tag{3.10}$$

It is seen that $(\mathcal{N}_0)^{-1}\mathcal{A}_0\mathcal{N}_0=\oplus_{1\le k\le d}\mathcal{B}_{k0}$, where $\mathcal{B}_{k0}$ has eigenvalues $\{\lambda_{kj}(t,x,\xi)\}_{j=1}^{m_k}$ $(1\le k\le d)$. Thus, by Proposition 2.3 (2), we have

$$\|(\mathcal{B}_{k0}(t,x,\xi))\|_{\varkappa,0,c(2)c(2)'R,\Gamma'}\le C'. \tag{3.11}$$

Let us set

$$\tilde{P}=(\mathcal{N}_0)^{-1}\circ P\circ\mathcal{N}_0=I_{\mathrm{N}}D_t+(\mathcal{N}_0)^{-1}\circ\mathcal{N}_{0(t)}-(\mathcal{N}_0)^{-1}\circ\mathcal{A}\circ\mathcal{N}_0$$
$$=I_{\mathrm{N}}D_t-\tilde{\mathcal{B}}, \tag{3.12}$$

where $\tilde{\mathcal{B}}_0=\oplus_{1\le k\le d}\mathcal{B}_{k0}$ and $\mathcal{N}_{0(t)}=D_t(\mathcal{N}_0)$. Regarding $\mathcal{N}_{0(t)}$ as the second element of a first order operator, the following estimate follows by Proposition 3.2.

$$\|(\tilde{\mathcal{B}}_i(t,x,\xi))\|_{\kappa,i,c(2)c(2)'R,\Gamma'} \leq C''R''^i, \tag{3.13}$$

where $R''=\max\{R',\ 2lc(2)^2c(2)'R^2\}$.

(*Step* 2) Let us seek for $\mathcal{N}(t,x,\xi)=\sum_{i=0}^{\infty}\mathcal{N}_i$ in $GL(\mathrm{N};S^0(\Gamma'))$, $\mathcal{N}_0=I_{\mathrm{N}}$ and $\mathcal{B}_k(t,x,\xi)=\sum_{i=0}^{\infty}\mathcal{B}_{ki}$ in $M_{m_k}(S^{\kappa}(\Gamma'))$ $(1\leq k\leq d)$ for which the follwing relation holds:

$$\mathcal{N}^{-1}\circ\tilde{P}\circ\mathcal{N} = \oplus_{1\leq k\leq d}P^k, \quad P^k = I_{m_k}D_t - \mathcal{B}_k. \tag{3.14}$$

The relation (3.14) is written as

$$\begin{aligned}\oplus_{1\leq k\leq d}\mathcal{B}_{ki}-(\mathcal{B}_0\mathcal{N}_i-\mathcal{N}_i\mathcal{B}_0) = &-\sum\nolimits_{i_1+i_2+|\gamma|=i,\ 0<i_1<i,\ i_2<i}\frac{1}{\gamma!}\mathcal{N}_{i_1}^{(\gamma)}(\oplus\mathcal{B}_{ki_2(\gamma)})\\ &+\sum\nolimits_{i_1+i_2+|\gamma|=i,\ i_2<i}\frac{1}{\gamma!}\tilde{\mathcal{B}}_{i_1}^{(\gamma)}\mathcal{N}_{i_2(\gamma)} - \mathcal{N}_{i-\kappa(t)},\end{aligned} \tag{3.15}$$

where $\mathcal{B}_0=\oplus_{1\leq k\leq d}\mathcal{B}_{k0}$. Let us decompose $\mathcal{N}_i$ and $\tilde{\mathcal{B}}_i$ corresponding to $\oplus_{1\leq k\leq d}\mathcal{B}_{k0}$:

$$\mathcal{N}_i = (\mathcal{N}_i^{<pq>})_{1\leq p,\ q\leq d}, \quad \tilde{\mathcal{B}}_i = (\tilde{\mathcal{B}}_i^{<pq>})_{1\leq p,\ q\leq d}.$$

We take $\mathcal{N}_i^{<kk>}=O$ for $i\geq 1$ and $1\leq k\leq d$. The relation (3.15) becomes

$$\mathcal{B}_{ki} = \sum\nolimits_{i_1+i_2+|\gamma|=i,\ i_2<i}\frac{1}{\gamma!}(\tilde{\mathcal{B}}_{i_1}^{(\gamma)}\mathcal{N}_{i_2(\gamma)})^{<kk>}, \tag{3.16}$$

$$\begin{aligned}\mathcal{B}_{p0}\mathcal{N}_i^{<pq>} - \tilde{N}_i^{<pq>}\mathcal{B}_{q0} = &\sum\nolimits_{i_1+i_2+|\gamma|=i,\ 0<i_1<i,\ i_2<i}\frac{1}{\gamma!}\mathcal{N}_{i_1}^{<pq>(\gamma)}\mathcal{B}_{qi_2(\gamma)}\\ &-\sum\nolimits_{i_1+i_2+|\gamma|=i,\ i_2<i}\frac{1}{\gamma!}[\tilde{B}_{i_1}^{(\gamma)}\tilde{N}_{i_2(\gamma)}]^{<pq>} + \mathcal{N}_{i-\kappa(t)}^{<pq>}, \quad (p\neq q).\end{aligned} \tag{3.17}$$

Lemma 3.4. *For $i\geq 1$, $C_5=C''R''$, $C_5'=\mathrm{M}c[3]^2C''C_2''R''$ $(\mathrm{M}=\max_{1\leq p,\ q\leq d}\{m_pm_q\})$, $R_4=c(2)c(2)'c(2)''R$ and a positive number R_4' determined by P, the following estimate hold*:

$$\begin{aligned}&\|\mathcal{B}_{ki}\|_{\kappa,i,R_4,\Gamma'}\leq C_5R_4'^{i-1},\\ &\|\mathcal{N}_i^{<pq>}\|_{0,i,R_4,\Gamma'}\leq C_5'R_4'^{i-1}.\end{aligned} \tag{3.18}$$

Proof. We denote the transposed matrix of A by A^T. Let $A=(a_{ij})$ be a $k\times l$ matrix and B be an $m\times n$ matrix. We set *vec* $A=(a_{11},\ a_{21},\ \cdots,\ a_{k1},\ a_{12},\ \cdots,\ a_{k2},\ \cdots,\ a_{1l},\ \cdots,\ a_{kl})^T$ and $A\otimes B=(a_{ij}B)_{1\leq i\leq k,\ 1\leq j\leq l}$: $km\times ln$ matrix and call them the associated vector of A and the Kronecker product of A and B, respectively.

By (3.16) and (3.17), $\mathcal{B}_{ki}$ and $\mathcal{N}_i^{<pq>}$ are determined step by step on i, respectively. Especially, on $\mathcal{N}_i^{<pq>}$, we need solve the equation $\mathcal{B}_{p0}X-X\mathcal{B}_{q0}=H$, where X and H are an unknown and a given $m_p\times m_q$ matrices. This is a linear equation on *vec* X and is written as the Kronecker form : $[(I_{m_p}\otimes\mathcal{B}_{p0})-(\mathcal{B}_{q0}^T\otimes I_{m_q})]\mathrm{vec}\ X=\mathrm{vec}\ H$. Its coefficient matrix has the eigenvalues $\{\lambda_{pj}(t,x,\xi)-\lambda_{qj'}(t,x,\xi)\}_{1\leq j\leq m_p,\ 1\leq j'\leq m_q}$. (See R.A. Horn and C.R. Johnson [4] 4.4.5.) Then, $\|[(I_{m_p}\otimes\mathcal{B}_{p0})-(\mathcal{B}_{q0}^T\otimes I_{m_q})]^{-1}$

$\|_{-\varkappa,0,c(2)c(2)'c(2)''R,\Gamma'} \leq C_2''$ holds by suitable constants $c(2)''$ and C_2'' by Proposition 2.5 replacing N by $m_p m_q$. We set $R_4 = c(2)c(2)'c(2)''R$.

Case of $i=1$) By (3.16) with $i=1$, $\mathscr{B}_{k1} = \mathscr{\hat{B}}_1^{<kk>}$. Then, taking $C_5 = C''R''$, the former of (3.18) holds for $i=1$ because $c(2)''>1$. The right-hand side of (3.17) with $i=1$ becomes $-\mathscr{\hat{B}}_1^{<pq>}$ and the latter estimate of (3.18) holds for $i=1$ for $C_5' = \mathrm{M}c[3]^2 C'' C_2'' R''$ by Proposition 2.3 (2).

Case of $i>1$) We assume (3.18) for i' less than i.

First, we consider $\mathscr{B}_{ki}$. We divide the sum in the right-hand side of (3.16) to I^1: $i_2=0$ and I^2: $i_2 \geq 1$. On I^1, as $\mathscr{N}_{(\gamma)}^0 = O$ for $\gamma \neq 0$, we have $I^1 = \mathscr{\hat{B}}_i^{<kk>}$ and

$$\|I^1\|_{\varkappa,i,R_4,\Gamma} \leq C''R''^i.$$

On I^2, as $i_2 \leq i-1$ in the right-hand side of (3.16), by virtue of Proposition 3.2 with $R''' = \max\{R'', 2lc(2)c(2)'RR_4\}$, we have

$$\|I^2\|_{\varkappa,i,R_4,\Gamma'} \leq \mathrm{N}c_0 C'' C_5' (R'''/R_4') R_4'^{i-1}.$$

Thus, we arrive at

$$\|B_{ki}\|_{\varkappa,i,R_4,\Gamma'} \leq (C''R'' + \mathrm{N}c_0 C'' C_5')(R'''/R_4') R_4'^{i-1}. \tag{3.19}$$

As we take $R_4' \geq (1+\mathrm{MN}c[3]^2 c_0 C_2'' C'')R'''$, the former of (3.18) holds also for i.

Now, we consider $\mathscr{\hat{N}}_i^{<pq>}$. On the first sum in (3.17), we divide it to I^3: $i_2=0$ and I^4: $i_2 \geq 1$. On I^3, as $i_1 \leq i-1$, we have

$$\|I^3\|_{\varkappa,i,R_4,\Gamma'} \leq mc_0 C' C_5' (R'''/R_4') R_4'^{i-1},$$

where m is $\max_{1 \leq k \leq d} m_k$. On I^4, also as $i_1 \leq i-1$, it follows that

$$\|I^4\|_{\varkappa,i,R_4,\Gamma'} \leq mc_0 C_5 C_5' R_4'^{-1} R_4'^{i-1}.$$

Thus, we arrive at

$$\|I^3 + I^4\|_{\varkappa,i,R_4,\Gamma'} \leq mc_0(C'R''' + C_5)/R_4' \cdot C_5' R_4'^{i-1}.$$

The second sum has the same estimate as (3.19).

On the last term in (3.17), we have

$$\|\mathscr{\hat{N}}_{i-\varkappa(t)}^{<pq>}\|_{\varkappa,i,R_4,\Gamma'} \leq \|\mathscr{\hat{N}}_{i-\varkappa(t)}^{<pq>}\|_{1,i+1-\varkappa,R_4,\Gamma'} \leq C_5'(R_4')^{-\varkappa} R_4'^{i-1}.$$

Therefore, setting the right-hand side of (3.17) as $I(i)$, we can see

$$\|I(i)\|_{\varkappa,i,R_4,\Gamma'} \leq (C(5)/R_4') C_5' R_4'^{i-1}, \tag{3.20}$$

where $C(5) = [c_0(mC' + \mathrm{N}C'') + C''R''/C_5']R''' + mc_0 C_5 + 1$. Finally, we obtain

$$\|\mathscr{\hat{N}}_i^{<pq>}\|_{0,i,R_4,\Gamma'} \leq (\mathrm{M}c[3]^2 C_2'' C(5)/R_4') C_5' R_4'^{i-1} \tag{3.21}$$

and taking $R_4' = [\mathrm{M}c[3]^2 c_0 C_2''(mC' + \mathrm{N}C'') + 1]R''' + \mathrm{M}c[3]^2 C''(mc_0 C''R'' + 1)$, the second estimate of (3.17) holds for i.

Then, (3.18) holds for arbitrary i in $\mathbf{N}$ and Lemma 3.4 has been shown. □

(*Step* 3) We set $C_4=\max\{C', C_5/R_4\}$ and $C_4'=\mathrm{N}c_0\hat{C}_2\max\{1, C_5'/R_4'\}$. As $\mathcal{N}_0=\mathcal{N}_0\circ\tilde{\mathcal{N}}$, we obtain the estimate (3.7) and then Theorem 1. □

Now, we give a normal form of Arnold-Petkov of systems in ultradifferentiable classes.

Theorem 2 (Normal form of Arnold-Petkov). *We assume that every entry of $\mathcal{A}$ satisfies* (2.1) *and that each eigenvalue $\lambda_k(t,x,\xi)$ $(1\le k\le d)$ of $\mathcal{A}_0$ has the constant multiplicity m_k, that is, $\sum_{k=1}^d m_k=\mathrm{N}$ and*

$$|\lambda_k(t,x,\xi)-\lambda_{k'}(t,x,\xi)|\ge c|\xi|^\kappa \quad \text{in } O \qquad (c>0,\ k\neq k')$$

Then, there exist finite disjoint open conical sets $\{O_j\}_j$ such that $\cup_j O_j$ is dense in O. On each O_j, there exist natural numbers d_k and $\{n_{kj}\}_{j=1}^{d_k}$ $(\sum_{j=1}^{d_k} n_{kj}=m_k)$. For every point (t_0,x_0,ξ_0) in O_j, there exist a conically compact neighborhood Γ, $\mathcal{N}^0(t,x,\xi)=\sum_{i=0}^\infty \mathcal{N}_i$ in GL $(\mathrm{N};S^0(\Gamma))$ and $\mathcal{C}_k(t,x,\xi)=\sum_{i=0}^\infty \mathcal{C}_{ki}$ in $M_{m_k}(S^\kappa(\Gamma))$, $\mathcal{C}_{ki}=(\mathcal{C}_{ki}^{<pq>})_{1\le p,q\le d_k}$, $\mathcal{C}_{ki}$: $n_{kp}\times n_{kq}$ such that

$$\mathcal{N}^{0^{-1}}(t,x,\xi)\circ P(t,x,D_t,\xi)\circ\mathcal{N}^0(t,x,\xi)=\oplus_{1\le k\le d}\hat{P}^k, \tag{3.22}$$

$$\hat{P}^k(t,x,D_t,\xi)=I_{m_k}(D_t-\lambda_k(t,x,\xi))-\mathcal{C}_k(t,x,\xi)$$

$$\mathcal{C}_{k0}=\oplus_{1\le j\le d_k}J(n_{kj})|\xi|^\kappa,\quad J(n)=\begin{pmatrix}0&1&&\\&\ddots&\ddots&\\&&\ddots&1\\&&&0\end{pmatrix}:\ n\times n$$

$$\mathcal{C}_{ki}^{<pq>}=\begin{pmatrix}&&\\&0&\\ *&\cdots&*\end{pmatrix}\ (p\le q),\qquad =\begin{pmatrix}*&&\\ \vdots&0&\\ *&&\end{pmatrix}\ (p>q)\quad \text{for } i\ge 1.$$

The entries of $\mathcal{N}_i$ and $\mathcal{C}_{ki}$ satisfy the same estimates as (3.7) *on Γ, replacing C_4, C_4', R_4 and R_4' by other positive constqants. They are also determined only by P.*

In case of meromorphic formal symbol, $\{O_j\}_j$ is composed by only one element and $O_1=O\backslash\Sigma'$ for a subvariety Σ'. $\mathcal{N}^0$ and $\mathcal{C}_k$ belong to GL $(\mathrm{N};S_M^0(O))$ and M_{m_k} $(S_M^\kappa(O))$, respectively.

Proof. The proof is almost same as that of Theorem 1 and all estimates are of same type replacing the constants by other ones. For the simplicity, we use the same notation.

(*Step* 1) We can find finite open conic sets $\{O_{kh}\}_h$ such that the Jordan structure of the generalized eigenspace of $\lambda_k(t,x,\xi)$ is stable on each O_{kh} and each $\cup_h O_{kh}$ is dense in O $(1\le k\le d)$. We can find Jordan chains on each O_{kh} using only the addition, subtraction and multiplication. (See Propositions 2.4 and 2.5 in W. Matsumoto [12].) Let us set $\{O_j\}=\{\cap_{1\le k\le d}O_{kh_k}\}_{(h_1,\cdots,h_d)}$. Then, for every point (t_0,x_0,ξ_0) in O_j, there exists a conically compact neighborhood Γ on which we can find invertible $\mathcal{N}_0$

which satisfies $(\mathscr{N}_0)^{-1}\mathscr{A}_0\mathscr{N}_0=\oplus_{1\le k\le d}(\lambda_k(t,x,\xi)I_{m_k}+\mathscr{C}_{k0})$, $\mathscr{C}_{k0}=\oplus_{1\le j\le d_k}J(n_{kj})|\xi|^{\varkappa}$ and $\mathscr{N}_0$ and $(\mathscr{N}_0)^{-1}$ satisfy similar estimates as (3.9) and (3.10), respectively. We set $(\mathscr{N}_0)^{-1}\circ P\circ\mathscr{N}_0=\hat{P}=I_ND_t-\tilde{\mathscr{C}}$. $\tilde{\mathscr{C}}_i$ satisfies similar estimates as (3.11) for $i=0$ and (3.13) for $i\ge 1$ on Γ, respectively.

(*Step* 2) Let us seek for $\tilde{\mathscr{N}}(t,x,\xi)=\sum_{i=0}^{\infty}\tilde{\mathscr{N}}_i$ in $GL(\mathrm{N};S^0(\Gamma))$, $\tilde{\mathscr{N}}_0=I_{\mathrm{N}}$ and $\mathscr{C}_k(t,x,\xi)=\sum_{i=0}^{\infty}\mathscr{C}_{ki}$ in $M_{m_k}(S^{\varkappa}(\Gamma))$ $(1\le k\le d)$. We decompose $\tilde{\mathscr{N}}_i$ and $\tilde{\mathscr{C}}_i$ corresponding to $\oplus_{1\le k\le d}\mathscr{C}_{k0}$:

$$\tilde{\mathscr{N}}_i=(\tilde{\mathscr{N}}_i^{<pq>})_{1\le p,\,q\le d},\quad \tilde{\mathscr{C}}_i=(\tilde{\mathscr{C}}_i^{<pq>})_{1\le p,\,q\le d}.$$

We have the relations

$$\begin{aligned}
&\mathscr{C}_{ki}-(\mathscr{C}_{k0}\tilde{N}_i^{<kk>}-\tilde{\mathscr{N}}_i^{<kk>}\mathscr{C}_{k0})\\
&=-\sum_{i_1+i_2+|\gamma|=i,\ 0<i_1<i,\ 0<i_2<i}\frac{1}{\gamma!}\tilde{\mathscr{N}}_{i_1}^{<kk>(\gamma)}\mathscr{C}_{ki_2(\gamma)}\\
&\qquad+\sum_{i_1+i_2+|\gamma|=i,\ i_2<i}\frac{1}{\gamma!}(\tilde{\mathscr{C}}_{i_1}^{(\gamma)}\tilde{\mathscr{N}}_{i_2(\gamma)})^{<kk>}-\tilde{\mathscr{N}}_{i-\varkappa(t)}^{<kk>},
\end{aligned}\tag{3.23}$$

$$\begin{aligned}
\mathscr{C}_{p0}\tilde{\mathscr{N}}_i^{<pq>}-\tilde{\mathscr{N}}_i^{<pq>}\mathscr{C}_{q0}&=\sum_{i_1+i_2+|\gamma|=i,\ 0<i_1<i,\ 0<i_2<i}\frac{1}{\gamma!}\tilde{\mathscr{N}}_{i_1}^{<pq>(\gamma)}\mathscr{C}_{qi_2(\gamma)}\\
&\quad-\sum_{i_1+i_2+|\gamma|=i,\ i_2<i}\frac{1}{\gamma!}[\tilde{\mathscr{C}}_{i_1}^{(\gamma)}\tilde{\mathscr{N}}_{i_2(\gamma)}]^{<pq>}+\tilde{\mathscr{N}}_{i-\varkappa(t)}^{<pq>},\quad (p\ne q).
\end{aligned}\tag{3.24}$$

We can show the similar estimate on $\mathscr{C}_{ki}$ and $\tilde{\mathscr{N}}_i^{<pq>}$ as in Lemma 3.4 also by the induction on i. In this case, as $\tilde{\mathscr{N}}_i^{<kk>}$ does not vanish, the differences from Lemma 3.4 are that we need to estimate the first sum and the third element in the right-hand side of (3.23) and that the decision of $\mathscr{C}_{ki}$ and $\tilde{\mathscr{N}}_i^{<kk>}$. Let us decompose $\tilde{\mathscr{N}}_i^{<kk>}$, $\mathscr{C}_{ki}$ and the right-hand side $H_i^{<kk>}$ of (3.23) corresponding to $\mathscr{C}_{k0}=\oplus_{1\le j\le d_k}J(n_{kj})|\xi|^{\varkappa}$:

$$\mathscr{C}_{ki}=(\mathscr{C}_i^{k<pq>})_{1\le p,\,q\le d_k}=(c_i^{kpq}(u,v))_{1\le u\le n_{kp},\ 1\le v\le n_{kq}},$$

$$\tilde{\mathscr{N}}_i^{<kk>}=(\tilde{\mathscr{N}}_i^{k<pq>})_{1\le p,\,q\le d_k}=(n_i^{kpq}(u,v))_{1\le u\le n_{kp},\ 1\le v\le n_{kq}},$$

$$H_i^{<kk>}=(H_i^{k<pq>})_{1\le p,\,q\le d_k}=(h_i^{kpq}(u,v))_{1\le u\le n_{kp},\ 1\le v\le n_{kq}}.$$

The relation (3.23) becomes

$$\mathscr{C}_i^{k<pq>}-(J(n_{kp})\tilde{\mathscr{N}}_i^{k<pq>}-\tilde{\mathscr{N}}_i^{k<pq>}J(n_{kq}))|\xi|^{\varkappa}=H_i^{k<pq>}.\tag{3.25}$$

For p≤q, this has the solution

$$c_i^{kpq}(u,v)=\begin{cases}0 & (1\le u\le n_{kp}-1)\\ \sum_{w=0}^{v-1}h_i^{kpq}(n_{kp}-w,\ v-w) & (u=n_{kp}),\end{cases}$$

and

$$n_i^{kpq}(u,v) = \begin{cases} 0 & (u=1) \\ \sum_{w=0}^{\min\{u-2,\ v-1\}} h_i^{kpq}(u-w-1,\ v-w)|\xi|^{-\varkappa} & (2 \leq u \leq n_{kp}), \end{cases}$$

where v runs from 1 to n_{kq}, and for $p>q$,

$$c_i^{kpq}(u,v) = \begin{cases} \sum_{w=0}^{n_{kq}-u} h_i^{kpq}(u+w,\ 1+w) & (v=1) \\ 0 & (2 \leq v \leq n_{kq}), \end{cases}$$

and

$$n_i^{kpq}(u,v) = \begin{cases} \sum_{w=0}^{\min\{n_{kq}-u,\ n_{kq}-v-1\}} h_i^{kpq}(u+w,\ v+w+1)|\xi|^{-\varkappa} & (1 \leq v \leq n_{kp}-1) \\ 0 & (v=n_{kq}), \end{cases}$$

where u runs from 1 to n_{kp}.
Using the above expressions of the solution of (3.25), we obtain the estimates of same type as (3.18).

(*Step* 3) The last step is just same as the proof of Theorem 1. □

3.4. Normal form of systems. As we showed in W.Matsumoto [12], applying Theorem 2 and changing order by $I_r|\xi|^\mu \oplus I_{N-r}$ or $I_r \oplus I_{N-r}|\xi|^\mu$ ($\mu>0$) finite times, we arrive at the following theorem. For the simplicity, we assume the differentiability condition on $\{M_n\}$ and $\{L_n\}$. Under this, we can say always the true order is its order.

Theorem 3 (Normal form of system in ultradifferentiable class, [12]). *We assume that every entry of $\mathscr{A}$ satisfies* (2.1) *and that the each eigenvalue $\lambda_k(t,x,\xi)$ $(1 \leq k \leq d)$ of $\mathscr{A}_0$ has the constant multiplicity m_k. Then, there exist finite disjoint open conical sets $\{O_h\}_h$ such that $\cup_h O_h$ is dense in O. On each O_h, there exist natural numbers d_k and $\{n_{kj}\}_{j=1}^{d_k}$ ($\sum_{j=1}^{d_k} n_{kj} = m_k$). For every point $(t_\circ,x_\circ,\xi_\circ)$ in O_h, there exist a conically compact neighborhood Γ, $\mathscr{N}(t,x,\xi)=\sum_{i=0}^{\infty}\mathscr{N}_i$ in $GL(\mathrm{N};S(\Gamma))$ and $\mathscr{D}_{kj}(t,x,\xi)=\sum_{i=0}^{\infty}\mathscr{D}_{kji}$ in $M_{n_{kj}}(S^\varkappa(\Gamma))$, such that*

$$\mathscr{N}^{-1}(t,x,\xi) \circ P(t,x,D_t,\xi) \circ \mathscr{N}(t,x,\xi) = \oplus_{1\leq k\leq d}\oplus_{1\leq j\leq d_k} Q_{kj},$$

$$Q_{kj}(t,x,D_t,\xi) = I_{n_{kj}}(D_t - \lambda_k(t,x,\xi)) - \mathscr{D}_{kj}(t,x,\xi),$$

(3.26)

$$\mathscr{D}_{kj0} = J(n_{kj})|\xi|^\varkappa, \quad D_{kji} = \begin{pmatrix} & 0 & \\ * & \cdots & * \end{pmatrix} \quad (i \geq 1).$$

The entries of $\mathscr{N}_i$ and $\mathscr{D}_{kji}$ satisfy the same estimates as (3.7) *on Γ, replacing C_4, C_4', R_4 and R_4' by other positive constants. Here, on $\mathscr{N}$ and $\mathscr{N}^{-1}$, the orders of those entries may be positive and the power $-i-|\beta|$ in* (3.7) *must be replaced by $\varkappa_\circ - i - |\beta|$ for a suitable non-negative number $\varkappa_\circ$. These constants are also* *determined only by P.*

In case of meromorphic formal symbol, $\{O_h\}_h$ is composed by only one element and $O_1=O\backslash\Sigma'$ for a subvariety Σ'. $\mathcal{N}$ and $\mathcal{D}_{kj}$ belong to $GL(N;S_M(O))$ and $M_{n_{kj}}(S^{\varkappa}_M(O))$, respectively.

Remark 3.1. Of course, the above d_k, n_{kj}, O_h and Σ' are different from those in Theorem 2. Further, in Theorem 3, d_k and n_{kj} on each O_h may be different each other.

Now, we consider the formal symbols which are partially ultradifferentiable. For the simplicity, we treat only formal symbols of C^{∞} class on t and of ultradifferentiable class on x.

Definition 4 (*Formal symbol of class $\{\infty, M_n, L_n\}$*). We say that the formal sum $a(t,x,\xi)=\sum_{i=0}^{\infty} a_i(t,x,\xi)$ is a formal symbol of class $\{\infty, M_n, L_n\}$ on $O=\bigcup_{t\in[T_1,T_2]}\{t\}\times O(t)$, $O(t)$ is an open conic set in $T^*\mathbf{R}^l$, when there exists a real number $\varkappa$ such that

1) $a_i(t,x,\xi)$ belongs to $C^{\infty}(O)$ and positively homogeneous of degree $\varkappa-i$ on ξ, $(i\in\mathbf{Z}_+)$.
2) For arbitrary conically compact subset Γ in O, there are positive constants $\{C_n\}$, R and R' and we have

$$|a_{i(\alpha)}^{(\beta)}(t,x,\xi)| \leq C_{\alpha_0}R'^i R^{|\alpha'|+|\beta|}M_{i+|\alpha'|-3}L_{i+|\beta|-3}(i-3)!^{-1}|\xi|^{\varkappa-i-|\beta|} \quad \text{on } \Gamma,\quad (i\in\mathbf{Z}_+,\ \alpha\in\mathbf{Z}_+{}^{1+l},\ \beta\in\mathbf{Z}_+{}^l), \tag{3.27}$$

where $\alpha=(\alpha_0, \alpha')$.

We consider also holomorphic formal symbol and a meromorphic one in C^{∞} on t.

Definition 5 (*Meromorphic and holomorphic formal symbols of C^{∞} class on t*).
I. We say that the formal sum $a(t,x,\xi)=\sum_{i=0}^{\infty} a_i(t,x,\xi)$ is a meromorphic formal symbol of C^{∞} class on t on $O=\bigcup_{t\in[T_1,T_2]}\{t\}\times O(t)$, $O(t)$ is an open conic set in $T^*\mathbf{C}^l$, when there exist a conic subvariety $\Sigma(t)$ for t in $[T_1, T_2]$ and a real number $\varkappa$ such that

1) For each fixed t, $a_i(t,x,\xi)$ is meromorphic in $O(t)$, holomorphic in $O(t)\backslash\Sigma(t)$ and positively homogeneous of degree $\varkappa-i$ on ξ, $(i\in\mathbf{Z}_+)$.
2) For arbitrary conically compact set Γ in $\bigcup_{t\in[T_1,T_2]}\{t\}\times(O(t)\backslash\Sigma(t))$, there are positive constants $\{C_n\}$, R and R' and we have

$$|a_{i(\alpha)}^{(\beta)}(t,x,\xi)| \leq C_{\alpha_0}R'^i R^{|\alpha'|+|\beta|}(i+|\alpha'|-3)!(i+|\beta|-3)!(i-3)!^{-1}|\xi_1|^{\varkappa-i} \quad \text{on } \Gamma,\quad (i\in\mathbf{Z}_+,\ \alpha\in\mathbf{Z}_+{}^{1+l},\ \beta\in\mathbf{Z}_+{}^l). \tag{3.28}$$

II. The formal sum $\sum_{i=0}^{\infty} a_i$ is called a holomorphic formal symbol of C^{∞} class on

t on $O=\bigcup_{t\in[T_1, T_2]}\{t\}\times O(t)$ when it is a meromorphic formal symbol with $\Sigma(t)=\emptyset$ for arbitrary t in $[T_1, T_2]$.

We denote the set of the formal symbols of class $\{\infty, M_n, L_n\}$ on $O=\bigcup_{t\in[T_1, T_2]}\{t\}\times O(t)$ by $C^\infty([T_1, T_2]; S\{M_n, L_n\}(O(t))$, that of the meromorphic formal symbol of C^∞ class in t by $C^\infty([T_1, T_2]; S_M(O(t)))$, etc.

We regard the estimate (3.28) is a special case of (3.27).

Theorem 4 (Normal form of system of C^∞ class on t and of u.d.'ble class in x). *We assume that every entry of $\mathscr{A}$ satisfies* (3.27) *and that the each eigenvalue $\lambda_k(t,x,\xi)$ $(1\leq k\leq d)$ of $\mathscr{A}_0$ has the constant multiplicity m_k. Then, the assertion in Theorem* 3 *also holds except the estimates. For arbitrary $i_\circ$ in $\mathbf{Z}_+$, the entries of $\mathscr{N}_i$ and $\mathscr{D}_{kji}$ satisfy the same estimates as* (3.27) *on Γ for $0\leq i\leq i_\circ$, replacing C_{α_0}, R and R' by other positive constants. They are determined only by P and $i_\circ$.*

In case of $C^\infty([T_1, T_2]; S_M(O(t)))$, $\{O_h\}$ becomes the following ; There exist finite disjoint open sets $\{o_h\}$ in $[T_1, T_2]$ such that $\bigcup_h o_h$ is dense in $[T_1, T_2]$ and exists a subvariety $\Sigma(t)$ in $O(t)$ for each t in $\bigcup_h o_h$. O_h is given by $\bigcup_{t\in o_h}\{t\}\times(O(t)\backslash\Sigma(t))$. Further, $\mathscr{N}$ and $\mathscr{D}_{kj}$ belong to $GL(\mathrm{N}; C^\infty(\bigcup o_h; S_M(O(t))))$ and $M_{n_{kj}}(C^\infty(\bigcup o_h; S^\varkappa_M(O(t))))$, respectively.

Proof. Applying Theorerm 2 and changing order by $W_j=I_r|\xi|^{\mu_j}\oplus I_{\mathrm{N}-r}$ or $=I_r\oplus I_{N-r}|\xi|^{\mu_j}$ $(\mu_j>0, 1\leq j\leq r)$ alternately, we arrive at the normal from. We can assume that $\{C_n\}$ is logarithmically convex and non-decreasing. Let us set $q=(r+1)i_\circ+\sum_{j=1}^r j\mu_j$ and $C'_n=C_{n+q}$ for n in $\mathbf{Z}_+$. On each conically compact Γ in O, every entries of $\mathscr{A}$ satisfies

$$(3.29)\quad \begin{aligned} |a_{i(\alpha)}^{(\beta)}(t,x,\xi)| \leq\ & C_{\max\{\alpha_0-q,\, 0\}}R'^i R^{|\alpha'|+|\beta|}M_{i+|\alpha'|-3}L_{i+|\beta|-3}(i-3)!^{-1}|\xi|^{\varkappa-i-|\beta|} \\ & \textit{on } [T_1, T_2]\times\Gamma, \quad (i\in\mathbf{Z}_+,\ \alpha\in\mathbf{Z}_+{}^{1+l},\ \beta\in\mathbf{Z}_+{}^{l}). \end{aligned}$$

After we transform P by $\mathscr{N}^\circ$ in Theorem 2, the entries of the transformed operator satisfy a similar estimate as (3.29) for $0\leq i\leq i_\circ+\sum_{j=1}^r\mu_j$ replacing $C_{\max\{\alpha_0-q,\, 0\}}$ by $C_{\max\{\alpha_0-q',\, 0\}}$, $q'=ri_\circ+\sum_{j=2}^r(j-1)\mu_j$. Further, after the transformation by W_1, it holds for $0\leq i\leq i_\circ+\sum_{j=2}^r\mu_j$. Repeating this r times, the estimate of the entries are like (3.29) for $0\leq i\leq i_\circ$ replacing $C_{\max\{\alpha_0-q,\, 0\}}$ by $C_{\max\{\alpha_0-i_\circ,\, 0\}}$. At last, once again we may apply Theorem 2 and the estimates of the entries in the normal form of type (3.27) hold for $0\leq i\leq i_\circ$. □

4. Appendix : Product and Division, Reconsideration

From now on, we assume the following :

Assumption$'$. $\{(M_n/n!^s)\}$ *is logarithmically convex and non-decreasing for some* $s>1$.

Gevrey weight $\{n!^s\}$ and many more rapidly increasing $\{M_n\}$ than $n!^s$ for $s>1$

satisfy this but $\{n!\}$ does not. The following proposition is easily obtained by virtue of Lemma 2.1 (8).

Proposition 4.1 (Product (2)). *We pose Assumption′. If the followings are satisfied on a compact set K by positive constants R and C_j $(j=1,2)$*

$$|f_j(x)_{(\alpha)}| \leq C_j R^{|\alpha|} M_{|\alpha|} \quad ,$$

the product of f_1 and f_2 satisfies

$$|(f_1)(x) f_2(x))_{(\alpha)}| \leq c_s C_1 C_2 R^{|\alpha|} M_{|\alpha|} \quad ,$$

where $\alpha \in \mathbf{Z}_+{}^l$ and c_s is that in Lemma 2.1 (8).

Further, $B\{M_n\}_R(K)$ is also closed on the division by non-zero element.

Proposition 4.2 (Division (2)). *We pose Assumption′. If, for some positive C, R and c_m, $f(x)$ satisfies the following on K*

$$|f(x)_{(\alpha)}| \leq C R^{|\alpha|} M_{|\alpha|}$$

and

$$|f(x)| \geq c_m,$$

then $1/f(x)$ satisfies

$$|(1/f(x))_{(\alpha)}| \leq C' R^{|\alpha|} M_{|\alpha|},$$

for c'_s in Lemma 2.1 (9), $c'(s)=\sum_{q=0}^{\infty}(c'_s C/c_m)^q/q!^{s-1}$ and $C'=c'(s)C/c_m{}^2$.

Proof. We use the majorant. We follow the proof for the composed function in H. Komatsu [6]. For two formal sum $G^j(x)=\sum_{\alpha\in\mathbf{Z}_+{}^l} G^j_\alpha x^\alpha$ $(j=1,2)$. We denote $G^1 << G^2$ when it holds that $|G^1_\alpha| \leq G^2_\alpha$ for all α. For $f(x)$ of C^∞ class, we identify it with its formal Taylor expansion. Let us set a formal sum $F(X)=\sum_{n=0}^{\infty} F_n X^n/n!$ of one variable X and take arbitrary point $x_\circ$ in K. The relations $|f_{(\alpha)}(x_\circ)| \leq F_{|\alpha|}$ for α in $\mathbf{Z}_+{}^l$ is equivalent to the relation $f(x) << F(\sum_{i=1}^{l}(x_i - x_{\circ i}))$. As

$$\frac{1}{f(x_\circ)\{1+(f(x)-f(x_\circ))/f(x_\circ)\}} = \frac{1}{f(x_\circ)} \sum_{q=0}^{\infty}\left(-\frac{f(x)-f(x_\circ)}{f(x_\circ)}\right)^q$$

$$<< \frac{1}{c_m}\sum_{q=0}^{\infty}\left(\frac{1}{c_m}\right)^q (F(X)-F(0))^q|_{X=x_1-x_{\circ 1}+\cdots+x_l-x_{\circ l}},$$

setting the coefficient of X^n as $G_n/n!$, we have $|(1/f(x))_{(\alpha)}| \leq G_{|\alpha|}$.

Let us calculate the following

$$\sum_{q=0}^{\infty}\left(\frac{1}{c_m}\right)^q (F(X) - F(0))^q = \sum_{q=0}^{\infty}\left(\frac{1}{c_m}\right)^q (\textstyle\sum_{p=1}^{\infty} C R^p M_p X^p/p!)^q$$

$$= 1 + \sum_{n=1}^{\infty} R^n X^n \sum_{q=1}^{n} \left(\frac{C}{c_m}\right)^q \sum_{p_1+\cdots+p_q=n,\ p_i\geq 1} \frac{M_{p_1}}{p_1!} \cdots \frac{M_{p_q}}{p_q!}.$$

As $\{(M_n/n!^s)^{1/(n-1)}\}$ is non-decreasing by lemma 2.1 (7) and $\{(M_n/n!)\}$ is logarithmically convex, it is seen

$$\frac{M_{p_i}}{p_i!} \leq \left(\frac{M_{n-q+1}}{(n-q+1)!^s}\right)^{(p_i-1)/(n-q)} p_i!^{s-1},$$

$$\frac{M_{p_1}}{p_1!} \cdots \frac{M_{p_q}}{p_q!} \leq \frac{M_{n-q+1}}{(n-q+1)!} \left(\frac{p_1! \cdots p_q!}{(n-q+1)!}\right)^{s-1}$$

$$\leq \frac{M_n}{n!} \frac{(q-1)!}{M_{q-1}} \left(\frac{p_1! \cdots p_q!}{(n-q+1)!}\right)^{s-1}$$

$$\leq \frac{M_n}{n!} \frac{1}{(q-1)!^{s-1}} \left(\frac{p_1! \cdots p_q!}{(n-q+1)!}\right)^{s-1}.$$

On the other hand, applying Lemma 2.1(9), for $q \geq 2$, we can see

$$\sum_{p_1+\cdots+p_q=n,\ p_i\geq 1} \left(\frac{p_1! \cdots p_q!}{(n-q+1)!}\right)^{s-1}$$

$$= \sum_{p_1=1}^{n-q+1} \left(\frac{p_1!(n-q+2-p_1)!}{(n-q+1)!}\right)^{s-1} \sum_{p_2=1}^{n-q+1-p_1} \left(\frac{p_2!(n-q+3-p_1-p_2)!}{(n-q+2-p_1)!}\right)^{s-1}$$

$$\cdots \sum_{p_{q-1}=1}^{n-q+1-p_1-\cdots-p_{q-2}} \left(\frac{p_{q-1}!(n-p_1-\cdots-p_{q-1})!}{(n-1-p_1-\cdots-p_{q-2})!}\right)^{s-1}$$

$$\leq c_s^{q-1}.$$

Thus, we arrive at

$$\sum_{q=1}^{\infty} \left(\frac{C}{c_m}\right)^q \sum_{p_1+\cdots+p_q=n,\ p_i\geq 1} \frac{M_{p_1}}{p_1!} \cdots \frac{M_{p_q}}{p_q!} < \frac{M_n}{n!} \frac{C}{c_m} \sum_{q=0}^{\infty} \frac{(c_s C/c_m)^q}{q!^{s-1}}$$

$$= \frac{c'(s)C}{c_m} \frac{M_n}{n!}.$$

Department of Applied Mathematics and Informatics
Faculty of Science and Technology
Ryukoku University
e-mail : waichiro@rins.ryukoku.ac.jp

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