J. Math. Kyoto Univ. (JMKYAZ)
40-3 (2000), 493-524

# On some related non homogeneous 3D Boltzmann models in the non cutoff case

By

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**Abstract**

We study some issues concerning the existence of weak solutions for two Boltzmann like equations : a Modified Boltzmann model and the Boltzmann Dirac model. The analysis of the collision operators rests on suitable decompositions. These are also provided for the Generalised Boltzmann operator. This study is performed without assuming Grad's angular cutoff hypothesis on the cross sections.

## 1. Introduction

In this work, we wish to show that the method introduced in our previous papers [Ale1, ..., 6] yields definite results on such issues as the existence, regularity ..., when looking to some related 3D Boltzmann models, for which Grad's usual cutoff hypothesis on the collision kernel $B$ fails to be true.

More precisely, the models studied in this paper are *non homogeneous ones*, that is they also depend on the position variable $x$ (via the free streaming operator) and the collision operators involved herein cannot be splitted into the usual gain and loss terms, in view of the high singularity of the collision cross sections.

For the sake of simplicity, we shall only consider 3D cases, although as already mentionned in [Ale1, ..., 5], the computations could be extended to other dimensions, once one knows the (more or less) explicit expressions for the cross-sections.

To explain our purpose, let us recall that the usual Boltzmann equation consists in looking for a solution $f=f(t,x,v)$ where $t$ (the time) is in $\mathbf{R}^+$, $x$(the position) in $\mathbf{R}^3$, $v$(the velocity) in $\mathbf{R}^3$, of the following non linear partial differential equation

$$(1.1)\qquad \begin{cases} \partial_t f(t,x,v)+v.\nabla_x f(t,x,v)=Q(f,f), \\ f(0,x,v)=f_0(x,v). \end{cases}$$

Here $f_0=f_0(x,v)$ is a given initial datum, and $Q$ is the so called collision operator

1991 *Mathematics Subject Classification*.-35D05, 35D10, 35K70, 82B40
Communicated by Prof. T. Nishida, December 4, 1998
Revised March 10, 1999

and acts on the variable $v$ as

(1.2)
$$Q(f,f)(v)\equiv\int_{\mathbf{R}^3_{v_1}}\int_{S^2_\omega}dv_1 d\omega\{f(v')f(v_1')-f(v)f(v_1)\}B\left(|v-v_1|,\left|\left(\frac{v-v_1}{|v-v_1|},\omega\right)\right|\right).$$

$v_1'$ and $v'$ are the post collisional velocities, which can be parametrised by $\omega\in S^2$, unit sphere of $\mathbf{R}^3$, as

$$v'=v+(v_1-v,\omega)\omega,\quad v_1'=v_1-(v_1-v,\omega)\omega. \tag{1.3}$$

Here $B$ is the given cross-section depending on the variables as pointed by (1.2). The physical meaning of all the above quantities is by now standard and may be found for instance in [ArBe, Cer, CIP, Gui].
Most of the mathematical works on (1.1) have been done under the so called Grad's angular cutoff hypothesis, which roughly means that

$$\omega\to B(.,.)\in L^1(S^2), \tag{1.4}$$

see also more precisely [Cer].
Let us mention that one main feature shared by the models considered herein consists in that we shall never use the concept of renormalised solutions of DiPerna and Lions [DiLi1, 2, Lio1].
This concept of solutions can be avoided for at least three models, which have a "clear" physical meaning: the Modified Boltzmann equation (MB) [CIP, DiLi], the Generalised Boltzmann equation (GB) [BePo, ArBe] and the Dirac Boltzmann model (DB) [Dol]. The mathematical theory is nearly clear for these models and again it has been done for cross sections $B$ such as (1.4), or less ...
The natural next step is therefore to ask for what happens if (1.4) fails to hold. That this question is indeed natural (physically) can be explained by turning to [Cer, Gui, Uka], where we are told that (1.4) never holds, at least for interaction potentials of the form $1/r^s(s>2)$. Indeed, in this case, $B$ is close to

$$B\left(|v-v_1|,\left|\left(\frac{v-v_1}{|v-v_1|},\omega\right)\right|\right)\equiv|v-v_1|^\gamma\frac{1}{\left|\left(\frac{v-v_1}{|v-v_1|},\omega\right)\right|^\nu}, \tag{1.5}$$

where the critical exponents are defined by

$$\gamma=\gamma(s)=\frac{s-5}{s-1},\quad \nu=\nu(s)=\frac{s+1}{s-1}. \tag{1.6}$$

For such $B$, most of the mathematical results are concerned with the non linear homogeneous account of (1.1), see for instance [Ark1, 2, Gou, Vil1], and regularity results are also proven to be true by [Des] in 2D cases and again in the homogeneous framework. In [Ale1, 2, 3], we provide different decompositions of $Q$ containing a principal part which may be thought as elliptic, and we apply this to various issues in [Ale4, 5]. Problem (1.1) is still outside this scope, with the aim at getting the existence of global solutions for general initial data, satisfying the usual entropic bounds. Indeed, the renormalisation method of Di Perna and Lions seems hard to fit, but not impossible, in view of the pdo like operators appearing in our previous works. In fact, we have been able [Ale6] to define renormalised solutions and thus to get ride of the assumption of average compacity of [Lio3], but I have only succeeded in showing that limits of such solutions are upper solutions (though formally, they are exact ones).

Note that the issue is similar to Landau's equation [Vil2], whose global solutions are unknown, in spite of the regularity results based on the entropy dissipation rate estimates [Lio2].

The aim of this work is to study the three above mentionned models, that is (in the order) (MB), (DB) and (GB).

Before introducing these ones, we would like to make some comments on some less known models. One such example is studied by [DeGo].

To begin with, we have shown [Ale2] that under (1.5) and (1.6), one could assume that the collision kernel $B$ takes the following form

$$B\left(|v-v_1|,\left|\left(\frac{v-v_1}{|v-v_1|},\omega\right)\right|\right)\equiv\frac{|v_1-v'|^{\gamma+\nu}}{|v'-v|^{\nu}}. \tag{1.7}$$

The difference between (1.7) and (1.5) corresponds to a cutoff cross section. Subsequently, we shall always assume the form (1.7)-(1.6). In fact, we shall even make a cutoff in velocity, but for the moment, let us keep this assumption. In addition, see [Ale2], we have shown that the operator $Q$ may be written (in a somehow simplified form) as

(1.8)
$$Q(f,f)(v)=\int_{\mathbf{R}^3_h}\int_{\mathbf{R}^3_\alpha}\frac{2dh}{|h|^{\nu+2}}d\alpha\delta_{\alpha.h=0}\{f(v-h)f(\alpha+v)-f(v)f(\alpha+v-h)\}|\alpha|^{\gamma+\nu},$$

where $\delta_{\alpha.h=0}$ denotes the Dirac measure over the surface $\{\alpha.h=0\}$.

Next, if we want some more accessible mathematical models, one may first mollify this measure. The simplest way is to change it by 1, and also replace $|\alpha|^{\gamma+\nu}$ by 1, so that we get

$$Q_{M_1}(f,f)(v)=\left\{\int_{\mathbf{R}^3_\alpha}f(\alpha)d\alpha\right\}\left\{\int_{\mathbf{R}^3_h}\frac{f(v-h)-f(v)}{|h|^{\nu+2}}dh\right\}. \tag{1.9}$$

This operator, not only leads to a very simple model, but also in connection with (1. 1), yields $L^p$ estimates. Note that up to constants, one has

$$Q_{M_1}(f,f)(v)=-\left\{\int_{\mathbf{R}^3_\alpha}f(\alpha)d\alpha\right\}(-\Delta)^{\frac{\nu-1}{2}}(f)(v). \tag{1.9'}$$

If we want to keep the weight $|\alpha|^{\gamma+\nu}$, one obtains

$$Q_{M_2}(f,f)(v)=\left\{\int_{\mathbf{R}^3_\alpha}f(\alpha+v)|\alpha|^{\gamma+\nu}d\alpha\right\}\left\{\int_{\mathbf{R}^3_h}\frac{f(v-h)-f(v)}{|h|^{\nu+2}}dh\right\}-$$

$$-f(v)\int_{\mathbf{R}_h^3}\int_{\mathbf{R}_\alpha^3}\frac{f(\alpha+v-h)|\alpha|^{\gamma+\nu}-f(\alpha+v)|\alpha|^{\gamma+\nu}}{|h|^{\nu+2}}, \tag{1.10}$$

that is also

$$Q_{M_2}(f,f)(v)=-\Big\{\int_{\mathbf{R}_\alpha^3}f(\alpha+v)|\alpha|^{\gamma+\nu}d\alpha\Big\}(-\Delta)^{\frac{\nu-1}{2}}(f)(v)+$$

$$+f(v)(-\Delta)^{\frac{\nu-1}{2}}\Big\{\int_{\mathbf{R}_\alpha^3}f(\alpha+v)|\alpha|^{\gamma+\nu}d\alpha\Big\}(v).$$

Next, if we want something nearest to the Dirac mass, we can approch it by a smooth function ... Finally, a last mathematical model can be obtained by changing in (1.8) $f(\alpha+v)$ and $f(\alpha+v-h)$ by their mean value.
In this paper, we shall not analyse these mathematical models, although they could be of interest. Nevertheless, one such study is provided by [DeGo] in one dimension.
Turning now to our objective, we modify assumption (1.7) as follows

$$B\Big(|v-v_1|,\Big|\Big(\frac{v-v_1}{|v-v_1|},\omega\Big)\Big|\Big)\equiv\theta(|v_1-v'|)\frac{|v_1-v'|^{\gamma+\nu}}{|v'-v|^{\nu}}, \tag{1.12}$$

where $\theta$ belongs to $\mathscr{S}^+$ and is null for small values. This hypothesis simplifies many of the computations displayed in this paper, but should be weakened by looking for moments estimates, something that we skip completely in this paper ....
Let us begin with the Modified Boltzmann model, see [CIP, DiLi]. In this case, the collision operator is given by

$$Q_{mb}(f,f)(v)=\frac{1}{1+\int fdv}Q(f,f)(v). \tag{1.13}$$

We will study it in Section II. Even if the physical meaning of (1.13) is not clear, we include it in order to introduce some earlier ideas and to make the paper self-content.
Next, we introduce (Section III) the collision operator for the Boltzmann Dirac model [Dol], (we set $\varepsilon=1$ with respect to this paper)

$$Q_{bd}(f,f)(v)=\int_{\mathbf{R}_{v_1}^3}\int_{S_\omega^2}dv_1d\omega\{f'f_1'(1-f)(1-f_1)-ff_1(1-f')(1-f_1')\}B(...), \tag{1.14}$$

again with B as in (1.12).
Then, for the Generalised Boltzmann operator (Section IV) studied by [BePo], we make some simplifications with respect to that paper. Let $R>0$, and $P=P(r)$, $|r|\leq$ R, a measurable function such that

$$P\equiv P(|r|),\quad 0<P^-\leq P(r)\leq P^+<\infty. \tag{1.15}$$

Now, as in the primitive variables, see (1.2) and (1.3), the corresponding operator $Q_{gb}$ acts on both variables $(x,v)$ as

$$Q_{gb}(f,f)(x,v)=\int_{-R}^{R}dr\int_{\mathbf{R}^3_{v_1}}\int_{\mathbf{S}^2_\omega}dv_1 d\omega\{f(x,v')f(x+r\omega,v_1')-$$

$$-f(x,v)f(x+r\omega,v_1)\}B(.,.)P(r), \tag{1.16}$$

with B still given by (1.12). As it is well known, we note that if $P=\delta_{r=0}$, then one recovers the usual Boltzmann operator.
Our aim is then to analyse the non homogeneous equation

$$(P)\qquad \begin{cases} \partial_t f(t,x,v)+v.\nabla_x f(t,x,v)=\mathcal{Q}(f,f), \\ f(0,x,v)=f_0(x,v), \end{cases}$$

where the operator $\mathcal{Q}$ is one choice among the above operators.
These are dealt with in Sections II to V respectively. However, we only provide the decompositions associated with $Q_{gb}$ in Section V, leaving out any other questions. To make the paper worth reading, I will recall the framework of this Section in each of the following ones. I also use freely some results provided by my earlier papers, for which readers are referred to (in order to limit the typesetting time spent herein ...). Clearly, many issues are not dealt with herein. Let us mention unicity, moment estimates, trends to equilibrium, existence in other functional spaces, other models such as the Povzner's one [Mor, Pov] ... There are also other methods which may prove more interesting, as for instance non linear semi-groups [Lun] (and references therein). All this is left for future research.

**Acknowledgements.** I would like to thank Nicola Bellomo for sending me several of his papers. As usual, I had many discussions with Kamal Hamdache.

## 2. Problem (MB)

This Section is devoted to the modified Boltzmann model as introduced in Section I. Recall that

$$B\left(|v-v_1|,\left|\left(\frac{v-v_1}{|v-v_1|},\omega\right)\right|\right)\equiv\theta(|v_1-v'|)\frac{|v_1-v'|^{\gamma+\nu}}{|v'-v|^{\nu}}, \tag{2.1}$$

where $\theta$ belongs $\mathscr{D}^+$, is null for small values, and

$$\gamma=\gamma(s)=\frac{s-5}{s-1},\quad \nu=\nu(s)=\frac{s+1}{s-1}. \tag{2.2}$$

The modified collision operator is given as

$$Q_{mb}(f,f)(v)=\frac{1}{1+\int_v f dv}Q(f,f)(v), \tag{2.3}$$

where $Q$ denotes the usual collision operator as given (1.2) of Section I.
We are interested in the following problem

$$(MB) \qquad \begin{cases} \partial_t f(t,x,v) + v.\nabla_x f(t,x,v) = Q_{mb}(f,f), \\ f(0,x,v) = f_0(x,v), \end{cases}$$

where $t \in (0,T)$, $T > 0$ fixed, $x \in \mathbf{R}^3$, $v \in \mathbf{R}^3$, and $f_0$ is the initial datum satisfying the usual entropic bounds, that is

$$(\mathscr{H}) \qquad \iint f_0(1+|x|^2+|v|^2+|\log f_0|)dxdv < \infty.$$

We will follow the easiest way to deal with (MB), and we will not focus on any questions of unicity ....
First, let us recall the following result from [Ale2], assuming in the sequel that $f \geq 0$, $f$ regular and satisfies the usual entropic bounds

**Lemma 2.1.** *With the above notations* (2.1)-(2.3), *the operator $Q_{mb}$ writes as*

$$Q_{mb}(f,f) = Q^1_{mb}(f,f) + Q^2_{mb}(f,f),$$

*where*

$$Q^1_{mb}(f,f)(v) = \frac{-C_s}{1+\int_v f dv} \int_{R^3} d\alpha f(\alpha+v) \bar{\theta}(|\alpha|) |S(\alpha).D_v|^{\nu-1}(f)(v),$$

*and*

$$Q^2_{mb}(f,f)(v) = \frac{C_s f(v)}{1+\int_v f dv} \int_{R^3} d\alpha \bar{\theta}(|\alpha|) |S(\alpha).D_v|^{\nu-1}(f)(\alpha+v),$$

*where $C_s$ is a constant depending on s, $\bar{\theta}$ denotes function $\theta$ multiplied by a power of $|\alpha|$, and $S(\alpha)$ is the projection over the hyperplane through* 0 *and orthogonal to $\alpha$.*

Next, we simplify the expression of the operator $Q^2_{mb}$ as follows

**Lemma 2.2.** *With the notations of Lemma* 2.1, *one has*

$$Q^2_{mb}(f,f)(v) = \frac{f(v)}{1+\int_v f dv} \int_{R^3} dk f(k) K(|v-k|),$$

*where $|K(|u|) \leq C_\nu$.*

*Proof.* In view of Lemma 2.1, all amounts to compute, see also our papers [Ale]

$$\int_{\mathbf{R}^3} d\alpha \bar{\theta}(|\alpha|) |S(\alpha).D_v|^{\nu-1}(f)(\alpha+v) =$$

$$=\int_k f(k)\int_\xi e^{-i\xi.(k-v)}|\xi|^{\nu-1}\psi(\xi), \tag{2.4}$$

where we have settled

$$\psi(\xi)=\int_\alpha \bar{\theta}(|\alpha|)|S(\alpha).\xi|^{\nu-1}e^{-i\xi.\alpha}. \tag{2.5}$$

Then, since $\psi$ is decreasing of any order wrt $\xi$ (as $\theta\in\mathcal{S}$)

$$\int_{\mathbf{R}^3} d\alpha\bar{\theta}(|\alpha|)|S(\alpha).D_v|^{\nu-1}(f)(\alpha+v)=\int_{\mathbf{R}^3_k} f(k)K(|k-v|), \tag{2.6}$$

with

$$K(|u|)=\int_\xi e^{-i\xi.(u)}\psi(\xi)|\xi|^{\nu-1}, \tag{2.7}$$

which directly leads to the Lemma.

From Lemma 2.2, one deduces easily the

**Lemma 2.3.** *With the above notations*

$$\|Q^2_{mb}(f,f)\|_{L^1_v}\le C\|f\|_{L^1_v},$$

$$\|Q^2_{mb}(f,f)-Q^2_{mb}(g,g)\|_{L^1_v}\le C\|f-g\|_{L^1_v}.$$

In particuliar, $Q^2_{mb}(f,f)$ has good functional properties, whereas we need to work a little more over $Q^1_{mb}(f,f)$. For this purpose, we will follow [Ale2, 6], setting the

**Definition 2.1.** *Let us write*

$$Q^1_{mb}(f,f)=Q^{11}_{mb}(f,f)+Q^{12}_{mb}(f,f),$$

$$Q^{11}_{mb}(f,f)=-a_{t,x}(v,D_v)(f)(v),$$

$$a_{t,x}(v,\xi)=\frac{1}{1+\int_v f}\int_{R^3} d\alpha f(t,x,\alpha)\,\tilde{\theta}(|\alpha-v|)\bar{\chi}(|\alpha-v)\wedge\xi|)|(\alpha-v)\wedge\Xi|^{\nu-1},$$

$$Q^{12}_{mb}(f,f)=\frac{1}{1+\int_v f}\int_{R^3} d\alpha f(t,x,\alpha+v)\chi(|\alpha\wedge D_v|)\,\tilde{\theta}(|\alpha|)|\alpha\wedge D_v|^{\nu-1}(f)(v),$$

*where* $\chi: R^+\to[0,1]$, *supported in* $[0,1]$, *smooth*, $=1$ *for* $t\le 2/3$ *and* $\bar{\chi}(t)=1-\chi(t)$.

*As in the above papers, one has*

**Lemma 2.4.** *With the above notations, one has*

$$|\partial_v^m \partial_\xi^n a_{t,x}(v,\xi)| \le C_{n,m}(1+|\xi|)^{(\nu-1)},$$

$$\|Q^{12}_{mb}(f,f)\|_{L^1_v} \le C\|f\|_{L^1_v}.$$

Now, we can introduce our definition of solutions for the $(MD)$ model, following earlier notations

**Theorem 2.1.** *Under the assumption* $(\mathscr{H})$, *there exists a weak solution* $f$ *of* $(MD)$, *that is* $f$ *satisfies*

$$\sup_{t\in(0,T)}\iint f[1+|v|^2+|x|^2|\log f|]dxdv \le C_T,$$

*the entropy dissipation rate*

$$\int_0^T\int_x\int_v\int_{v_1}\int_\omega \frac{B(.,.)}{1+\int_v f}\{f'f'_1-ff_1\}\log\left\{\frac{f'f'_1}{ff_1}\right\} \le C_T,$$

*and* $f$ *satisfies in distribution sense*

$$\begin{cases} \partial_t f(t,x,v)+v.\nabla_x f(t,x,v)=Q^{11}_{mb}(f,f)+Q^{12}_{mb}(f,f)+Q^2_{mb}(f,f), \\ f(0,x,v)=f_0(x,v). \end{cases}$$

*Furthermore, there is weak stability of such solutions.*

In view of the previous statements on the operator $Q_{mb}$, this result follows at once from [Lio3] and the approximated problem considered by [DiLi], see also related issues in Section III. However, we would like to end this Section by showing how one could deduce regularity results from the entropic dissipation rate estimate (stated by Theorem 2.1), without resting on [BoDe] as done by [Lio3]. In the following, we let $g=\sqrt{f}$, $\rho=\int f$, so that one starts from

$$\int_0^T\int_x\int_v\int_{v_1}\int_\omega \frac{B}{1+\rho}|g'g'_1-gg_1|^2 \le C_T. \tag{2.8}$$

Next, we proceed as follows (note that the decomposition in [Vil1] is slightly different). Write first

$$|g'g'_1-gg_1|^2=(g_1g'-gg'_1)^2+(g'^2-g^2)(g'^2_1-g^2_1). \tag{2.9}$$

Then, consider the estimate (2.8) involving the second term in (2.9). We claim that it is bounded. By the usual change of variables, this is equivalent to show that

$$\left|\int_0^T\int_x\int_v\int_{v_1}\int_\omega \frac{B}{1+\int_v f} f(f'_1-f_1)\right| \le C. \tag{2.10}$$

Indeed, letting $\mathscr{B}$ for the l.h.s of (2.10), one has first

$$\mathcal{B} \leq \int_{t,x} \sup_v \left| \int_{v_1} \int_\omega B(f_1' - f_1) \right|, \tag{2.11}$$

and as $\int_{v_1}\int_\omega B(f_1'-f_1)$ is nothing else than the operator $Q^2$ of [Ale6], it follows $\mathcal{B} \leq C\|f\|_{L^1_{t,x,v}}$. In view of this estimate, one deduces from (2.8)-(2.9)

$$0 \leq \int_0^T \int_x \int_v \int_{v_1} \int_\omega \frac{B}{1+\rho} |g_1 g' - g g_1'|^2 \leq C_T. \tag{2.12}$$

Next, we use the Carleman's representation as in [Ale] to get from this

$$\int_0^T \int_x \int_v \int_h \frac{dh}{|h|^{\nu+2}} \int_{E_{0,h}} \frac{\bar{\theta}(|\alpha|)}{1+\rho} \{g(\alpha+v-h)g(v-h) - g(\alpha+v)g(v)\}^2 \leq C. \tag{2.13}$$

Setting

$$j(z,\alpha) = g(\alpha+z)g(z), \tag{2.14}$$

and using the Parseval's relation with respect to the variable $v$, one gets

$$\int_0^T \int_x \int_h \frac{dh}{|h|^{\nu+2}} \int_{E_{0,h}} \frac{\bar{\theta}(|\alpha|)}{1+\rho} \int_k |\hat{j}^1(k,\alpha)|^2 |e^{-ih.k} - 1|^2 \leq C, \tag{2.15}$$

($\hat{j}^1$ denotes the F-transform w.r.t to the variable $z$) that is also

$$\int_0^T \int_x \int_{S^2_\omega} \int_{E_{0,\omega}} \frac{\bar{\theta}(|\alpha|)}{1+\rho} \int_k |\hat{j}^1(k,\alpha)|^2 |k.\omega|^{\nu-1} \leq C, \tag{2.16}$$

or, using previous notations

$$\int_0^T \int_x \int_\alpha \frac{\bar{\theta}(|\alpha|)}{1+\rho} \int_k |\hat{j}^1(k,\alpha)|^2 |S(\alpha).k|^{\nu-1} \leq C. \tag{2.17}$$

We claim that

$$\int_0^T \int_x \frac{1}{1+\rho} \int_k \left[ \int_\alpha \bar{\theta}(|\alpha|) |\hat{j}^1(k,\alpha)| |k|^{\frac{\nu-1}{2}} \right]^2 \leq C. \tag{2.18}$$

Indeed, letting $\mathcal{A}$ for the left hand side of (2.18), one has

$$\mathcal{A} = \int_0^T \int_x \frac{1}{1+\rho} \int_k \left[ \int_\alpha \bar{\theta}(|\alpha|) |\hat{j}^1(k,\alpha)| |S(\alpha).k|^{\frac{\nu-1}{2}} \cdot \frac{|k|^{\frac{\nu-1}{2}}}{|S(\alpha).k|^{\frac{\nu-1}{2}}} \right]^2,$$

which, using Cauchy-Scharwz inequality with respect to the variable $\alpha$ gives

$$\mathcal{A} \leq \int_0^T \int_x \frac{1}{1+\rho} \int_k \left\{ \int_\alpha \bar{\theta}(|\alpha|) |\hat{j}^1(k,\alpha)|^2 |S(\alpha).k|^{\nu-1} \right\} \times$$

$$\times \left\{ \int_\alpha \bar{\theta}(|\alpha|) \frac{|k|^{\nu-1}}{|S(\alpha).k|^{\nu-1}} \right\}. \tag{2.19}$$

But

$$\int_{\alpha}\bar{\theta}(|\alpha|)\frac{|k|^{\nu-1}}{|S(\alpha).k|^{\nu-1}}=\int_{\alpha}\bar{\theta}(|\alpha|)\frac{1}{|S(k).\alpha|^{\nu-1}}\leq C,$$

by assumptions on $\theta$ (note also that $0<\nu-1<2$). Therefore, it follows from (2.19) that

$$\mathcal{A}\leq C\int_0^T\int_x\frac{1}{1+\rho}\int_k\int_{\alpha}\bar{\theta}(|\alpha|)|\hat{f}^1(k,\alpha)|^2|S(\alpha).k|^{\nu-1}, \tag{2.20}$$

and the right hand side of (2.20) is bounded in view of (2.17), obtaining (2.18). From this, it follows

$$\int_0^T\int_x\frac{1}{1+\rho}\int_k|k|^{\nu-1}\Big|\int_{\alpha}\bar{\theta}(|\alpha|)\hat{f}^1(k,\alpha)\Big|^2\leq C. \tag{2.21}$$

Note that $|\hat{f}^1(k,\alpha)|$ is up to dilatation in $\alpha$ the modulus of the Wigner transform of $g$ (and thus bounded in $L^2_{k,\alpha}\cap L^{\infty}_{k,\alpha}$). *Finally, we obtain*

$$\frac{1}{\sqrt{1+\rho}}(\sqrt{f}*_v\tilde{\theta}).\sqrt{f}\in L^2((0,T)\times\mathbf{R}^3_x;\ H^{\frac{\nu-1}{2}}(\mathbf{R}^3_v)). \tag{2.22}$$

Note that this result improves on [Lio3]. Furthermore, one advantage is that this scheme of proof adapts to completely different collision operators. Let us note that we did not deduce regularity with respect to the variables $(x,v)$, see Section III for related results. However, it should be possible to get some results by looking to a renormalised form of the problem. Finally note that we asked for a somehow weaker formulation of solutions than possible, in that we can use instead the stronger notion of H-solutions as introduced by [Vil1]. This is done in the next Section for model (BD).

## 3. Problem (BD)

This Section is devoted to the Boltzmann-Dirac model. As previously, let us introduce the notations to be used herein.

We consider for $f=f(v)$, the (BD) collision operator acting on the variable $v$ as follows

$$Q_{bd}(f,f)(v)=\int_{\mathbf{R}^3_{v_1}}\int_{S^2_{\omega}}dv_1d\omega\{f'f'_1(1-f)(1-f_1)-ff_1(1-f')(1-f'_1)\}B(.,.). \tag{3.1}$$

We have used the classical notations of kinetic theory, and

$$B\Big(|v-v_1|,\Big|\Big(\frac{v-v_1}{|v-v_1|},\omega\Big)\Big|\Big)\equiv\theta(|v_1-v'|)\frac{|v_1-v'|^{\gamma+\nu}}{|v'-v|^{\nu}}, \tag{3.2}$$

where $\theta$ belongs $\mathcal{S}^+$, is null for small values, and

$$\gamma=\gamma(s)=\frac{s-5}{s-1},\quad \nu=\nu(s)=\frac{s+1}{s-1}. \tag{3.3}$$

Later on, we shall comment about possible weakenings on $\theta$, and in particuliar

allow for $\theta\equiv 1$. As in the former Section, the main advantage is to make easier some "classical" pdo analysis.
Among other issues, we want to solve the (BD) equation

$$(BD)\qquad \begin{cases} \partial_t f(t,x,v)+v.\nabla_x f(t,x,v)=Q_{bd}(f,f), \\ f(0,x,v)=f_0(x,v), \end{cases}$$

in whole phase space.
First, let us recall quickly the known facts on problem (BD) in the cutoff case [Dol]. To fix the ideas, let for $n\geq 1$, $\chi_n(t)=1$ for $t\leq n$, 0 elsewhere and

$$(3.4)\qquad B^n(.,.)=B(.,.)\chi_n\left(\frac{|v-v'|}{|v_1-v'|}\right).$$

Clearly, for any fixed $n\in\mathbf{N}$, it belongs to $L^1$. Then consider the following (cutoff type) problem

$$(BD^n)\qquad \begin{cases} \partial_t f(t,x,v)+v.\nabla_x f(t,x,v)=Q^n_{bd}(f,f), \\ f(0,x,v)=f_0(x,v), \end{cases}$$

where $Q^n_{bd}$ denotes the BD operator with the cross section $B^n$ of (3.4). Naturally $f=f^n$ ...
Dolbeault [Dol] has shown that under the following natural assumption on $f_0$

$$(3.5)\qquad 0\leq f_0\leq 1 \;\; a.e, \iint_{\mathbf{R}^6_{x,v}} f_0(x,v)\{1+|v|^2+|x|^2+|\log f_0|\}<\infty,$$

(of course, last part is superfluous), then there exists an unique solution $f^n$ to problem $(BD^n)$ satisfying

$$(3.6)\qquad f^n\in L^\infty(\mathbf{R}^+\times\mathbf{R}^3_x\times\mathbf{R}^3_v),\quad 0\leq f^n\leq 1 \;\; a.e,$$

with $f^n$ absolutely continuous with respect to $t$ and

$$(3.7)\qquad \begin{cases} f^n\in C^0(\mathbf{R}^+\,;\,L^1(\mathbf{R}^3_x\times\mathbf{R}^3_v)), \\ \iint_{\mathbf{R}^6_{x,v}} f^n(t,x,v)dxdv=\iint_{\mathbf{R}^6_{x,v}} f_0(x,v)dxdv, \quad \forall t\in\mathbf{R}^+, \end{cases}$$

$$(3.8)\qquad \begin{cases} (t,x,v)\to f^n(t,x,v)|v|^2\in C^0(\mathbf{R}^+\,;\,L^1(\mathbf{R}^3_x\times\mathbf{R}^3_v)), \\ \iint_{\mathbf{R}^6_{x,v}} f^n(t,x,v)|v|^2dxdv=\iint_{\mathbf{R}^6_{x,v}} f_0(x,v)|v|^2dxdv, \end{cases}$$

(3.9)
$$\begin{cases} (t,x,v)\to f^n(t,x,v)|x|^2\in C^0(\mathbf{R}^+\,;\,L^1(\mathbf{R}^3_x\times\mathbf{R}^3_v)),\\ \iint_{\mathbf{R}^6_{x,v}} f^n(t,x,v)|x-tv|^2dxdv=\iint_{\mathbf{R}^6_{x,v}} f_0(x,v)|x|^2dxdv, \end{cases}$$

With the notations

(3.10)
$$s(u)=u\log u+(1-u)\log(1-u),$$

$$e(f)(t,x,v)=\frac{1}{4}\int_{\mathbf{R}^3_{v_1}}\int_{S^2_\omega}dv_1d\omega\{f'f_1'(1-f)(1-f_1)-ff_1(1-f')(1-f_1')\}\times$$

(3.11)
$$\times\log\left\{\frac{f'f_1'(1-f)(1-f_1)}{ff_1(1-f')(1-f_1')}\right\}B^n(.,.),$$

one has also the following entropic type estimates

(3.12)
$$\iint_{\mathbf{R}^6_{x,v}}|f^n\log f^n|(t,x,v)dxdv\leq C+\iint_{\mathbf{R}^6_{x,v}}f_0(x,v)(|v|^2+|x|^2)dxdv,$$

(3.13)
$$s(f^n)\in L^\infty(\mathbf{R}^+\,;\,L^1(\mathbf{R}^3_x\times\mathbf{R}^3_v)),\quad e(f^n)\in L^1(\mathbf{R}^+\times\mathbf{R}^6_{x,v}),$$

with the estimate, for a.e positive $t$

(3.14)
$$\iint_{\mathbf{R}^6_{x,v}}s(f^n)(t,x,v)+\int_0^t\iint_{\mathbf{R}^6_{x,v}}e(f^n)(s,x,v)=\iint_{\mathbf{R}^6_{x,v}}s(f_0)(x,v).$$

The main point is to note, although it is important in the proofs by [Dol], that all these bounds do not involve the $L^1$ norm of $B^n$.
Now what can we do to get (suitable) weak solutions of the non cutoff problem (BD) ? One idea would be to pass to the limit as $n\to\infty$ directly in problem $(BD^n)$, in view of the above uniform bounds (3.6) to (3.14).
However, while this method is suitable for the homogeneous problems, see [Gou, Vil1], it breaks down for the non homogeneous ones, since we have to account for the products involving functions of the variable $x$. The method choosen hereafter will need some compactness and for this purpose, the entropic rate bound, that is (3. 11) with the non cutoff kernel, is essential as shown in [Lio3], see also[Ale3], in the pure Boltzmann case.
Therefore, instead of the problem $(BD^n)$, we introduce the following modification $(BD^n_m)$, $n$, $m\geq 1$

$(BD^n_m)$
$$\begin{cases} \partial_t f^n_m(t,x,v)+v.\nabla_x f^n_m(t,x,v)-\frac{1}{m}\Delta_v f^n_m=Q^n_{bd}(f^n_m,f^n_m),\\ f^n_m(0,x,v)=f_0(x,v). \end{cases}$$

First, we shall fix $m\geq 1$, pass to the limit as $n\to\infty$, then send $m\to\infty$. The problem $(BD^n_m)$ is one method for achieving the entropic dissipation bound, and note for instance the work of [Ale5] in the pure homogeneous Boltzmann case. Also note

that one could change $-\Delta_v$ by $(-\Delta_v)^\alpha$ with a suitable $\alpha$, in fact $\alpha \le \frac{1}{s-1}$, whose advantage is clear, in view of some computations [Ale2]. However, we do not insist on this point.
Next, following [Dol], one has

**Lemma 3.1.** *For $f_0$ satisfying (3.5), there exists an unique solution $f^n_m$ of problem $(BD^n_m)$, with the uniform bounds*

$$f^n_m \in L^\infty(\boldsymbol{R}^+ \times \boldsymbol{R}^3_x \times \boldsymbol{R}^3_v), \quad 0 \le f^n_m \le 1 \ a.e, \tag{3.15}$$

*with $f^n_m$ absolutely continuous with respect to $t$ and*

$$\begin{cases} f^n_m \in C^0(\boldsymbol{R}^+ ; L^1(\boldsymbol{R}^3_x \times \boldsymbol{R}^3_v)), \\ \iint_{\boldsymbol{R}^6_{x,v}} f^n_m(t,x,v)\,dxdv = \iint_{\mathbf{R}^6_{x,v}} f_0(x,v)\,dxdv, \quad \forall t \in \boldsymbol{R}^+, \end{cases} \tag{3.16}$$

$$\begin{cases} (t,x,v) \to f^n_m(t,x,v)|v|^2 \in C^0(\boldsymbol{R}^+ ; L^1(\boldsymbol{R}^3_x \times \boldsymbol{R}^3_v)), \\ \iint_{\boldsymbol{R}^6_{x,v}} f^n_m(t,x,v)|v|^2 dxdv = \iint_{\mathbf{R}^6_{x,v}} f_0(x,v)|v|^2 dxdv, \end{cases} \tag{3.17}$$

$$\begin{cases} (t,x,v) \to f^n_m(t,x,v)|x|^2 \in C^0(\boldsymbol{R}^+ ; L^1(\boldsymbol{R}^3_x \times \boldsymbol{R}^3_v)), \\ \iint_{\boldsymbol{R}^6_{x,v}} f^n_m(t,x,v)|x-tv|^2 dxdv = \iint_{\boldsymbol{R}^6_{x,v}} f_0(x,v)|x|^2 dxdv. \end{cases} \tag{3.18}$$

$$\iint_{\boldsymbol{R}^6_{x,v}} |f^n_m \log f^n_m|(t,x,v)\,dxdv \le C + \iint_{\mathbf{R}^6_{x,v}} f_0(x,v)(|v|^2 + |x|^2)\,dxdv, \tag{3.19}$$

*and for almost every $t \ge 0$*

$$\frac{1}{m}\int_0^t \iint_{\boldsymbol{R}^6_{x,v}} |\nabla_v \sqrt{f}|^2 dxdv +$$

$$+ \int_0^t \iint_{\boldsymbol{R}^6_{x,v}} \int_{\boldsymbol{R}^3_{v_1}} \int_{S^2_\omega} dv_1 d\omega \{f'f'_1(1-f)(1-f_1) - ff_1(1-f')(1-f'_1)\} \times$$

$$\times \log\left\{\frac{f'f'_1(1-f)(1-f_1)}{ff_1(1-f')(1-f'_1)}\right\} B^n(.,.) \le C, \tag{3.20}$$

*where we set $f = f^n_m$ to simplify.*

Note that from the first part of (3.20) and usual arguments, one has, at least

$$\frac{1}{m}\int_0^t\iint_{\mathbf{R}^6_{x,v}}|\nabla_v\beta(f^n_m)|^2dxdv\leq C, \tag{3.21}$$

for all $\beta\in C^1(\mathbf{R})$.
Now, as a preliminary step, we want to send $n$ to $\infty$, while keeping $m$ fixed. Note then, we fall into the non cutoff BD operator and therefore, we must first define suitable weak solutions for problem $(BD_m)$. One way is to proceed as in [Ark2, Gou, Vil1].

**Definition 3.1.** *We say $g$ is an H-solution of problem $(BD_m)$ if $g$ satisfies* (3.15) *to* (3.20), *without the index n and thus in the non cutoff case, with B such as in the beginning of this Section, and if g satisfies in $\mathcal{D}'$ sense*

$$(BD_m)\qquad \begin{cases} \partial_t g(t,x,v)+v.\nabla_x g(t,x,v)-\frac{1}{m}\Delta_v g=Q_{bd}(g,g),\\ g(0,x,v)=f_0(x,v).\end{cases}$$

Let us note that $Q_{bd}(g,g)$, with $g$ such as (3.15) to (3.20), is well defined. Indeed

**Lemma 3.2.** *Assume that g satisfies* (3.15) *to* (3.20), *where $Q_{bd}$ is given by* (3.1), (3.2) *and* (3.3). *Then one has*

$$Q_{bd}(g,g)\in L^2(0,T)\times \boldsymbol{R}^3_x\,;\,H^{-\frac{\nu-1}{2}}(\boldsymbol{R}^3_v)),$$

*and more precisely*

$$|<Q_{bd}(g,g)\,;\,\phi>|\leq C\|\phi\|_{L^2((0,T)\times\boldsymbol{R}^3_x\,;\,H^{\frac{\nu-1}{2}}(\boldsymbol{R}^3_v))},$$

*with C a constant only depending on the above uniform bounds, and not on n.*

*Proof.* Using the above entropic estimate, note that $g$ satisfies (since $g$ is bouded)

$$\int_0^T\iint_{\mathbf{R}^6_{x,v}}\int_{\mathbf{R}^3_{v_1}}\int_{S^2_\omega}dv_1d\omega\{g'g'_1(1-g)(1-g_1)-gg_1(1-g')(1-g'_1)\}^2B(.,.)\leq C. \tag{3.22}$$

Next, for smooth $\phi$

$$<Q_{bd}(g,g)\,;\,\phi>=-\int_0^T\iint_{\mathbf{R}^6_{x,v}}\int_{\mathbf{R}^3_v}\int_{S^2_\omega}dv_1d\omega\{g'g'_1(1-g)(1-g_1)-$$

$$-gg_1(1-g')(1-g'_1)\}B(.,.)\{\phi'-\phi\}dtdxdv.$$

We have used the classical symmetries associated with the operator $Q_{bd}$. Therefore, using (3.22), we find

$$|<Q_{bd}(g,g)\,;\,\phi>|\leq C\left\{\int_0^T\iint_{\mathbf{R}^6_{x,v}}\int_{\mathbf{R}^6_{v_1}}\int_{S^2_\omega}B(.,.)|\phi'-\phi|^2\right\}^{\frac{1}{2}}. \tag{3.23}$$

Using the same computations as in [Ale2], one may write

$$\{.\}=\int_0^T\iint_{\mathbf{R}^6_{x,v}}\int_{\mathbf{R}^3_h}\frac{2dh}{|h|^{\nu+2}}\int_{E_{0,h}}\{\phi(x,v-h)-\phi(x,v)\}^2\,\bar{\theta}(|\alpha|)d\alpha$$

$$\leq C\int_0^T\iint_{\mathbf{R}^6_{x,v}}\int_{\mathbf{R}^3_h}\frac{2dh}{|h|^{\nu+2}}\{\phi(x,v-h)-\phi(x,v)\}^2 dvdxdt$$

$$\leq C\int_0^T\iint_{\mathbf{R}^3_x}\{|\hat{\phi}^2(x,\xi)|^2|\xi|^{\nu-1}d\xi\},$$

by classical results, see for instance [Ste1, 2] and as $0<\nu-1<2$. Therefore, we have shown

$$|<Q_{bd}(g,g)\,;\,\phi>|\leq C\|\phi\|_{L^2((0,T)\times\mathbf{R}^3_x;\,H^{\frac{\nu-1}{2}}(\mathbf{R}^3_v))}. \tag{3.24}$$

Note that $Q_{bd}$ is also well defined on $C^2$ functions, with suitable decay estimates. Now, let us turn to the sequence $f^n_m$, $m$ fixed, of solutions of problem $(BD^n_m)$ as given by Lemma 3.1, that is

$$(BD^n_m)\qquad \begin{cases}\partial_t f^n_m(t,x,v)+v.\nabla_x f^n_m(t,x,v)-\dfrac{1}{m}\Delta_v f^n_m=Q^n_{bd}(f^n_m,f^n_m),\\ f^n_m(0,x,v)=f_0(x,v).\end{cases}$$

In view of Lemma 3.2 and all previous estimates, one has the following uniform bound, with respect to $n$

$$|<Q^n_{bd}(f^n_m,f^n_m)\,;\,\phi>|\leq C\|\phi\|_{L^2((0,T)\times\mathbf{R}^3_x;\,H^{\frac{\nu-1}{2}}(\mathbf{R}^3_v))}. \tag{3.25}$$

Recalling that $m$ is fixed for the moment, and using (3.20) and [Lio2, 3], this is enough to conclude that the sequence $\{f^n_m\}_n$ is strongly compact in $L^p((0,T)\times\mathbf{R}^6_{x,v})$, for any $1\leq p<\infty$, and we let $f_m$ be a limit point.
Obviously, it satisfies estimates (3.15) to (3.20), with $B^n$ turned to $B$ as results from the above strong convergence and Fatou's lemma.
In conclusion, we prove

**Theorem 3.1.** *Under the assumption* (3.5) *on* $f_0$, *there exists an H-solution* $f_m$ *of problem* $(BD_m)$ *that is satisfying in weak sense*

$$(BD_m)\qquad \begin{cases}\partial_t f_m(t,x,v)+v.\nabla_x f_m(t,x,v)-\dfrac{1}{m}\Delta_v f_m=Q_{bd}(f_m,f_m),\\ f_m(0,x,v)=f_0(x,v),\end{cases}$$

*see Definition* 3.1 *and Lemma* 3.2, *and the following bounds*

$$f_m\in L^\infty(\boldsymbol{R}^+\times\boldsymbol{R}^3_x\times\boldsymbol{R}^3_v),\quad 0\leq f_m\leq 1\ a.e, \tag{3.26}$$

$$(3.26)' \qquad \begin{cases} f_m \in L^\infty(\boldsymbol{R}^+ ; L^1(\boldsymbol{R}^3_x \times \boldsymbol{R}^3_v)), \\ \iint_{\boldsymbol{R}^6_{x,v}} f_m(t,x,v)\,dxdv = \iint_{\boldsymbol{R}^6_{x,v}} f_0(x,v)\,dxdv, \quad \forall\, t \in \boldsymbol{R}^+, \end{cases}$$

$$(3.27) \qquad \begin{cases} (t,x,v) \to f_m(t,x,v)|v|^2 \in L^\infty(\boldsymbol{R}^+ ; L^1(\boldsymbol{R}^3_x \times \boldsymbol{R}^3_v)), \\ \iint_{\boldsymbol{R}^6_{x,v}} f_m(t,x,v)|v|^2 dxdv \le \iint_{\boldsymbol{R}^6_{x,v}} f_0(x,v)|v|^2 dxdv, \end{cases}$$

$$(3.28) \qquad \begin{cases} (t,x,v) \to f_m(t,x,v)|x|^2 \in L^\infty(\boldsymbol{R}^+ ; L^1(\boldsymbol{R}^3_x \times \boldsymbol{R}^3_v)), \\ \iint_{\boldsymbol{R}^6_{x,v}} f_m(t,x,v)|x-tv|^2 dxdv \le \iint_{\boldsymbol{R}^6_{x,v}} f_0(x,v)|x|^2 dxdv. \end{cases}$$

$$(3.29) \qquad \iint_{\boldsymbol{R}^6_{x,v}} |f_m \log f_m|(t,x,v)\,dxdv \le C + \iint_{\boldsymbol{R}^6_{x,v}} f_0(x,v)(|v|^2+|x|^2)\,dxdv,$$

*and for almost every* $t \ge 0$

$$\frac{1}{m}\int_0^t \iint_{\boldsymbol{R}^6_{x,v}} |\nabla_v \sqrt{f_m}|^2 dxdv +$$

$$+\int_0^t \iint_{\boldsymbol{R}^6_{x,v}} \int_{\boldsymbol{R}^3_{v_1}} \int_{S^2_\omega} dv_1 d\omega \{f'f'_1(1-f)(1-f_1) - ff_1(1-f')(1-f'_1)\} \times$$

$$(3.30) \qquad \times \log\left\{\frac{f'f'_1(1-f)(1-f_1)}{ff_1(1-f')(1-f'_1)}\right\} B(.,.) \le C.$$

*where we set* $f = f_m$ *to simplify.*

Having at our disposal a sequence $\{f_m\}_m$ of H solutions of problem $(BD_m)$, (possibly non unique), we want to pass to the limit as $m \to \infty$, so that we loose the compactifying properties of the viscosity term $-\frac{1}{m}\Delta_v$.

However, as shown in the pure Boltzmann case, see [Ale3, Lio3], we will see that the entropy dissipation rate bound, that is the uniform estimate in $m$ of the second term of the left hand side of (3.30), leads to some regularity and compactness. The following are modelled on the steps of Section II.

Omitting the index $m$ in the remainder, we start from the estimate ($T>0$ fixed)

$$(3.31) \qquad \int_0^T \iint_{\mathbf{R}^6_{x,v}} \int_{\mathbf{R}^3_{v_1}} \int_{S^2_\omega} B\{f'f'_1(1-f)(1-f_1) - ff_1(1-f')(1-f'_1)\}^2 \le C,$$

that we write as follows

(3.32) $$\int_0^T\iint_{\mathbf{R}^6_{x,v}}\int_{\mathbf{R}^3_{v_1}}\int_{S^2_\omega} B\{f'_1(1-f_1)(f'-f)+f(1-f')(f'_1-f_1)\}^2\leq C.$$

Using the (classical) change of variables $(v,v_1)\rightarrow(v',v'_1)$, one gets also

(3.33) $$\int_0^T\iint_{\mathbf{R}^6_{x,v}}\int_{\mathbf{R}^3_{v_1}}\int_{S^2_\omega} B\{f'_1(1-f'_1)(f-f')+f'(1-f)(f_1-f'_1)\}^2\leq C.$$

From (3.32) and (3.33), we get finally the following estimate

$$\int_0^T\iint_{\mathbf{R}^6_{x,v}}\int_{\mathbf{R}^3_{v_1}}\int_{S^2_\omega} B|\{(1-f'_1)f_1+(1-f_1)f'_1\}(f'-f)+$$

(3.34) $$+\{(1-f')f+(1-f)f'\}(f'_1-f_1)|^2\leq C.$$

Next, we make the change of variables due to Carleman [Car], and provided by [Ale1, 2, Wen] to get, where we still use the notations $f'$, $f'_1$, $f$ and $f_1$ this time for $f(v-h)$, $f(\alpha+v)$, $f(v)$ and $f(\alpha+v-h)$ respectively,

$$\int_0^T\iint_{\mathbf{R}^6_{x,v}}\int_{\mathbf{R}^3_h}\frac{2dh}{|h|^{\nu+2}}\int_{E_{0,h}}\bar{\theta}(|\alpha|)|\{(1-f'_1)f_1+(1-f_1)f'_1\}(f'-f)+$$

(3.35) $$+\{(1-f')f+(1-f)f'\}(f'_1-f_1)|^2 d\alpha dv dx dt\leq C.$$

*We keep the same notations as in our previous papers, in particuliar $E_{0,h}$ denotes the hyperplane through* 0 *and orthogonal to h.*

In view of the properties of function $\theta$, we can use Jensen's inequality with respect to the variable $\alpha$ to get

$$\int_0^T\iint_{\mathbf{R}^6_{x,v}}\int_{\mathbf{R}^3_h}\frac{1}{|h|^{\nu+2}}\left|\left[\int_{E_{0,h}}\bar{\theta}(|\alpha|)|\{(1-f'_1)f_1+(1-f_1)f'_1\}\right](f'-f)+\right.$$

(3.36) $$\left.+\{(1-f')f+(1-f)f'\}\left[\int_{E_{0,h}}\bar{\theta}(|\alpha|)(f'_1-f_1)\right]\right|^2 dh dv dx dt\leq C.$$

Recall the notations, and in particuliar that $f$ and $f'$ do not depend on $\alpha$. Next, we want to show the estimate

(3.37)
$$\int_0^T\iint_{\mathbf{R}^6_{x,v}}\int_{\mathbf{R}^3_h}\frac{1}{|h|^{\nu+2}}\left[\int_{E_{0,h}}\bar{\theta}(|\alpha|)\{(1-f'_1)f_1+(1-f_1)f'_1\}\right]^2(f'-f)^2 dh dv dx dt\leq C.$$

In view of (3.36), it is enough to show

(3.38)
$$\int_0^T\iint_{\mathbf{R}^6_{x,v}}\int_{\mathbf{R}^3_h}\frac{1}{|h|^{\nu+2}}\{(1-f')f+(1-f)f'\}^2\left[\int_{E_{0,h}}\bar{\theta}(|\alpha|)(f'_1-f_1)\right]^2 dv dx dt\leq C.$$

The left hand side of (3.38), denoted by B hereafter, reads as

$$B \leq C\int_0^T\int_{\mathbf{R}_x^3} \sup\{(1-f)^2 f^2\}\int_{\mathbf{R}_v^3}\int_{\mathbf{R}_h^3}\frac{1}{|h|^{\nu+2}}\times$$

$$\times\Big|\int_{E_{0,h}}\bar{\theta}(|\alpha|)\{f(\alpha+v)-f(\alpha+v-h)\}d\alpha\Big|^2.$$

If one denotes

$$\beta\Big(z,\frac{h}{|h|}\Big)\equiv\int_{E_{0,h}}f(\alpha+z)\,\bar{\theta}(|\alpha|)d\alpha,$$

and by $\widehat{\beta}^1$ its Fourier transform with respect to the first variable, we are led to

$$B \leq C\int_0^T\int_{\mathbf{R}_x^3} \sup\{(1-f)^2 f^2\}\int_{\mathbf{R}_v^3}\int_{\mathbf{R}_h^3}\frac{1}{|h|^{\nu+2}}\Big\{\beta\Big(v,\frac{h}{|h|}\Big)-\beta\Big(v-h,\frac{h}{|h|}\Big)\Big\}^2$$

which, using Parseval's relation, leads to

$$B \leq C\int_0^T\int_{\mathbf{R}_x^3} \sup\{(1-f)^2 f^2\}\int_{\mathbf{R}_h^3}\frac{1}{|h|^{\nu+2}}\int_{\mathbf{R}_k^3}\Big|\widehat{\beta}^1\Big(k,\frac{h}{|h|}\Big)\Big|^2\{e^{ih.k}-1\}^2.$$

Shifting $h=r\omega$ in polar coordinates, one gets also

$$B \leq C\int_0^T\int_{\mathbf{R}_x^3} \sup\{(1-f)^2 f^2\}\int_{S_\omega^2}\int_{\mathbf{R}_k^3}|\widehat{\beta}^1(k,\omega)|^2|k.\omega|^{\nu-1}.$$

Now, recalling that $S(\omega)$ denotes the orthogonal projection over $E_{0,h}$ one obtains

$$\widehat{\beta}^1(k,\omega)=\hat{f}(k)\,\hat{\theta}(|S(\omega).k|),$$

and therefore

$$B \leq C\int_0^T\int_{\mathbf{R}_x^3} \sup\{(1-f)^2 f^2\}\int_{\mathbf{R}_k^3}|\hat{f}(k)|^2 \leq C.$$

As the estimate (3.38) holds true, we arrive at the final estimate (3.37). Next, since

$$(1-f_1')f_1+(1-f_1)f_1'\geq(1-f_1')f_1',$$

one gets

$$\int_0^T\iint_{\mathbf{R}_{x,v}^6}\int_{\mathbf{R}_h^3}\frac{1}{|h|^{\nu+2}}\Big[\int_{E_{0,h}}\bar{\theta}(|\alpha|)(1-f(\alpha+v))f(\alpha+v)\Big]^2\times$$

$$\times|f(v-h)-f(v)|^2 dv dx dt \leq C. \tag{3.39}$$

We want to work this estimate. For this purpose, let us set

$$g(z)=\{1-f(z)\}f(z), \tag{3.40}$$

so that the term inside estimate (3.39) is $\{g(\alpha+v)f(v-h)-f(v)g(\alpha+v)\}$, that we write as

(3.41)
$$\{.\}=g(\alpha+v-h)f(v-h)-f(v)g(\alpha+v)+f(v-h)\{g(\alpha+v)-g(\alpha+v-h)\}.$$

The second term of (3.41) may be estimated as above (for $B$), so that we are led to the estimate

$$\int_0^T\int_{\mathbf{R}_x^3}\int_{\mathbf{R}_h^3}\frac{1}{|h|^{\nu+2}}\int_{\mathbf{R}_v^3}\left|\int_{E_{0,h}}\bar{\theta}(|\alpha|)\{g(\alpha+v-h)f(v-h)-\right.$$

(3.42)
$$\left.-f(v)g(\alpha+v)\}d\alpha\right|^2 dxdt\leq C.$$

Setting

(3.43)
$$j\left(z,\frac{h}{|h|}\right)=\left\{\int_{E_{0,h}}\bar{\theta}(|\alpha|)g(\alpha+z)\right\}f(z),$$

the estimate (3.42) reads as

(3.44)
$$\int_0^T\int_{\mathbf{R}_x^3}\int_{\mathbf{R}_h^3}\frac{1}{|h|^{\nu+2}}\int_{\mathbf{R}_v^3}\left|j\left(v-h,\frac{h}{|h|}\right)-j\left(v,\frac{h}{|h|}\right)\right|^2 dvdxdt\leq C,$$

which again using Parseval's relation leads to

(3.45)
$$\int_0^T\int_{\mathbf{R}_x^3}\int_{\mathbf{R}_h^3}\frac{1}{|h|^{\nu+2}}\int_{\mathbf{R}_k^3}\left|\hat{j}^1\left(k,\frac{h}{|h|}\right)\right|^2|e^{-ih.k}-1|^2 dkdxdt\leq C.$$

Using again polar coordinates in $h=r\omega$, one gets

(3.46)
$$\int_0^T\int_{\mathbf{R}_x^3}\int_{\mathbf{R}_k^3}\int_{S_\omega^2}|\hat{j}^1(k,\omega)|^2\int_0^\infty dr\frac{|e^{-ir\omega.k}-1|^2}{r^\nu}drdkdxdt\leq C,$$

and from previous computations, we finally have

(3.47)
$$\int_0^T\int_{\mathbf{R}_x^3}\int_{\mathbf{R}_k^3}\int_{S_\omega^2}|\hat{j}^1(k,\omega)|^2|k.\omega|^{\nu-1}dxdt\leq C.$$

Using again ... Jensen's inequality in variable $\omega$ leads to

(3.48)
$$\int_0^T\int_{\mathbf{R}_x^3}\int_{\mathbf{R}_k^3}\left|\int_{S_\omega^2}\hat{j}^1(k,\omega)|k.\omega|^{\frac{\nu-1}{2}}d\omega\right|^2 dxdt\leq C.$$

Let us compute the term inside the term $|.|$. We obtain

$$\int_{S_\omega^2}\hat{j}^1(k,\omega)|k.\omega|^{\frac{\nu-1}{2}}d\omega=$$

$$=\int_{S_\omega^2}\int_{\mathbf{R}_z^3}\left\{\int_{E_{0,\omega}}\bar{\theta}(|\alpha|)g(\alpha+z)\right\}f(z)e^{-ik.z}|k.\omega|^{\frac{\nu-1}{2}}$$

$$=\int_{\mathbf{R}_z^3}\int_{\mathbf{R}_\alpha^3}\bar{\theta}(|\alpha|)g(\alpha+z)f(z)e^{-ik.z}\left\{\int_{S_\omega^2,\omega.\alpha=0}|k.\omega|^{\frac{\nu-1}{2}}d\omega\right\}$$

$$=C_s\int_{\mathbf{R}_z^3}\int_{\mathbf{R}_\alpha^3}\bar{\theta}(|\alpha|)g(\alpha+z)f(z)e^{-ik.z}|S(\alpha).k|^{\frac{\nu-1}{2}}$$

$$=C_s\int_{\mathbf{R}_z^3}\int_{\mathbf{R}_\alpha^3}\bar{\theta}(|\alpha|)g(\alpha+z)f(z)e^{-ik.z}|\alpha\wedge k|^{\frac{\nu-1}{2}}.$$

With this expression, turning to (3.48) and using Parseval's inequality for the variable $k$, we find

$$\mathcal{A}_{t,x}^*(f)(v)\in L^2((0,T)\times\mathbf{R}_x^3\times\mathbf{R}_v^3), \tag{3.48$'$}$$

where $\mathcal{A}_{t,x}^*$ is the operator given by (up to unimportant constants)

$$\mathcal{A}_{t,x}^*(f)(v)\equiv\int_{\mathbf{R}_z^3}\int_{\mathbf{R}_k^3}\left\{\int_{\mathbf{R}_\alpha^3}d\alpha\,\bar{\theta}(|\alpha|)g(\alpha+z)|\alpha\wedge k|^{\frac{\nu-1}{2}}\right\}f(z)e^{-i(z-v).k}. \tag{3.49}$$

$\mathcal{A}_{t,x}^*$ is nothing else than the adjoint of the operator

$$\mathcal{A}_{t,x}(f)(v)\equiv\int_{\mathbf{R}_\alpha^3}\bar{\theta}(|\alpha|)g(\alpha+v)|\alpha\wedge D_v|^{\frac{\nu-1}{2}}(f)(v), \tag{3.50}$$

see for instance our previous papers.
If there were no time and space dependence, one could deduce, exactly as in [Ale1], and for non trivial $f$, that the symbol of $\mathcal{A}$ satisfies, for all $v\in\Omega$, $\Omega$ being any bounded open set of $\mathbf{R}_v^3$

$$\int_{\mathbf{R}_\alpha^3}\tilde{\theta}(|\alpha|)g(\alpha+v)|\alpha\wedge k|^{\frac{\nu-1}{2}}\geq C_K|k|^{\frac{\nu-1}{2}}\int_{\mathbf{R}_\alpha^3}\bar{\bar{\theta}}(|\alpha|)g(\alpha), \tag{3.51}$$

and therefore $f\in H_{loc}^{\frac{\nu-1}{2}}(\mathbf{R}_v^3)$. But unfortunatly (!) and as in Section II, we must account for these variables, so at this point, we just state the result obtained (which could be interesting by itself)

**Lemma 3.3.** *If $f\geq 0$ satisfies the above bounds, and in particuliar the entropic estimate* (3.31), *then*

$$\mathcal{A}_{t,x}^*(f)(v)\in L^2((0,T)\times\boldsymbol{R}_{x,v}^6), \tag{3.52}$$

*where the operators $\mathcal{A}_{t,x}^*$ and $\mathcal{A}_{t,x}$ are given by* (3.49) *and* (3.50) *respectively, with* $g(\alpha)\equiv(1-f(\alpha))f(\alpha)$.

**Remark 3.1.** *In the above computations, there is some kind of Winger transform involved therein.*

**Remark 3.2.** *As in any Sections of the paper, the cutoff* (*in velocity function*) *$\theta$, or $\bar{\theta}$, .... reduces the computations.*
*Of course, it is likely that the same results will hold true in the general case, see Section I, by putting enough moments in the entropy dissipation rate bound, so as to use Jensen's inequality.*

*For instance, we have used the fact that the measure* $\bar{\theta}(|\alpha|)$ *has finite mass (over* $E_{0,h}$*). In the general case where* $\theta(|\alpha|)=1$*, see also the examples in [Ale1], then* $\bar{\theta}(|\alpha|)=|\alpha|^{\nu+\gamma}$*, and the entropy dissipation rate bound reads as, forgetting variables* $(t,x)$ *(with obvious notations)*

$$\int_{\mathbf{R}_v^3}\int_{\mathbf{R}_h^3}\frac{2dh}{|h|^{\nu+2}}\int_{E_{0,h}} d\alpha|\alpha|^{\nu+\gamma}...dv\leq C. \tag{3.53}$$

*Recall* $\nu+\gamma>0$*. Next, letting* $\delta>1$*, the use of Jensen's inequality yields*

$$\int_{\mathbf{R}_v^3}\int_{\mathbf{R}_h^3}\frac{2dh}{|h|^{\nu+2}}\Big|\int_{E_{0,h}} d\alpha|\alpha|^{\frac{\nu+\gamma}{2}}\frac{1}{(1+|\alpha|^2)^{\frac{\delta}{2}}}..\Big|^2 dv\leq C, \tag{3.54}$$

*and one may proceed as earlier. In fact, it is even simplier from* (3.53) *to note that it is bigger than our estimate with* $\theta$*.*

As said previously, the uneasiness is to keep track of the variables $(t,x)$.
Recall that our main intention is to prove the compactness of the sequence $\{f_m\}$ of H-solutions of problem $(BD_m)$. Although it should be possible to adapt the arguments of [Lio3], note that we have to work out the computations from [BoDe] in the Boltzmann Dirac case. Instead, we provide a direct method, following exactly Section II, which I believe will prove usefull for different collision operators. At this point, let us say that we have'nt looked for optimal estimates below ....
Starting from (3.47), let us first deal *with the case* $0<\nu-1<1$ *that is* $s>3$.
We claim that

$$\int_0^T\int_x\int_k\Big[\int_{S_\omega^2}|\hat{j}^1(k,\omega)||k|^{\frac{\nu-1}{2}}\Big]^2\leq C. \tag{3.55}$$

Indeed, letting $\mathscr{C}$ for the left hand side of (3.55), one has

$$\mathscr{C}=\int_0^T\int_x\int_k\Big[\int_{S_\omega^2}|\hat{j}^1(k,\omega)||k.\omega|^{\frac{\nu-1}{2}}\frac{|k|^{\frac{\nu-1}{2}}}{|k.\omega|^{\frac{\nu-1}{2}}}\Big]^2\leq C, \tag{3.56}$$

and by Cauchy-Scharwz inequality in $\omega$

$$C\leq\int_0^T\int_x\int_k\Big[\int_{S_\omega^2}|\hat{j}^1(k,\omega)|^2|k.\omega|^{\nu-1}\Big]\Big[\int_{S_\omega^2}\frac{1}{|\frac{k}{|k|}\cdot\omega|^{\nu-1}}\Big].$$

Since $\omega\to\frac{1}{|\frac{k}{|k|}\cdot\omega|^{\nu-1}}$ is integrable over $S^2$, we deduce that $\mathscr{C}$ is bounded in view of (3.47) and thus (3.55) is proven.
Next, from (3.55), we use again Jensen's inequality in $\omega$ (as for (3.47) to (3.48)), and finally we obtain

**Lemma 3.4.** *If* $f\geq 0$ *satisfies the above bounds, and in particuliar the entropic estimate* (3.31), *with the assumption* $s>3(0<\nu-1<1)$ *then*

$$\left\{\int_{R^3_\alpha} d\alpha\,\bar{\theta}(|\alpha|)g(\alpha+v)\right\}f(v)\in L^2((0,T)\times R^3_x\,;\,H^{\frac{\nu-1}{2}}(R^3_v)).$$

*In the case* $2<s\leq 3$, note that one has (3.47) with the weight $|k.\omega|^{\nu-1}$ changed to $|k.\omega|^{\nu-2}$ ... , and therefore, proceeding as above, we get

**Lemma 3.5.** *If* $f\geq 0$ *satisfies the above bounds, and in particuliar the entropic estimate* (3.31), *with the assumption* $2<s\leq 3 (0<\nu-2<1)$ *then*

$$\left\{\int_{R^3_\alpha} d\alpha\,\bar{\theta}(|\alpha|)g(\alpha+v)\right\}f(v)\in L^2((0,T)\times R^3_x\,;\,H^{\frac{\nu-1}{2}}(R^3_v)).$$

In view of [Lio2, 3], the Lemmas 3.4 and 3.5 are enough to conclude that the sequence $\{f_m\}$ of H-solutions of problem $(BD_m)$ is strongly compact in any $L^p((0,T)\times R^6_{x,v})$, $1\leq p<\infty$.
We finally obtain the following existence theorem for the (initial !) problem $(BD)$. Note we can pass to the limit in $Q_{bd}$ by "symmetrising" classically the inner integral.

**Theorem 3.2.** *Under the assumption* (3.5) *on* $f_0$, *there exists an H-solution* $f$ *of problem* $(BD)$, *that is satisfying in weak sense*

$$(BD)\qquad \begin{cases} \partial_t f+v.\nabla_x f=Q_{bd}(f,f),\\ f(0,x,v)=f_0, \end{cases}$$

*see also Definition* 3.1 *and Lemma* 3.2, *and the following bounds*

$$0\leq f\leq 1 \ \ a.e,\ f\in L^\infty(R^+\,;\,L^1(R^3_x\times R^3_v)),$$

$$\iint_{R^6_{x,v}} f(t,x,v)dxdv=\iint_{R^6_{x,v}} f_0(x,v)dxdv,\quad \forall t\in R^+, \tag{3.57}$$

$$\iint_{R^6_{x,v}} f(t,x,v)dxdv\{|v|^2+|x-vt|^2\}\leq\iint_{R^6_{x,v}} f_0(x,v)\{|x|^2+|v|^2dxdv, \tag{3.58}$$

$$\iint_{R^6_{x,v}} |f\log f|(t,x,v)dxdv\leq C+\iint_{R^6_{x,v}} f_0(x,v)(|v|^2+|x|^2)dxdv, \tag{3.59}$$

$$\int_0^t\iint_{R^6_{x,v}}\int_{R^3_{v_1}}\int_{S^2_\omega} dv_1 d\omega\{f'f'_1(1-f)(1-f_1)-ff_1(1-f')(1-f'_1)\}\times$$

$$\times\log\left\{\frac{f'f'_1(1-f)(1-f_1)}{ff_1(1-f')(1-f'_1)}\right\}B(.,.)\leq C. \tag{3.60}$$

*It also satisfies the conclusions of Lemma* 3.4 *and Lemma* 3.5.

Having provided an entropic solution satisfying some "regularity" in the variable $v$,

we want to bootstrap this information on all variables $(t,x,v)$, at least in $(x,v)$. One instance of this is the compactness of sequences of entropic solutions, as shown by P.L. Lions [Lio2, 3]. Another one is the fact that the velocity averages of $f$ has some regularity in $(t,x)$. Let us show how we can use this. In the sequel, we will simplify our study by assuming that $s>3$, so that one may use Lemma 3.4.
In view of Lemma 3.2, using [DiLi2, DiLiMe], one has for all $\psi\in\mathcal{D}(\mathbf{R}_v^3)$

$$\int_{\mathbf{R}_v^3} f(t,x,v)\psi(v)dv\in H^{\frac{1}{1+\frac{\nu-1}{2}}\frac{1}{2}}((0,T)\times\mathbf{R}_x^3). \tag{3.61}$$

Therefore, in view of the above bounds, one has also

$$\rho*\left\{\int_{\mathbf{R}_\alpha^3}\bar{\bar{\theta}}(|\alpha|)(1-f(\alpha))f(\alpha))d\alpha\right\}\int_{\mathbf{R}_v^3} f(t,x,v)\psi(v)dv\in H^{\frac{1}{1+\frac{\nu-1}{2}}\frac{1}{2}}((0,T)\times\mathbf{R}_x^3), \tag{3.62}$$

where $\rho$ is a smooth function of the variables $(t,x)$. Next, let (note that we use the same letter $g$ ...)

$$g(t,x,v)\equiv\rho*\left\{\int_{\mathbf{R}_\alpha^3}\bar{\bar{\theta}}(|\alpha|)(1-f(\alpha))f(\alpha)d\alpha\right\}\phi(v)f(t,x,v)\psi(v), \tag{3.63}$$

where $\phi\in\mathcal{D}(\mathbf{R}_v^3)$, omitting cutoff in the variables $(t,x)$ ... , and if we denote $(\tau,k,\xi)$ for the dual variables of $(t,x,v)$ respectively, then from [DiLi2, DiLiMe], the information (3.62) can be translated into, where $\hat{g}$ denotes the Fourier transform in all variables

$$(1+|\tau|^2+|k|^2)^{\frac{1}{1+\frac{\nu-1}{2}}\frac{1}{2}}\hat{g}\in L^\infty(\mathbf{R}_\xi^3\,;\,L^2(\mathbf{R}_\tau\times\mathbf{R}_k^3)). \tag{3.64}$$

According to the Lemma 3.4, one has ($g$ there is $(1-f)f$ ....)

$$(1+|\xi|^2)^{\frac{\nu-1}{2}}\hat{g}\in L^2(\mathbf{R}_\xi^3\,;\,L^2(\mathbf{R}_\tau\times\mathbf{R}_k^3)). \tag{3.65}$$

In particuliar, one has also

$$\frac{(1+|\tau|^2+|k|^2)^{\frac{1}{1+\frac{\nu-1}{2}}\frac{1}{2}}}{(1+|\xi|^2)^\delta}\hat{g}\in L^2(\mathbf{R}_\xi^3\,;\,L^2(\mathbf{R}_\tau\times\mathbf{R}_k^3)). \tag{3.66}$$

for all $\delta>3/4$, and finally

$$\{(1+|\xi|^2)^{\frac{\nu-1}{2}}+\frac{(1+|\tau|^2+|k|^2)^{\frac{1}{1+\frac{\nu-1}{2}}\frac{1}{2}}}{(1+|\xi|^2)^\delta}\hat{g}\in L^2(\mathbf{R}_\xi^3\,;\,L^2(\mathbf{R}_\tau\times\mathbf{R}_k^3)). \tag{3.67}$$

Since for all $p>1$, $a$, $b>0$, one has $\frac{1}{p}(a^{\frac{1}{p}})^p+\frac{1}{p'}(b^{\frac{1}{p'}})^{p'}\geq a^{\frac{1}{p}}b^{\frac{1}{p'}}$, where $\frac{1}{p}+\frac{1}{p'}=1$, one deduces from (3.67) that

$$(1+|\xi|^2)^{\frac{\nu-1}{2p}-\frac{\delta}{p'}}(1+|\tau|^2+|k|^2)^{\frac{1}{1+\frac{\nu-1}{2}}\frac{1}{2p'}}\hat{g}\in L^2(\mathbf{R}_\xi^3\,;\,L^2(\mathbf{R}_\tau\times\mathbf{R}_k^3)). \tag{3.68}$$

Setting

$$\beta=\inf\left\{\frac{\nu-1}{2p}-\frac{\delta}{p'}\,;\,\frac{1}{1+\frac{\nu-1}{2}}\frac{1}{2p'}\right\}, \tag{3.69}$$

which is positive by choosing p sufficiently close to 1, we get finally

$$(1+|\xi|^2+|\tau|^2+|k|^2)^\beta \hat{g}\in L^2, \tag{3.70}$$

and therefore, we have arranged for

**Theorem 3.3.** *For all* $\phi$, $\psi$, $\rho\in\mathcal{D}$, *set*

$$g(t,x,v)\equiv\rho*\left\{\int_{R^3_\alpha}\bar{\theta}(|\alpha|)(1-f(\alpha))f(\alpha)d\alpha\right\}\phi(v)f(t,x,v)\psi(v).$$

*where* $f$ *is the solution provided by Theorem* 3.2, *with* $s>3$. *Then, one has*

$$g\in H^\beta(\boldsymbol{R}_t\times\boldsymbol{R}^3_x\times\boldsymbol{R}^3_v),$$

*where* $\beta$ *is defined as*

$$\beta=\inf\left\{\frac{\nu-1}{2p}-\frac{\delta}{p'}\,;\,\frac{1}{1+\frac{\nu-1}{2}}\frac{1}{2p'}\right\},$$

*for all* $\delta>3/4$, $p>1$, *such that* $\beta\geq0$.

**Remark 3.3.** *In particuliar, note that for all* $\mu>0$ *small enough, one has g in* $H^{\frac{\nu-1}{2(1+\mu)}-\frac{\delta\mu}{1+\mu}}(\boldsymbol{R}_t\times\boldsymbol{R}^3_x\times\boldsymbol{R}^3_v)$, *that is something which is close to* $H^{\frac{\nu-1}{4}}$. *For instance, choosing* $\delta=1$, *then for* $1\leq p\leq 1+\frac{1}{2}\frac{(\nu+1)(\nu-1)}{\nu+2}$, *one obtains* $\beta=\frac{\nu-1}{2p}-\frac{1}{p'}$. *However this regularity result is likely not optimal. One may also wonder if* $H^\beta$ *spaces are the good spaces, and if instead we should not take some kind of Orlicz's type spaces, see for instance [PiSi]. Finally, it seems possible to deduce (small) regularity on* $f$ *as in [DeGo].*

Now that we have produced $H$-solutions with some (partial) regularity, can we bootstrap this type of result ?

It seems that we cannot do this, since we rested upon the entropic dissipation rate bound and used "only" the non linear pde (that is problem $(BD)$) to get the regularity of the velocity averages.

So, we must turn to the pde, and notice (at this point) that the weak form of problem $(BD)$ is not (or at least I do not know) well suited for (micro) local analysis. This is where the decompositions, as those provided for the pure Boltzmann case, could be usefull as we believe. In the following, we just explain these decompositions, leaving out any possible applications.

A first step is to write the curly brackets term of (3.1) as follows

$$\{.\}=(1-f_1)f'_1(f'-f)+(1-f')f(f'_1-f_1). \tag{3.71}$$

However, I have not succeeded in using this form, one reason being that, when going over to the representation in variables $(h,\alpha)$ for the operator (3.1), then $f', f'_1, f'$ and $f$ stand for $f(\alpha+v-h)$, $f(\alpha+v)$, $f(v-h)$ and $f(v)$ respectively, and looking to [Ale2], note that a significant step there was that the change of variables $h\to -h$ left "invariant" the pure Boltzmann operator. In our case (3.71), note that $Q_{bd}$ "transforms" as, where $\bar{f}_1$, $\bar{f}'$ mean $f(\alpha+v+h)$, $f(v+h)$ respectively,

$$\{.\}=(1-\bar{f}_1)f'_1(\bar{f}'-f)+(1-\bar{f}')f(f'_1-\bar{f}_1), \tag{3.72}$$

and thus we cannot factorize terms such as $\bar{f}'+f'-2f$ ....
However, I believe that this can be made, as it is suggested by the pure Boltzmann case, note also in [Dol] that $f$ and $1-f$ are somehow linked.
Since we are only interested from now in bootstrapping some regularity, I shall simply write $\{.\}$ as

$$\{.\}=(1-f'_1)f'_1(f'-f)+(1-f)f(f'_1-f_1)+(f'_1-f_1)(f'-f)(f'_1-f), \tag{3.73}$$

from which follows

**Theorem 3.4.** *With $Q_{bd}$ as given by* (3.1) *and the previous assumptions, one may write*

$$Q_{bd}(f,f)(v)=Q^1_{bd}(f,f)(v)+Q^2_{bd}(f,f)(v)+Q^3_{bd}(f,f)(v),$$

*where*

$$Q^1_{bd}(f,f)(v)\equiv\int_{R^3_{v_1}}\int_{S^2_\omega}\frac{\bar{\theta}(|v_1-v'|)}{|v-v'|^\nu}(1-f'_1)f'_1(f'-f),$$

$$Q^2_{bd}(f,f)(v)\equiv\int_{R^3_{v_1}}\int_{S^2_\omega}\frac{\bar{\theta}(|v_1-v'|)}{|v-v'|^\nu}(1-f)f(f'_1-f_1),$$

$$Q^3_{bd}(f,f)(v)\equiv\int_{R^3_{v_1}}\int_{S^2_\omega}\frac{\bar{\theta}(|v_1-v'|)}{|v-v'|^\nu}(f'_1-f_1)(f'-f)(f'_1-f),$$

In fact, I should have written the last term as

$$(f'_1-f_1)(f'-f)(f'_1-f)=(f'_1-f_1)(f'-f)f'_1-(f'_1-f_1)(f'-f)f,$$

and in view of the previous estimates, one notes that

$$Q^3_{bd}\in L^2((0,T)\times\mathbf{R}^3_x\,;\,H^{-\frac{\nu-1}{2}}(\mathbf{R}^3_v)),$$

so in the following we shall just concetrate on the two first terms $Q^1_{bd}(f,f)$ and $Q^2_{bd}(f,f)$. Oncemore, let us say that we do not believe that this is the right way, as we think that a more symetric form could be possible as in [Ale] and in the previous Section. Anyway, for these operators, we just refer to our papers where the following results are shown

**Lemma 3.6.** *With the above assumptions, the operator $Q^1_{bd}$ of Theorem* 3.4 *writes as*

$$Q^1_{bd}(f,f)(v) = -C_s \int_{R^3_\alpha} d\alpha f(\alpha+v)[1-f(\alpha+v)]\,\bar{\theta}(|\alpha|)|S(\alpha).D|^{\nu-1}(f)(v),$$

*where $C_s$ is a fixed constant (depending only on $\nu$). Recall that $S(\alpha)$ denotes the orthogonal projection over $E_{0,\alpha}$, the hyperplane passing through* 0 *and orthogonal to $\alpha$.*

**Lemma 3.7.** *With the above assumptions, the operator $Q^2_{bd}$ of Theorem* 3.4 *writes as*

$$Q^2_{bd}(f,f)(v) = C_s(1-f(v))f(v)\int_{R^3_\alpha} d\alpha\,\bar{\theta}(|\alpha|)|S(\alpha).D|^{\nu-1}(f)(\alpha+v),$$

*with same notations as in Lemma* 3.6.

Let us show how to expound suitably these terms, beginning with the simpliest $Q^2_{bd}$

**Lemma 3.8.** *With the above definition of $Q^2_{bd}(f,f)$, one has*

$$Q^2_{bd}(f,f)(v) = (1-f(v))f(v)\int_{R^3_k} f(k)K(|k-v|)dk,$$

*where $K \in L^\infty$.*

The proof follows that of the previous section and is omitted.
Next, for $Q^1_{bd}$ as given by Lemma 3.6, we let, in view of considerations already explained earlier, $\chi(t)$, $t \in \mathbf{R}^+$, a smooth function positive, with support in $[0,1]$, 1 for $t \leq 2/3$, $\bar{\chi} = 1-\chi$, and decompose

$$Q^1_{bd}(f,f)(v) = Q^{11}_{bd}(f,f)(v) + Q^{12}_{bd}(f,f)(v), \tag{3.74}$$

with

$$Q^{11}_{bd}(f,f)(v) = -C_s \int_{\mathbf{R}^3_\alpha} d\alpha f(\alpha+v)[1-f(\alpha+v)]\theta(|\alpha|)|\alpha \wedge D|^{\nu-1}\chi(|\alpha \wedge D|)(f)(v), \tag{3.75}$$

and

$$Q^{12}_{bd}(f,f)(v) = -C_s \int_{\mathbf{R}^3_\alpha} d\alpha f(\alpha+v)[1-f(\alpha+v)]\theta(|\alpha|)|\alpha \wedge D|^{\nu-1}\bar{\chi}(|\alpha \wedge D|)(f)(v). \tag{3.76}$$

For these expressions, one has

**Lemma 3.9.** *$Q^{11}_{bd}(f,f)(v)$ defined by* (3.75) *belongs to $L^2((0,T)\times \mathbf{R}^6_{x,v})$.*

*Proof.* Omitting the variables $(t,x)$, if we denote

$$\tau_\alpha(f)(v) \equiv |\alpha \wedge D|^{\nu-1} \chi(|\alpha \wedge D|)(f)(v),$$

then $\tau_\alpha$ : $L^2 \to L^2$ is bounded for each $\alpha$ non null and uniformly. Therefore, the generalised Minkowski's inequality yields the result.

There remains to study $Q_{bd}^{12}(f,f)$ as given by (3.76), and we deal with it as follows. If one sets, omitting $(t,x)$ dependance,

(3.77)
$$a(v,\xi) \equiv -C_s \int_{\mathbf{R}_\alpha^3} d\alpha f(\alpha)[1-f(\alpha)]\theta(|\alpha-v|)|(\alpha-v)\wedge\xi|^{\nu-1}\bar{\chi}(|(\alpha-v)\wedge\xi|),$$

then one checks that this is a good symbol within the class $\mathcal{S}_{0,0}^{\nu-1}$, with $(t,x)$ as parameters, and thus we can define $Q_{bd}^{12}(f,f)$ in $\mathcal{D}'$ sense as

$$<Q_{bd}^{12}(f,f) ; \phi> = <f ; a^*(v,D_v)(\phi)>. \tag{3.78}$$

using the whole calculus of [Mar1, 2, 3, Tay]. Recalling Theorem 3.2, one has finally

**Theorem 3.5.** *Under the assumption* (3.5) *on* $f_0$, *there exists a weak solution* $f$ *of problem* (*BD*), *satisfying the conclusions of Theorem* 3.2 *and* (*BD*) *in distribution sense where*

$$Q_{bd}(f,f) = Q_{bd}^{11}(f,f) + Q_{bd}^{12}(f,f) + Q_{bd}^{2}(f,f) + Q_{bd}^{3}(f,f).$$

## 4. Problem (BG)

In this Section, we will only provide the decompositions for the $(BG)$ operator and leave out any other issues. Let us again recall quickly the setting of Section I. We let $\mathcal{Y}$ for the unit periodic box, and introduce a measurable bounded function $P=P(r)$, $|r|\leq R(R>0$ fixed ), such that

$$P \equiv P(|r|), \quad 0 < P^- \leq P(r) \leq P^+ < \infty, \tag{4.1}$$

and we set

$$B\left(|v-v_1|, \left|\left(\frac{v-v_1}{|v-v_1|}, \omega\right)\right|\right) \equiv \theta(|v_1-v'|)\frac{|v_1-v'|^{\gamma+\nu}}{|v'-v|^{\nu}}, \tag{4.2}$$

where $\theta$ belongs to $\mathcal{S}^+$ and is null for small values, and

$$\gamma = \gamma(s) = \frac{s-5}{s-1}, \quad \nu = \nu(s) = \frac{s+1}{s-1}. \tag{4.3}$$

The other quantities are defined as in Section I. Furthermore, we still denote by $P$ the extension by 0 for $|r|\geq R$.
Next, we define, for $f=f(t,x,v)$, the operator

$$J(f,f)(x,v) = \int_{-R}^{R} dr \int_{\mathbf{R}_{v_1}^3} \int_{S_\omega^2} dv_1 d\omega \{f(x,v')f(x+r\omega,v_1') -$$

(4.4) $$-f(x,v)f(x+r\omega,v_1)\}B(.,.)P(r).$$

The different expression for $J(f,f)$ starts as follows

**Lemma 4.1.** *With the assumptions* (4.1) *to* (4.3), *the operator $J$ given by* (4.4) *writes as*

$$J(f,f)(x,v)=\int_{-R}^{R}dr\int_{\mathbf{R}_h^3}\frac{2dh}{|h|^{\nu+2}}\int_{\mathbf{R}_\alpha^3}\delta_{\alpha.h=0}\left\{f(x,v-h)f\left(x+r\frac{h}{|h|},\alpha+v\right)-\right.$$

$$\left.-f(x,v)f\left(x+r\frac{h}{|h|},\alpha+v-h\right)\right\}\theta(|\alpha|)|\alpha|^{\gamma+\nu}P(r).$$

The proof is omitted as it is infered from [Ale2] by an easy inspection.
*In the sequel, we shall use the notation $\bar{\theta}$ ...* to denote any function $\theta$ multiplied by a power of $|\alpha|$. In fact below, $\bar{\theta}(|\alpha|)=\theta(|\alpha|).|\alpha|^{\gamma+\nu}$, ...
Still as in [Ale2], we split $J$ according to the

**Definition 4.1.** *One has*

$$J(f,f)(x,v)=J_1(f,f)(x,v)+J_2(f,f)(x,v),$$

*where $J_1(f,f)$ and $J_2(f,f)$ are defined by*

$$J_1(f,f)(x,v)=\int_{-R}^{R}dr\int_{\mathbf{R}_h^3}\frac{2dh}{|h|^{\nu+2}}\int_{\mathbf{R}_\alpha^3}\delta_{\alpha.h=0}\{f(x,v-h)-f(x,v)\}\times$$

$$\times f\left(x+r\frac{h}{|h|},\alpha+v\right)\bar{\theta}(|\alpha|)P(r),$$

*and*

$$J_2(f,f)(x,v)=f(x,v)\int_{-R}^{R}dr\int_{\mathbf{R}_h^3}\frac{2dh}{|h|^{\nu+2}}\int_{\mathbf{R}_\alpha^3}\delta_{\alpha.h=0}\left\{f\left(x+r\frac{h}{|h|},\alpha+v\right)-\right.$$

$$\left.-f\left(x+r\frac{h}{|h|},\alpha+v-h\right)\right\}\bar{\theta}(|\alpha|)P(r).$$

The second step consists in an explicit expression for these operators, using Fourier transform as in [Ale2]. For $J_1$ one has

**Lemma 4.2.** *With the notations and hypothesis of Definition* 4.1, *one has*

$$J_1(f,f)(x,v)=-a(x,v,D_v)(f)(x,v),$$

*where the symbol $a$ is given explicitely by*

$$a(x,v,\xi)=\int_{R_h^3}\int_{R_\alpha^3}C_s'\delta_{\alpha.h=0}\frac{P(|h|)|\xi.h|^{\nu-1}}{|h|^{\nu+1}}\bar{\theta}(|\alpha|)f(x+h,\alpha+v)d\alpha dh.$$

*Proof.* Making the change of variables $h\to -h$ and $r\to -r$, one finds that

$$J_1(f,f)(x,v)=\int_{-R}^{R}dr\int_{\mathbf{R}^3_h}\frac{dh}{|h|^{\nu+2}}\int_{\mathbf{R}^3_\alpha}\delta_{\alpha.h=0}\{f(x,v-h)+f(x,v+h)-2f(x,v)\}\times$$

$$\times f\Big(x+r\frac{h}{|h|},\alpha+v\Big)\bar{\theta}(|\alpha|)P(r).$$

Next, let us define

$$\beta_f\Big(x,\frac{h}{|h|},v\Big)=\int_{-R}^{R}dr\int_{\mathbf{R}^3_\alpha}\delta_{\alpha.h=0}\,\bar{\theta}(|\alpha|)P(r)f\Big(x+r\frac{h}{|h|},\alpha+v\Big)d\alpha. \tag{4.5}$$

$J_1(f,f)(x,v)$ writes as

$$J_1(f,f)(x,v)=\int_{\mathbf{R}^3_h}\frac{\mathrm{dh}}{|h|^{\nu+2}}\beta_f\Big(x,\frac{h}{|h|},v\Big)\{f(x,v-h)+f(x,v+h)-2f(x,v)\}.$$

Thus, writing $h$ in polar coordinates $h=u\omega$, $u=|h|$, we get

$$J_1(f,f)(x,v)=\int_0^{+\infty}du\int_{S^2_\omega}d\omega\frac{1}{u^\nu}\{f(x,v+u\omega)+f(x,v-u\omega)-2f(x,v)\}\beta_f(x,\omega,v).$$

Setting $\hat{f}^2(x,\xi)$ for the Fourier transform of with respect to the variable $v$, one gets

$$J_1(f,f)(x,v)=\int_{\mathbf{R}^3_\xi}d\xi\hat{f}^2(x,\xi)e^{i\xi.v}\int_{S^2_\omega}d\omega\beta_f(x,\omega,v)\Big\{\int_0^{+\infty}du\frac{1}{u^\nu}\{e^{iu\xi.\omega}-2\}\Big\}$$

$$=-C_s\int_{\mathbf{R}^3_\xi}d\xi\hat{f}^2(x,\xi)e^{i\xi.v}\Big\{\int_{S^2_\omega}d\omega\beta_f(x,\omega,v)|\xi.\omega|^{\nu-1}\Big\}. \tag{4.6}$$

In view of (4.5), note that

$$\int_{S^2_\omega}\beta_f(x,\omega,v)|\xi.\omega|^{\nu-1}=\int_{S^2_\omega}\int_{-R}^{R}\int_{\mathbf{R}^3_\alpha}\delta_{\alpha.\omega=0}\,\bar{\theta}(|\alpha|)P(r)f(x+r\omega,\alpha+v)|\xi.\omega|^{\nu-1},$$

which putting $h=r\omega$ with the identification $r=|h|$, $\omega=\frac{h}{|h|}$, gives us

$$\int_{S^2_\omega}\beta_f(x,\omega,v)|\xi.\omega|^{\nu-1}=\int_{\mathbf{R}^3_h}\int_{\mathbf{R}^3_\alpha}\delta_{\alpha.h=0}\frac{P(|h|)|\xi.h|^{\nu-1}\bar{\theta}(|\alpha|)}{|h|^{\nu-1}}f(x+h,\alpha+v).$$

We get the Lemma setting

$$a(x,v,\xi)=\int_{\mathbf{R}^3_h}\int_{\mathbf{R}^3_\alpha}\delta_{\alpha.h=0}\frac{P(|h|)|\xi.h|^{\nu-1}\bar{\theta}(|\alpha|)}{|h|^{\nu+1}}f(x+h,\alpha+v).$$

Let us come to $J_2$ for which we have similarly

**Lemma 4.3.** *With the notations and hypothesis of Definition* 4.1,

$$J_2(f,f)(x,v)=C_s'''f(x,v)\int_{R_h^3}\int_{R_k^3}\frac{P(|h|)|h|^{\nu-4}}{|h.k|^{\nu-2}}\bar{\theta}\Big(\frac{|h\wedge k|}{|h|}\Big)f(x+h,v+k).$$

*Proof.* From

$$J_2(f,f)(x,v)=f(x,v)\int_{|r|\leq R}dr\int_{\mathbf{R}_h^3}\frac{2dh}{|h|^{\nu+2}}\int_{\mathbf{R}_\alpha^3}d\alpha\delta_{\alpha.h=0}$$

$$\Big\{f\Big(x+r\frac{h}{|h|},\alpha+v\Big)-f\Big(x+r\frac{h}{|h|},\alpha+v-h\Big)\Big\}\,\bar{\theta}(|\alpha|)P(r),$$

and setting $h=u\omega$ in polar coordinates, we get

$$J_2(f,f)(x,v)=f(x,v)\int_{|r|\leq R}dr\int_0^{+\infty}du\int_{S_\omega^2}d\omega\frac{1}{u^\nu}\int_{\mathbf{R}_\alpha^3}\delta_{\alpha.\omega=0}$$

$$\{f(x+r\omega,\alpha+v)-f(x+r\omega,\alpha+v-u\omega)\}\,\bar{\theta}(|\alpha|)P(r)=$$

$$=f(x,v)\int_{|r|\leq R}dr\int_{\mathbf{R}_\alpha^3}d\alpha\delta_{\alpha.\omega=0}\int_{\mathbf{R}_\xi^3}\hat{f}^2(x+r\omega,\xi)\times$$

$$\times e^{i(\alpha+v)\xi}\int_0^{+\infty}du\frac{1}{u^\nu}(2-e^{iu\omega.\xi}-e^{-iu\omega.\xi})\,\bar{\theta}(|\alpha|)P(r)=$$

$$=f(x,v)\,C_s'''\int_{|r|\leq R}dr\int_{S_\omega^2}d\omega P(r)\int_{\mathbf{R}_k^3}f(x+r\omega,k)\times$$

$$\times\Big\{\int_{\mathbf{R}_\xi^3}\int_{\mathbf{R}^3_{\alpha,\alpha.\omega=0}}|\xi.\omega|^{\nu-1}\,\bar{\theta}(|\alpha|)e^{i(\alpha+v).\xi}e^{-ik.\xi}\Big\}.$$

As

$$\int_{\mathbf{R}^3_{\alpha,\alpha.\omega=0}}\bar{\theta}(|\alpha|)e^{i\alpha.\xi}d\alpha=\bar{\theta}(|S(\omega).\xi|),$$

(after some computations), the above curly brackets term reads

$$\{.\}=C_s'''\frac{1}{|(k-v).\omega|^{\nu-2}}\,\bar{\theta}(|S(\omega).(k-v)|).$$

To sum up

$$J_2(f,f)(x,v)=f(x,v)\,C_s'''\int_{|r|\leq R}dr\int_{S_\omega^2}d\omega P(r)\int_{\mathbf{R}_k^3}f(x+r\omega,k)\times$$

$$\times\frac{1}{|(k-v).\omega|^{\nu-2}}\,\bar{\theta}(S(\omega).(k-v)|),$$

and thus setting $h=r\omega$ .... , we are done.

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## References

[Agr] M.S. Agranovich, Elliptic Singular Integro Differential Operators, Russ. Maths. Surveys, **20** (5-6)(1965), 1-121.

[Ale1] R. Alexandre, Sur l'opérateur de Boltzmann linéaire en dimension 3, sans troncature angulaire, Note C.R.A.S Paris, **325** (1997) Série I, 959-962.

[Ale2] R. Alexandre, Sur l'opérateur de Boltzmann non linéaire 3D, sans troncature angulaire, Note C.R.A.S Paris, **326** (1998) Série I, 165-168, and paper to appear in $M^2AN$.

[Ale3] R. Alexandre, Sur le taux de dissipation d'entropie sans troncature angulaire, Note C.R.A.S Paris, **326** (1998) Série I, 311-315.

[Ale4] R. Alexandre, The non cutoff 3D Boltzmann linearised operator and applications, work in preparation.

[Ale5] R. Alexandre, Some Remarks on $3D$ Boltzmann linear operator without cutoff, Transp. Th. and Stat. Phys., **28**-5 (1999), 433-473.

[Ale6] R. Alexandre, Une définition des solutions renormalisées pour l'équation de Boltzmann sans troncature angulaire, Note C.R.A.S Paris, **328**, Série I, 987-991(1999).

[Ark1] L. Arkeryd, On the Boltzmann equation, Arch. Rat. Mech. Anal., **45** (1972), 1-34.

[Ark2] L. Arkeryd, Intermolecular forces of infinite range and the Boltzmann equation, Arch. Rat. Mech. Anal., **77** (1981), 11-21.

[ArBe] L. Arlotti and N. Bellomo, Lectures Notes in the Mathematical Theory of the Boltzmann Equation. Ed. N. Bellomo, Vol 33, Worl Sc., 1995.

[BePo] N. Bellomo and J. Polewczak, On the Generalised Boltzmann equation. Existence and exponentional trend to equilibrium, Transp. Theory and Stat. Phys., 1997.

[BoDe] F. Bouchut and L. Desvillettes, On the regularity of the positive part of Boltzmann operator, preprint Univ. Orléans 97 (1997).

[Cal] A.P. Calderon, Algebras of Singular Integro-Differential Operators, In Proc. of Symp. in Pure Maths., Vol 10, AMS, Providence, 1967.

[Car] T. Carleman, Problèmes Mathématiques dans la Théorie cinétique des gaz, Ed by Almquist and Wiksell, Uppsala, 1957.

[Cer] C. Cercignani, Mathematical Methods in Kinetic Theory, 2nd Ed. Plenum, 1990

[CIP] C. Cercignani, R. Illner and M. Pulvirenti, The Mathematical Theory of Dilute Gases, Series in Appl. Sc., Vol 106, Springer Verlag, New York, 1994.

[ChPi] J. Chazarain and A. Piriou, Introduction à la théorie des e.d.p linéaires,Gauthier-Villars, Paris, 1981

[Cor] H.O. Cordes, The Technique of PdO, London Math. Soc. Lectures Note Series,Vol 202, Cambridge Univ. Press, 1995.

[Des] L. Desvillettes, Regularisation properties of the 2D homogeneous Boltzmann equation, preprint 95-06, Univ. Orléans, France (1995).

[DeGo] L. Desvillettes and F. Golse, On a model Boltzmann equation without angular cutoff, prépublication Univ. Orléans 98-01 (1998).

[DiLi1] R.J. DiPerna and P.L. Lions, On the Cauchy problem for Boltzmann equation; Global existence and weak stability, Ann. Maths., **130** (1989), 321-366.

[DiLi2] R.J. DiPerna and P.L. Lions, Global weak solutions for Vlasov-Maxwell system, Com. Pure Appl. Maths., **42** (1989), 729-757.

[DLM] R.J. DiPerna, P.L. Lions and Y. Meyer, $L^p$ regularity of velocity averages, Ann.IHP Anal. non lin., **8** (1991), 271-287.

[Dol] J. Dolbeault, Kinetic models and quantum effcts. A modified Boltzmann equation for Fermi Dirac particles, Arch. Rat. Mech. Anal., **127** (1994), 101-131.
[Gou] T. Goudon, On Boltzmann Equations and Fokker-Planck Asymptotics: Influence of grazing collisions, J. of Stat. Phys., **89** (1997), 3-4.
[Gui] J.P. Guiraud, The Boltzmann equation in kinetic theory. A survey. Fluid Dyn. Trans. Vol 7, Part II, 37-84 (1973).
[Ham] K. Hamdache, Sur l'existence globale de quelques solutions de l'équation de Boltzmann, Thèse de Doctorat d'Etat. Univ. Paris VI (1986).
[Kum] H. Kumano-go, Pseudo-Differential Operators, MIT Press, Cambridge, 1981.
[Lio1] P.L. Lions, Compactness in Boltzmann's equation, via FIO and applic. J. Math. Kyoto Univ. Part I, **34**, 391-427, Part II, **34**, 429-461, Part III, **34**, 539-584(1994).
[Lio2] P.L. Lions, On Boltzmann and Landau equations, Phil. Trans. Roy. Soc. London, **346** (1994), 191-202.
[Lio3] P.L. Lions, Régularité et compacité pour des noyaux de collision de Boltzmann sans troncature angulaire, Note C.R.A.S Paris, **326** (1998) Série I, 37-41.
[Lun] A. Lunardi, Analytic semi-groups. Progress in Nonlin. Diff. Eq., Birkhauser, Berlin, 1995.
[Mar1] J. Marschall, Pdo with non regular symbols of the class $S^m_{\rho,\delta}$,Com. Part. Diff. Equ., **12**-8 (1987), 921-965.
[Mar2] J. Marschall, Pdo with coefficients in Sobolev spaces, Trans. AMS, **307**-1(1988), 335-361.
[Mar3] J. Marschall, Parametrices for non regular elliptic Pdo, Math. Nachr., **159** (1992), 175-188.
[MiPr] S.G. Mikhlin and S. Prossdorf, Singular Integral Operators, Springer Verlag, Berlin, 1986.
[Mor] D. Morgenstern, Analytic studies related to the Maxwell Boltzmann equation, Arch. Rat. Mech. Anal., **4** (1955), 533-555.
[Pao] Y.P. Pao, Boltzmann Collision Operator with Inverse power Intermolecular potentials, C.P.A. M, Part I, 27, 407-428, Part II, 27, 559-581 (1974).
[PiSi] L. Pick and W. Sickel, Several types of Intermediate Besov Orlicz Spaces, Math. Nachr., **164** (1993), 141-165.
[Pov] A.J. Povzner, About the Boltzmann equation in kinetic gas theory, Math. SSSR Sb., **58** (1962), 65-86.
[RuSi] T. Runst and W. Sickel, Sobolev Spaces of fractional order, .... Series in Nonlin. Anal. and Appl. Vol 3, DeGruyter, New-York, 1996.
[Sch] E. Schrohe, Boundedness and spectral invariance... Int. Equ. and Oper. Theory, **13** (1990), 271-284.
[ScTr] H.J. Schmeisser and H. Triebel, Topics in Fourier Analysis and Functions Spaces, Wiley, New-York, 1987.
[Ste1] E.M. Stein, Singular Integrals and Differentiability properties of functions, Princeton Univ. Press, 1970.
[Ste2] E.M. Stein, Harmonic Analysis, Real variables methods, Princeton Univ. Press, 1993.
[Tay1] M.E. Taylor, Pseudo-Differential Operators, Princeton Univ. Press, 1981.
[Tay2] M.E. Taylor, Partial Differential Equations. Vol I, II and III, Appl. Math. Sc., Vol 115, 116, 117, Springer Verlag, 1996.
[Tay3] M.E. Taylor, PDO and non linear PDE, Birkhauser, Boston, 1991.
[Tri1] H. Triebel, Theory of Functions Spaces. Monog. in Maths., Vol 78, Birkhauser, 1983.
[Tri2] H. Triebel, Theory of Functions Spaces II. Monog. in Maths., Vol 84, Birkhauser, 1992.
[Tri3] H. Triebel, Interpolation theory, Functions Spaces, Differential Operators, North-Holland, Amsterdam, 1978.
[Uka] S. Ukai, Solutions of the Boltzmann equation. In Patters and Waves, North-Holland, 1985.
[Vil1] C. Villani, Thèse Université Paris-Dauphine, Paris, 1998.
[Vil2] C. Villani, On the Cauchy problem for Landau equation, Adv, in Diff. equ., **1**-5(1996), 793-816.
[Wen] B. Wennberg, Regularity in the Boltzmann equation and the Radon transform, CPDE, **19** (1994), 2057-2074.