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Behavior of rational curves of the minimal degree in projective space bundle in any characteristic

By

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In this paper we investigate the behavior of rational curves of the minimal degree in a projective space bundle. Particularly we try to generalize the theory of adjoint bundle of ample vector bundles to any characteristic. In characteristic zero the classifications of their adjoint bundles are made by many authors [F2], [F3], [Io], [PW], [YZ]. Then a main tool is the contraction theorem due to Mori-Kawamata and Kobayashi-Ochiai theorem. In particular they heavily depend on generalized Kodaira-Vanishing theorem in characteristic zero.
On the other hand considering Theorem by [YZ] in any characteristic, we have

Theorem 2.6. *Let X be an n-dimensional smooth projective variety defined over an algebraically closed field of any characteristic and let E be an ample vector bundle on X. Assume that $K_X + c_1(E)$ is not nef. Then we have*

1) *If* $\operatorname{rank} E = n \geq 2$*, then each line bundle of X is numerically equivalent to aL with an integer a where $L := -K_X - c_1(E)$ is an ample line bundle. In particular $-K_X$ is numerically equivalent to $(n+1)L$.*

2) *If* $\operatorname{rank} E = n - 1 \geq 3$*, X has two cases*:

I) $NS(X) \times_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$ *and X is Fano.*

II) *There is a generically bijective and finite morphism $\phi : D \to X$ from a projective variety D to X where $\psi : D \to C$ is a fiber space over a projective curve C. Moreover for each point c in C $\phi(\psi^{-1}(c))$ is a divisor swept out by rational curves parameterised by $(2n-4)$-dimensional divisor of Y.* (*See Proposition* 2.3.2 *for Y*)

Note that if E is spanned then the same conclusion (Theorems 4.7.2 and 6.16) as in Theorems 1 and 2 [YZ] is obtained. Namely X is one of $\mathbf{P}^n$, hyperquadric and scroll over a smooth curve. It is a corollary of Theorem 4.1 A, B stated just below.

Now when we study the adjoint bundle of ample vector bundle (X, E), we see that the essential point is 1) to show the existence of extremal rational

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curve in $\mathbf{P}(E)$ which is not in a fiber of a canonical projection $\mathbf{P}(E) \to X$, 2) to investigate the locus D consisting of extremal rational curves which intersect with each other and 3) to determine the base space dominated by D or a family of D.

Then note that 2) is very important and in characteristic zero that D is a fiber of contraction map induced by the extremal rational curve by virtue of a powerful tool (= a contraction map). On the other hand since we have no contraction map in positive characteristic, we analyze the structure of D directly as in Section 3. From the viewpoint we can state the usual problem of the adjoint bundle in the more general form stated in Theorem 4.1 A, B below, for example, we need not the condition of $\operatorname{rank} E < \dim X + 1$ there. Thus we get the following

Theorem 4.1.A. *Let X be an n-dimensional smooth projective variety defined over an algebraically closed field of any characteristic and let E be a* rank *r-vector bundle on X. Assume that E is ample and spanned. Moreover suppose that there is a rational curve $\bar{C}$ on $\mathbf{P}(E)$ so that $(\mathcal{O}_{P(E)}(1).\bar{C}) = 1$ and that $\bar{C}$ is not in a fiber of the canonical projection $\mathbf{P}(E) \to X$. Then if $-(K_{P(E)}.\bar{C}) = n+1$, (X, E) is isomorphic to $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus r})$.*

Theorem 4.1.B. *Let the notations and assumptions be as in Theorem* 4.1.A. *When $-(K_{P(E)}.\bar{C}) = n > 4$, h^*E is either* I) *$\mathcal{O}_{\mathbf{P}^1}(1)^{\oplus r}$ or* II) *$\mathcal{O}_{\mathbf{P}^1}(1)^{\oplus r-1} \oplus \mathcal{O}_{\mathbf{P}^1}(s)$ with $s > 1$ where $h : \mathbf{P}^1 \to \bar{C}$ is the normalization of $\bar{C}$. Moreover in case* I) *(X, E) is either*

1) a $\mathbf{P}^{n-1}$-bundle $\phi : X \to C$ over a smooth projective curve C and $E|_{\phi^{-1}(c)} \cong \mathcal{O}_{\mathbf{P}^{n-1}}(1)^{\oplus r}$; or

2) $(Q^n, \mathcal{O}_{Q^n}(1)^{\oplus r})$.

In case II) *(X, E) is either*

3) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus r-1} \oplus \mathcal{O}_{\mathbf{P}^n}(2))$; or

4) $NS(X) \times_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$ and $\mathbf{P}(E)$ is swept out by $(n-1)$-dimensional projective spaces P with $\mathcal{O}_{P(E)}(1)|_P \cong \mathcal{O}_{\mathbf{P}^{n-1}}(1)$.

Finally suppose that $\mathcal{O}_{P(E)}(1)$ is very ample. Then (X, E) in 4) is $(\mathbf{P}^n, T_{\mathbf{P}^n} \oplus \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus r-n})$ with $r \geq n$.

As a byproduct of the above Theorem we get

Theorem 6.17. *Let L be an ample line bundle on an n-dimensional smooth projective variety X defined over an algebraically closed field of any characteristic. Assume that L is spanned. Then we have the following*

1) $K_X + nL$ is nef unless (X, L) is $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$.

2) Assume that $K_X + nL$ is nef. If $n > 4$, $K_X + (n-1)L$ is nef unless

(a) X is a hyperquadric and $L = \mathcal{O}_X(1)$).

(b) (X, L) is a scroll over a smooth curve.

Moreover Theorom 4.1.A yields

Thereom 7.1. *Let X be an $n(\geq 4)$-dimensional smooth projective variety defined over an algebraically closed field of any characteristic and E an ample* *vector bundle of rank n on X. Assume that E is spanned. Then the following conditions are equivalent:*

(a) $c_n(E) = 1$

(b) $K_X + c_1(E)$ *is not nef.*

(c) (X, E) *is isomorphic to* $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus n})$.

The above result is a generalization to any characteristic of a result of J. Wisniewski [W]. At the same time it is an answer of Corollary given by [LS].

We organize this paper as follows.

In Section 1 in any characteristic we study the behavior of rational curves with the minimal degree on rational connected varieties and obtain a sufficient condition (= Corollary 1.5) for the Picard group of a rational connected variety to be isomorphic to $\mathbf{Z}$ modulo the numerical equivalence. This corollary and its argument make it possible to treat our problem easily and, under the additional assumption that E is spanned, to give a classification in positive characteristic. In this case the direct consideration of the locus D itself takes an almost same part in the dealing contraction theory.

In Section 2 without assuming that E is spanned we study (X, E) with not nef bundle $K_X + \det E$ to get Theorem 2.6.

In Section 3 we investigate the behavior of rational curves with the minimal degree in the projective space bundle. The observation gives rise to the detailed information with respect to the property of rational curves and the structure of X.

From Sections 4 to 6 the investigation of (X, E) is made under the assumption that E is spanned.

In Section 7 we state an application of Theorem 4.1. Theorem 7.1 yields a partial answer (Theorem 7.6) of the conjecture Ballico posed in [B].

Conventions and Notations. Throughout all the sections we work over the algebraically closed field of any characteristic. We use the customary terminology of algebraic geometry. $\mathcal{O}(a)$ denotes the line bundle $\mathcal{O}_{\mathbf{P}^1}(a)$ on $\mathbf{P}^1$. $\mathbf{Gr}(n.1)$ denotes the Grassmann variety parameterising lines on $\mathbf{P}^n$. $\check{E}$ is dual to a vector bundle E.

1. Preliminaries

Let A, B and T be projective varieties and let $p : B \to A$ and $q : B \to T$ surjective morphism where a general fiber of q is irreducible and of dimension ≥ 1 and where every fiber is connencted.

(1.1) Assume that for each point t in T $p : q^{-1}(t) \to A$ is a finite morphism and for each point x in A $p : q^{-1}(q(p^{-1}(x))) \to A$ is a generically finite morphism.

In this section these notations and assumption are maintained.

(1.2) For a closed subset $*$ in A a closed subscheme $S_1(*)$ of A denotes the reduced structure of $p(q^{-1}(q(p^{-1}(*))))$. We inductively define $S_m(*)$ by $S_1(S_{m-1}(*))$ where $S_0(*) = *$. Then for a point x in A a sequence $\{S_m(x)\}_{m=1,2,3,\ldots}$ is the increasing one of closed subschems in A.

Remark 1.2.1. When a point y in A is contained in $S_m(x)$, there is a subset $\{t_1, t_2, ...t_m\}$ in T so that $x \in C_{t_1}$, $y \in C_{t_m}$ and so that if $m \geq 2$, then $C_{t_i} \cap C_{t_{i+1}}$ is not empty for $i = 1, ...m$ where C_* denotes $p(q^{-1}(*))$.

From now on we study a condition for $NS(A) \times_{\mathbf{Z}} \mathbf{Q}$ to be isomorphic to $\mathbf{Q}$. For the purpose we make a preparation.

Lemma 1.3. *Let the notations and the assumptions be as in* (1.1) *and* (1.2)*. Let L and M be line bundles on A and T respectively. Assume that $p^*L = q^*M$. Then we have*

1) *Let W be a closed subscheme of A. If $L|_W$ is numerically equivalent to zero (hereafter written as $L|_W \approx \mathcal{O}$), then $M|_{q(p^{-1}(W))} \approx \mathcal{O}$.*

2) *Let V be a closed subscheme of T. If $M|_V \approx \mathcal{O}$, then $L|_{p(q^{-1}(V))} \approx \mathcal{O}$.*

Proof. This proposition is obtained from the following
Fact: Let $f : X \to Y$ be a morphism between complete algebraic schemes and D, E line bundles on X and Y respectively with $f^*E = D$ If $E \approx \mathcal{O}$, then $D \approx \mathcal{O}$. Moreover the converse holds if f is surjective. □

Therefore we get

Corollary 1.4. *Let the line bundles L and M be as in Lemma* 1.3*. Let x be a point in A and for a point t in T set a closed subscheme $p(q^{-1}(t))(\subset A)$ as C_t. Then for each positive integer m $L|_{S_m(x)}$ and $L|_{S_m(C_t)}$ are numerically equivalent to zero. Similarly $M|_{q(p^{-1}(S_{m-1}(x)))}$ and $M|_{q(p^{-1}(S_{m-1}(C_t)))}$ are numerically equivalent to zero.*

Moreover we get

Corollary 1.5. *Let us maintain the condition and the notation in* (1.1) *and* (1.2)*. Let A, B, T, p and q be as above. Assume that q is a fiber bundle with the fiber F and that Picard group of F is isomorphic to* $\mathbf{Z}$*. Assume, moreover, that there are a point x in A and an integer m so that $S_m(x) = A$. Then $NS(A) \times_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$.*

Proof. Let N be an ample line bundle and L a line bundle on A. Then $p^*N|_F \cong \mathcal{O}_F(a)$ and $p^*L|_F \cong \mathcal{O}_F(b)$ with integers $a(> 0), b$. By virtue of base change theorem we infer that $p^*(L^{\otimes a} \otimes N^{\otimes -b}) = q^*M$ with some line bundle M on B. Thus $L^{\otimes a}$ is numerically equivalent to $N^{\otimes -b}$ from Corollary 1.4. □

Next we state a condition for A to have a fiber structure.

Let $\bar{p}(:= p \times \mathrm{id}_A) : B \times A \to A \times A$ and $\bar{q}(:= q \times \mathrm{id}_A) : B \times A \to T \times A$ where Δ is the diagonal of A. Then we take Δ as $*$ in (1.1) and do the same as (1.1) for $\bar{p}$ and $\bar{q}$. Then we can define the closed subscheme $S_m(\Delta)$ of $A \times A$ for each positive integer m in the same way as above. Remark that for each point x in A $S_m(\Delta) \cap (A \times x) = S_m(x) \times \{x\}$

Thus we pose the following conditions:

(1.6) 1) for each point x in A and for each positive integer m, $S_m(x)$ is a proper closed set in A.
2) There are two integers $a(\leq n-1), m_0$ and an open set U in A so that for each point x in U, dim $S_m(x) = a$ for each integer $m \geq m_0$.

From the above condition we can take an open set $\bar{A} \subset U$ in A and an irreducible component $\mathcal{D}'$ of $S_{m_0}(\Delta) \subset A \times A$ satisfying the following:

1) for each point x in $\bar{A}$ D_x is an irreducible component of $S_{m_0}(x)$ which is of a-dimension where $\mathcal{D}' \cap (A \times \{x\}) = D_x \times \{x\}$.

2) an induced morphism $\mathcal{D}' \cap A \times \bar{A} \to \bar{A}$ is flat.

Remark that D_x is contained in $S_m(x)$ for each point x in A.

We assume that

(1.7) A is smooth and $2a \geq \dim A$.

Then the flat family $\{D_x\}_{x \in \bar{A}}$ forms an algebraic family.

Moreover we have a

Claim. Under the condition (1.6) and the assumption (1.7) there are two points $x_1, x_2 \in \bar{A}$ (by shrinking $\bar{A}$ further) so that D_{x_1} and D_{x_2} do not intersect.

In fact if otherwise, we see from $2a \geq \dim A$ for any two points $x_1, x_2 \in \bar{A}$ D_{x_1} and D_{x_2} intersect. Take a point u in D_{x_1} and D_{x_2}. Since u is contained in both $S_{m_0}(x_1)$ and $S_{m_0}(x_2)$, the point x_1 is contained in $S_{2m_0}(x_2)$ from Remark (1.2.1). Thus for any point $x \in \bar{A}$ $S_{2m_0}(x)$ is a closed subset in X which contains $\bar{A}$ and therefore coincides X, which induces a contradiction to 1.6.1.

(1.8) Take a Hilbert scheme $\bar{\Lambda}$ induced by the algebraic family $\{D_x\}_{x \in \bar{A}}$. Moreover let $\bar{\mathcal{D}}$ be the universal space over $\bar{\Lambda}$. Then by the universality we get a morphism $f : \bar{A} \to \bar{\Lambda}$. Set the closure of $f(\bar{A})$ of $\bar{\Lambda}$ as Λ. Then let $\mathcal{D}$ be $\bar{\mathcal{D}} \times_{\bar{\Lambda}} \Lambda$. Let $\phi : \mathcal{D} \to A$ and $\psi : \mathcal{D} \to \Lambda$ be canonical projections and $D_\lambda = \phi\psi^{-1}(\lambda)$.

From the above argument we have

Proposition 1.9. *Let us maintain the condition* (1.1) *and the notations* (1.2) *and* (1.6)*. Assume that A is smooth and $2a \geq \dim A$. Then there are projective varieties $\mathcal{D}, \Lambda$ and surjective morphisms $\phi : \mathcal{D} \to A$, $\psi : \mathcal{D} \to \Lambda$ so that ϕ is a generically bijective morphism which is injective on $\psi^{-1}(\Lambda_0)$ with some open set Λ_0 in Λ. Moreover if the characteristic of the base field is zero, then ϕ is a birational morphism which is an isomorphism on $\psi^{-1}(\Lambda_0)$.*

Proof. Take an open set Λ_0 of $f(\bar{A})$ $(\subset \Lambda)$. The remainder is easily shown. $\square$

Moreover we get

Corollary 1.10. *Let the condition and notations be as in Proposition* 1.9*. Assume that A is smooth and of n-dimension with $a = n-1$. Then for each point x in A $\dim S_m(x) = n-1$ for each positive integer $m \geq 1$ and a canonical morphism $\phi : \mathcal{D} \to A$ is finite and generically bijective. Moreover if the characteristic of the base field is zero, then ϕ is an isomorphism. Thus for each couple λ, λ' in Λ, $D_\lambda \cap D_{\lambda'}$ is empty.*

Proof. We have only to show that ψ is finite. But since $\dim \Lambda = 1$ it is obvious from 1.5. $\square$

Remark 1.11. When the characteristic of the base field is arbitrary, we cannot check whether the ϕ is an isomorphism. Even if the morphism ϕ is an isomorphism, a fiber of $\psi : A(\cong D) \to \Lambda$, is possibly singular in positive characteristic. When the characteristic is 2 or 3, it is known that there exists a smooth surface which is a fiber space over a smooth curve where every fiber is a singular curve.

2. $K_X + c_1(E)$ is not nef

In this section we construct a family of rational curves in X induced by an extremal rational curve and investigate the property of the family.

Let X be an n-dimensional smooth projective variety and E an ample vector bundle on X of rank $r(\leq n)$. These notations X, E are maintained hereafter.

First we begin with the following

Proposition 2.1. *Let C be a rational curve on X. Then C does not deform to a sum of effective 1-cycles, if one of the following conditions about X or E holds*

1) *C is an extremal rational curve on X where* $(C. - K_X) = \min\{(\bar{C}. - K_X)|\bar{C}$ *is a rational curve on $X\}$. See* [Io]

2) *C is an extremal rational curve on X where* $(K_X + c_1(E).C) < 0$ *and* $\operatorname{rank} E \geq n/2 + 1$.

3) *There is an ample line bundle L on X with $(C.L) = 1$.*

Proof. 1) is shown in [Io]. We show 2). First the following is well-known.

Sublemma. *Let F be an ample vector bundle of rank r on a rational curve. Then deg $F \geq r$.*

Assume that C deforms to a sum of effective 1-cycles $\Sigma a_i C_i$ where a cycle C_i is an irreducible and reduced curve with integer a_i. Then C_i is a rational curve and $(C_i. \det E) \geq n/2 + 1$ from sublemma. Thus $(C. - K_X) \geq n + 2$, which yields a contradiction to $(C. - K_X) \leq n + 1$ by Theorem 4 in [Mo]. 3) is trivial. □

Hereafter in this section we assume that

(2.2) C is a rational curve on X which does not deform to a sum of effective 1-cycles.

(2.3) Let $\phi : \mathbf{P}^1 \to C$ be the normalisation of C and let us take a Hilbert scheme $\operatorname{Hom}(\mathbf{P}^1, X)$ of the morphism ϕ. Then it is known by virtue of Grothendieck that

(2.3.1) $\dim_{[\phi]} \operatorname{Hom}(\mathbf{P}^1, X) \geq \chi(\mathbf{P}^1, \phi^* T_X) = (-K_X.C) + n.$

Letting e the left-hand side of the above inequality, we have an e dimensional irreducible component V containing the morphism ϕ so that the automorphism group G of $\mathbf{P}^1$ acts naturally on the normalisation $\bar{V}$ of V. Similarly G acts naturally on $V \times \mathbf{P}^1$ and therefore on $\bar{V} \times \mathbf{P}^1$.

Thus we can define a morphism
$\dot{\pi} : V \to \mathrm{Chow}_X^{\bar{d}}$ by $\dot{\pi}(v) =$ the cycle of $v(\mathbf{P}^1)$. Here $\bar{d} = (v(\mathbf{P}^1). \det E)$ and Chow_X^d is Chow variety parameterizing 1-dimensional effective cycles C such that $(C, \det E) = d$. Thus let Y be the normalisation of the closure of $\dot{\pi}(V)$ in $\mathrm{Chow}_X^{\bar{d}}$.

In [Mo] the following is shown:

Proposition 2.3.2. *Let C be a rational curve on a smooth projective variety X. Assume that C does not deform to a sum of effective 1-cycles. Then there exists a geometric quotient of $\bar{V}$ by G which is isomorphic to Y and a geometric quotient of $\bar{V} \times \mathbf{P}^1$ by G written by Z.*

(2.3.3) Let $p : Z \to X$ and $q : Z \to Y$ be canonical morphisms where q is a $\mathbf{P}^1$-bundle. Here $\dim Y = e - \dim \mathrm{Aut}\, \mathbf{P}^1 = e - 3$.

Let $C_y = p(q^{-1}(y))$. We state a property of the morphism p.

Proposition 2.4. *Let conditions and notations be as in 2.2 and 2.3. Assume that $\dim p^{-1}(x) \geq 1$ for each point x in $p(Z)$. Then for each point x in $p(Z)$, the morphism p restricted to $q^{-1}(q(p^{-1}(x))) - (p^{-1}(x))(=: D) \to X$ is quasi-finite. Particularly $\dim p^{-1}(x) \leq n - 1$.*

Proof. Assume that $p|_D$ is not quasi-finite. When $\dim p(D) = 1$, $p(D) = p(q^{-1}(y))$ for each point y in $q(p^{-1}(x))$, which yields a contradiction. Next consider the case of $\dim p^{-1}(x) \geq 1$. Then $\dim p(D) \geq 2$. Thus we can take a point A in $p(D) - x$ and a projective curve C in D such that $p(C) = A$. The ruled surface $Z \times_Y q(C)(:= S)$ has two curves $S \cap p^{-1}(x), C$ which do not intersect and which go to two points x, A via p. This yields a contradiction by virtue of Theorem 4 in [Mo]. The last statement is trivial. □

Moreover we have

Proposition 2.5. *Let X and E be as above and $r = \mathrm{rank}\, E$. Assume that $K_X + c_1(E)$ is not nef. Let C be an extremal rational curve on X satisfying the inequality $(K_X + c_1(E).C) < 0$ and the assumption 2.2. Then under the notations Y, Z in Proposition 2.3.2, we have*

1) *If $-(K_X.C) \geq n$, then p is surjective and $2n - 3 \leq \dim Y \leq 2n - 2$.*

2) *When $\dim Y = 2n - 2$, the morphism p has an equi-dimensional fiber whose dimension is $n - 1$ and for each point x in X, $p(q^{-1}(q(p^{-1}(x)))) = X$. Hence $NS(X) \times_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$.*

3) *When $\dim Y = 2n - 3$, we have the following two cases:*

3.1) *There is a point x in X and an integer m such that $S_m(x) = X$.*

3.2) *$\dim S_m(x) < n$ for each point x in X and for each positive integer m.*

i) *In case of 3.1 $NS(X) \times_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$.*

ii) *In case of 3.2 there is a finite and generally bijective morphism $\phi : \mathcal{D} \to X$ from a projective variety $\mathcal{D}$ to X where $\psi : \mathcal{D} \to C$ is a fiber space over a* *projective curve C. Moreover for each point c in C $\phi(\psi^{-1}(c))$ is a divisor swept out by rational curves parameterised by $(2n-4)$-dimensional divisor of Y.*

Proof. We show 1). First $(-K_X.C) \geq n$ yields $2n-3 \leq \dim Y$ by (2.3.1) and Proposition 2.3.2. Thus $\dim Z \geq 2n-2$. First assume that p is not surjective, namely $\dim p(Z) \leq n-1$. Take a point x in $p(Z)$. Then $\dim p^{-1}(x) \geq n-1$ and therefore $\dim q^{-1}(q(p^{-1}(x))) \geq n$. Hence $q^{-1}(q(p^{-1}(x))) - p^{-1}(x)(=: D) \to p(Z)(\subset X)$ is not quasi-finite, a contradiction to Proposition 2.4. Similarly we have the inequality: $\dim Y \leq 2n-2$. Next let us consider 2). Then p is surjective from the above argument of 1). The condition implies $\dim Z \geq 2n-2$. Thus the first statement of 2) is shown from Proposition 2.4. The rest is obtained from Corollary 1.5. Finally we show 3). First note that p is surjective from the proof of the above 1). When X has a property of $S_m(x) = X$ for a point x in X an integer m, i) is obtained by Corollary 1.5. As for the other case, since $\dim q^{-1}(q(p^{-1}(x))) \geq n-1$ for any point x in X, we get $\dim S_m(x) = n-1$. Thus ii) follows from Proposition 1.10. □

Theorem 2.6. *Let X be an n-dimensional smooth projective variety defined over an algebraically closed field and E an ample vector bundle on X of* rank $r(\leq n)$. *Assume that $K_X + c_1(E)$ is not nef. Then we have*

1) *If $r = n > 1$, then $NS(X) \times_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$ and $-K_X - c(E)$ is an ample line bundle. Moreover $-K_X$ is numerically equivalent to $(n+1)L$ where $L = -K_X - c(E)$. Therefore X is Fano.*

2) *If $r = n-1$ and $n \geq 4$, then X is* i) *or* ii) *of* 3) *in Proposition* 2.5.

Proof. First since rank $E \geq n/2+1$, (2.2) is satisfied by Proposition 2.1. We consider 1). The assumption implies that there is an extremal rational curve C on X so that $(K_X + c_1(E).C) < 0$ and that $n+1 \geq -(K_X, C)$. Since $-(K_X, C) > (c_1(E).C)$ and $(c_1(E).C) \geq n$ from sublemma of Proposition 2.1 we see that $n+1 = -(K_X, C)$ and $(c_1(E).C) = n$. Thus under the notations in Proposition 2.3.2 we get $\dim Y = 2n-2$ from (2.3.1) and Proposition 2.5, which implies the former part. On the other hand since $(K_X + c_1(E).C) = -1$, we get the latter part.

Next we consider 2). In the same way as 1) we get $-(K_X, C) = n+1, n$ and the desired facts by Proposition 2.5. □

3. Rational curves on projective space bundle

Let E be an ample vector bundle of rank r on an n-dimensional smooth projective variety X. Let $\bar{C}$ be a rational curve on $\mathbf{P}(E)$ and $\pi : \mathbf{P}(E) \to X$ the canonical projection.

Assume that

(3.1) $(\bar{C}.\mathcal{O}_{P(E)}(1)) = 1$ and $\bar{C}$ is not in a fiber of π.

Our aim of this section is to investigate the deformation of $\bar{C}$ in $\mathbf{P}(E)$ and to study the property of (X, E) with $-(K_{\mathcal{X}}.\bar{C}) = n+1$ or n. As a consequence we can determine the structure of (X, E) with $-(K_X.C) = n{+}1$ or n in question stated in Sections 1 and 2.

Let $\mathcal{X} = \mathbf{P}(E)$, $\xi = \mathcal{O}_{P(E)}(1)$ and $g : \tilde{C} \to \bar{C}$ be the normalization of $\bar{C}$. Then considering a Hilbert scheme $\operatorname{Hom}(\mathbf{P}^1, \mathcal{X})$ at the morphism g, we see that
(3.2) $\dim_{[g]} \operatorname{Hom}(\mathbf{P}^1, \mathcal{X}) \geq \chi(\mathbf{P}^1, g^*T_{\mathcal{X}}) = -(K_{\mathcal{X}}.\bar{C}) + n + r - 1$
Here [g] means a point corresponding to the morphism g in $\operatorname{Hom}(\mathbf{P}^1, \mathcal{X})$

Thus take an irreducible component $V_{\mathcal{X}}$ of $\operatorname{Hom}(\mathbf{P}^1, \mathcal{X})$ containing the morphism $[g]$ where $\dim V_{\mathcal{X}} \geq \chi(\mathbf{P}^1, g^*T_{\mathcal{X}})$. Let $\tilde{V}_{\mathcal{X}}$ be the normalization of $V_{\mathcal{X}}$. Then by Mori's method (see Lemma 9 in [Mo]) $(\bar{C}.\xi) = 1$ provides us, from Proposition 2.1 and 2.3.2, with

Proposition 3.3. *Let the conditions and notations be as in* 3.1 *and* 3.2. *Then there are projective varieties $\mathcal{Y}, \mathcal{Z}$ and morphisms $a : \mathcal{Z} \to \mathcal{X}, b : \mathcal{Z} \to \mathcal{Y}$ where b is $\mathbf{P}^1$-bundle and* $\dim \mathcal{Y} \geq -(K_{\mathcal{X}}.\bar{C}) + n + r - 4$. *Here $\mathcal{Y}, \mathcal{Z}$ are geometric quotients of $\tilde{V}_{\mathcal{X}}$ and $\tilde{V}_{\mathcal{X}} \times \mathbf{P}^1$ by* $\operatorname{Aut} \mathbf{P}^1$ *respectively.*

Remark 3.3.1. Let $\operatorname{Chow}^{d'}_{\mathcal{X}}$ be Chow variety parameterizing 1-dimensional effective cycles C of $\mathcal{X}$ such that $(C.\xi) = d'$. Then we have a canonical morphism $\pi' : V_{\mathcal{X}} \to \operatorname{Chow}^1_{\mathcal{X}}$ by $\pi'(v) =$ the cycle of $v(\mathbf{P^1}))$ modulo rational equivalence where $(v(\mathbf{P}^1).\xi) = 1$. Then note that $\mathcal{Y}$ is the normalization of the image of π' in $\operatorname{Chow}^1_{\mathcal{X}}$.

Moreover we have

Corollary 3.4. *Let $\bar{C}_t = a(b^{-1}(t))$ for a point t in $\mathcal{Y}$. Then we have*
1) *for each point t in $\mathcal{Y}$, $\bar{C}_t$ is an irreducible rational curve with $(\bar{C}_t.\xi) = 1$ and a canonical morphism $b^{-1}(t) \to \bar{C}_t$ induced by a is birational.*
2) *$\pi : \bar{C}_t \to \pi(\bar{C}_t)$ is birational.*
3) *Moreover $(\bar{C}_t.\pi^*E)$ and $(\bar{C}_t.\pi^*K_X)$ are independent of a choice of the point t in $\mathcal{Y}$.*

Proof. 1) is trivial from $(\bar{C}.\xi) = 1$. 2) follows from the following

Sublemma. *Let F be an ample vector bundle of* rank r *on $\mathbf{P}^1$. Let C be a rational curve on $\mathbf{P}(F)$ which is not in a fiber of a canonical projection $\pi : \mathbf{P}(F) \to \mathbf{P}^1$. Assume that $(\mathcal{O}_{\mathbf{P}(F)}(1).C) = 1$. Then F has $\mathcal{O}_{\mathbf{P}^1}(1)$ as a direct summand and C is a section induced by a line bundle $\mathcal{O}_{\mathbf{P}^1}(1)$. Namely $\pi : C \to \pi(C)$ is an isomorphism.*

In fact we assume that C is not a section of π. Taking the normalisation of $\phi : \bar{C} \to C$ the fiber product $\mathbf{P}(F) \times_{\mathbf{P}^1} \bar{C} (\cong \mathbf{P}((\phi\pi)^*F))$ has a section $\tilde{C}$ of a canonical projection $\mathbf{P}((\phi\pi)^*F) \to \bar{C}$, which is induced by the curve C. Hence we infer that $(\mathcal{O}_{P((\phi\pi)*F)}(1).\tilde{C}) = (\mathcal{O}_{\mathbf{P}(F)}(1).C) = 1$, which means that $(\phi\pi)^*F$ has a quotient line bundle $\mathcal{O}_{\mathbf{P}^1}(1)$, which turn out to be a direct summand, because $(\phi\pi)^*F$ is ample. On the other hand since the ample vector bundle F is isomorphic to $\oplus_{i=1}^r \mathcal{O}_{\mathbf{P}^1}(b_i)$ where $1 \leq b_1 \leq b_2 \leq \ldots \leq b_r$, $(\phi\pi)^*F$

is isomorphic to $\oplus_{i=1}^{r}\mathcal{O}_{\mathbf{P}^1}(db_i)$. Here d is the degree of $\phi\pi \geq 2$. Thus we get a contradiction, as desired.

3) is trivial. $\square$

(3.4.1) G denotes $\mathrm{Aut}\,\mathbf{P}^1$. G acts natually on $\mathrm{Hom}(\mathbf{P}^1, \mathcal{X})$ and $\mathrm{Hom}(\mathbf{P}^1, X)$ respectively. By 2) of Corollary 3.4 we see that for the G-stable component $V_{\mathcal{X}}$ there are a G-stable component V_X of $\mathrm{Hom}(\mathbf{P}^1, X)$ and a canonical morphism $\tilde{\pi} : V_{\mathcal{X}} \to V_X$ defined by mapping g in $V_{\mathcal{X}}$ to $\tilde{\pi}(g)$ in V_X where $\tilde{\pi}(g) : \mathbf{P}^1 \to (\pi g)(\mathbf{P}^1)$ is the normalization of $(\pi g)(\mathbf{P}^1)$ by virtue of the universality of $\mathrm{Hom}(\mathbf{P}^1, X)$. Particularly $\tilde{\pi}$ is a G-morphism. Then we have a canonical morphism $e : \tilde{\pi}(V_{\mathcal{X}}) \to \mathrm{Chow}_X^d$ by $\dot{\pi}(v) =$ the cycle of $v(\mathbf{P}^1)$ modulo rational equivalence where $d = (\pi v(\mathbf{P}^1).\det E)$ and where $\mathrm{Chow}_X^{d'}$ is Chow variety parameterizing 1-dimensional effective cycles C of X such that $(C.\det E) = d'$. Let Y be the normalization of $e(\tilde{\pi}(V_{\mathcal{X}}))$ in Chow_X^d. Then Y is normal projective variety which is a geometric quotient of the normalization V' of $\tilde{\pi}(V_{\mathcal{X}})$ by G by virtue of Lemma 9 in [Mo2]. Moreover G acts natually on $V' \times \mathbf{P}^1$ and therefore we have the geometric quotient Z of $V' \times \mathbf{P}^1$ by G and a canonical morphism $\bar{\pi} : \mathcal{Y} \to Y$. Let $p : Z \to X$ and $q : Z \to Y$ be canonical morphisms where p is surjective and q a $\mathbf{P}^1$-bundle.

Remark 3.5. 1) For a point t in $\mathcal{Y}$ $\bar{C}_t$ is induced by a direct summand $\mathcal{O}_{\mathbf{P}^1}(1)$ of $p^*E|_{q^{-1}(\bar{\pi}(y))}$ where $p^*E|_{q^{-1}(\bar{\pi}(y))} \cong \oplus_{i=1}^{r}\mathcal{O}_{\mathbf{P}^1}(a_i)$. Thus $\dim \mathcal{Y} - \dim \bar{\pi}(y) \leq r-1$.

2) $\dim \mathcal{Y} - \dim \bar{\pi}(\mathcal{Y}) = r-1$ if and only if $p^*E|_{q^{-1}(y)} \cong \oplus^r \mathcal{O}_{\mathbf{P}^1}(1)$ for any point $y \in \bar{\pi}(\mathcal{Y})$. Then the morphism $\mathcal{Y} \to \bar{\pi}(\mathcal{Y})$ is of equi-(r-1) dimension.

3) If $\dim \mathcal{Y} - \dim \bar{\pi}(\mathcal{Y}) = r-2$, then for any point $y \in \bar{\pi}(\mathcal{Y})$ $p^*E|_{q^{-1}(y)} \cong \oplus^{r-1}\mathcal{O}_{\mathbf{P}^1}(1)\oplus\mathcal{O}_{\mathbf{P}^1}(s)$ with some positive integer s.
In fact 1) and 2) are trivial by the construction of $\bar{\pi}$. 3) is obtained by the ampleness of E.

Next we state the facts about $\dim \mathcal{Y}$ and the properties of a closed subscheme $S_m(x)$ in $a(\mathcal{Z})$ defined in (1.2) for a general point x on $\mathcal{X}$.

Proposition 3.6. *Let the condition and notation be as in* 3.1 *and* 3.3. *Then for each x in $\mathcal{X}$, we have the following*:

1) *for each positive integer m a canonical morphism $\pi^{\cdot} : S_m(x) \to X$ induced by π is a finite morphism. Hence* $\dim S_m(x) \leq n$.

2) $\dim \mathcal{Z} - \dim a(\mathcal{Z}) \leq n-1$. *Moreover* $\dim \mathcal{Z} - \dim a(\mathcal{Z}) = n-1$ *if and only if the morphism $a; \mathcal{Z} \to a(\mathcal{Z})$ is the one of* $(n-1)$ *equi-dimensional fiber. Consequently if* $\dim \mathcal{Z} - \dim a(\mathcal{Z}) = n-1$, *for each point x in $\mathcal{X}$ and for each positive integer m* $\dim S_m(x) = n$ *and a canonical morphism $\pi^{\cdot}$ of* 1) *is a finite surjective morphism.*

3) $\dim \mathcal{Y} \leq 2n+r-3$. *If* $\dim \mathcal{Y} = 2n+r-3$, *then the morphism $a; \mathcal{Z} \to \mathcal{X}$ is surjective and* $\dim \mathcal{Z} - \dim a(\mathcal{Z}) = n-1$.

4) $-(K_{\mathcal{X}}.\bar{C}) \leq n+1$.

5) *If* $-(K_{\mathcal{X}}.\bar{C}) = n+1$, *then "if" parts in both* 2) *and* 3) *hold.*

Proof. Let $j = (\pi^*\det E.\bar{C})$ and let us set $j\xi - \pi^*\det E$ as $\mathcal{L}$. Then we see from Corollary 1.4 that $\mathcal{L}|_{S_m(x)}$ is numerically equivalent to zero. On the other hand, assume the contrary of 1), namely there is a curve D in $S_m(x)$ which is contained in some fiber of π. Then since $\mathcal{L}|_D = j\xi|_D$, it is ample, a contradiction. As for 2) assume that $\dim \mathcal{Z} - \dim a(\mathcal{Z}) \geq n$. Since $\dim b^{-1}(b(a^{-1}(x))) = n+1$, $b^{-1}(b(a^{-1}(x))) - a^{-1}(x) \to X$ is not quasi-finite. Noting that $(\xi, \bar{C}) = 1$, we have a contradiction to Proposition 2.4. Thus we get all of 2) at the same time. 3) is shown in the same manner as in 2). As for 4) assume that $-(K_{\mathcal{X}}.\bar{C}) \geq n+2$. Then $\dim \mathcal{Y} \geq \chi(\mathbf{P}^1, g^*T_{\mathcal{X}}) - \dim \operatorname{Aut} \mathbf{P}^1 = -(K_{\mathcal{X}}.\bar{C}) + (n+r-1)\chi(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}) - 3 \geq 2n+r-2$. Then we get the same contradiction as in 2), which implies 5). $\square$

The above argument of Proposition 3.6 says

(3.6.1) Note from Proposition 3.3 that
If $-(K_{\mathcal{X}}.\bar{C}) = n$ and if $a(\mathcal{Z})$ is a proper subvariety of $\mathcal{X}$, then $\dim \mathcal{Z} - \dim a(\mathcal{Z}) = n-1$, $\dim a(\mathcal{Z}) = n+r-2$ and the last part of 2) in Proposition 3.6 holds for any point $x \in a(\mathcal{Z})$.

Corollary 3.6.2. *Let $u := \dim \mathcal{Z} - \dim a(\mathcal{Z})$. Then we have*

1) $\dim S_m(x) \geq u+1$ *for each point x in $a(\mathcal{Z})$ and for each positive integer m.*

2) *Assume that there is a point x in $a(\mathcal{Z})$ where* $\dim S_2(x) = u+1$. *Then there are a $(u+1)$-dimensional irreducible component S_x of $S_2(x)$ and an $2u$-dimensional irreducible component T_x of $ba^{-1}(S_1(x))$ enjoying the following: $S_x = a(b^{-1}(T_x))$ and for two points x_1, x_2 in S_x there exists a point y of T_x with $a(b^{-1}(y)) \ni x_1, x_2$ Namely S_x is swept out by a family of rational curves parameterized by a $2u$-dimensional closed subvariety of $\mathcal{Y}$.*

Proof. 1) is trivial by Proposition 2.4. Consider 2). Since $S_m(x) \subset S_{m+1}(x) = a(b^{-1}(b(a^{-1}(S_m(x)))))$, we can take an irreducible component V of $S_1(x)$ with $\dim V = u+1$. We have a
Claim: $\dim b(a^{-1}(V)) = 2u$.
In fact it is clear from Proposition 2.4 that $\dim b(a^{-1}(V))$ is $2u$ or $2u+1$. Assume that $\dim b(a^{-1}(V)) = 2u+1$. Then since $S_1(V) \subset S_2(x)$, we get $\dim b^{-1}(b(a^{-1}(V))) = 2u+2$. Hence we infer that for a point $x' \in V(\subset S_1(V))$ $\dim a^{-1}(x') = u+1$ and therefore $\dim b(a^{-1}(x')) = u+1$ and $\dim b^{-1}(b(a^{-1}(x'))) = u+2$. On the other hand since $S_1(x') \subset S_2(x)$ and therefore $\dim S_1(x') = u+1$, the induced morphism $b^{-1}(b(a^{-1}(x'))) - a^{-1}(x') \to S_1(x')$ is not quasi-finite, a contradiction from Proposition 2.4.

Now take an irreducible component W of $b(a^{-1}(V))$ which is of $2u$-dimension. Then $a(b^{-1}W)$ is the desired irreducible component of $S_2(x)$ which is of $u+1$ dimension. Thus we have only to set V and W as S_x and T_x. $\square$

Next we state the relation between $\mathcal{Y}$ and $\bar{\pi}(\mathcal{Y})$.

Proposition 3.7. *Let $\mathcal{Y}$ and $\bar{\pi}(\mathcal{Y})$ be as stated above.*

1) *If* $\dim \mathcal{Y} = 2n + r - s$ *with* $s \geq 3$, *then* $\dim \mathcal{Y} - \dim \bar{\pi}(\mathcal{Y}) \geq r - s + 2$.

2) *If* $\dim \mathcal{Y} = 2n + r - 3$, *then* $(c(E).C) = r$, $\dim \bar{\pi}(\mathcal{Y}) = 2n - 2$ *and for any point* $y \in \bar{\pi}(\mathcal{Y})$ $p^*E|_{q^{-1}(y)} \cong \oplus^r \mathcal{O}_{\mathbf{P}^1}(1)$. *Moreover if* $\dim \mathcal{Y} = 2n + r - 4$, *then* $\dim \mathcal{Y} - \dim \bar{\pi}(\mathcal{Y}) = r - 1$ *or* $r - 2$ *and therefore* $\dim \bar{\pi}(\mathcal{Y}) = 2n - 3$ *or* $2n - 2$ *respectively.*

Proof. Assume that $\dim \mathcal{Y} - \dim \bar{\pi}(\mathcal{Y}) < r - s + 2$. From 1) of Remark 3.5 $\dim \mathcal{Y} - \dim \bar{\pi}(\mathcal{Y}) < r - 1$, namely $\dim \bar{\pi}(\mathcal{Y}) > 2n - 2$. Therefore it turns out that there is a at least (2n-1)-dimensional family of irreducible rational curves on X. By bend and break theory of rational curves there is a curve C_y with some point $y \in \bar{\pi}(\mathcal{Y})$, so that C_y decomposes to a sum of rational curves, a contradiction to the property of $\bar{\pi}(\mathcal{Y})$. The remainder is obtained from Remark 3.5. □

Let $C = \pi(\bar{C})$. From now on we consider the case of
(3.8) $-(K_{\mathcal{X}}.\bar{C}) = n + 1$.

We immediately get from Proposition 3.6

Corollary 3.8. *Assume that* $-(K_{\mathcal{X}}.\bar{C}) = n + 1$. *Then we have*

1) $-(K_X.C) = n + 1$ *and* $(c_1(E).C) = r$. *Hence* $p^*E|q^{-1}(t) \cong \oplus^r \mathcal{O}_{\mathbf{P}^1}(1)$ *for each* t *in* Y.

2) $\dim \bar{\pi}(\mathcal{Y}) = 2n - 2$ *and moreover each fiber of the morphism* $\bar{\pi} : \mathcal{Y} \to \bar{\pi}(\mathcal{Y})$ *is of (r-1)-dimension.*

3) *For every point* x *in* $\mathcal{X}$, *the conclusion in Corollary* 3.6.2 *holds with* $u = n - 1$. *Consequently* $NS(X) \times_{\mathbf{Z}} \mathbf{Q}$ *is isomorphic to* $\mathbf{Q}$.

Proof. Since $-K_{\mathcal{X}} = r\xi - \pi^*(K_X + c(E))$, we get $-\pi^*(K_X + c(E).\bar{C}) = n+1-r$ and therefore $-(K_X + c(E).C) = n+1-r$ from 2) of Corollary 3.4. On the other hand from 5) of Proposition 3.6 we infer that $\dim \mathcal{Y} = 2n+r-3$. Thus from 2) of Proposition 3.7 we get $(c(E).C) = r$ and therefore $-(K_X.C) = n+1$ and others. 2) follows from 2) of Remark 3.5 and the above 1). The former of 3) is obtained from 3) of Proposition 3.6. The final part of 3) is obtained from Corollary 1.5 by letting $A = X, B = \mathcal{Z}$ and $T = \mathcal{Y}$. □

Next we consider the case of
(3.9) $-(K_{\mathcal{X}}.\bar{C}) = n$.

Then we get
(3.9.1) $-(K_X + c(E).C) = n - r$.

Hence we have $0 \geq r - (c(E).C) = n + (K_X.C)$.

In this case we have, from Proposition 3.3 and from 3) in Proposition 3.6, following two cases:

$\dim \mathcal{Y} = 2n + r - 3$ and $= 2n + r - 4$.

First we consider the case of
(3.10.1) $\dim \mathcal{Y} = 2n + r - 3$.

From Proposition 3.7 we get $(c(E).C) = r$ and therefore $-(K_X.C) = n$. Moreover for each point x in $\mathcal{X}$ $\dim S_m(x) = \dim \mathcal{Z} - \dim a(\mathcal{Z}) + 1$ by 2) and 3) in Proposition 3.6. This case does not happen under the assumption that E is spanned. (see (4.8))

Next we consider the case
(3.10.2) $\dim \mathcal{Y} = 2n + r - 4$.

We see by (2) of Proposition 3.7 that $\dim \mathcal{Y} - \dim \bar{\pi}(\mathcal{Y})$ is $r - 1$ or $r - 2$ and that $\dim \bar{\pi}(\mathcal{Y})$ is $2n - 3$ or $2n - 2$ respectively. In the former case we have $(c(E).C) = r$ and in the latter case $(c(E).C) > r$ by 2) of Remark 3.5. Consequently for any point $y \in \bar{\pi}(\mathcal{Y})$ $p^*E|_{q^{-1}(y)} \cong \oplus^{r-1}\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(s)$ with some positive integer s by 2) of Remark 3.5. Moreover $\dim S_m(x) \geq n - 1$ for each point x in $\mathcal{X}$ and for each positive integer m.

Here we divide the morphism $a : \mathcal{Z} \to \mathcal{X}$ into two cases:
(3.10.2.1) a is not surjective

We see from 2) of Proposition 3.6 and $\dim a(\mathcal{Z}) = n + r - 2$ therefore that $a : \mathcal{Z} \to \mathcal{X}$ is of an (n-1)-dimensional fiber. Hence for each point x in $a(\mathcal{Z})$ $\dim S_m(x) = n$ for each positive integer m.
(3.10.2.2) a is surjective. (Thus a general fiber of a is of dimension $n - 2$.)

Then noting the properties of $S_m(\Delta)$ before (1.6), we have the following table of (3.10.2.2) for a general point x in $\mathcal{X}$.

	$\dim \bar{\pi}(\mathcal{Y})$	$\dim S_2(x)$	$(c(E).C)$	$p^*E\|_{q^{-1}(y)}$	$-(K_X.C)$
$\gamma.1$	$2n-3$	$n-1$ *or* n	r	$\mathcal{O}(1)^{\oplus r}$	n
$\gamma.2$	$2n-2$	$n-1$ *or* n	$\geq r+1$	$\mathcal{O}(1)^{\oplus r-1} \oplus \mathcal{O}(s)$	$\geq n+1$

where $s \geq 2$. We give a

Remark 3.10.3. Let us consider the case of $\dim S_2(x) = n$ in the table before. Similarly in the proof in 2) of Corollary 3.6.2 there is an irreducible subvariety T_x of $\mathcal{Y}$ so that 1) $\dim T_x = 2n - 3$ and 2) $S_x(:= pq^{-1}(T_x))$ is an n-dimensional irreducible component of $S_2(x)$. There is an open set R in S_x so that S_x is smooth around R and contains an $(n-1)$-dimensional irreducible component of $S_1(x')$ for each point x' in R.

4. Proof of Theorem 4.1 (I)

In this section let E be an ample vector bundle of rank r on a smooth projective variety X. We maintain the notations $X, E, \mathcal{X}(= \mathbf{P}(E))$ and so on in Section 3. Moreover throughout in this section we assume that
(4.0) E is spanned.

We determine the structure (X, E) and get the following:

Theorem 4.1.A. *Let X be an n-dimensional smooth projective variety defined over the algebraically closed field in any characteristic and let E be a* rank *r-vector bundle on X. Assume that E is ample and spanned. Moreover* *suppose that there is a rational curve* $\bar{C}$ *on* $\mathbf{P}(E)$ *so that* $(\mathcal{O}_{P(E)}(1).\bar{C}) = 1$ *and that* $\bar{C}$ *is not in a fiber of a canonical projection* $\mathbf{P}(E) \to X$ *. Then when* $-(K_{P(E)}.\bar{C}) = n+1$ (3.8), (X, E) *is isomorphic to* 1) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus r})$.

Theorem 4.1.B. *Let the notations and assumptions be as in Theorem* 4.1.A. *When* $-(K_{P(E)}.\bar{C}) = n > 4$ (3.9), h^*E *is either* I) $\mathcal{O}_{\mathbf{P}^1}(1)^{\oplus r}$ *or* II) $\mathcal{O}_{\mathbf{P}^1}(1)^{\oplus r-1} \oplus \mathcal{O}_{\mathbf{P}^1}(s)$ *with* $s > 1$ *where* $h : \mathbf{P}^1 \to \bar{C}$ *is the normalization of* $\bar{C}$. *Moreover in case* I) (X, E) *is either*

1) $\phi : X \to C$ *has a* $\mathbf{P}^{n-1}$*-bundle structure over a smooth projective curve* C *and* $E|_{\phi^{-1}(c)} \cong \mathcal{O}_{\mathbf{P}^{n-1}}(1)^{\oplus r}$. (3.10.2.2. γ.1) *or*

2) $(Q^n, \mathcal{O}_{Q^n}(1)^{\oplus r})$. (3.10.2.2. γ.1)

In case II) (X, E) *is either*

3) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus r-1} \oplus \mathcal{O}_{\mathbf{P}^n}(2))$. (3.10.2.1) *or*

4) $NS(X) \times_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$ *and* $\mathbf{P}(E)$ *is swept out by* $(n-1)$*-dimensional projective spaces* P *with* $\mathcal{O}_{P(E)}(1)|_P \cong \mathcal{O}_{\mathbf{P}^{n-1}}(1)$. (3.10.2.2. γ.2)

Finally suppose that $\mathcal{O}_{P(E)}(1)$ *is very ample. Then* (X, E) *in* 4) *is* $(\mathbf{P}^n, T_{\mathbf{P}^n} \oplus \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus r-n})$ *with* $r \geq n$. (3.10.2.2. γ.2)

See (4.7) for the case of (3.8), (4.10) for (3.10.2.1), and (5.15), (6.15) for (3.10.2.2. γ.1 γ.2) respectively.

By the assumption that E is spanned, the tautological line bundle $\mathcal{O}_{P(E)}(1)$ $(=: \xi)$ of E is base points free.

Let $\phi : \mathcal{X} \to \mathbf{P}^N$ be a morphism with $h^0(X, \xi) = N+1$. Then by $(\bar{C}_t.\xi) = 1$, we see that $\bar{C}_t$ is a smooth rational curve for each point t in $\mathcal{Y}$. Then to determine the structure of S_x, we begin with the following

Proposition 4.2. *Let* U, V *and* W *be projective varieties and* $p : W \to U$ *surjectivre morphism and* $q : W \to V$ $\mathbf{P}^1$*-bundle. Let* L *be an ample spanned line bundle on* U *which induces a morphism* $\phi : U \to \mathbf{P}^N$ *with* $h^0(U, L) = N+1$. *Assume that for each* v *in* V,

1) *the set* $\{v' | C_v = C'_v\}$ *is a finite set with* $C_v = p(q^{-1}(v))$,

2) $(L.C_v) = 1$, *and*

3) $\dim V = 2\dim U - 2$.

Then ϕ *is an isomorphism and* $(U, L) \cong (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$.

Proof. First we have

Claim 1. For any two points u_1, u_2 in U there are finitely many points $\{v\}$ in V so that u_1, u_2 are contained in a rational curve C_v.
Remarking that C_v does not decompose to a sum of 1-cycles by the condition 2, we can easily show it in the same way as in Proposition 2.4.

Let $S = \phi(U)$ and $l_v = \phi(C_v)$. Then we have

Claim 2. $S = \mathbf{P}^n$ and $n = N$.
Indeed, take a general and smooth point A in S. Then the tangent space $T_{S,A}$ at the point A in S is an n-dimensional linear subspace in $\mathbf{P}^N$ and it contains a line l_v passing through the point A. On the other hand the closed subscheme $\{v \in V | l_v \ni A\} (= \bar{V})$ of V is of $n-1$ dimension from the assumption and

the finiteness of ϕ. Hence by claim 1 the set $\bigcup\{l_v | v \in \bar{V}\}$ coincides $T_{S,A}$ and therefore S itself, required.
(4.2.1) By the universality of Hilbert scheme of lines on $\mathbf{P}^n$ we have a canonical morphism $\bar{\phi} : V \to \mathbf{Gr}(n,1)$ which is finite and surjective by the assumption of 1), 3) and claim 1.

Let U_{n-1} be a general member of $|L|$.

Claim 3. U_{n-1} is irreducible and reduced.
From the assumptions 2) and claim 1 the reducedness of U_{n-1} is obvious. Moreover when $n > 1$, the irreducibility is obtained by virtue of Bertini's Theorem, as desired.

For the proof we have only to show that the degree of ϕ is 1. We can assume that U is normal. We show this proposition by induction on n.

When $n = 1$, it is trivial.

Next take a general member U_{n-1} of $|L|$. Then it is irreducible and reduced by claim 3. Moreover the $(n-1)$-dimensional linear space $\phi(U_{n-1})$ induces a closed subvariety $\mathbf{Gr}(n-1,1)$ in $\mathbf{Gr}(n,1)$. Then we have an irreducible component V_{n-1} of $\bar{\phi}^{-1}(\mathbf{Gr}(n-1,1))$ with $pq^{-1}(V_{n-1}) = U_{n-1}$ and let $W_{n-1} = q^{-1}(V_{n-1})$. Thus by induction assumption $\phi|_{U_{n-1}}$ is an isomorphism and $(U_{n-1}, L|_{U_{n-1}}) \cong (\mathbf{P}^{n-1}, \mathcal{O}_{\mathbf{P}^{n-1}}(1))$. Consequently degree of ϕ is 1, as required $\square$

Thus combining 2) of Corollary 3.6.2 and Proposition 4.2, we get

Corollary 4.3. *Let the condition and notation be as in* 3.1 *and* 3.3. *Assume that there is a point x in $\mathcal{X}$ so that* $\dim S_1(x) = \dim S_2(x) = \dim \mathcal{Z} - \dim a(\mathcal{Z}) + 1 (:= d)$. *Moreover suppose that E is spanned. Then there is a d-dimensional irreducible component S_x of $S_2(x)$ as in* 2) *of Corollary* 3.6.2 *where* $(S_x, \xi_{|S_x}) \cong (\mathbf{P}^d, \mathcal{O}_{\mathbf{P}^d}(1))$. *Namely the morphism $\phi_{|S_x}$ is isomorphism.*

Proof. $S_2(x)$ provides us with S_x, T_x used in the notations of Corollary 3.6.2. Since ξ is base points free, the assumptions in Proposition 4.2 are satisfied, as desired. $\square$

Next to show that the image of above S_x via the canonical projection $\pi : \mathcal{X} \to X$ is a projective space we make several preliminaries to give a sufficient condition for the image of $\bar{C}_t$ via the canonical projection π to be $\mathbf{P}^1$.

Proposition 4.4. *Let F be a vector bundle on a rational curve D. Assume that* $g^*F \cong \oplus_{i=1}^m (\mathcal{O}_{\mathbf{P}^1}(a_i))^{\oplus r_i}$ *with the normalisation* $g : \mathbf{P}^1 \to D$ *of D where* $1 = a_1 < a_2 <, \ldots < a_r$. *Assume that F is spanned. Then we have*

1) *if D is a singular curve and A is a singular point in D, then* $h(\pi^{-1}(A)) (\cong \mathbf{P}^{\Sigma_{i=1}^m r_i - 1})$ *contains* $h(\bar{g}(\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1))^{\oplus r_1}))$ *which is of dimension r_1. Here* $\pi : \mathbf{P}(F) \to D$ *and* $\bar{g} : \mathbf{P}(g^*F) \to \mathbf{P}(F)$ *are canonical morphisms and* $h : \mathbf{P}(F) \to \mathbf{P}^N$ *a morphism induced by a line bundle* $\mathcal{O}_{\mathbf{P}(F)}(1)$.

2) *D is smooth either if $m = 1$ or if $\mathcal{O}_{\mathbf{P}(F)}(1)$ is very ample.*

Proof. Take a section $\bar{D}(:= \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1)))$ of $\mathbf{P}(g^*F)$ corresponding to a direct summand $\mathcal{O}_{\mathbf{P}^1}(1)$ of g^*F. Since $(\mathcal{O}_{\mathbf{P}(g^*F)}(1).\bar{D}) = 1$, we see that an induced morphism : $\bar{D} \to h\bar{g}(\bar{D})$ via $h\bar{g}$ is an isomorphism and that $h\bar{g}(\bar{D})$ is a line. Thus $\bar{g} : \bar{D} \to \bar{g}(\bar{D})$ is an isomorhism and $\bar{g}(\bar{D})$ is a smooth rational curve.

Then we have two cases: the morphism g is a) not a homeomorphism around $g^{-1}(A)$ b) a homeomorphism around $g^{-1}(A)$. Let $r = \Sigma_{i=1}^m r_i$.

In case of a) there are two points R_1, R_2 on $\mathbf{P}^1$ with $g(R_i) = A$. Then we infer that each fiber $\bar{\pi}^{-1}(R_i)$ for $i = 1, 2$ goes to the same $(r-1)$-plane P in $\mathbf{P}^N$ via the morphism $h\bar{g}$. On the other hand $h(\bar{g}(\bar{D}))$ is a line. Thus this line is contained in the $(r-1)$-plane P. Consequently $h(\bar{g}(\mathbf{P}(M))$ is contained in P where M is a subbundle $\mathcal{O}_{\mathbf{P}^1}(1)^{\oplus r_1}$ of g^*F. In case of b) take a point R on $\mathbf{P}^1$ with $g(R) = A$. Then the section $\bar{D}$ tangents to the fiber $\bar{\pi}^{-1}(R)$. Thus we get the phenomena as the first case a). Hence 1) is shown. Remarking that $\dim \mathbf{P}(M) = r_1$ because of the finiteness of $h\bar{g}$ we can show 2) by 1). □

The above immediately yields

Corollary 4.4.1. *Under the condition* 3.1 *let* $\bar{C}_t$ *be as in Corollary* 3.4 *and* $\bar{\pi}$ *as in* 3.5. *Assume that* $p^*E|_{q^{-1}(y)}(\cong \oplus_{i=1}^m \mathcal{O}_{\mathbf{P}^1}(a_i)^{\oplus r_i})$ *is independent of a choice of a point* y *in* $\bar{\pi}(\mathcal{Y})$. *Moreover assume that* E *is spanned. Then we have the following:*

1) *the image of* $\bar{C}_t$ *via the canonical projection* π *is* $\mathbf{P}^1$ *and* $\pi : \bar{C}_t \to \pi(\bar{C}_t)$ *is an isomorphism either if* $m = 1$ *or if* $\mathcal{O}_{\mathbf{P}(E)}(1)$ *is very ample.*

2) *Let* $m = 2, r_2 = 1$ *and* $\mathrm{Sing}(Y) = \{y \in \bar{\pi}(\mathcal{Y}) | p(q^{-1}(y))$ *is singular*$\}$ *a closed set in* $\bar{\pi}(\mathcal{Y})$. *Then for each point* x *in* X $\mathrm{Sing}(Y, x)\{:= \{y \in \mathrm{Sing}(Y)| p(q^{-1}(y))$ *is singular at* $x\}$ *is at most finite many points.*

Proof. 1) is trivial. For 2) remark that for each point y in $\mathrm{Sing}(Y)$ the set $\{y' \in \mathrm{Sing}(Y) | p(q^{-1}(y) = p(q^{-1}(y')\}$ is at most finite many points. Assume the contrary of 2) namely there is a point x in X so that $\mathrm{Sing}(Y, x)$ is of positive dimension. Then noting that $p^*E|_{q^{-1}(y)} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus r-1} \oplus \mathcal{O}_{\mathbf{P}^1}(a_2)$, we see that for each point y in $\mathrm{Sing}(Y, x)$ a divisor in $\mathbf{P}(E|_{p(q^{-1}(y))})$ corresponding to a divisor $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1)^{\oplus r-1})$ in $\mathbf{P}(p^*E|_{q^{-1}(y)})$ is tranformed via ϕ to a unique fiber $\pi^{-1}(x)(\cong \mathbf{P}^{r-1})$, a contradiction to the finiteness of ϕ by 1) of Proposition 4.4. □

Now we give an easy

Proposition 4.5. *Let* A, B *be varieties and* $f : B \to A$ *a morphism. Assume that there is a smooth point* x *in* B *whose image* $f(x)$ *is a smooth point in* A *and that there is no non-zero tangent vector* v *at the point* x *which goes to zero vector via the linear map* $df_{*,x}$. *Then* $f : B \to f(B)$ *is separable.*

Proof. Assume that $f : B \to f(B)$ is not separable. Then for the point x there is a non-zero tangent vector v of $T_{B,x}$ so that the differential map $df_{*,x}v$ is zero which yields a contradiction to the assumption. □

Note that each non-zero tangent vector on $\mathbf{P}^n$ is obtained by the one induced by a line. Thus we have

Corollary 4.6. *Let A be a smooth variety and $f : P^n \to A$ a quasi-finite morphism. Assume that for each line l on $\mathbf{P}^n$ an induced morphism $f : l \to f(l)$ is an isomorphism. Then $f(\mathbf{P}^n)$ is smooth and $f : \mathbf{P}^n \to f(\mathbf{P}^n)$ is isomorphism.*

Proof. First since there is a line passing through any two point, the assumption implies that f is injective. From the above argument we see that for each point x in $\mathbf{P}^n$ the differential map $df_{*,x}v : T_{\mathbf{P}^n}, x \to T_{A,f(x)}$ is injective. Hence from the injectivity of f we infer that $f(\mathbf{P}^n)$ is smooth and $f : \mathbf{P}^n \to f(\mathbf{P}^n)$ is etale. Thus f is an isomorphism. $\square$

Corollary 4.6.1. *Let us maintain the condition* (3.10.2.1) *and* (γ.2) *of* (3.10.2.2)*. Assume that E is spanned. Then for a general point x in $a(\mathcal{Z}) \subset \mathcal{X}$ S_x is a n (or, $n-1$)-dimensional projective space respectively. An induced morphism $\pi : S_x \to \pi(S_x)$ is separable.*

Proof. We have an argument of the case (3.10.2.1). The other case follows in the same way. The former part is trivial by Corollary 4.3 since $\dim S_m(x) = \dim \mathcal{Z} - \dim a(\mathcal{Z}) + 1 = n$ for each positive integer m. Next we consider the latter part. Let $S = S_x$. Then there is an (2n-2)-dimensional closed subvariety T in $\mathcal{Y}$ where $a(b^{-1}(T)) = S$ and T is isomorphic to $\mathbf{Gr}(n.1)$ by 2) of Corollary 3.6.2. Note that for each point t in T a curve $\bar{C}_t$ is a line on S and that $\bar{\pi}(T) = \bar{\pi}(\mathcal{Y})$. Now we assume the contrary of the conclusion. (#) For each point x in S there is a point t in T so that $\pi(\bar{C}_t)$ is a cuspidal curve passing through cuspidal point $\pi(x)$ by Proposition 4.5. The image of such a point t via $\bar{\pi}$ is contained in $\mathrm{Sing}(Y)$ of 2) in Corollary 4.4.1. Here a cuspidal curve C means the one so that there is a birational morphism $j : < \to C$ where $<$ is a cubic plane curve with the cusp singularity R and the cuspidal point means the image $j(R)$. We have a

Claim: For each point x in S there is a unique point t satisfying (#).

In fact assume that there is another point t' in T so that $\pi(\bar{C}_{t'})$ is a cuspidal curve passing through cuspidal point $\pi(x)$ Then we see easily that any line $\bar{C}_{t''}$), through the point x on the 2- plane ($\subset S$) generated by two line $\bar{C}_t$, $\bar{C}_{t'}$ is projected via the morphism π to a cuspidal curve with the cuspidal point $\pi(x)$, a contradiction to 2) in Corollary 4.4.1.

For the inseparable morphism $\pi : S \to X$ we consider the homomorphism $h : T_{\mathbf{P}^n} \to \pi^* T_X$. Then Claim implies that there is an exact sequence:

$0 \to L \to T_{\mathbf{P}^n} \to F \to 0$

where L is the kernel of h and F the image of h. Then L is a line bundle and F a rank (m-1) vector bundle on $\mathbf{P}^n$. But there does not exist such an exact sequence on $\mathbf{P}^n$. $\square$

At last we have come to the proof of Theorem 4.1. We investigate the case

(4.7) $-(K_{\mathcal{X}}.\bar{C}) = n+1$.

Take a point x in $\mathcal{X}$. Then by 2) and 5) of Proposition 3.6, we infer that $\dim S_m(x) = n$ for each positive integer m. Thus Corollary 4.3 implies that $S_x \cong \mathbf{P}^n$ and $\xi|_{S_x} \cong \mathcal{O}_{\mathbf{P}^n}(1)$. Moreover 1) of Corollaries 3.8, 4.4.1 and 4.6 imply that the morphism $\pi' : S_x \to X$ is isomorphism. Moreover we infer that E is a uniform vector bundle from Corollary 3.8, where for each line l, $E|_l \cong \oplus^r \mathcal{O}_{\mathbf{P}^n}(1)$. By Proposition 4.2 in [Sa1]. we get

Theorem 4.7.1. *Let $X, E, \bar{C}$ be as in Theorem* 4.1 *and let us maintain the assimption in Theorem* 4.1. *If $-(K_{\mathcal{X}}.\bar{C}) = n+1$, then (X, E) is isomorphic to $(\mathbf{P}^n, \oplus^r \mathcal{O}_{\mathbf{P}^n}(1))$.*

The above immediately yields

Theorem 4.7.2. *Let E be an ample vector bundle of* rank $r = n+1$ *or n on an n-dimensional smooth projective variety X. Assume that E is spanned. Moreover assume that if $r = n+1$, then $K_X + c_1(E)$ is numerically trivial and that if $r = n$, then $K_X + c_1(E)$ is not nef. Then (X, E) is isomorphic to $(\mathbf{P}^n, \oplus^r \mathcal{O}_{\mathbf{P}^n}(1))$.*

Proof. First we consider the first case. Then since $-K_X$ is numerically equivalent to an ample line bundle $c(E)$, X is Fano. Thus there is an extremal rational curve C on X so that $-(K_X.C) \leq n+1$. On the other hand by sublemma in Proposition 2.1 we infer that $-(K_X.C) = n+1$ and $(c(E).C) = n+1$. Take the nomarization $\phi : \bar{C} \to C$ of C. Then $\phi^* E \cong \mathcal{O}(1)^{\oplus n+1}$, whose direct summand $\mathcal{O}(1)$ yields a rational curve $\tilde{C}$ on $\mathcal{X}$ with $(\tilde{C}.\mathcal{O}_{\mathbf{P}(E)}(1)) = 1$. Thus $-K_{\mathcal{X}}$ is numerically equivalent to $(n+1)\mathcal{O}_{\mathbf{P}(E)}(1)$ and therefore $-(K_{\mathcal{X}}.\bar{C}) = n+1$. Since $\tilde{C}$ is not in the fiber of the projection $\mathbf{P}(E) \to X$ we get the desired fact by Theorem 4.7.1.

Next we consider the second case. By Theorem 2.6 $-K_X$ is numerically equivalent to $(n+1)(-K_X - c(E))$. Thus there is an extremal rational curve C on X so that $-(K_X.C) \leq n+1$. In the same manner as above we see that $(c(E).C) = n$, $-(K_X.C) = n+1$ and therefore $\phi^* E \cong \mathcal{O}(1)^{\oplus n}$. Consequently $-(K_{\mathcal{X}}.\bar{C}) = n+1$. Hence we get the desired result. $\square$

Next under the assumption (4.0) we investigate the case $-(K_{\mathcal{X}}.\bar{C}) = n$.

(4.8) We show that the case (3.10.1) does not happen.

First by $\dim \mathcal{Y} = 2n + r - 3$, we see that $p^* E|_{q^{-1}(y)} \cong \mathcal{O}_{\mathbf{P}^1}(1)^r$ by 2) of Proposition 3.7. Since $\dim S_m(x) = \dim \mathcal{Z} - \dim a(\mathcal{Z}) + 1 = n$ for each positive integer m. we see X is isomorphic to $\mathbf{P}^n$ from the above argument to obtain Theorem 4.7.1. Thus it follows that $-(K_X.C) = -(K_{\mathbf{P}^n}.C) = n+1$, a contradiction to $-(K_X.C) = n$.

Next we give a

Lemma 4.9. *G is a* rank r *uniform vector bundle on $\mathbf{P}^n$ where for a line l $G|_l = \oplus_{i=1}^r \mathcal{O}(a_i)$ with $0 = a_1 \geq a_2 \geq, ..., \geq a_r$. Assume that G has a non-zero* *section. Then G has a trivial line bundle and the quotient vector bundle G' is a uniform vector bundle with $G'|_l = \oplus_{i=2}^r \mathcal{O}(a_i)$.*

Proof. A non-zero section s yields a homomorphism $s : \mathcal{O} \to G$. Assume that there is a point x in $\mathbf{P}^n$ so that $s(x) = 0$. Then take a line C passing through the point x such that the induced homomorphism $s|_C : \mathcal{O}_C \to G_C$ is a non-zero homomorphism. On the other hand we see easily that a non-zero section of $G|_l = \oplus_{i=1}^r \mathcal{O}(a_i)$ with $0 = a_1 \geq a_2 \geq, ..., \geq a_r$ vanishes nowhere, we get a contradiction as for the induced homomorphism $s|_C$. $\square$

(4.10) Here we treat the case (3.10.2.1).

We take a rational curve $\bar{C}_t$ in $t \in \mathcal{Y}$ and construct S_x as above. We infer that S_x is a projective space $\mathbf{P}^n$. Let $S = S_x$. Next if $s = 1$ in (3.10.2), $\pi|_S : S \to X$ is an isomorphism in the same way as in the proof of Theorem 4.7.1. We get $-(K_X.C) = n + 1$, a contradiction. If $s > 1$, $\pi|_S : S \to X$ is separable by Corollary 4.6.1.

From now on we show that π is etale and therefore an isomorphism. For the sake we consider the following function:

$g : X \ni x \mapsto \#(\pi|_S^{-1}(x)) \in \mathbf{N}$

Then the function is lower semi-continuous in the Zariski topology. Next note that $S = \cup_{y \in T} \bar{C}_y$. Moreover as shown in the proof of Proposition 3.6 the restriction of the line bundle $\mathcal{L} = (r + s - 1)\xi - \pi^* \det E$ of $\mathcal{X}(=: P(E))$ on S is numerically equivalent to zero. Thus for each point $y \in T$, $\mathcal{L}|_{\pi^{-1}(\pi(\bar{C}_y)) \cap S}$ is numerically equivalent to zero. Here remark that $E|_{\pi(\bar{C}_y)} \times_{\pi(\bar{C}_y)} \bar{C}_y \cong \pi^* E|_{\bar{C}_y} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus r-1} \oplus \mathcal{O}_{\mathbf{P}^1}(s)$. Moreover let $i_y : \mathbf{P}(\pi^* E|_{\bar{C}_y}) \to \mathbf{P}(E)$ be a natural morphism induced by a canonical morphism $\bar{C}_y \to X$. Then ${i_y}^* \mathcal{L} = \mathcal{O}_{\mathbf{P}(\mathcal{O}^{\oplus r-1} \oplus \mathcal{O}(s-1))}(r + s - 1)$. Thus $\pi^{-1}(\pi(\bar{C}_y)) \cap S$ is a union of finite many sections $\bar{C}_y$ and (probably) at most finite many points. Thus the function g is constant. By purity property we see that π is etale and therefore an isomorphism by the following

Sublemma 4.10.1. *Let U, V be smooth projective varieties and $f : U \to V$ etale and finite. Assume that $\chi(U, \mathcal{O}_U) = 1$. Then f is an isomorhism.*

See Proposition 1.4 in [Sa2].

By 3.9.1 E is a uniform vector bundle where $E|_l \cong \mathcal{O}(1)^{\oplus r-1} \oplus \mathcal{O}(2)$. Note that each S_x corresponds to a quotient line bundle $\mathcal{O}_{\mathbf{P}^n}(1)$ of E. Moreover the above argument says that there is a set $\{S_x\}_{x \in T}$ so that $S_x \cong \mathbf{P}^n$ via π and that the closure of $\cup_{x \in T} S_x$ is a divisor $a(\mathcal{Z})$ in $\mathcal{X}$. Now consider a homomorphism $f : E \to \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus m}$ induced by the above S_x's. This f induces a rational map $\bar{f}$ from $\mathbf{P}(\mathcal{O}_{\mathbf{P}^n}(1)^{\oplus m})$ to $\mathbf{P}(E)$. From the property of $\mathbf{P}(E)$ we see that the image of $\bar{f}$ is equal to $a(\mathcal{Z})$. Thus we infer from Lemma 4.9 that f is surjective with the following eaxct sequence:

$0 \to L \to E \to \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus r-1} \to 0$.

where L is a line bundle. Thus $L = \mathcal{O}_{\mathbf{P}^n}(2)$ and E is $\mathcal{O}_{\mathbf{P}^n}(1)^{\oplus r-1} \oplus \mathcal{O}_{\mathbf{P}^n}(2)$.

Remark 4.11. 1) Let $E = \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus r-1} \oplus \mathcal{O}_{\mathbf{P}^n}(2)$ be a vector bundle on $\mathbf{P}^n$. Then under the natations $a, \mathcal{Z}, \mathcal{X}$ as in section 3, we have $a(\mathcal{Z}) = \mathbf{P}(\mathcal{O}_{\mathbf{P}^n}(1)^{\oplus r-1})(\subset \mathcal{X})$.

2) It is known in [Sa1] that the above uniform vector bundle with rank $E < n$ is a direct sum of line bundles.

5. Proof of Theorem 4.1 (II)

In this section we determine (X, E) of the table of (3.10.2.2). Let us maintain the notations in Section 3. It is assumed that E is spanned.

In case of $\gamma.1$ and $\gamma.2$ we have the following condition:
(5.0) For each point x in an open set of $a(\mathcal{Z})$ so that $\dim S_2(x) = n-1 (\geq 2)$. Note that $\dim S_1(x) = n-1$.

Moreover since a is surjective, $\dim \mathcal{Z} - \dim a(\mathcal{Z}) = \dim \mathcal{Z} - \dim \mathcal{X} = n-2$. As a consequence we get $\dim S_2(x) = \dim \mathcal{Z} - \dim \mathcal{X} + 1$ and therefore we infer by Corollary 4.3 that $S_2(x)$ contains its irreducible component S_x which is isomorphic to $\mathbf{P}^{n-1}$.

Remark 5.0.1. If (X, E) is in the case $\gamma.2$ or if $\mathcal{O}_{\mathbf{P}(E)}(1)$ is very ample, the induced morphism $\pi|_{S_x} : S_x \to \pi(S_x)$ is an isomorphism by 1) of Corollary 4.4.1 and Corollary 4.6. Moreover a smooth rational curve $\bar{C}$ stated in section 4 is a line on such an $(n-1)$-dimensional projective space and $C = \pi(\bar{C})$ is a line on the $(n-1)$-dimensional projective space $\pi(\mathbf{P}^{n-1})$.

From now on let us construct an algebraic family of $S_x (\cong \mathbf{P}^{n-1})$ in $\mathcal{X}$, which is a relative version in 2) of Corollary 3.6.2 to study the structure of (X, E).

Let $\bar{a} : \mathcal{Z} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ and $\bar{b} : \mathcal{Z} \times \mathcal{X} \to \mathcal{Y} \times \mathcal{X}$ be canonical morphisms as in (1.6) and moreover let $a' : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $b' : \mathcal{Y} \times \mathcal{X} \to \mathcal{Y}$ be the first projections respectively.

(5.1) In view of Corollary 3.6.2 we can take an irreducible component J of $\bar{b}(\bar{a}^{-1}(S_1(\Delta)))(\subset \mathcal{Y} \times \mathcal{X})$ satisfying the following conditions:
1) $\dim J = 3n + r - 5$.
2) For each point x in $\mathcal{X}$ $\dim J_x \geq 2n-4$ where $J_x := b'((\mathcal{Y} \times \{x\}) \cap J)$.
3) There is an open set V' in $\mathcal{X}$ so that J_x is a $(2n-4)$-dimensional closed subvariety in $\mathcal{X}$ for each point x in V'.

Let $J_{\mathcal{X}} = \bar{a}(\bar{b}^{-1}(J))$ and for each point x in $\mathcal{X}$ let $J(x)$ to be a closed subscheme $a'((\mathcal{X} \times \{x\}) \cap J_{\mathcal{X}})$ of $\mathcal{X}$.

Then we infer that
(5.2) there is an open set V in $V' \subset \mathcal{X}$ so that
1) an induced morphism $(\mathcal{X} \times V) \cap J_{\mathcal{X}} \to V$ via the second projection $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is flat
2. $(J(x), \xi|_{J(x)}) \cong (\mathbf{P}^{n-1}, \mathcal{O}_{\mathbf{P}^{n-1}}(1))$ for each point x in V by the condition 3) of (5.1) and Corollary 4.3.

(5.3) In $\mathcal{X}$ let U be an irreducible component of the Hilbert scheme containing $J(x)$ for a general point x in V. Moreover let $W \subset \mathcal{X} \times U$ the universal space of U where $j : W \to \mathcal{X}$ and $t : W \to U$ canonical projections. Then there is a canonical morphism $\sigma : V \to U$ by the universality of Hilbert scheme.

For a point u in U let $J[u] = j(t^{-1}(u))$. Note that
(5.3.1) (1) $J[\sigma(x)] \cong J(x)(\subset \mathcal{X})$ for each point x in V. The set of lines on such $J[\sigma(x)]$ is a $(2n-4)$-dimensional Grassmann variety and can be canonically taken as a closed subscheme in $\mathcal{Y}$. Therefore

(2) Let U_σ be the closure of $\sigma(V)$. $J[u]$ is an $(n-1)$-dimensional closed subscheme in $\mathcal{X}$ and the set of lines on such $J[u]$ in $\mathcal{Y}$ is at least $(2n-4)$-dimensional closed subscheme.

Taking account of the fact that $j^*\xi$ is t-ample, we have a

Proposition 5.4. *$(J[u], \xi|_{J[u]}) \cong (\mathbf{P}^{n-1}, \mathcal{O}_{\mathbf{P}^{n-1}}(1))$ for any point u of U_σ.*

Proof. First for a general point u in U_σ $(J[u], \xi|_{J[u]})$ is known to be isomorphic to $(\mathbf{P}^{n-1}, \mathcal{O}_{\mathbf{P}^{n-1}}(1))$ by (2) of 5.2. For each point u in U_σ $J[u]$ has an irreducible component $J[u]_1$ so that $\dim J[u]_1 = n-1$ and $J[u]_1$ is swept out by an $(2n-4)$-dimensional closed subvariety of $\mathcal{Y}$. Thus by Proposition 4.2 we see that $(J[u]_1, \xi|_{J[u]_1}) \cong (\mathbf{P}^{n-1}, \mathcal{O}_{\mathbf{P}^{n-1}}(1))$. Since the morphism t is flat, $J[u]$ is scheme-theoretically equal to $J[u]_1$. $\square$

We summerize

Fact. 5.5. 1) $\mathcal{X}$ is swept out by $(n-1)$-planes, namely $j : W \to \mathcal{X}$ is surjective.
2) Under the notations of (5.3) $t|_{t^{-1}(U_\sigma)} : t^{-1}(U_\sigma) \to U_\sigma$ is $\mathbf{P}^{n-1}$-bundle and $\dim U \geq \dim U_\sigma \geq r$.

Moreover we have

Proposition 5.6. *Let us maintain the condition* 5.0. *Then $j : W \to \mathcal{X}$ is generically bijective.*

Proof. It suffices to show that $j : j^{-1}(P) \to P$ is generically bijective and finite. Thus if otherwise there is an $(n-1)$-plane $P = J[u]$ containing a general point x in $\mathcal{X}$ so that $t(j^{-1}(P))$ contains a curve D. Then $S_2(x)$ contains $j(t^{-1}(t(j^{-1}(P))))$ which is of n-dimension, a contradiction to (5.0). $\square$

Let us begin with the case of $\gamma.1$. For each point u in U_σ recall that $\pi(J[u]) = \mathbf{P}^{n-1}$ by Remark 5.0.1. Moreover we have

Proposition 5.7. *In case of* $\gamma.1$ *assume $n > 2$. Then we have*

1) $\pi^{-1}(\pi(J[u]))$ *is isomorphic to* $\mathbf{P}^{n-1} \times \mathbf{P}^{r-1}$. *Thus* $N_{J[u]/\pi^{-1}(\pi(J[u]))} \cong \mathcal{O}^{\oplus r-1}$.

2) $N_{\pi(J[u])/X} \cong \mathcal{O}$.

3) $N_{J[u]/\mathcal{X}} \cong \mathcal{O}^{\oplus r}$. *Thus $j : W \to \mathcal{X}$ in* $\gamma.1$ *is an isomorphism.*

Proof. Let $P = J[u]$. Remarking from $\gamma.1$ that $E|_{\pi(P)}$ is a uniform vector bundle where $E|_l \cong \mathcal{O}(1)^{\oplus r}$ on each line on $\pi(P)$ from 3.10.2.2, we get 1) the former part by [Sa1]. By the construction $\pi^{-1}(\pi(J[u])) \cong \mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$ and $J[u]$ is $\mathbf{P}^{n-1} \times Q$ in $\mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$ with some point Q in $\mathbf{P}^{n-1}$. Hence we get the latter part.

For 2) in view of the fact that X is swept out by an algebraic family $\{\pi(J[u])\}_{u \in U}$ where $\pi(J[u]) \cong \mathbf{P}^{n-1}$, we see that the normal bundle $N_{\pi(P)/X}$ is $\mathcal{O}_{\mathbf{P}^{n-1}}(a)$ with a non-negative integer a. Then we have $a = 0$. In fact if a was positive, then $NS(X) \times_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$ by Corollary 1.5 because such $\mathbf{P}^{n-1}$'s intersect each other, namely $S_2(x_1) \cap S_2(x_1)$ is not empty for two points x_1, x_2 in X. In the same way as in the proof of Step 2 in Proposotion 5.6 a smooth projective variety X is $\mathbf{P}^n$ by [Mo] and therefore $-(K_X.C) = n+1$, a contradiction. Thus we get $a = 0$.

The former of 3) is obtained from the following exact sequence:

$0 \to N_{P/\pi^{-1}(\pi(P))} \to N_{P/\mathcal{X}} \to N_{\pi^{-1}(\pi(P))/\mathcal{X}|P} \to 0.$

Since $H^1(P, N_{P/\mathcal{X}}) = 0$, we infer that j is separable. Thus the latter of 3) follows from Proposition 5.6. $\square$

Thus 2) of Proposition 5.7 implies that there is a morphism $g : X \to T$ where T is a smooth curve and a general fiber is $\pi(P) \cong \mathbf{P}^{n-1}$. Moreover we see that every fiber is a finite union of $\pi(P)$'s which is connected. On the other hand for any two points t_1, t_2 $(g\pi)^{-1}(t_1)$ is algebraically equivalent to $(g\pi)^{-1}(t_1)$. Since $\xi|_{\pi^{-1}(\pi(P))} \cong \mathcal{O}_{\mathbf{P}(\mathcal{O}_{\mathbf{P}^{n-1}}(1)^{\oplus r})}(1)$, we infer that g is $\mathbf{P}^{n-1}$-bundle over T by taking the intersection number $(\xi.\xi...\xi(n\text{–times}),(g\pi)^{-1}(t))$ in $\mathcal{X}$.

Summarizing the above argument, we get

Corollary 5.8. *Let the condition as in* (5.0). *Assume that* $\dim \bar{\pi}(\mathcal{Y}) = 2n - 3(\gamma.1)$. *Then there is a morphism* $g : X \to T$ *which is a* $\mathbf{P}^{n-1}$*-bundle over* T.

Next we study the case of

(5.9) $\gamma.2$) $\dim \bar{\pi}(\mathcal{Y}) = 2n - 2$.

(5.9.1) By Corollary 1.4 and Proposition 2.5 we have $NS(X) \times_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$.

Hereafter till the end of this section we assume that

(5.9.2) $\mathcal{O}_{\mathbf{P}(E)}(1)$ is very ample.

Note from Remak 5.0.1 that the induced morphism $\pi|_{S_x} : S_x \to \pi(S_x)$ is an isomorphism. Thus since $\pi(P)(\cong \mathbf{P}^{n-1})$ is an ample divisor in X we infer that X is $\mathbf{P}^n$ and $\pi(P)$ is a hyperplane by [Mo1]. Note that $\pi(\bar{C})$ is a line on $X(\cong \mathbf{P}^n)$ and therefore $-(K_X.C) = n + 1$. Thus we see that $(c(E).C) = r + 1$ from (3.9) and by the ampleness of E that E is unform of the type that $E|_l \cong \mathcal{O}(1)^{\oplus r-1} \oplus \mathcal{O}(2)$ for a line l on P.

Thus when $r \leq n - 1$, then $E \cong \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus r-1} \oplus \mathcal{O}_{\mathbf{P}^n}(2)$ and when $r = n$, E is either $\mathcal{O}_{\mathbf{P}^n}(1)^{\oplus n-1} \oplus \mathcal{O}_{\mathbf{P}^n}(2)$ or $T_{\mathbf{P}^n}$ by [Sa1] and [Ein]. But the former case does not occur by Remark 4.10.

Hereafter we assume $r \geq n + 1$.

For a point u in U_σ let P be $(n-1)$-plane $J[u]$ in $\mathcal{X}$ and let us fix it. Now we have the following exact sequence:

$0 \to \Omega_\pi \otimes \xi \to \pi^* E \to \xi \to 0$

where Ω_π is the relative cotangent bundle of π. Note that an induced morphism $\pi : P \to \pi(P)$ is an isomorphism by Corollary 4.6 and that $\xi|_P = \mathcal{O}_P(1)$. Thus the above exact sequence yields the following:

(5.10) $0 \to \Omega_\pi \otimes \xi|_P \to E|_{\pi(P)} \to \mathcal{O}(1) \to 0.$

Let $F_P := \Omega_\pi \otimes \xi|_P$. Since E is a uniform vector bundle stated in (5.9), we have

Fact (5.10.1) F_P is a rank $(r-1)$ uniform vector bundle on $\pi(P)(\cong \mathbf{P}^{n-1})$ where for each line l on $\pi(P)$ $F_P|_l = \mathcal{O}(1)^{\oplus r-2} \oplus \mathcal{O}(2)$ and the type of a vector bundle $F_{J(u)}$ is independent of a choice of a point u in U_σ. On the other hand the inclusion $P \subset \pi^{-1}(\pi(P)) \subset \mathcal{X}$ yields the following exact sequence of normal bundles:

(5.11.1) $0 \to T_P \xrightarrow{i} T_{\mathcal{X}}|_P \xrightarrow{j} N_{P/\mathcal{X}} \to 0.$

(5.11.2) $0 \to N_{P/\pi^{-1}(\pi(P))} \to N_{P/\mathcal{X}} \to N_{\pi^{-1}(\pi(P))/\mathcal{X}}|_P \to 0.$

Here we see that $N_{\pi^{-1}(\pi(P))/\mathcal{X}}|_P = N_{\pi(P)/X} = \mathcal{O}_{\mathbf{P}^{n-1}}(1)$ and $N_{P/\pi^{-1}(\pi(P))} = T_\pi|_P$. Here $T_\pi|_P$ is dual to Ω_π. The last fact is obtained from the following

Sublemma. *Let $0 \to E_2 \to E_1 \to E_3 \to 0$ be an exact sequence of vector bundles on a smooth variety A with a linebundle E_3. Let $\pi : \mathbf{P}(E_1) \to A$ be a canonical projection and π. Then the normal bundle $N_{\mathbf{P}(E_3)/\mathbf{P}(E_1)}$ of $\mathbf{P}(E_3)$ in $\mathbf{P}(E_1)$ is $E_2 \otimes E_3$*

It is left as an exercise.

(5.11.2) turns to be the following exact sequence:

(5.12) $0 \to T_\pi|_P \to N_{P/\mathcal{X}} \to \mathcal{O}_P(1) \to 0.$

From now on we determine the structure of F_P. Now we know that Zariski tangent space $T_{U,[P]}$ of the Hilbert scheme U at $[P]$ is isomorphic to $h^0(P, N_{P/\mathcal{X}}) \geq \dim U \geq \dim U_\sigma \geq r$ and by 2) of Fact 5.5. Therefore $h^0(P, T_\pi|_P) = h^0(P, \check{F}_P(1)) \geq r-n$.

Remarking from 5.10.1 that $\check{F}_P(1) = \mathcal{O}^{\oplus r-2} \oplus \mathcal{O}(1)$. Thus we have by Lemma 4.9

(5.13) $0 \to F_{P,1} \to F_P \to \mathcal{O}(1)^{\oplus r-n} \to 0.$

where $F_{P,1}$ is an uniform vector bundle on P where $F_{P,1}|_l = \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}(2)$. Thus $F_{P,1}$ is either $T_{\mathbf{P}^{n-1}}$ or $\mathcal{O}_{\mathbf{P}^{n-1}}(1)^{\oplus n-2} \oplus \mathcal{O}_{\mathbf{P}^{n-1}}(2)$ by [Ein]. Moreover from (5.13) and (5.10) we see that $F|_{P,1}$ is either $T_{\mathbf{P}^{n-1}} \oplus \mathcal{O}_{\mathbf{P}^{n-1}}(1)^{\oplus r-n}$ or $\mathcal{O}_{\mathbf{P}^{n-1}}(1)^{\oplus r-2} \oplus \mathcal{O}_{\mathbf{P}^{n-1}}(2)$ and therefore that $E|_{\pi(P)} = T_{\mathbf{P}^{n-1}} + \mathcal{O}_{\mathbf{P}^{n-1}}(1)^{\oplus r-n+1}$ or $\mathcal{O}_{\mathbf{P}^{n-1}}(1)^{\oplus r-1} \oplus \mathcal{O}_{\mathbf{P}^{n-1}}(2)$.

Next to determine E we will show

Lemma 5.14. *M is a* rank r *uniform vector bundle on $\mathbf{P}^n$ where for a line l $M|_l = \mathcal{O}^{\oplus(r-1)} \oplus \mathcal{O}(-1)$ with $r > n \geq 2$. Then we have*

1) If there is a $(n-1)$-plane P on $\mathbf{P}^n$ so that $M|_P = \Omega_P(1) \oplus \mathcal{O}_P^{\oplus r-n+1}$, then M is $\Omega_{\mathbf{P}^n}(1) \oplus {\mathcal{O}_{\mathbf{P}^n}}^{\oplus r-n}$.

2) If there is a $(n-1)$-plane P on $\mathbf{P}^n$ so that $M|_P = \mathcal{O}_P(-1) \oplus \mathcal{O}_P^{\oplus r-1}$, then M is $\mathcal{O}_{\mathbf{P}^n}(-1) \oplus {\mathcal{O}_{\mathbf{P}^n}}^{\oplus r-1}$.

Proof. First we show 1). Let $M(t) = M \otimes \mathcal{O}_{\mathbf{P}^n}(t)$. We have an exact sequence:

$0 \to M(t-1) \to M(t) \to M(t)|_P \to 0.$

The above yields a long exact sequence:

(5.14.1) $0 \to H^0(\mathbf{P}^n, M(t-1)) \to H^0(\mathbf{P}^n, M(t)) \to H^0(P, M(t)|_P)$
$\to H^1(\mathbf{P}^n, M(t-1)) \to H^1(\mathbf{P}^n, M(t)) \to H^1(P, M(t)|_P) \to$

Claim 1). When $t \leq -2$, $H^1(\mathbf{P}^n, M(t))$ vanishes.

2) $\dim H^1(\mathbf{P}^n, M(-1)) = 0$ or 1.

3) $\dim H^0(\mathbf{P}^n, M) = r - n + 1$ or $r - n$.

In fact $H^1(P, M(t)|_P) = 0$ for $t \leq -2$ by the assumption. $\dim H^1(\mathbf{P}^n, M(t))$ is a monotone-decreasing function with respect to $t(\leq -2)$. Thanks to Serre's duality and Serre vanishing theorem, we get $H^1(\mathbf{P}^n, M(t)) = H^{n-1}(\mathbf{P}^n, \check{M}(-t) \otimes K_{\mathbf{P}^n}) = 0$. Hence we complete the proof of 1).

For 2) we consider the exact sequence of $t = -1$. Since $\dim H^1(\mathbf{P}^n, M(-2)) = 0$ and $\dim H^1(P, M(-1)|_P) = 1$, we get 2). Since $M(-1)|_l = \mathcal{O}(-1)^{\oplus(r-1)} \oplus \mathcal{O}(-2)$, we see $\dim H^0(P, M(-1)) = 0$. Thus 3) is trivial.

Hence M has a subbundle ${\mathcal{O}_{\mathbf{P}^n}}^{\oplus r-n}$ (or, ${\mathcal{O}_{\mathbf{P}^n}}^{\oplus r-n+1}$) by Lemma 4.9 and the quotient bundle M' where M' is a uniform vector bundle with $M'|_l = \mathcal{O}(-1)\mathcal{O}^{\oplus(n-1)}$ (or, $\mathcal{O}(-1)\mathcal{O}^{\oplus(n-2)}$ resp.). By [Ein] M' is one of $\Omega_{\mathbf{P}^n}(1)$, (or, $\mathcal{O}_{\mathbf{P}^n}(-1) \oplus {\mathcal{O}_{\mathbf{P}^n}}^{\oplus n-2}$). By the assumption the only first case occurs. Thus we can show 1) of this lemma. As for 2) we can show in the same manner. $\square$

Consequently combining Corollary 5.8, (5.9) and Lemma 5.14, we get

Proposition 5.15. *Let (X, E) be as in the table of* (3.10.2.2). *Assume that E is spanned. Moreover for a general point x in $a(\mathcal{Z})$ assume that* $\dim S_2(x) = n - 1 (\geq 2)$. *Then we have*

1) *when* $\dim \bar{\pi}(\mathcal{Y}) = 2n - 3$, *$g : X \to C$ has a $\mathbf{P}^{n-1}$-bundle structure over a smooth projective curve C and $E|_{g^{-1}(c)} \cong \mathcal{O}_{\mathbf{P}^{n-1}}(1)^{\oplus r}$.* (3.10.2.2. γ.1), *or*

2) *when* $\dim \bar{\pi}(\mathcal{Y}) = 2n - 2$, *$NS(X) \times_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$ and $\mathbf{P}(E)$ is swept out by $(n-1)$-dimensional projective spaces P with $\mathcal{O}_{P(E)}(1)|_P \cong \mathcal{O}_{\mathbf{P}^{n-1}}(1)$. Moreover suppose that $\mathcal{O}_{P(E)}(1)$ is very ample. Then (X, E) is $(\mathbf{P}^n, T_{\mathbf{P}^n} \oplus \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus r-n})$.* (3.10.2.2. γ.2)

6. Theorem 4.1 (III)

In this section we determine (X, E) of the table (3.10.2.2). Let us maintain the notations in Section 3. It is assumed that E is spanned and $n > 4$.

In case of γ.1 and γ.2 we have the following condition:

(6.0) For each point x in an open set $\mathcal{X}_0$ of $a(\mathcal{Z})(= \mathcal{X})$ so that $\dim S_1(x) = n-1$ and that $\dim S_2(x) = n$.

Hereafter fix $\mathcal{X}_0$. As a consequence it is shown that only the case γ.1 happens. (Proposition 6.15)

Remark that the arguments until Proposition 6.5 are made without using the properties of table (3.10.2.2) except for the ones of $\dim S_2(x) = n$.

(6.1) For a point x in $\mathcal{X}$ let $a^{-1}(x) = \cup_{i\geq 1} A_i$ be the decomposition of $a^{-1}(x)$ into the irreducible components. Note that $\dim A_i = n-2$.

We study the situation of the cone $a(b^{-1}(b(A_i)))$ around the vertex x. We maintain the notations in (3.3). Let $\bar{C}_y = a(b^{-1}(y))$. We recall the following facts from the statement before (4.2) and 2) of Proposition 4.4:
(6.2) Since the vector bundle E is spanned, for each point y in $\mathcal{Y}$

1) $b^{-1}(y)$ is mapped biregularly to $\bar{C}_y$ via the morphism $a : \mathcal{Z} \to \mathcal{X}$.
2) $\bar{C}_y$ is a smooth rational curve.
3) $\bar{C}_y$ is mapped biregularly to a line $\phi(\bar{C}_y)$ via the morphism $\phi : \mathcal{Z} \to \mathbf{P}^N$.
4) $\bar{C}_y$ is mapped biregularly to $\pi(\bar{C}_y)$ via the $\pi : \mathcal{X} \to X$.

In view of 1) of (6.2) for a morphism $\mathcal{Z} \to \mathcal{X}$ we define a natural morphism $D_a : \mathcal{Z} \to \mathbf{P}(\Omega_{\mathcal{X}})$ as follows:

For a point (v, x) in $\tilde{V}_{\mathcal{X}} \times \mathbf{P}^1$ (see Proposition 3.3) we have a natural morphism $\tilde{D} : \tilde{V}_{\mathcal{X}} \times \mathbf{P}^1 \to \cup_{x\in\mathcal{X}} T_{\mathcal{X},x}$ where $\tilde{D}(v, x)$ is defined as a tangent vector $dv_{*,x}(\frac{\partial}{\partial t})$ in $T_{\mathcal{X},x}$ and t is a local parameter of $\mathbf{P}^1$. $\operatorname{Aut}\mathbf{P}^1$ acts on $\tilde{V}_{\mathcal{X}} \times \mathbf{P}^1$ naturally and Φ is $\operatorname{Aut}\mathbf{P}^1$-invariant morphism. Thus an induced morphism $D_a : \mathcal{Z} \to \mathbf{P}(\Omega_{\mathcal{X}})$ is defined.

Similarly for a morphism $a\phi : \mathcal{Z} \xrightarrow{a} \mathcal{X} \xrightarrow{\phi} \mathbf{P}^N$ we define a morphism: $D_{\phi a}$ factors as $\mathcal{Z} \xrightarrow{D_a} D(\mathcal{Z}) \xrightarrow{D_\phi} \mathbf{P}(\Omega_{\mathbf{P}^N})$. For a morphism $\pi a : \mathcal{Z} \xrightarrow{a} \mathcal{X} \xrightarrow{\pi} X$ we define a morphism: $D_{\pi a}$ factors as $\mathcal{Z} \xrightarrow{D_a} D(\mathcal{Z}) \xrightarrow{D_\pi} \mathbf{P}(\Omega_X)$.

We come back to A_i stated in (6.1). First we have

Claim 6.3. An induced morphism $D_{\phi a} : A_i \to \mathbf{P}(\Omega_{\mathbf{P}^N,\phi(x)}) (\cong \mathbf{P}^{N-1})$ is a finite morphism. Consequently $\dim D_a(A_i) = \dim A_i = n-2$.

Remark that $A_i \to b(A_i)$ is an isomorphism via q from 1) of (6.2). We assume the contrary namely, there is a curve H in A_i so that $D_{\phi a}(H)$ is a point. From 3) of (6.2) and the property of a line it follows that a curve $\bar{C}_{q(z)}$ for each point z in H is mapped to a unique line in $\mathbf{P}^N$ via the morphism ϕ. Thus yields a contradiction to the finiteness of $\phi : \mathcal{X} \to \mathbf{P}^N$.

Next we study the dimension of $D_{\pi a}(A_i)$ in $\mathbf{P}(\Omega_{X,x})$ to determine the structure of X.

(6.3.1) Starting with each point x in $\mathcal{X}$, we have an n-dimensional irreducible component S_x of $S_2(x)$ and its open set R in S_x enjoying Remark 3.10.3. For every x' in R take an (n-2)-dimensional irreducible component B_i of $a^{-1}(x')$. Hereafter without confusion we use the notation A_i in place of B_i. Recall that x' is a smooth point in S_x so that $a(b^{-1}(b(A_i)))$ is a cone with the vertex x' which contained in S_x.

Then $D_a(A_i)$ in $\mathbf{P}(\Omega_{\mathcal{X},x'}) (\cong \mathbf{P}^{n+r-1})$ is already contained in $\mathbf{P}(\Omega_{S_x,x'}) (\cong \mathbf{P}^{n-1})$. Thus by Claim 6.3 we have a

Remark 6.4. $D_a : A_i \to D_a(A_i)$ is finite and $D_a(A_i)$ is its divisor in $\mathbf{P}(\Omega_{S_x,x'})$.

Proposition 6.5. *Let x, x' be points as in* (6.3.1). *Assume that* $n > 4$. *Then* $\dim D_{\pi a}(A_i) = \dim A_i = n-2$.

Proof. Since $\dim D_a(A_i)$ is a hypersurface in $\mathbf{P}^{n-1}(n-1 \geq 4)$, we see that $PicD_a(A_i) \cong \mathbf{Z}$. Thus by $D_{\pi a}(A_i) = D_\pi(D_a(A_i))$ $\dim(D_{\pi a}(A_i))$ is $\dim A_i$ or 0. Now we assume that $\dim D_{\pi a}(A_i) = 0$. A canonical morphism $\pi : S \to X$ with $S = S_x$ yields a homomorphism $d\pi_{*,x'}(:= d) : T_{S,x'} \to T_{X,\pi(x')}$. Let $g : T_{S,x'} - \{0\} \to \mathbf{P}(\Omega_{S,x'})$ be a canonical projection. Then $g^{-1}(D_a(A_i)) \cup \{0\}$ is an $(n-1)$-dimensional affine cone in $T_{S,x'}$. The assumption that $\dim D_{\pi a}(A_i) = 0$ says that $g^{-1}(D_{\pi a}(A_i)) \cup \{0\}$ is tramsformed via the linear map d to a one-dimensional subspace in $T_{X,\pi(x')}$, namely the rank of d is 1. Thus we can take a point z in A_i whose curve $\bar{C}_{b(z)}$ induces a non-zero tangent vector v in $T_{X,\pi(x')}$ with $dv = 0$. On the other hand note from 4) of (6.2) that for each z in A_i each smooth rational curve $\pi(\bar{C}_{\pi(z)})$ yields a non-zero tangent vector of $T_{X,\pi(x')}$, a contradiction. □

Hereafter until the end of this section it is assumed that $n > 4$.

Next we make several preparations to show that $\pi_S : S \to X$ is separable.

(6.6) Now we interpret the above situation in terms of normal bundle of a smooth rational curve $\pi(\bar{C}_y)$ of X.

We take a general point x in the open set $\mathcal{X}_0$ of $\mathcal{X}$ so that $\dim S_2(x) = n$. Thus $\dim D_{\pi a}(A_i) = \dim A_i = n-2$. Hereafter let $C(y) = \pi(\bar{C}_y)$ for y of $\mathcal{Y}$.

Remarking that each curve $C(y)$ for a point y in $b(A_i)$ yields a tangent vector through the point x as stated above namely a global section of the normal bundle $N_{C(y)/X} \otimes \mathcal{O}(-1)$, we see by Proposiotion 6.5 that
(6.6.1) global sections of the normal bundle $N_{C(y)/X} \otimes \mathcal{O}(-1)$ of $C(y)$ generate rank $\geq (n-2)$ subsheaf of $N_{C(y)/X} \otimes \mathcal{O}(-1)$.

Letting $N_{C(y)/X} \otimes \mathcal{O}(-1) = \oplus_{i=2}^n \mathcal{O}(a_i)$ with $a_2 \geq a_3 \geq ... \geq a_n$, we have $a_{n-1} \geq 0$.

(6.7) Let us set S_x and T_x in (6.3.1) as S and T respectively. Let $Z_T := b^{-1}(T)$. Then we have canonical morphisms $b_T : Z_T \to T, a_T : Z_T \to S$ induced by a, b. Moreover let $a_T(b_T^{-1}(b_T(a_T^{-1}(x'))))$ be a cone with the vertex x' as Cone$[x'](\subset S)$.

From the two diagrams $b_T : Z_T \to T$, $a_T : Z_T \to S$ and $\pi a_T : Z_T \to X$ we first have
(6.7.0) $NS(S) \times_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$ by Corollary 1.5 and by $\dim S_2(x) = n$ (6.0). Similarly $NS(S) \times_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$ by 2) of Corollary 3.4.

From now on we show that a natural morphism $\pi_S : S \to X$ via π is separable and consequently π_S is etale and S is smooth.

First since $D_{\pi a}(Z_T)$ is a divisor in $\mathbf{P}(\Omega_X)$, we divide into two cases:
(6.7.1) $D_{\pi a}(Z_T)$ is a divisor of the tautological line bundle of Ω_X namely for a general point u in X $D_{\pi a}(Z_T) \cap c^{-1}(u)$ is an (n-2)-dimensional linear subspace in $\mathbf{P}(\Omega_{X,u})(\cong \mathbf{P}^{n-1})$ where $c : \mathbf{P}(\Omega_X) \to X$ is a canonical projection.
(6.7.2) $D_{\pi a}(Z_T)$ is not a divisor of the tautological line bundle. It follows from the remark in 6.6 that $\pi_S : S \to X$ is separable.

We treat the case (6.7.1).

Claim. For each point u in X $D_{\pi a}(Z_T) \cap c^{-1}(u)$ is an (n-2)-dimensional hyperplane in $\mathbf{P}(\Omega_{X,u})$.
In fact assume that there is a point u in X so that $D_{\pi a}(Z_T) \cap c^{-1}(u)$ is equal

to $\mathbf{P}(\Omega_{X,u})(\cong \mathbf{P}^{n-1})$. Letting $N_{C(z)/X} \otimes \mathcal{O}(-1) = \oplus_{i=2}^{n}\mathcal{O}(a_i)$ with $a_2 \geq a_3 \geq ... \geq a_n$, we see that a_i is non-negative for $2 \leq i \leq n$ by remark in (6.6). Thus $N_{C(z)/X} \cong \oplus_{i=2}^{n}\mathcal{O}(a_i+1)$ with a non-negative integer a_i and $(-K_X.C(y)) = n+1+\Sigma_{i=2}^{n}a_i \geq n+1$, a contradiction to $(-K_X.C(y)) = n$.

The above claim implies that there is the following exact sequence:
$0 \to E_1 \to T_X \to L \to 0$.
where E_1 is a rank n-1 subbundle of T_X and L a line bundle.

Here moreover we divide into two cases:

For a general point x in X

α) For a general point $x_1(\neq x)$ in Cone(x) there is a point y in T so that Cone(x) intersects $C(y)$ at x_1 transversally. Here Cone$(x) := \pi(\text{Cone}[x])$.

In this case a natural morphism $\pi_S : S \to X$ via π is separable.

β) Each line $C(y)$ $(y \in T)$ passing through a general point $x_1 \neq x$ in Cone(x) is tangent to the Cone(x) at x_1.

We show that this case does not happen.

Since E_1 yields a foliation on X, for a point x in U $E_1|_{\text{Cone}(x)_{reg}} = T_{\text{Cone}(x)_{reg}}$ and Cone(x) is smooth. Thus we see that the normal bundle $NB(x)$ of Cone(x) in X is $L|_{\text{Cone}(x)}$.

(6.7.3) Remark from (6.6.1) that

1) Since $E_1|_{C(y)} \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-2}$, $(L.C(y)) = 0$. Thus L is numerically trivial by (6.7.0).

2) For some open set U in X there is an algebraic family $\{\text{Cone}(x)\}_{x\in U}$ in X (by choosing Cone(x) induced by a suitable component of $a_T^{-1}(x)$ if necessary). Then we see from the construction that the normal bundle of the cone in X is nef and not numerically trivial.

Therefore 1) and 2) of Remark (6.7.3) contradict each other.

Consequently we get

Proposition 6.8. *A natural morphism $\pi_S : S \to X$ via π is separable.*

Next we show etaleness of $\pi_S : S \to X$ and the smoothness of X in the same way as in (4.10).
The function
(6.9) $g : X \ni x \mapsto \#(\pi_S^{-1}(x)) \in \mathbf{N}$
is lower semi-continuous in the Zariski topology. Next note that in this case $\pi(\bar{C}_y) = \mathbf{P}^1$ (4 of (6.2)) and $S = \cup_{y\in T}\bar{C}_y$. Moreover as shown in the proof of Proposition 3.6 the restriction of the line bundle $\mathcal{L} = r\xi - \pi^* \det E$ of $\mathcal{X}(=: P(E))$ on S is numerically equivalent to zero. Thus for each point $y \in T$ $\mathcal{L}|_{\pi^{-1}(\pi(\bar{C}_y))\cap S}$ is numerically equivalent to zero. Here $E|_{\pi(\bar{C}_y)} \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus r}$ and $\mathcal{L}|_{\pi^{-1}(\pi(\bar{C}_y))} = \mathcal{O}_{P(\mathcal{O}_{\mathbf{P}^1}(1)^{\oplus r})}(r)$. Thus $\pi^{-1}(\pi(\bar{C}_y)) \cap S$ is a union of finite many sections $\bar{C}_y$ and (probably) at most finite many points. Thus the function g is constant. By purity property we get

Proposition 6.10. *A natural morphism $\pi_S : S \to X$ via π is etale and*

S is smooth. Moreover every line bundle on S is numerically equivalent to $b\xi_S$ with an integer b. Particularly $-K_S$ is numerically equivalent to $n\xi_S$.

Proof. The first part is already shown. The second part is obtained by $PicS \times_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$ (6.7.0) and by $(\xi.\bar{C}_y) = 1$. The final part is trivial by the property that $(-K_X.C(y)) = n$ and $\pi(\bar{C}_y) = C(y)$. $\square$

Let $\phi_S : S \to \mathbf{P}^m$ be a morphism by the complete linear system by ξ_S with $m = h^0(\xi_S) - 1$. From now on we show
(6.11) $\phi_S : S \to \phi_S(S)$ is separable and $\phi_S(S)$ is a hyperquadric.

First $\phi_S(\text{Cone}[x])$ is a cone with the vertex $\phi(x)$ in $\mathbf{P}^m$. When $\phi_S(S)$ is smooth at the point $\phi(x)$, the tangent space of $\phi_S(S)$ at the vertex contains the cone $\phi_S(\text{Cone}[x])$. Thus any hyperplane H in $\mathbf{P}^m$ containing the tangent space contains the cone $\phi_S(\text{Cone}[x])$. This implies that the cone $\text{Cone}[x]$ in S is contained in $\phi^{-1}(H)$, the zero locus of a section of ξ_S. Thus by the second fact of Proposition 6.10 we get

Step.1 For a general point x in S $\xi_S = \mathcal{O}_S(\text{Cone}[x])$ and $\text{Cone}[x]$ is irreducible. Moreover m is $n+1$ or n.

Moreover we have

Step.2 ϕ_S is generically injective.
In fact assume the contrary, for a general point u in $\phi(S)$ there are two points v, w in S so that $\phi(v) = \phi(w) = u$ and that the point w is on the cone $\text{Cone}[v]$ with the vertex v by the former part of Step 1. Since there is a smooth rational curve $\bar{C}_y$ pasing through two points v, w which is a generator of the cone and since $\bar{C}_y$ goes to a line via ϕ we have a contradiction to the assumption.

Recall the notations D_a, D_ϕ in (6.1) and S, T, Z_T in (6.7). First an induced morphism $Z_T \to S$ yields a canonical morphism $D_S : Z_T \to P(\Omega_S)$. Moreover the composite morphism of $Z_T \to S$ and $\phi_S : S \to \mathbf{P}^m$ yields a canonical morphism $e : Z_T \to P(\Omega_{\mathbf{P}^m})$ where $\theta : P(\Omega_{\mathbf{P}^m}) \to \mathbf{P}^m$ is a canonical projection in the same way as $D_{\phi a}$ stated above. Let $\bar{T} = e(Z_T)$. Then $\bar{T}$ is a (2n-2)-dimensional subvariety in $P(\Omega_{\mathbf{P}^m})$ since e is a finite morphism by the former part of Claim 6.3. For a general point u in $\phi(S)$ $\bar{T} \cap \theta^{-1}(u)$ is a (n-2)-dimensional subvariety in $P(\Omega_{\mathbf{P}^m,u})$.

Hence we have a

Remark 6.11.1. 1) Suppose there is a point x in S so that $\bar{T} \cap \theta^{-1}(\phi(x))$ is a (n-2)-dimensional linear subspace in $P(\Omega_{\mathbf{P}^m,\phi(x)})$. Then $\phi(\text{Cone}[x])$ is $(n-1)$-plane in $\phi(S)$ and $m = n$ by Step 1 and the argument just before Step 1.

2) When there is a point u in $\phi(S)$ so that $\bar{T} \cap \theta^{-1}(u)$ is not a (n-2)-dimensional linear subspace in $P(\Omega_{\mathbf{P}^m,u})$, $\phi_S : S \to \phi(S)$ is separable.

Thus we get
Step.3 The case $m = n$ does not occur.

Assume $m = n$. First we suppose ϕ_S is separable. By Step 2 ϕ_S is birational and therefore an isomorphism $S \cong \mathbf{P}^n$, a contradiction to the assumption $(-K_S.\bar{C}) = n$. Next we suppose ϕ_S is not separable. By 1) and 2) of Remark (6.11.1) $\bar{T} = e(Z_T)$ is an $\mathbf{P}^{n-2}$-bundle which induecs a tautological line bundle in $P(\Omega_{\mathbf{P}^n})$. Thus we have the following exact sequence:

$0 \to F \to T_{\mathbf{P}^n} \to L \to 0$

where F is a rank $n-1$ bundle and L a line bundle. But there exists no such exact sequence.

(6.12) Thus we get $m = n+1$. Moreover by 1) of Remark 6.11.1 we see that for each point x in S $\bar{T} \cap \theta^{-1}(\phi(x))$ is a hypersurface in $P(\Omega_{\mathbf{P}^m,\phi(x)})$ of degree > 1. Thus $\phi_S : S \to \mathbf{P}^{n+1}$ is umramified and therefore it is a closed embedding by virtue of Theorem due to Fulton and Hansen [FH].

By $K_S.\bar{C}_y = -n$ we have

Proposition 6.13. *$\phi(S)$ is a smooth hyperquadric. Moreover S is a smooth hyperquadric.*

Since S is a smooth hyperquadric, we get $\chi(S, \mathcal{O}_S) = 1$. Thus sublemma 4.10.1 yields

Corollary 6.14. *π_S is an isomorphism and X is a smooth hyperquadric.*

Consequently the vector bundle E on smooth hyperquadric is uniform vector bundle where the restriction of $E \otimes \mathcal{O}(-1)$ on any line on X is a trivial vector bundle. Thus by Proposition 3.6.1 in [W] (whose proof is characteristic-free) we have

Proposition 6.15. *Let us maintain the notations $\mathcal{X}, \mathcal{X}_0$ and $S_1(x), S_2(x)$ used in Section 3. Assume that E is spanned and $n > 4$. Moreover for each point x in some open set $\mathcal{X}_0$ of $a(\mathcal{Z})(= \mathcal{X})$ $\dim S_1(x) = n-1$ and that $\dim S_2(x) = n$. Then (X, E) is isomorphic to $(Q, \mathcal{O}_Q(1)^{\oplus r})$.*

Hence we complete the proof of Theorem 4.1 by Thereoms 4.7, 4.10, Propositions 5.15 and 6.15.

Moreover we have

Theorem 6.16. *Let E be an ample vector bundle of* $\operatorname{rank} r = n-1$ *on an n-dimensional smooth projective variety X. Assume that $n > 4$ and E is spanned. Then $K_X + \det E$ is nef unless (X, E) is one of the follwing*

1) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus(n-1)})$.

2) *$\phi : X \to C$ has a $\mathbf{P}^{n-1}$-bundle structure over a smooth projective curve C and $E|_{\phi^{-1}(c)} \cong \mathcal{O}_{\mathbf{P}^{n-1}}(1)^{\oplus(n-1)}$.*

3) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))^{\oplus(n-2)} \oplus \mathcal{O}_{\mathbf{P}^n}(2))$.

4) $(\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(1)^{\oplus(n-1)})$.

Proof. We do the same thing as in Theorem 4.7.2. Then by the assumption there is an extremal rational curve C on X so that $(-(K_X.C), (c(E).C) = (n+1, n-1), (n, n-1)$ or $(n+1, n)$. Take the nomarization $\phi : \bar{C} \to C$ of C. Then ϕ^*E is $\mathcal{O}(1)^{\oplus n-1}$ or $\mathcal{O}(1)^{\oplus n-2}\mathcal{O}(2)$. Thus a direct summand $\mathcal{O}(1)$ yields a rational curve $\tilde{C}$ on $\mathbf{P}(E)$ with $(\tilde{C}.\mathcal{O}_{\mathbf{P}(E)}(1)) = 1$. Therefore we have two cases: $-(K_{\mathbf{P}(E)}.\bar{C}) = n+1, n$. Note that $\tilde{C}$ is not in the fiber of the projection $\mathbf{P}(E) \to X$. Thus by Theorem 4.1 we complete the proof. □

As a byproduct we get

Theorem 6.17. *Let L be an ample line bundle on an n-dimensional smooth projective variety X defined over an algebraically closed field of any characteristic. Assume that L is spanned. Then we have the following*

1) *$K_X + nL$ is nef unless (X, L) is $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$.*
2) *Assume that $K_X + nL$ is nef. If $n > 4$, $K_X + (n-1)L$ is nef unless*
   (a) *X is a hyperquadric and $L = \mathcal{O}_X(1)$).*
   (b) *(X, L) is a scroll over a smooth curve.*

Proof. We consider (1). Let $E = L^{\oplus n}$. Then since L is spanned, so is E. Moreover $K_X + nL = K_X + \det E$. When $K_X + nL$ is not nef, we see that by Theorem 4.7.2 that $(X, E) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus n})$. Thus L is $\mathcal{O}_{\mathbf{P}^n}(1$ by Krull-Schmit Theorem. Next considering case (2) in the same way as in (1), we infer by Theorem 6.16 that (X, L) is one of $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$ and (a) (b) stated after 'unless' in the conclusion. But the first case is ruled out by the assumption. □

The above result is a partial generalization of the one by Fujita [Fu1] to positive characteristic under the assumption that E is spanned.

7. Applications

In this section we show

Thereom 7.1. *Let X be an n-dimensional smooth projective variety defined over an algebraically closed field of any characteristic and E an ample vector bundle of rank n on X. Assume that E is spanned. Then the following conditions are equivalent*:

(a) $c_n(E) = 1$.
(b) *$K_X + c_1(E)$ is not nef.*
(c) *(X, E) is isomorphic to $(\mathbf{P}^n, \oplus^n \mathcal{O}_{\mathbf{P}^n}(1))$.*

Remark 7.2. 1) To show (a) → (b) and (a) → (c) in Theorem 7.1 the assumption that E is spanned is neccesary even in characteristic zero. In fact as an counter example we can state an ample line bundle of degree 1 on an elliptic curve.

2) Theorem 7.1 is a generalization of Theorem 3.4 in [W] in any characteristic.

We begin with the proof of Theorem 7.1.
If b) follows from a) in Theorem 7.1, we see b) $\to$ c) easily by Theorem 4.7.2. Thus hereafter we show that b) is obtained from a). (See [LS] also.)

Since E is spanned, there is the following exact sequence:

$0 \to \check{F} \to \mathcal{O}_X^{\oplus h^0(X,E)} \to E \to 0.$
where $\check{F}$ is dual to the vector bundle F.

Let $\pi : P(F) \to X$ be the canonical projection. Since F is spanned, $\phi : P(F) \to \mathbf{P}^{f-1}$ denotes a morphism induced by the complete linear system of the tautological linebundle ξ of F with $f = h^0(X, F)$. Since $\xi^{n+r-1} = c_n(E) = 1$ with $r = \operatorname{rank} F$, ϕ is birational and $\phi(P(F))$ is a projective space $\mathbf{P}^{f-1}$. Thus we have
(7.3) $r + n = h^0(X, F)$.

Moreover take general (r-1) hyperplanes $H_1, ...H_{r-1}$ in $\mathbf{P}^{r+n-1}$ and let $Y = H_1 \cap ... \cap H_{r-1}$. Consequently we infer that
(7.4) 1) The induced $\phi_Y : Y \to \mathbf{P}^n$ via ϕ is birational and therefore Y is reduced and irreducible.

2)As for the projection π Y is a rational section over X and therefore Y is a rational variety.

3)The singular part $\operatorname{Sing} Y$ of Y is at least codimension 2 by Zariski Main Theorem. Thus it is normal.

Moreover we have (7.5) $\omega_Y + \xi|_Y = \pi^*(K_X + \det E)$.

By 3) of (7.4) we have a smooth rational curve $C(\subset Y)$ off $\operatorname{Sing} Y$ so that $\phi(C)$ is a lines in $\mathbf{P}^{r+n-1}$ and $\pi : C \to \pi(C)$ is an isomorphism. From (7.5) we get $(\omega_Y.C) + (\xi|_Y.C) = (\pi^*(K_X + \det E).C)$ and consequently $\pi(C).(K_X + detE) = -n$. Thus we are done.

Theorem 7.1 yields the following result which is a partial answer of conjecture by Ballico [B].

Theorem 7.6. *Let X be an n-dimensional smooth projective variety defined over an algebraically closed field of any characteristic and $E_1, E_2, ...E_s$ ample vector bundles of rank $r_1, ...r_s$ on X respectively with $\Sigma_{i=1}^s r_i = n$. Assume that for each i E_i is spanned and $c_{r_1}(E_1).c_{r_2}(E_2).,,,.c_{r_s}(E_s) = 1$. Then X is $\mathbf{P}^n$ and E_i is isomorphic to $\oplus^{r_i}\mathcal{O}_{\mathbf{P}^n}(1)$.*

Proof. We consider the vector bundle $\oplus_{i=1}^s E_i (=: E)$, which is easily shown that the bundle E is spanned and it satisfies the condition (a) in Theorem 7.1. Thus by virtue of Krull- Schmit's Theorem we are done. □

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