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Capelli type identities on certain scalar generalized Verma modules

By

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Abstract

We give an analogue of the Capelli identity. The analogue is constructed on certain scalar generalized Verma modules of the complex simple Lie algebras of Hermitian symmetric type, and has stronger compatibility with group actions than the original Capelli identity. As in the original case, the analogue expresses certain invariant operators as operators from the center of the universal enveloping algebra. We also give examples for all the possible cases except for E_7.

1. Introduction

The Capelli identity plays an important role in invariant theory. In [3], it was studied in the context of multiplicity-free actions, and analogues of the Capelli identity were given for these actions when they exist. We call both these analogues and the Capelli identity [2] in the nineteenth century, the *classical* Capelli identities. To avoid confusion, we call the analogue which will be constructed on certain scalar generalized Verma modules, a Ψ_λ-*analogue*. Let $\mathfrak{p}$ be a parabolic subalgebra of a complex simple Lie algebra $\mathfrak{g}$ with a commutative nilpotent radical $\mathfrak{n}^+$. In this case, we say $(\mathfrak{g}, \mathfrak{p})$ to be of *commutative parabolic type* or *Hermitian symmetric type*. Then $(L, \mathrm{Ad}, \mathfrak{n}^+)$ becomes a prehomogeneous vector space and the action is multiplicity-free, where L is the closed subgroup of G corresponding to a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$. Note that the classical Capelli identity for this case was investigated in [3]. Set $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbf{C}_\lambda$ for $\lambda \in \mathrm{Hom}(\mathfrak{p}, \mathbf{C})$, where $\mathbf{C}_\lambda$ is the representation space of λ. This $U(\mathfrak{g})$-module $M(\lambda)$ is called a *scalar generalized Verma module* induced from a character λ. There is a natural linear isomorphism between $M(\lambda)$ and $\mathbf{C}[\mathfrak{n}^+]$. Hence we can give the structure of $U(\mathfrak{g})$-module to $\mathbf{C}[\mathfrak{n}^+]$ through this linear isomorphism and let $\Psi_\lambda : U(\mathfrak{g}) \longrightarrow \mathrm{End}\, \mathbf{C}[\mathfrak{n}^+]$ denote this representation. The main purpose of this paper is to give an analogue of the Capelli identity for Ψ_λ.

Let us state the Ψ_λ-analogue concretely. The classical Capelli identity shows that an $\mathrm{Ad}(L)$-invariant operator $f\,{}^t\!f(\partial)$ is in $\mathrm{ad}(Z(\mathfrak{l}))$, where $f \in \mathbf{C}[\mathfrak{n}^+]$

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is a relative invariant of $(L, \mathrm{Ad}, \mathfrak{n}^+)$ and ${}^t f(\partial)$ denotes the constant coefficient differential operator with the weight opposite to that of f (see (2.2)). Our main theorem and its corollary are (1.1) and (1.2) below. If there exists the classical Capelli identity $f\, {}^t f(\partial) = \mathrm{ad}(u)$ for $u \in U(\mathfrak{l})$, then we have (Theorem 3.3)

$$\Psi_\lambda(f\, {}^t f) = \pm \Psi_{2\lambda+2\rho}(u)\, \mathrm{ad}(u), \tag{1.1}$$

where ${}^t f \in S(\mathfrak{n}^+) \subset U(\mathfrak{g})$ is defined in Definition 2.3 (1) and the sign is explicitly determined by $(\mathfrak{g}, \mathfrak{p})$ and f. We note that u is independent of λ in this presentation of the Ψ_λ-analogue. Under the same condition, there exists $u_\lambda \in U(\mathfrak{l})$ determined by u and λ such that (Corollary 3.7)

$$\Psi_\lambda(f\, {}^t f) = \Psi_\lambda(u_\lambda). \tag{1.2}$$

We call this formula the Ψ_λ-analogue again.

The Ψ_λ-analogue of the classical Capelli identity was originally motivated by the study on the irreducibility criterion of $M(\lambda)$ in terms of b-functions [7]. The relation between this study and the Ψ_λ-analogue is as follows: By using the classical Capelli identity, we can easily calculate the b-function defined by $f\, {}^t f(\partial) f^s = b(s-1) f^s$. Similarly we can calculate $\beta_\lambda(s)$ defined by $\Psi_\lambda(f\, {}^t f) f^s = \beta_\lambda(s-1) f^s$ by using the Ψ_λ-analogue (1.1). Here we know [7, main theorem]

$$\beta_\lambda(s) = \pm b(s) b(s + \text{ constant}). \tag{1.3}$$

We can regard the Ψ_λ-analogue as an operator version of (1.3). In fact, (1.3) immediately follows from the Ψ_λ-analogue and we will prove the Ψ_λ-analogue using (1.3). Thus the Ψ_λ-analogue of the Capelli identity is closely related

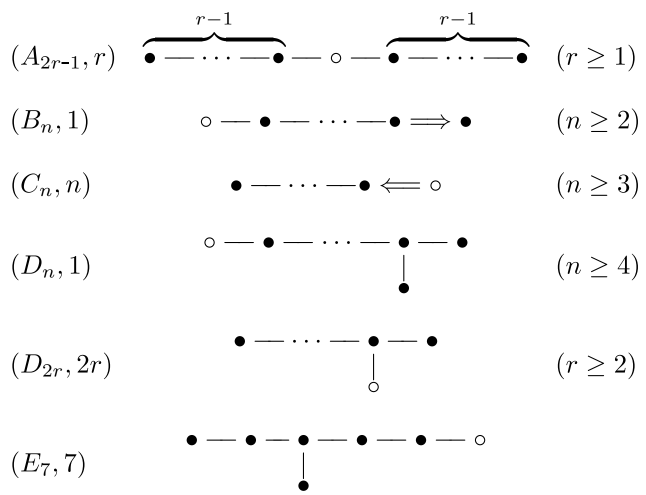

Figure 1: Regular type

to the structure of generalized Verma modules via the b-functions. We can find another importance of the Ψ_λ-analogue from the expression (1.2). The Ψ_λ-analogue (1.2) lives in the world of the representation of $U(\mathfrak{g})$ while the classical Capelli identity lives in that of $U(\mathfrak{l})$. Namely the Ψ_λ-analogue has stronger compatibility with group actions than the classical Capelli identity.

The contents of this paper is as follows: In Section 2, we prepare basic definitions to state the main theorem. In Section 3, we show the main theorem. In Section 4, we give examples of types $(A_{2r\text{-}1}, r)$, $(B_n, 1)$, (C_n, n), $(D_n, 1)$ and $(D_{2r}, 2r)$ in the notation of Figure 1, which are $GL_r \otimes GL_r$, $O_{2n} \otimes GL_1$, S^2GL_n, $O_{2n-1} \otimes GL_1$ and $\wedge^2 GL_{2r}$ in the notation of [3], respectively.

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2. Scalar generalized Verma modules

In this section we give a realization of a scalar generalized Verma module as a representation on a certain polynomial ring. Then we define three anti-involutions which have close relation to the realization, and show some of their properties. Throughout this section we assume that $(\mathfrak{g}, \mathfrak{p})$ is of commutative parabolic type.

We give some notations and definitions needed later. Let $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{l}$ and $\mathfrak{n}^+$ be as in Section 1. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let Δ be the root system of $\mathfrak{g}$, Δ^+ be the positive root system containing all the roots occurring in $\mathfrak{n}^+$, $\{\alpha_1, \ldots, \alpha_n\}$ be the simple system, and $\{\varpi_1, \ldots, \varpi_n\}$ be the set of corresponding fundamental weights. We assume that $\mathfrak{p}$ contains $\mathfrak{h}$ and all the positive root spaces, and that $\mathfrak{l}$ contains $\mathfrak{h}$. Let Δ_L and Δ_N^+ be the sets of roots occurring in $\mathfrak{l}$ and $\mathfrak{n}^+$, respectively. Set $\mathfrak{n}^- = \sum_{\alpha \in \Delta_N^+} \mathfrak{g}^{-\alpha}$, where $\mathfrak{g}^\alpha$ denotes the α-root space in $\mathfrak{g}$. Then $\mathfrak{l}$ acts on $\mathfrak{n}^+$ by the adjoint representation and accordingly $U(\mathfrak{l})$ acts on a polynomial ring $\mathbf{C}[\mathfrak{n}^+]$. We denote this representation by ad again. In this setting, $\mathfrak{p}$ becomes a maximal parabolic subalgebra of $\mathfrak{g}$, and there exists the unique number i_0 such that the simple root α_{i_0} is not a root of $\mathfrak{l}$. We sometimes denote a pair $(\mathfrak{g}, \mathfrak{p})$ by $(\mathfrak{g}, i_0)$. We will always use Bourbaki's simple root numbering for i_0 ([1]). We fix an invariant bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$, which must be a multiple of the Killing form.

For $(\mathfrak{g}, \mathfrak{p})$ of commutative parabolic type, and $\lambda \in \mathrm{Hom}(\mathfrak{p}, \mathbf{C})$, we set

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbf{C}_\lambda,$$

where $\mathbf{C}_\lambda$ denotes the representation space of λ. We have isomorphisms of vector spaces $M(\lambda) \simeq U(\mathfrak{n}^-) = S(\mathfrak{n}^-) \simeq \mathbf{C}[\mathfrak{n}^+]$, since both $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are commutative. We obtain the second isomorphism above by identifying $\mathfrak{n}^-$ with the dual space $(\mathfrak{n}^+)^*$ of $\mathfrak{n}^+$ via the fixed invariant bilinear form. Hence $U(\mathfrak{g})$ acts on $\mathbf{C}[\mathfrak{n}^+]$ and we denote this representation by $\Psi_\lambda : U(\mathfrak{g}) \longrightarrow \mathrm{End}\,\mathbf{C}[\mathfrak{n}^+]$. We can check the following explicit forms of $\Psi_\lambda(X)$ for $X \in \mathfrak{g}$ by direct calculations.

Lemma 2.1. *Let $\{F_k\}$ be a basis of $\mathfrak{n}^-$. Then we have*

$$\begin{aligned}
&(1) \quad \Psi_\lambda(X) = X && (X \in \mathfrak{n}^-),\\
&(2) \quad \Psi_\lambda(X) = \mathrm{ad}(X) + \lambda(X) \\
&\qquad\qquad\; = \sum_k [X, F_k] \frac{\partial}{\partial F_k} + \lambda(X) && (X \in \mathfrak{l}),\\
&(3) \quad \Psi_\lambda(X) = \frac{1}{2} \sum_{k,l} [[X, F_k], F_l] \frac{\partial}{\partial F_k} \frac{\partial}{\partial F_l} + \sum_k \lambda([X, F_k]) \frac{\partial}{\partial F_k} \\
&\qquad\qquad\; = \frac{1}{2} \sum_k \Psi_{2\lambda}([X, F_k]) \frac{\partial}{\partial F_k} && (X \in \mathfrak{n}^+).
\end{aligned}$$

In the case $X \in \mathfrak{n}^-$, the operator $\Psi_\lambda(X) = X \in \mathfrak{n}^- \simeq (\mathfrak{n}^+)^* \subset \mathbf{C}[\mathfrak{n}^+]$ is just a multiplying operator on $\mathbf{C}[\mathfrak{n}^+]$. In the latter expression of $\Psi_\lambda(X)$ for $X \in \mathfrak{n}^+$, we can use $\Psi_{2\lambda}$ recursively, since $[X, F_k] \in \mathfrak{l}$.

Remark 2.2. We have realized the scalar generalized Verma module on $\mathbf{C}[\mathfrak{n}^+]$ and represented the action in $D_{\mathfrak{n}^+}$. In fact, for every finite dimensional irreducible representation λ of $\mathfrak{p}$, we can realize the generalized Verma module induced from λ on $\mathbf{C}[\mathfrak{n}^+] \otimes V_\lambda$ and represent the actions as differential operators. Here V_λ denotes the representation space of λ. We do not, however, give the proof in this paper.

Next we fix a Chevalley basis of $\mathfrak{g}$. Let $\{X_\alpha \mid \alpha \in \Delta\} \cup \{H_i \mid i = 1, \ldots, n\}$ be a Chevalley basis of $\mathfrak{g}$ such that $X_\alpha \in \mathfrak{g}^\alpha$ and that $H_i \in \mathfrak{h}$ is the coroot of the simple root α_i. Namely, it satisfies (1) $\varpi_j(H_i) = \delta_{ij}$, (2) $[X_\alpha, X_{-\alpha}]$ is equal to the coroot of α for $\alpha \in \Delta^+$, and (3) if we define constants $c_{\alpha\beta}$ by $[X_\alpha, X_\beta] = c_{\alpha\beta} X_{\alpha+\beta}$, then $c_{-\alpha,-\beta} = -c_{\alpha\beta}$.

We compute $\langle X_\alpha, X_{-\alpha} \rangle$ for later use. We have $2\langle X_\alpha, X_{-\alpha} \rangle = \langle [H_\alpha, X_\alpha], X_{-\alpha} \rangle = \langle H_\alpha, [X_\alpha, X_{-\alpha}] \rangle = \langle H_\alpha, H_\alpha \rangle = (2\alpha/(\alpha,\alpha), 2\alpha/(\alpha,\alpha)) = 4/(\alpha,\alpha)$, where $(\ ,\ )$ is the bilinear form on $\mathfrak{h}^*$ induced from the fixed invariant bilinear form $\langle\ ,\ \rangle$. Hence we have

$$\langle X_\alpha, X_{-\alpha} \rangle = \frac{2}{(\alpha, \alpha)}. \tag{2.1}$$

Let us define three anti-involutions. One of them is on $U(\mathfrak{g})$, and the rest are on $D_{\mathfrak{n}^+}$, the ring of polynomial coefficient differential operators on $\mathfrak{n}^+$. We can identify $D_{\mathfrak{n}^+}$ with $\mathbf{C}[\mathfrak{n}^+] \otimes S(\mathfrak{n}^+)$ naturally. Indeed, $S(\mathfrak{n}^+)$ is identified with the ring of constant coefficient differential operators on $\mathfrak{n}^+$ as follows: For $P \in S(\mathfrak{n}^+)$, we define a constant coefficient differential operator $P(\partial)$ on $\mathfrak{n}^+$ as an operator such that

$$P(\partial) \exp\langle x, y \rangle = P(y) \exp\langle x, y \rangle \quad (x \in \mathfrak{n}^+, y \in \mathfrak{n}^-). \tag{2.2}$$

Here $P(y)$ on the right hand side is the polynomial function on $\mathfrak{n}^-$ by the identification $S(\mathfrak{n}^+) \simeq \mathbf{C}[\mathfrak{n}^-]$ via the fixed invariant bilinear form $\langle\ ,\ \rangle$. In particular, for $F \in \mathfrak{n}^-$ and $P \in \mathfrak{n}^+$

$$P(\partial)(F) = \langle P, F \rangle. \tag{2.3}$$

Definition 2.3. Let $D_{\mathfrak{n}^+}$ denote the ring of polynomial coefficient differential operators on $\mathfrak{n}^+$.

(1) Define an anti-involution ${}^t\cdot$ on $U(\mathfrak{g})$ by

$$\begin{aligned} {}^tX_\alpha &= X_{-\alpha} \qquad (\alpha \in \Delta), \\ {}^tH_i &= H_i \qquad (i \in \{1, \ldots, n\}), \end{aligned}$$

where X_α and H_i are the elements of the fixed Chevalley basis. These formulas define an anti-automorphism of the Lie algebra $\mathfrak{g}$, and we extend it to $U(\mathfrak{g})$ as an anti-homomorphism of a ring.

(2) Define an anti-involution σ on $D_{\mathfrak{n}^+}$ by

$$\begin{aligned} \sigma(F_j) &= F_j, \\ \sigma\left(\frac{\partial}{\partial F_j}\right) &= -\frac{\partial}{\partial F_j}, \end{aligned}$$

where $\{F_j\}$ is a basis of $\mathfrak{n}^-$. It is obvious that σ does not depend on the choice of a basis.

(3) Define an anti-involution τ on $D_{\mathfrak{n}^+}$ by

$$\begin{aligned} \tau(X_{-\alpha}) &= X_\alpha(\partial), \\ \tau(X_\alpha(\partial)) &= X_{-\alpha}, \end{aligned}$$

for $\alpha \in \Delta_N^+$. Here $\{X_\alpha\}$ is a part of the fixed Chevalley basis. Note that

$$X_\alpha(\partial) = \langle X_\alpha, X_{-\alpha} \rangle \frac{\partial}{\partial X_{-\alpha}} = \frac{2}{(\alpha, \alpha)} \frac{\partial}{\partial X_{-\alpha}},$$

because of (2.1) and (2.3).

Next we show some of the properties of the anti-involutions defined above.

Lemma 2.4. (1) (cf. Humphreys [4], Section 6 Exercise) *The anti-involution ${}^t\cdot$ on $U(\mathfrak{g})$ is the identity on the center $Z(\mathfrak{l})$ of $U(\mathfrak{l})$.*

(2) *The anti-involution σ on $D_{\mathfrak{n}^+}$ satisfies*

$$\sigma(\Psi_\lambda(u)) = \Psi_{-\lambda-2\rho}(s(u)) \quad (u \in U(\mathfrak{g})),$$

where s is the anti-involution on $U(\mathfrak{g})$ defined by

$$s(X) = \begin{cases} -X & (X \in \mathfrak{l}), \\ X & (X \in \mathfrak{n}^+ + \mathfrak{n}^-), \end{cases}$$

and $\rho \in \mathrm{Hom}(\mathfrak{p}, \mathbf{C})$ is the half sum of roots in Δ_N^+, or equivalently $2\rho(X) = \mathrm{Tr}_{\mathfrak{n}^+} \mathrm{ad}(X)$ for $X \in \mathfrak{p}$.

(3) *The anti-involution τ on $D_{\mathfrak{n}^+}$ satisfies*

$$\tau(\Psi_\lambda(u)) = \Psi_\lambda({}^tu) \quad (u \in U(\mathfrak{l})).$$

Proof. (1) $\mathfrak{l}$ is a direct sum of the semisimple part $[\mathfrak{l},\mathfrak{l}]$ and the one-dimensional center contained in $\mathfrak{h}$. Since the anti-involution ${}^t\cdot$ is the identity on $U(\mathfrak{h})$, we have only to check that ${}^t\cdot$ is the identity on $Z([\mathfrak{l},\mathfrak{l}])$.

Let $\varphi : Z([\mathfrak{l},\mathfrak{l}]) \longrightarrow U(\mathfrak{h} \cap [\mathfrak{l},\mathfrak{l}])$ be the Harish-Chandra homomorphism, which maps $Z([\mathfrak{l},\mathfrak{l}])$ isomorphically onto its image. In particular, φ is injective. The explicit form of φ is a composite of the projection from $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{g}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{g}^+)$ to $U(\mathfrak{h})$ and the isomorphism on $U(\mathfrak{h})$ defined by $H \mapsto H - \delta(H)$. Here $\mathfrak{g}^{\pm} = \sum_{\alpha\in\Delta^+} \mathfrak{g}^{\pm\alpha}$ and $\delta = (1/2)\sum_{\alpha\in\Delta_L^+} \alpha$.

Since $[X, {}^t z] = -{}^t[{}^t X, z]$, the anti-involution ${}^t\cdot$ stabilizes $Z([\mathfrak{l},\mathfrak{l}])$. Because of the construction of φ, it follows that $\varphi({}^t u) = \varphi(u)$ for $u \in Z([\mathfrak{l},\mathfrak{l}])$. Hence the injectivity of φ proves the assertion.

(2) It suffices to show the assertion for $u = X \in \mathfrak{g}$. It is obvious for $X \in \mathfrak{n}^-$.

Let $X \in \mathfrak{l}$ and $\{F_k\}$ be a basis of $\mathfrak{n}^-$.

$$\begin{aligned}
\sigma(\Psi_\lambda(X)) &= \sigma\left(\sum_k [X, F_k]\frac{\partial}{\partial F_k} + \lambda(X)\right) \\
&= -\sum_k \frac{\partial}{\partial F_k}[X, F_k] + \lambda(X) \\
&= -\sum_k \left\{[X, F_k]\frac{\partial}{\partial F_k} + \frac{\partial}{\partial F_k}([X, F_k])\right\} + \lambda(X) \\
&= -\operatorname{ad}(X) + \lambda(X) - \sum_k \frac{\partial}{\partial F_k}(\operatorname{ad}(X)(F_k)) \\
&= -\operatorname{ad}(X) + \lambda(X) - \operatorname{Tr}_{\mathfrak{n}^-} \operatorname{ad}(X) \\
&= -\operatorname{ad}(X) + \lambda(X) + 2\rho(X).
\end{aligned}$$

We have proved the assertion for $X \in \mathfrak{l}$.

Let $X \in \mathfrak{n}^+$ to finish proving (2).

$$\begin{aligned}
\sigma(2\Psi_\lambda(X)) &= \sigma\left(\sum_k \Psi_{2\lambda}([X, F_k])\frac{\partial}{\partial F_k}\right) \\
&= \sum_k \frac{\partial}{\partial F_k}\Psi_{-2\lambda-2\rho}([X, F_k]) \\
&= \sum_k \Psi_{-2\lambda-2\rho}([X, F_k])\frac{\partial}{\partial F_k} + \sum_k \left[\frac{\partial}{\partial F_k}, \Psi_{-2\lambda-2\rho}([X, F_k])\right], \qquad (2.4)
\end{aligned}$$

where we used the (proved) assertion for $X \in \mathfrak{l}$ in the second equality. We compute the second term in the last expression.

$$\begin{aligned}
\sum_k \left[\frac{\partial}{\partial F_k}, \Psi_{-2\lambda-2\rho}([X, F_k])\right] &= \sum_k \left[\frac{\partial}{\partial F_k}, \operatorname{ad}([X, F_k])\right] \\
&= \sum_{k,j} \left[\frac{\partial}{\partial F_k}, [[X, F_k], F_j]\frac{\partial}{\partial F_j}\right]
\end{aligned}$$

$$= \sum_{k,j} \frac{\partial}{\partial F_k}(\mathrm{ad}([X, F_j])(F_k))\frac{\partial}{\partial F_j}$$
$$= \sum_j \{\mathrm{Tr}_{\mathfrak{n}^-}\ \mathrm{ad}([X, F_j])\} \frac{\partial}{\partial F_j}$$
$$= -\sum_j 2\rho([X, F_j])\frac{\partial}{\partial F_j},$$

where we used the identity $[[X, F_k], F_j] = [[X, F_j], F_k]$ which is followed from $[F_k, F_j] = 0$ and the Jacobi identity. Then (2.4) is equal to

$$\sum_k \{\Psi_{-2\lambda-2\rho}([X, F_k]) - 2\rho([X, F_k])\} \frac{\partial}{\partial F_k} = \sum_k \Psi_{-2\lambda-4\rho}([X, F_k])\frac{\partial}{\partial F_k}$$
$$= 2\Psi_{-\lambda-2\rho}(X).$$

This completes the proof of (2).

(3) It suffices to show the assertion for $u = X \in \mathfrak{l}$ and $\lambda = 0$. We define $c_\alpha = \langle X_\alpha, X_{-\alpha}\rangle^{-1}$ for $\alpha \in \Delta_N^+$. Then $\partial/\partial X_{-\alpha} = c_\alpha X_\alpha(\partial)$ and $\{c_\alpha X_\alpha \mid \alpha \in \Delta_N^+\}$ is a dual basis of $\{X_{-\alpha} \mid \alpha \in \Delta_N^+\}$ with respect to $\langle\ ,\ \rangle$.

$$\tau(\mathrm{ad}(X)) = \tau\left(\sum_{\alpha\in\Delta_N^+} [X, X_{-\alpha}]\frac{\partial}{\partial X_{-\alpha}}\right)$$
$$= \tau\left(\sum_{\alpha\in\Delta_N^+}\sum_{\beta\in\Delta_N^+} \langle c_\beta X_\beta, [X, X_{-\alpha}]\rangle X_{-\beta}\cdot c_\alpha X_\alpha(\partial)\right)$$
$$= \sum_{\alpha,\beta} c_\alpha c_\beta \langle X_\beta, [X, X_{-\alpha}]\rangle X_{-\alpha}X_\beta(\partial)$$
$$= \sum_{\alpha,\beta} \langle c_\alpha X_{-\alpha}, [X_\beta, X]\rangle X_{-\alpha}\cdot c_\beta X_\beta(\partial)$$
$$= \sum_\beta {}^t[X_\beta, X] c_\beta X_\beta(\partial)$$
$$= \sum_\beta [{}^tX, X_{-\beta}]\frac{\partial}{\partial X_{-\beta}}.$$

This shows (3) and we have proved the lemma. □

Remark 2.5. In the actual calculation of examples, it is slightly complicated to use anti-automorphisms. We will accordingly use the automorphism $\sigma\circ\tau$ on $D_{\mathfrak{n}^+}$ instead of σ and τ. It maps

$$X_{-\alpha} \mapsto -X_\alpha(\partial),$$
$$X_\alpha(\partial) \mapsto X_{-\alpha},$$

for $\alpha \in \Delta_N^+$. In particular, we have

$$\sigma \circ \tau(\mathrm{ad}(u)) = \Psi_{-2\rho}(s({}^t u)),$$

for $u \in U(\mathfrak{l})$.

3. Main theorem

In this section we first introduce a relative invariant of a prehomogeneous vector space $(L, \mathrm{Ad}, \mathfrak{n}^+)$, and then state the main theorem. The theorem says that if there exists the classical Capelli identity for the relative invariant corresponding to some $(\mathfrak{g}, \mathfrak{p})$ then there also exists its Ψ_λ-analogue of the classical Capelli identity. Although there exists the classical Capelli identity not only for the relative invariant [3], its Ψ_λ-analogue does not exist in general.

Definition 3.1. (1) Let G be an algebraic group. A finite dimensional G-module V is called a *prehomogeneous vector space* if there exists an open G-orbit on V.

(2) A nonzero function f on V is called a *relative invariant* of (G, V), if there exists a character χ of G such that $f(gv) = \chi(g)f(v)$ for all $g \in G$ and $v \in V$.

(3) A prehomogeneous vector space (G, V) is said to be *regular* if there exists a relative invariant f of (G, V) and the Hessian $\det(\partial^2 f/\partial x_i \partial x_j)$ is not identically zero, where $\{x_i\}$ is a linear coordinate system on V.

We will consider the pairs $(\mathfrak{g}, i_0)$ of commutative parabolic type such that $(L, Ad, \mathfrak{n}^+)$ becomes a regular prehomogeneous vector space. Such pairs are $(A_{2r\text{-}1}, r)$, $(B_n, 1)$, (C_n, n), $(D_n, 1)$, $(D_{2r}, 2r)$ and $(E_7, 7)$ and their Dynkin diagrams are listed in Figure 1. We assume that $(\mathfrak{g}, \mathfrak{p})$ is of commutative parabolic type and $(L, Ad, \mathfrak{n}^+)$ is a regular prehomogeneous vector space until the end of this section.

Since $(L, Ad, \mathfrak{n}^+)$ is regular, there exists the relative invariant $f \in \mathbf{C}[\mathfrak{n}^+]$ with weight $-2\varpi_{i_0}$ ([7, Lemma 6.4]). This relative invariant f is unique up to a constant multiple and has the smallest degree among non-trivial relative invariants in $\mathbf{C}[\mathfrak{n}^+]$. Moreover all the relative invariants in $\mathbf{C}[\mathfrak{n}^+]$ are of the form f^k ($k \in \mathbf{Z}_{\geq 0}$) up to constant multiples.

Lemma 3.2. *Let $f \in \mathbf{C}[\mathfrak{n}^+]$ be the relative invariant of $(L, Ad, \mathfrak{n}^+)$ with weight $-2\varpi_{i_0}$. Then we have*

$$f^\alpha \Psi_\lambda(u) f^{-\alpha} = \Psi_{\lambda + 2\alpha\varpi_{i_0}}(u) \quad (\alpha \in \mathbf{C}, u \in U(\mathfrak{l})).$$

Proof. We have only to show the equality for $u = X \in \mathfrak{l}$. We note that $-2\varpi_{i_0}$ is not only a weight, but also a character of $\mathfrak{p}$. We have

$$\begin{aligned}[\Psi_\lambda(X), f^{-\alpha}] &= [\mathrm{ad}(X), f^{-\alpha}] \\ &= \mathrm{ad}(X)(f^{-\alpha}) \\ &= 2\alpha\varpi_{i_0}(X) f^{-\alpha}.\end{aligned}$$

Hence we have

$$\Psi_\lambda(X)f^{-\alpha} = f^{-\alpha}(\Psi_\lambda(X) + 2\alpha\varpi_{i_0}(X)) = f^{-\alpha}\Psi_{\lambda+2\alpha\varpi_{i_0}}(X).$$

This immediately gives the equality. □

Now we can state the main theorem.

Theorem 3.3. *Let* $(\mathfrak{g}, \mathfrak{p})$ *be of commutative parabolic type such that* $(L, \mathrm{Ad}, \mathfrak{n}^+)$ *is a regular prehomogeneous vector space. Let* $f \in \mathbf{C}[\mathfrak{n}^+]$ *be the relative invariant of* $(L, \mathrm{Ad}, \mathfrak{n}^+)$ *with weight* $-2\varpi_{i_0}$*, and* r *be the degree of* f*.*

(1) *For* $k \in \mathbf{Z}_{>0}$*, if there exists* $u \in U(\mathfrak{l})$ *such that*

$$f^k\, {}^tf^k(\partial) = \mathrm{ad}(u), \tag{3.1}$$

then we have

$$\Psi_\lambda(f^k\, {}^tf^k) = (-1)^{kr}\Psi_{2\lambda+2\rho}(u)\Psi_0(u). \tag{3.2}$$

(2) *For* $k \in \mathbf{Z}_{>0}$*, if there exists* $v \in U(\mathfrak{l})$ *such that*

$${}^tf^k(\partial)f^k = \mathrm{ad}(v), \tag{3.3}$$

then we have

$$\Psi_\lambda({}^tf^k f^k) = (-1)^{kr}\Psi_{2\lambda+2\rho}(v)\Psi_0(v). \tag{3.4}$$

Here $\rho \in \mathrm{Hom}(\mathfrak{p}, \mathbf{C})$ *is the half sum of roots in* Δ_N^+*.*

Remark 3.4. (1) In the theorem above, the classical Capelli identity is the $k = 1$ case of (3.1). Among the pairs listed in Figure 1, only in the case $(\mathrm{E}_7, 7)$, the classical Capelli identity does not exist ([3]). There may be, however, a possibility that $\Psi_\lambda({}^tf^k f^k)$ is in the image $\Psi_\lambda(Z(\mathfrak{l}))$ for the case $(\mathrm{E}_7, 7)$.

(2) Even in the case where the classical Capelli identity exists, either u or v in the theorem above is not unique. In Section 4, we give several choices for u and v.

Proof. We prove only (2). We can prove (1) similarly or applying Lemma 3.8 to (2).

[Step 1] In this step, we recall the $\mathrm{ad}(U(\mathfrak{l}))$-irreducible decomposition of $\mathbf{C}[\mathfrak{n}^+]$, or equivalently its $\Psi_\lambda(U(\mathfrak{l}))$-irreducible decomposition.

By Schmid's theorem ([5]), we can take the maximal set of mutually strongly orthogonal roots $\gamma_1, \ldots, \gamma_r \in \Delta_N^+$ such that

$$\mathbf{C}[\mathfrak{n}^+] = \bigoplus_\mu I_\mu \quad (\mu = k_1\lambda_1 + \cdots + k_r\lambda_r, k_j \in \mathbf{Z}_{\geq 0}),$$

where $\lambda_j = -(\gamma_1 + \cdots + \gamma_j)$ and $I_\mu \subset \mathbf{C}[\mathfrak{n}^+]$ is the unique $\mathrm{ad}(U(\mathfrak{l}))$-irreducible submodule with highest weight μ. Moreover I_μ has a homogeneous degree

$k_1+2k_2+\cdots+rk_r$. Since $\Psi_\lambda(X)=\mathrm{ad}(X)+\lambda(X)$ for $X\in\mathfrak{l}$, it is obvious that $\mathbf{C}[\mathfrak{n}^+]=\bigoplus I_\mu$ is also the $\Psi_\lambda(U(\mathfrak{l}))$-irreducible decomposition.

We take a highest weight vector $f_j\in I_{\lambda_j}$ for each j. Then f_j is of degree j and f_r is equal to the relative invariant f up to a constant multiple. Set $f_\mu=f_1^{k_1}\cdots f_r^{k_r}$. Then f_μ is a highest weight vector of I_μ and $\{f_\mu\}$ is a complete set of $\mathrm{ad}(U(\mathfrak{l}))$-maximal vectors in $\mathbf{C}[\mathfrak{n}^+]$.

[Step 2] In this step we remark that there exists $P_\lambda\in D^L_{\mathfrak{n}^+}$ such that $\Psi_\lambda({}^tf^k)=P_\lambda\,{}^tf^k(\partial)$, where $D^L_{\mathfrak{n}^+}$ denotes the subring of $D_{\mathfrak{n}^+}$ consisting of all $\mathrm{Ad}(L)$-invariant elements. Indeed, we can prove this by modifying the proof of [7, Lemma 14.1], that is, by replacing f_r in the proof by f^k.

For the theorem, we thus have only to prove that P_λ is equal to $(-1)^{kr}\Psi_{2\lambda+2\rho}(v)$.

[Step 3] In this step we define polynomials $b(\mu)$ and $\beta_\lambda(\mu)$ in $k_1,\dots,k_r$, and we show that $\beta_\lambda(\mu)=(-1)^rb(\mu)b(\mu-(\lambda^0+\rho^0)\lambda_r)$. Here λ^0 and ρ^0 are the complex number defined by $\lambda=\lambda^0\varpi_{i_0}$ and $\rho=\rho^0\varpi_{i_0}$, respectively.

Periods in the following formulas mean the action of $D_{\mathfrak{n}^+}$ on $\mathbf{C}[\mathfrak{n}^+]$. An operator ${}^tf(\partial)f$ is $\mathrm{Ad}(L)$-invariant, since ${}^tf$ has the character χ^{-1}, where $\chi\in\mathrm{Hom}(L,\mathbf{C}^\times)$ is the character corresponding to the relative invariant f. Hence ${}^tf(\partial)f.f_\mu$ has the same weight as that of f_μ, and it is still an $\mathrm{ad}(U(\mathfrak{l}))$-maximal vector. Since $\mathbf{C}[\mathfrak{n}^+]$ is multiplicity free, ${}^tf(\partial)f.f_\mu$ is a constant multiple of f_μ. We define $b(\mu)$ by

$$ {}^tf(\partial)f.f_\mu=b(\mu)f_\mu. $$

Similarly $\Psi_\lambda({}^tff)$ is $\mathrm{Ad}(L)$-invariant, since Ψ_λ is $\mathrm{Ad}(L)$-equivariant ([7, Lemma 6.5]). We define $\beta_\lambda(\mu)$ by

$$ \Psi_\lambda({}^tff)f_\mu=\beta_\lambda(\mu)f_\mu. $$

The main theorem of [7] asserts $\beta_\lambda(\mu)=(-1)^rb(\mu)b(\mu-(\lambda^0+\rho^0)\lambda_r)$. Note that $b(\mu)$ and $\beta_\lambda(\mu)$ are denoted by $b_r(\mu)$ and $\beta_{\lambda,r}(\mu)$, respectively, in [7].

[Step 4] In this step we show that $P_\lambda=(-1)^{kr}\Psi_{2\lambda+2\rho}(v)$.

We will compute $\Psi_\lambda({}^tf^kf^k)f_\mu$ and $\Psi_{2\lambda+2\rho}(v)f_\mu$ in turn. First we have

$$ \begin{aligned} \Psi_\lambda({}^tf^kf^k)f_\mu &= \Psi_\lambda({}^tf^{k-1})\Psi_\lambda({}^tff)f^{k-1}f_\mu \\ &= \Psi_\lambda({}^tf^{k-1})\Psi_\lambda({}^tff)f_{\mu+(k-1)\lambda_r} \\ &= \beta_\lambda(\mu+(k-1)\lambda_r)\Psi_\lambda({}^tf^{k-1}f^{k-1})f_\mu \\ &= \cdots \\ &= \beta_\lambda(\mu+(k-1)\lambda_r)\beta_\lambda(\mu+(k-2)\lambda_r)\cdots\beta_\lambda(\mu)f_\mu. \end{aligned} $$

Similarly we have

$$ {}^tf^k(\partial)f^k.f_\mu=b(\mu+(k-1)\lambda_r)b(\mu+(k-2)\lambda_r)\cdots b(\mu)f_\mu. $$

It follows from the above two equalities and Step 2 that

$$ \begin{aligned} (3.5)\quad P_\lambda f_\mu = (-1)^{kr}&b(\mu+(k-1-\lambda^0-\rho^0)\lambda_r)b(\mu+(k-2-\lambda^0-\rho^0)\lambda_r)\cdots \\ &\cdots b(\mu+(-\lambda^0-\rho^0)\lambda_r)f_\mu. \end{aligned} $$

Second, we have

$$\Psi_{2\lambda+2\rho}(v)f_\mu = b(\mu+(k-1-\lambda^0-\rho^0)\lambda_r)\cdots b(\mu+(-\lambda^0-\rho^0)\lambda_r)f_\mu. \tag{3.6}$$

Indeed, we can prove this formula by using Lemma 3.2 as follows:

$$\begin{aligned}
&\Psi_{2\lambda+2\rho}(v)f_\mu \\
&= f^{\lambda^0+\rho^0}\Psi_0(v)f^{-\lambda^0-\rho^0}f_\mu \\
&= f^{\lambda^0+\rho^0}\,{}^t\!f^k(\partial)f^k.f_{\mu-(\lambda^0+\rho^0)\lambda_r} \\
&= f^{\lambda^0+\rho^0}\times b(\mu-(\lambda^0+\rho^0)\lambda_r+(k-1)\lambda_r)\cdots b(\mu-(\lambda^0+\rho^0)\lambda_r)f_{\mu-(\lambda^0+\rho^0)\lambda_r} \\
&= b(\mu+(k-1-\lambda^0-\rho^0)\lambda_r)\cdots b(\mu+(-\lambda^0-\rho^0)\lambda_r)f_\mu.
\end{aligned}$$

Here two operators P_λ and $\Psi_{2\lambda+2\rho}(v)$ are $\mathrm{Ad}(L)$-invariant. Hence they are scalar operators on each $\mathrm{ad}(U(\mathfrak{l}))$-irreducible submodule of $\mathbf{C}[\mathfrak{n}^+]$. The values of the scalars are given by (3.5) and (3.6), and they coincide (up to sign). We therefore have $P_\lambda = (-1)^{kr}\Psi_{2\lambda+2\rho}(v)$ and we have proved the theorem. □

Remark 3.5. Under the notation in the proof of Theorem 3.3, let $\{g_j\}$ be a basis of I_μ and $\{g_j^*\}$ be the dual basis of ${}^tI_\mu$ with respect to the invariant perfect pairing of $\mathbf{C}[\mathfrak{n}^+]$ and $S(\mathfrak{n}^+)$ defined by $(g,p)\mapsto(p(\partial)g)(0)$. Set $z_\mu = \sum_j g_j g_j^*(\partial)$. Then z_μ is in $D^L_{\mathfrak{n}^+}$ and $\{z_\mu\}$ spans $D^L_{\mathfrak{n}^+}$ thanks to Schur's lemma. The classical Capelli identity says that we can express z_{λ_r} by an operator in $\mathrm{ad}(Z(\mathfrak{l}))$, since $z_\mu = {}^t\!ff(\partial)$ for $\mu=\lambda_r$. In fact, z_μ is in $\mathrm{ad}(Z(\mathfrak{l}))$ for all μ in the types except for $(\mathrm{E}_7, 7)$ where the classical Capelli identity does not hold ([3]). In other words, the classical Capelli identity can be generalized to all $\mathrm{ad}(\mathfrak{l})$-invariant operators z_μ. By contrast the Ψ_λ-analogue does not have such a generalization in general.

We can describe the relation between u and v in Theorem 3.3.

Definition 3.6. Define an automorphism ι_ν on $U(\mathfrak{l})$ for $\nu\in\mathrm{Hom}(\mathfrak{p},\mathbf{C})$ by

$$\iota_\nu(X) = X+\nu(X) \quad (X\in\mathfrak{l}).$$

Then we have

$$\Psi_\lambda(\iota_\nu(u)) = \Psi_{\lambda+\nu}(u) \quad (u\in U(\mathfrak{l})),$$

since $\Psi_\lambda(X) = \mathrm{ad}(X)+\lambda(X)$ for $X\in\mathfrak{l}$.

Thereby we can express the main theorem in the following way:

Corollaly 3.7. *Under the same notations and the same conditions as in Theorem 3.3, we have*

$$\Psi_\lambda(f^k\,{}^t\!f^k) = \Psi_\lambda(u_\lambda),$$
$$\Psi_\lambda({}^t\!f^k f^k) = \Psi_\lambda(v_\lambda),$$

by setting

$$u_\lambda = (-1)^{kr}\iota_{\lambda+2\rho}(u)\iota_{-\lambda}(u),$$
$$v_\lambda = (-1)^{kr}\iota_{\lambda+2\rho}(v)\iota_{-\lambda}(v).$$

The following lemma is useful in Section 4.

Lemma 3.8. *Let $f \in \mathbf{C}[\mathfrak{n}^+]$ be the relative invariant with weight $-2\varpi_{i_0}$. The anti-involution s is the same as in Lemma 2.4(2). Then we have*

(1) *For $k \in \mathbf{Z}_{>0}$, if there exists $u \in U(\mathfrak{l})$ such that*

$$f^k\, {}^t\! f^k(\partial) = \mathrm{ad}(u),$$

then we have

$${}^t\! f^k(\partial) f^k = \mathrm{ad}(\iota_{-2k\varpi_{i_0}}(u)) = (-1)^{kr}\, \mathrm{ad}(\iota_{-2\rho}(s({}^t u))).$$

(2) *For $k \in \mathbf{Z}_{>0}$, if there exists $v \in U(\mathfrak{l})$ such that*

$${}^t\! f^k(\partial) f^k = \mathrm{ad}(v),$$

then we have

$$f^k\, {}^t\! f^k(\partial) = \mathrm{ad}(\iota_{2k\varpi_{i_0}}(v)) = (-1)^{kr}\, \mathrm{ad}(\iota_{-2\rho}(s({}^t v))).$$

Proof. If $f^k\, {}^t\! f^k(\partial) = \mathrm{ad}(u)$ for some $u \in U(\mathfrak{l})$, then it follows from Lemma 3.2 that ${}^t\! f^k(\partial) f^k = f^{-k}\, \mathrm{ad}(u) f^k = \Psi_{-2k\varpi_{i_0}}(u) = \Psi_0(\iota_{-2k\varpi_{i_0}}(u))$.

Moreover by Remark 2.5, the automorphism $\sigma \circ \tau$ maps $f^k\, {}^t\! f^k(\partial)$ to $(-1)^{kr}\, {}^t\! f^k(\partial) f^k$ and $\mathrm{ad}(u)$ to $\Psi_{-2\rho}(s({}^t u)) = \mathrm{ad}(\iota_{-2\rho}(s({}^t u)))$. These prove (1). We can prove (2) similarly. □

4. Examples

In this section, we give examples of the Ψ_λ-analogue of the Capelli identity. We deal with $(\mathfrak{g}, \mathfrak{p})$ of commutative parabolic type, where $(L, \mathrm{Ad}, \mathfrak{n}^+)$ is a regular prehomogeneous vector space, and, in addition, the classical Capelli identity for the relative invariant holds. Such a pair $(\mathfrak{g}, \mathfrak{p})$ is (A_{2r-1}, r), $(\mathrm{B}_n, 1)$, (C_n, n), $(\mathrm{D}_n, 1)$ or $(\mathrm{D}_{2r}, 2r)$. These pairs correspond to $GL_r \otimes GL_r$, $O_{2n} \otimes GL_1$, $S^2 GL_n$, $O_{2n-1} \otimes GL_1$ and $\wedge^2 GL_{2r}$, respectively, in the notation of [3, (11.0.1)].

For each type, we first give a realization of $\mathfrak{g}$, give an explicit form of the relative invariant f (f^2 in $(\mathrm{D}_{2r}, 2r)$), and state the classical Capelli identity for the relative invariant and its Ψ_λ-analogue.

In the following subsections, E_{ij} denotes the matrix unit whose (i, j)-entry is equal to one, Π and Π_L denote the simple systems of $\mathfrak{g}$ and $\mathfrak{l}$, respectively, and the other notations are the same as in the preceding sections.

For a matrix whose entries belongs to not necessarily commutative ring, we recall the definition of a determinant.

Definition 4.1. Let $A = (a_{ij})_{ij}$ be a matrix with n rows and n columns. Define $\det A$ by

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n},$$

where $\varepsilon(\sigma)$ is the sign of σ.

4.1. Type (A_{2r-1}, r) or $GL_r \otimes GL_r$

In [3], the $\mathrm{ad}(\mathfrak{l})$-invariant operators on $\mathfrak{n}^+$ such as the Euler operator were expressed as operators in $\mathrm{ad}(Z(\mathfrak{l}))$. For the type (A_{2r-1}, r), the group action on $\mathfrak{n}^+$ corresponding to $(\mathfrak{gl}(2r, \mathbf{C}), r)$ were used in [3] instead of that corresponding to $(\mathfrak{sl}(2r, \mathbf{C}), r)$, since we can express the Euler operator on $\mathfrak{n}^+$ not within $\mathrm{ad}(Z(\text{Levi subalgebra of } \mathfrak{sl}))$, but within $\mathrm{ad}(Z(\text{Levi subalgebra of } \mathfrak{gl}))$. We accordingly use $(\mathfrak{gl}(2r, \mathbf{C}), r)$ for the Ψ_λ-analogue. Set $\mathfrak{g} = \mathfrak{gl}(2r, \mathbf{C})$. Let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$. We define $\varepsilon_i \in \mathfrak{h}^*$ $(i \in \{1, \ldots, 2r\})$ by $\varepsilon_i(E_{jj}) = \delta_{ij}$.

We list the information about the root system, the Chevalley basis (C. B.) which we fix, the invariant bilinear form which we fix, and so on.

$$
\begin{array}{lcl}
\Pi & = & \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{2r-1} - \varepsilon_{2r}\}, \\
\Delta^+ & = & \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le 2r\}, \\
E_{ij} & : & (\varepsilon_i - \varepsilon_j)\text{-root vector for } i \ne j, \\
\Pi_L & = & \Pi \setminus \{\varepsilon_r - \varepsilon_{r+1}\}, \\
\Delta_L^+ & = & \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le r\} \cup \{\varepsilon_i - \varepsilon_j \mid r+1 \le i < j \le 2r\}, \\
\varpi_{i_0} & = & \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r - \varepsilon_{r+1} - \cdots - \varepsilon_{2r}), \\
\rho & = & r\varpi_{i_0}, \\
\langle X, Y \rangle & = & \mathrm{Tr}(XY) \quad (X, Y \in \mathfrak{g}), \\
\text{C. B.} & : & \{E_{ij} \mid i \ne j\} \cup \{E_{ii} - E_{i+1,i+1} \mid 1 \le i < 2r\}.
\end{array}
$$

The subalgebras $\mathfrak{p}$, $\mathfrak{n}^+$ and $\mathfrak{l}$ are given explicitly by

$$
\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathfrak{g} \;\middle|\; A, B, D \in \mathfrak{gl}(r, \mathbf{C}) \right\},
$$

$$
\mathfrak{n}^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \;\middle|\; B \in \mathfrak{gl}(r, \mathbf{C}) \right\},
$$

$$
\mathfrak{l} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{g} \;\middle|\; A, D \in \mathfrak{gl}(r, \mathbf{C}) \right\}.
$$

Set $x_{ij} = E_{r+j,i}$ and $\partial_{ij} = \partial/\partial x_{ij}$ for $i, j \in \{1, \ldots, r\}$. Then $\{x_{ij}\}$ is a linear coordinate system on $\mathfrak{n}^+$. The relative invariant $f \in \mathbf{C}[\mathfrak{n}^+]$ with weight $-2\varpi_{i_0}$ is given by

$$
f = \det(x_{ij}).
$$

The celebrated (classical) Capelli identity ([2], [3]) is

$$
\det(x_{ij}) \det(\partial_{ij}) = \det \left[\sum_{k=1}^r x_{ki} \partial_{kj} + (r - j)\delta_{ij} \right]_{1 \le i,j \le r}. \tag{4.1}
$$

It follows from Lemma 2.1 (2) that

$$
\mathrm{ad}(E_{ij}) = -\sum_{k=1}^r x_{jk} \partial_{ik},
$$

$$
\mathrm{ad}(E_{r+i,r+j}) = \sum_{k=1}^r x_{ki} \partial_{kj}.
$$

We can therefore translate (4.1) into our setting, and obtain

$$f\,{}^t\!f(\partial) = \mathrm{ad}(\det[E_{r+i,r+j} + (r-j)\delta_{ij}]_{ij}). \tag{4.2}$$

Moreover we obtain the following proposition.

Proposition 4.2. *Set*

$$\begin{aligned}
u_r^R &= \det\left[E_{r+i,r+j} + (r-j)\delta_{ij}\right]_{ij},\\
u_r^{RT} &= \det\left[E_{r+j,r+i} + (j-1)\delta_{ij}\right]_{ij},\\
v_r^R &= \det\left[E_{r+i,r+j} + (r+1-j)\delta_{ij}\right]_{ij},\\
v_r^{RT} &= \det\left[E_{r+j,r+i} + j\delta_{ij}\right]_{ij},\\
u_r^{LT} &= \det\left[-E_{ji} + (r-j)\delta_{ij}\right]_{ij},\\
u_r^{L} &= \det\left[-E_{ij} + (j-1)\delta_{ij}\right]_{ij},\\
v_r^{LT} &= \det\left[-E_{ji} + (r+1-j)\delta_{ij}\right]_{ij},\\
v_r^{L} &= \det\left[-E_{ij} + j\delta_{ij}\right]_{ij}.
\end{aligned}$$

Then we have

$$\begin{aligned}
f\,{}^t\!f(\partial) &= \mathrm{ad}(u),\\
{}^t\!f(\partial)f &= \mathrm{ad}(v),
\end{aligned}$$

where u is $u_r^R, u_r^{RT}, u_r^{LT}$ or u_r^L, and v is similar. Hence we have

$$\begin{aligned}
\Psi_\lambda(f\,{}^t\!f) &= (-1)^r\Psi_{2\lambda+2\rho}(u)\Psi_0(u),\\
\Psi_\lambda({}^t\!f f) &= (-1)^r\Psi_{2\lambda+2\rho}(v)\Psi_0(v).
\end{aligned}$$

Proof. The second assertion is just the result of Theorem 3.3. Thus we have only to prove the first assertion.

In the equality (4.1), we change variables x_{ij} to x_{ji}, and then we obtain

$$\det(x_{ij})\det(\partial_{ij}) = \det\left[\sum_{k=1}^r x_{ik}\partial_{jk} + (r-j)\delta_{ij}\right]_{1\le i,j\le r}.$$

This can be translated into

$$f\,{}^t\!f(\partial) = \mathrm{ad}(\det\left[-E_{ji} + (r-j)\delta_{ij}\right]_{ij}).$$

In this way we can obtain 'L-version' from 'R-version'. Thus we have only to show the assertions of the 'R-version'.

We use Lemma 3.8. We have

$$\iota_{-2\varpi_{i_0}}(u_r^R) = \det[E_{r+i,r+j} + \delta_{ij} + (r-j)\delta_{ij}]_{ij} = v_r^R,$$

and we also have $\iota_{-2\varpi_{i_0}}(u_r^{RT}) = v_r^{RT}$. Moreover

$$(-1)^r \iota_{-2\rho}(s({}^t v_r^R)) = (-1)^r \det[-E_{r+j,r+i} - r\delta_{ij} + (r+1-j)\delta_{ij}]_{ij} = u_r^{RT},$$

and we also have $(-1)^r \iota_{-2\rho}(s({}^t v_r^{RT})) = u_r^R$. Then by Lemma 3.8 we have proved all the assertions of the 'R-version', since we already know the equality (4.2) for u_r^R. □

4.2. Type (C_n, n) or $S^2 GL_n$

We set

$$\mathfrak{g} = \mathfrak{sp}(n, \mathbf{C}) = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \in \mathfrak{gl}(2n, \mathbf{C}) \;\middle|\; \begin{array}{l} A, B, C \in \mathfrak{gl}(n, \mathbf{C}) \\ {}^tB = B,\ {}^tC = C \end{array} \right\}.$$

Let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$. Set

$$\begin{aligned} H_{ij} &= E_{ij} - E_{n+j,n+i}, \\ G_{ij} &= E_{i,n+j} + E_{j,n+i}, \\ F_{ij} &= E_{n+i,j} + E_{n+j,i}, \end{aligned}$$

for $i, j \in \{1, \ldots, n\}$. We define $\varepsilon_i \in \mathfrak{h}^*$ $(i \in \{1, \ldots, n\})$ by $\varepsilon_i(H_{jj}) = \delta_{ij}$.

We list the information about the root system, the Chevalley basis (C. B.) which we fix, the invariant bilinear form which we fix, and so on.

$$\begin{array}{lcl} \Pi & = & \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}, \\ \Delta^+ & = & \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\} \cup \{2\varepsilon_i\}, \\ H_{ij} & : & (\varepsilon_i - \varepsilon_j)\text{-root vector for } i \ne j, \\ G_{ij} & : & (\varepsilon_i + \varepsilon_j)\text{-root vector}, \\ F_{ij} & : & -(\varepsilon_i + \varepsilon_j)\text{-root vector}, \\ \Pi_L & = & \Pi \setminus \{2\varepsilon_n\}, \\ \Delta_L^+ & = & \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}, \\ \varpi_{i_0} & = & \varepsilon_1 + \cdots + \varepsilon_n, \\ \rho & = & \frac{n+1}{2}\varpi_{i_0}, \\ \langle X, Y \rangle & = & \frac{1}{2}\operatorname{Tr}(XY) \quad (X, Y \in \mathfrak{g}), \\ \text{C. B.} & : & \{H_{ij}\} \cup \{G_{ij} \mid i < j\} \cup \{\frac{1}{2}G_{ii}\} \cup \{F_{ij} \mid i < j\} \cup \{\frac{1}{2}F_{ii}\}. \end{array}$$

The subalgebras $\mathfrak{p}$, $\mathfrak{n}^+$ and $\mathfrak{l}$ are given explicitly by

$$\begin{aligned} \mathfrak{p} &= \left\{ \begin{pmatrix} A & B \\ 0 & -{}^tA \end{pmatrix} \in \mathfrak{g} \;\middle|\; A, B \in \mathfrak{gl}(n, \mathbf{C}),\ {}^tB = B \right\}, \\ \mathfrak{n}^+ &= \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \;\middle|\; B \in \mathfrak{gl}(n, \mathbf{C}),\ {}^tB = B \right\}, \\ \mathfrak{l} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} \in \mathfrak{g} \;\middle|\; A \in \mathfrak{gl}(n, \mathbf{C}) \right\}. \end{aligned}$$

Set $x_{ij} = F_{ij}$ and $\partial_{ij} = \partial/\partial x_{ij}$ for $i, j \in \{1, \ldots, n\}$. Then $\{x_{ij} \mid i \le j\}$ is a linear coordinate system on $\mathfrak{n}^+$. The relative invariant $f \in \mathbf{C}[\mathfrak{n}^+]$ with weight $-2\varpi_{i_0}$ is given by

$$f = \det(x_{ij}).$$

For $i \neq j$, ${}^t F_{ij}(\partial) = G_{ij}(\partial) = \langle G_{ij}, F_{ij}\rangle \partial_{ij} = \partial_{ij}$. By contrast, ${}^t F_{ii}(\partial) = \langle G_{ii}, F_{ii}\rangle \partial_{ii} = 2\partial_{ii}$. For this reason we set $\tilde{\partial}_{ij} = (1+\delta_{ij})\partial_{ij}$ and thereby obtain

$$ {}^t f(\partial) = \det(\tilde{\partial}_{ij}). $$

The classical Capelli identity for the type (C_n, n) ([6], [3]) is

$$ \det(x_{ij}) \det(\tilde{\partial}_{ij}) = \det \left[\sum_{k=1}^{n} x_{ik}\tilde{\partial}_{jk} + (n-j)\delta_{ij} \right]_{1 \le i,j \le n}. \tag{4.3} $$

It follows from Lemma 2.1 (2) that

$$ \mathrm{ad}(H_{ij}) = -\sum_{k=1}^{n} x_{jk}\tilde{\partial}_{ik}. $$

We therefore translate (4.3) into our setting, and obtain

$$ f\,{}^t f(\partial) = \mathrm{ad}\left(\det[-H_{ji} + (n-j)\delta_{ij}]_{ij}\right). \tag{4.4} $$

Moreover we obtain the following proposition.

Proposition 4.3. *Set*

$$ \begin{aligned} u_n^T &= \det[-H_{ji} + (n-j)\delta_{ij}]_{ij}, \\ u_n &= \det[-H_{ij} + (j-1)\delta_{ij}]_{ij}, \\ v_n^T &= \det[-H_{ji} + (n+2-j)\delta_{ij}]_{ij}, \\ v_n &= \det[-H_{ij} + (j+1)\delta_{ij}]_{ij}. \end{aligned} $$

Then we have

$$ \begin{aligned} f\,{}^t f(\partial) &= \mathrm{ad}(u), \\ {}^t f(\partial) f &= \mathrm{ad}(v), \end{aligned} $$

where u is u_n^T or u_n, and v is similar. Hence we have

$$ \begin{aligned} \Psi_\lambda(f\,{}^t f) &= (-1)^n \Psi_{2\lambda+2\rho}(u)\Psi_0(u), \\ \Psi_\lambda({}^t f f) &= (-1)^n \Psi_{2\lambda+2\rho}(v)\Psi_0(v). \end{aligned} $$

Proof. The equalities for the Ψ_λ-analogues are the result of Theorem 3.3. We will prove only the first assertion.

We use Lemma 3.8 as in Section 4.1. We easily have

$$ \begin{aligned} \iota_{-2\varpi_{i_0}}(u_n^T) &= v_n^T, \\ \iota_{-2\varpi_{i_0}}(u_n) &= v_n, \end{aligned} $$

and also have

$$(-1)^n \iota_{-2\rho}(s({}^t v_n^T)) = u_n,$$
$$(-1)^n \iota_{-2\rho}(s({}^t v_n)) = u_n^T.$$

Then by Lemma 3.8 we have proved the proposition, since we already know the equality (4.4) for u_n^T. □

4.3. Type $(\mathrm{D}_{2r}, 2r)$ or $\wedge^2 GL_{2r}$

We set $n = 2r$ and

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \in \mathfrak{gl}(2n, \mathbf{C}) \;\middle|\; \begin{array}{l} A, B, C \in \mathfrak{gl}(n, \mathbf{C}) \\ {}^tB = -B,\ {}^tC = -C \end{array} \right\}.$$

Let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$. Set

$$H_{ij} = E_{ij} - E_{n+j,n+i},$$
$$G_{ij} = E_{i,n+j} - E_{j,n+i},$$
$$F_{ij} = E_{n+j,i} - E_{n+i,j},$$

for $i, j \in \{1, \dots, n\}$. We define $\varepsilon_i \in \mathfrak{h}^*$ $(i \in \{1, \dots, n\})$ by $\varepsilon_i(H_{jj}) = \delta_{ij}$. We list the information about the root system, the Chevalley basis (C. B.) which we fix, the invariant bilinear form which we fix, and so on.

$$\begin{array}{lcl}
\Pi & = & \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}, \\
\Delta^+ & = & \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\}, \\
H_{ij} & : & (\varepsilon_i - \varepsilon_j)\text{-root vector for } i \ne j, \\
G_{ij} & : & (\varepsilon_i + \varepsilon_j)\text{-root vector for } i \ne j, \\
F_{ij} & : & -(\varepsilon_i + \varepsilon_j)\text{-root vector for } i \ne j, \\
\Pi_L & = & \Pi \setminus \{\varepsilon_{n-1} + \varepsilon_n\}, \\
\Delta_L^+ & = & \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}, \\
\varpi_{i_0} & = & \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n), \\
\rho & = & (n-1)\varpi_{i_0}, \\
\langle X, Y \rangle & = & \frac{1}{2}\operatorname{Tr}(XY) \quad (X, Y \in \mathfrak{g}), \\
\text{C. B.} & : & \{H_{ij}\} \cup \{G_{ij} \mid i < j\} \cup \{F_{ij} \mid i < j\}.
\end{array}$$

The subalgebras $\mathfrak{p}$, $\mathfrak{n}^+$ and $\mathfrak{l}$ are given explicitly by

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ 0 & -{}^tA \end{pmatrix} \in \mathfrak{g} \;\middle|\; A, B \in \mathfrak{gl}(n, \mathbf{C}),\ {}^tB = -B \right\},$$
$$\mathfrak{n}^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \;\middle|\; B \in \mathfrak{gl}(n, \mathbf{C}),\ {}^tB = -B \right\},$$
$$\mathfrak{l} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} \in \mathfrak{g} \;\middle|\; A \in \mathfrak{gl}(n, \mathbf{C}) \right\}.$$

Set $x_{ij} = F_{ij}$ and $\partial_{ij} = \partial/\partial x_{ij}$ for $i \ne j$ in $\{1, \dots, n\}$. Then $\{x_{ij} \mid i < j\}$ is a linear coordinate system on $\mathfrak{n}^+$. The relative invariant $f \in \mathbf{C}[\mathfrak{n}^+]$ with weight $-2\varpi_{i_0}$ is given by the Pfaffian

$$f = \mathrm{Pf}(x_{ij}).$$

Then we have

$${}^t f(\partial) = \mathrm{Pf}(\partial_{ij}).$$

Since the classical Capelli identity for the Pfaffian ([3]) is complicated to state, we give an example only for the determinant which is the square of the Pfaffian and has weight $-4\varpi_{i_0}$.

The classical Capelli identity for the determinant in the type $(\mathrm{D}_{2r}, 2r)$ ([3]) is

$$f^2\, {}^t f^2(\partial) = \mathrm{ad}\left(\det[-H_{ji} + (n-1-j)\delta_{ij}]_{1\le i,j\le n}\right). \tag{4.5}$$

Here we have from Lemma 2.1 (2),

$$\mathrm{ad}(H_{ij}) = -\sum_{k\neq i,j} x_{jk}\partial_{ik}.$$

Moreover we obtain the following proposition.

Proposition 4.4. *Set*

$$\begin{aligned} u_n^T &= \det[-H_{ji} + (n-1-j)\delta_{ij}]_{ij}, \\ u_n &= \det[-H_{ij} + (j-2)\delta_{ij}]_{ij}, \\ v_n^T &= \det[-H_{ji} + (n+1-j)\delta_{ij}]_{ij}, \\ v_n &= \det[-H_{ij} + j\delta_{ij}]_{ij}. \end{aligned}$$

Then we have

$$\begin{aligned} f^2\, {}^t f^2(\partial) &= \mathrm{ad}(u), \\ {}^t f^2(\partial) f^2 &= \mathrm{ad}(v), \end{aligned}$$

where u is u_n^T or u_n, and v is similar. Hence we have

$$\begin{aligned} \Psi_\lambda(f^2\, {}^t f^2) &= \Psi_{2\lambda+2\rho}(u)\Psi_0(u), \\ \Psi_\lambda({}^t f^2 f^2) &= \Psi_{2\lambda+2\rho}(v)\Psi_0(v). \end{aligned}$$

Proof. The equalities for the Ψ_λ-analogues are the result of Theorem 3.3. We will prove the first assertion.

We use Lemma 3.8 as in Section 4.1. We easily have

$$\begin{aligned} \iota_{-4\varpi_{i_0}}(u_n^T) &= v_n^T, \\ \iota_{-4\varpi_{i_0}}(u_n) &= v_n, \end{aligned}$$

and also have

$$\begin{aligned} \iota_{-2\rho}(s({}^t v_n^T)) &= u_n, \\ \iota_{-2\rho}(s({}^t v_n)) &= u_n^T. \end{aligned}$$

Then by Lemma 3.8 we have proved the proposition, since we already know the equality (4.5) for u_n^T. □

4.4. Type $(\mathrm{D}_n, 1)$ or $O_{2n-1} \otimes GL_1$

We set

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \in \mathfrak{gl}(2n, \mathbf{C}) \;\middle|\; \begin{array}{l} A, B, C \in \mathfrak{gl}(n, \mathbf{C}) \\ {}^tB = -B,\ {}^tC = -C \end{array} \right\}.$$

Let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$. Here we define basis elements of $\mathfrak{g}$ in a slightly tricky way. Set

$$H_{ij} = E_{\bar{i}\,\bar{j}} - E_{\overline{n+j},\overline{n+i}} \quad (i, j \in \mathbf{Z}_{>0}),$$

where $\bar{i}$ denotes the integer such that $1 \leq \bar{i} \leq 2n$ and $i \equiv \bar{i} \pmod{2n}$. Then every H_{ij} belongs to $\mathfrak{g}$ and it follows that

$$\begin{aligned} H_{n+i,n+j} &= -\,{}^tH_{ij} = -H_{ji}, \\ H_{i,n+i} &= 0, \end{aligned}$$

where ${}^tH_{ij}$ means the transposed matrix of H_{ij}, not the image of the anti-involution on $U(\mathfrak{g})$, although we will find that they coincide when we fix a certain Chevalley basis. We define $\varepsilon_i \in \mathfrak{h}^*$ $(i \in \{1, \ldots, n\})$ by $\varepsilon_i(H_{jj}) = \delta_{ij}$ $(j \in \{1, \ldots, n\})$.

We list the information about the root system, the Chevalley basis (C. B.) which we fix, the invariant bilinear form which we fix, and so on.

$$\begin{array}{lcl}
\Pi & = & \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}, \\
\Delta^+ & = & \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}, \\
H_{ij} & : & (\varepsilon_i - \varepsilon_j)\text{-root vector } (1 \leq i, j \leq n,\ i \neq j), \\
H_{i,n+j} & : & (\varepsilon_i + \varepsilon_j)\text{-root vector } (1 \leq i, j \leq n,\ i \neq j), \\
H_{n+j,i} & : & -(\varepsilon_i + \varepsilon_j)\text{-root vector } (1 \leq i, j \leq n,\ i \neq j), \\
\Pi_L & = & \Pi \setminus \{\varepsilon_1 - \varepsilon_2\}, \\
\Delta_L^+ & = & \{\varepsilon_i \pm \varepsilon_j \mid 1 < i < j \leq n\}, \\
\varpi_{i_0} & = & \varepsilon_1, \\
\rho & = & (n-1)\varpi_{i_0}, \\
\langle X, Y \rangle & = & \frac{1}{2}\operatorname{Tr}(XY) \quad (X, Y \in \mathfrak{g}), \\
\text{C. B.} & : & \{H_{ij} \mid 1 \leq i, j \leq n\} \cup \{H_{i,n+j} \mid 1 \leq i < j \leq n\} \\
 & & \cup \{H_{n+j,i} \mid 1 \leq i < j \leq n\}.
\end{array}$$

The subalgebras $\mathfrak{p}$, $\mathfrak{n}^+$ and $\mathfrak{l}$ are as follows:

$$\begin{aligned}
\mathfrak{l} &= \operatorname{span}_{\mathbf{C}}\{H_{11}, H_{ij}(1 < i, j \leq n), H_{i,n+j}(1 < i, j \leq n), H_{n+j,i}(1 < i, j \leq n)\}, \\
\mathfrak{n}^+ &= \operatorname{span}_{\mathbf{C}}\{H_{1j}(1 < j \leq n), H_{1,n+j}(1 < j \leq n)\}, \\
\mathfrak{p} &= \mathfrak{l} + \mathfrak{n}^+.
\end{aligned}$$

Set $x_i = H_{i1}$ and $\partial_i = \partial/\partial x_i$ for $i \in \{2,\dots,n,n+2,\dots,2n\}$. Then $\{x_i \mid 1 < i \le n \text{ or } n+1 < i \le 2n\}$ forms a linear coordinate system on $\mathfrak{n}^+$. The relative invariant $f \in \mathbf{C}[\mathfrak{n}^+]$ with weight $-2\varpi_{i_0}$ is given by

$$f = x_2 x_{n+2} + x_3 x_{n+3} + \cdots + x_n x_{2n}.$$

Moreover we have

$${}^t f(\partial) = \partial_2 \partial_{n+2} + \cdots + \partial_n \partial_{2n}.$$

The classical Capelli identity for the type $(\mathrm{D}_n, 1)$ ([3]) is

$$f\,{}^t f(\partial) = \mathrm{ad}\left(\frac{1}{4} H_{11}(H_{11} - (2n-4)) - \frac{1}{4} c\right), \tag{4.6}$$

where c is the (universal) Casimir element of $[\mathfrak{l},\mathfrak{l}]$, the semisimple part of $\mathfrak{l}$, with respect to the invariant bilinear form $\langle\ ,\ \rangle$, and

$$\mathrm{ad}(H_{11}) = -\sum_j x_j \partial_j \quad (j \in \{2,\dots,n,n+2,\dots,2n\}).$$

Note that (4.6) is one-fourth of [3, (11.4.12)] because of the difference of settings. Moreover we obtain the following proposition.

Proposition 4.5. *Set*

$$u_{2n} = \frac{1}{4} H_{11}(H_{11} - (2n-4)) - \frac{1}{4} c,$$
$$v_{2n} = \frac{1}{4} (H_{11} - 2)(H_{11} - (2n-2)) - \frac{1}{4} c.$$

Then we have

$$f\,{}^t f(\partial) = \mathrm{ad}(u_{2n}),$$
$${}^t f(\partial) f = \mathrm{ad}(v_{2n}),$$

and hence,

$$\Psi_\lambda(f\,{}^t f) = \Psi_{2\lambda+2\rho}(u_{2n}) \Psi_0(u_{2n}),$$
$$\Psi_\lambda({}^t f f) = \Psi_{2\lambda+2\rho}(v_{2n}) \Psi_0(v_{2n}).$$

Proof. As in the preceding cases, we have only to show the classical equalities. We already know the equality (4.6) corresponding to u_{2n}. We have $\iota_{-2\varpi_{i_0}}(u_{2n}) = v_{2n}$, since the Casimir element c belongs to $U([\mathfrak{l},\mathfrak{l}])$ and hence $\varpi_{i_0}(c) = 0$. Then the proposition is proved by Lemma 3.8. □

4.5. Type $(\mathrm{B}_n, 1)$ or $O_{2n} \otimes GL_1$

We set

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & a & b \\ -{}^t b & A & B \\ -{}^t a & C & -{}^t A \end{pmatrix} \in \mathfrak{gl}(2n+1,\mathbf{C}) \;\middle|\; \begin{array}{l} A, B, C \in \mathfrak{gl}(n,\mathbf{C}) \\ a, b \in \mathbf{C}^n \\ {}^t B = -B,\ {}^t C = -C \end{array} \right\}.$$

Let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$. Here we define basis elements of $\mathfrak{g}$ by extending that in the type $(\mathrm{D}_n, 1)$. We use the convention that the indices of rows and columns of matrices in $\mathfrak{g}$ begin with zero. Set

$$\begin{aligned} H_{ij} &= E_{\bar{i}\bar{j}} - E_{\overline{n+j},\overline{n+i}} \quad (i, j \in \mathbf{Z}_{>0}), \\ g_i &= E_{0\bar{i}} - E_{\overline{n+i}0} \quad (i \in \mathbf{Z}_{>0}), \end{aligned}$$

where $\bar{i}$ is the same as in the type $(\mathrm{D}_n, 1)$. Then every H_{ij} and g_i belongs to $\mathfrak{g}$ and it follows that

$$\begin{aligned} H_{n+i,n+j} &= -{}^tH_{ij} = -H_{ji}, \\ H_{i,n+i} &= 0, \\ g_{n+i} &= -{}^tg_i. \end{aligned}$$

We define $\varepsilon_i \in \mathfrak{h}^*$ $(i \in \{1, \ldots, n\})$ by $\varepsilon_i(H_{jj}) = \delta_{ij}$ $(j \in \{1, \ldots, n\})$.

We list the information about the root system, the Chevalley basis (C. B.) which we fix, the invariant bilinear form which we fix, and so on.

$$\begin{array}{lcl} \Pi & = & \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}, \\ \Delta^+ & = & \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{\varepsilon_i \mid 1 \leq i \leq n\}, \\ H_{ij} & : & (\varepsilon_i - \varepsilon_j)\text{-root vector } (1 \leq i, j \leq n,\ i \neq j), \\ H_{i,n+j} & : & (\varepsilon_i + \varepsilon_j)\text{-root vector } (1 \leq i, j \leq n,\ i \neq j), \\ H_{n+j,i} & : & -(\varepsilon_i + \varepsilon_j)\text{-root vector } (1 \leq i, j \leq n,\ i \neq j), \\ g_{n+i} & : & \varepsilon_i\text{-root vector } (1 \leq i \leq n), \\ g_i & : & -\varepsilon_i\text{-root vector } (1 \leq i \leq n), \\ \Pi_L & = & \Pi \setminus \{\varepsilon_1 - \varepsilon_2\}, \\ \Delta_L^+ & = & \{\varepsilon_i \pm \varepsilon_j \mid 1 < i < j \leq n\} \cup \{\varepsilon_i \mid 1 < i \leq n\}, \\ \varpi_{i_0} & = & \varepsilon_1, \\ \rho & = & \frac{2n-1}{2}\varpi_{i_0}, \\ \langle X, Y \rangle & = & \frac{1}{2}\operatorname{Tr}(XY) \quad (X, Y \in \mathfrak{g}), \\ \text{C. B.} & : & \{H_{ij} \mid 1 \leq i, j \leq n\} \cup \{H_{i,n+j} \mid 1 \leq i < j \leq n\} \\ & & \quad \cup \{H_{n+j,i} \mid 1 \leq i < j \leq n\} \cup \{\sqrt{-2}g_i \mid 1 \leq i \leq 2n\}. \end{array}$$

The subalgebras $\mathfrak{p}$, $\mathfrak{n}^+$ and $\mathfrak{l}$ are given explicitly by

$$\begin{aligned} \mathfrak{l} &= \operatorname{span}_{\mathbf{C}}\{H_{11}, H_{ij}(1 < i, j \leq n), H_{i,n+j}(1 < i, j \leq n), H_{n+j,i}(1 < i, j \leq n), \\ &\qquad g_i(1 < i \leq n), g_{n+i}(1 < i \leq n)\}, \\ \mathfrak{n}^+ &= \operatorname{span}_{\mathbf{C}}\{H_{1j}(1 < j \leq n), H_{1,n+j}(1 < j \leq n), g_{n+1}\}, \\ \mathfrak{p} &= \mathfrak{l} + \mathfrak{n}^+. \end{aligned}$$

Set $x_i = H_{i1}$, $x_0 = g_1$, $\partial_i = \partial/\partial x_i$ and $\partial_0 = \partial/\partial x_0$ for $i \in \{2, \ldots, n, n+2, \ldots, 2n\}$. Then $\{x_i \mid i = 0, 1 < i \leq n \text{ or } n+1 < i \leq 2n\}$ forms a linear coordinate system on $\mathfrak{n}^+$. The relative invariant $f \in \mathbf{C}[\mathfrak{n}^+]$ with weight $-2\varpi_{i_0}$ is given by

$$f = x_2x_{n+2} + x_3x_{n+3} + \cdots + x_nx_{2n} + \frac{1}{2}x_0^2.$$

Moreover we have

$$ {}^t f(\partial) = \partial_2 \partial_{n+2} + \cdots + \partial_n \partial_{2n} + \frac{1}{2} \partial_0 \partial_0. $$

The classical Capelli identity for the type $(\mathrm{B}_n, 1)$ ([3]) is

$$ f\, {}^t f(\partial) = \operatorname{ad}\left(\frac{1}{4} H_{11}(H_{11} - (2n-3)) - \frac{1}{4} c \right), \tag{4.7} $$

where c is the Casimir element of $[\mathfrak{l}, \mathfrak{l}]$ with respect to the invariant bilinear form $\langle\ ,\ \rangle$, and

$$ \operatorname{ad}(H_{11}) = -\sum_j x_j \partial_j \quad (j \in \{0, 2, \dots, n, n+2, \dots, 2n\}). $$

Note again that (4.7) is one-fourth of [3, (11.4.12)] because of the difference of settings. Moreover we obtain the following proposition.

Proposition 4.6. *Set*

$$ u_{2n+1} = \frac{1}{4} H_{11}(H_{11} - (2n-3)) - \frac{1}{4} c, $$
$$ v_{2n+1} = \frac{1}{4} (H_{11} - 2)(H_{11} - (2n-1)) - \frac{1}{4} c. $$

Then we have

$$ f\, {}^t f(\partial) = \operatorname{ad}(u_{2n+1}), $$
$$ {}^t f(\partial) f = \operatorname{ad}(v_{2n+1}), $$

and hence,

$$ \Psi_\lambda(f\, {}^t f) = \Psi_{2\lambda+2\rho}(u_{2n+1}) \Psi_0(u_{2n+1}), $$
$$ \Psi_\lambda({}^t f f) = \Psi_{2\lambda+2\rho}(v_{2n+1}) \Psi_0(v_{2n+1}). $$

Proof.
We can prove the proposition in a similar way to Proposition 4.5. □

Remark 4.7. We can state examples of type $(\mathrm{D}_n, 1)$ and $(\mathrm{B}_n, 1)$ together. Define a complex number ρ^0 by $\rho = \rho^0 \varpi_{i_0}$. Namely, $2\rho^0 = 2n - 2$ in $(\mathrm{D}_n, 1)$ and $2\rho^0 = 2n - 1$ in $(\mathrm{B}_n, 1)$. We also define $m \in \mathbf{Z}_{>0}$ as the number of rows of matrices in $\mathfrak{g}$. Namely, $m = 2n$ in $(\mathrm{D}_n, 1)$ and $m = 2n + 1$ in $(\mathrm{B}_n, 1)$. Then we can define u and v to combine Proposition 4.5 with Proposition 4.6.

$$ u_m = \frac{1}{4} H_{11}(H_{11} - 2\rho^0 + 2) - \frac{1}{4} c, $$
$$ v_m = \frac{1}{4} (H_{11} - 2)(H_{11} - 2\rho^0) - \frac{1}{4} c. $$

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References

[1] N. Bourbaki, Groupes et algèbres de Lie, Chapitres 4, 5, et 6, Hermann, Paris, 1968.

[2] A. Capelli, Sur les opérations dans la théorie des formes algébriques, Math. Ann., **37** (1890), 1–37.

[3] R. Howe, and T. Umeda, The Capelli identity, the double commutant theorem, and multiplicity-free actions, Math. Ann., **290** (1991), 565–619.

[4] J. E. Humphreys, Finite and infinite dimensional modules for semisimple Lie algebra, Lie theories and their applications, Queen's Papers in Pure and Appl. Math. no. 48, Queen's Univ., Kingston, Ont., 1978, 1–64.

[5] W. Schmid, Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen, Invent. Math., **9** (1969), 61–80.

[6] H. W. Turnbull, Symmetric determinants and the Cayley and Capelli operators, Proc. Edinb. Math. Soc. Ser. II, **8** (1947), 76–86.

[7] A. Wachi, Contravariant forms on generalized Verma modules and b-functions, Hiroshima Math. J., **29** (1999), 193–225.