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A Topological Proof of Real and Symplectic Bott Periodicity Theorem

By

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1. Introduction

The purpose of this paper is, as in title, to prove real and symplectic Bott periodicity. For introduction I'll tell, roughly, the way of the proof. Because the goal of this paper is that $\mathrm{BSp} \simeq (\Omega^4\mathrm{BO})\langle 0\rangle$ and $\mathrm{BO} \simeq (\Omega^4\mathrm{BSp})\langle 0\rangle$ (For a topological space X, $X\langle n\rangle$ means n-connected fiber space of X and let $pr_n : X\langle n\rangle \to X$ be an ordinary projection.), we must construct maps that is $\lambda : S^4 \wedge \mathrm{BO} \to \mathrm{BSp}$ and $\mu : S^4 \wedge \mathrm{BSp} \to \mathrm{BO}$. (To do that, we use the word of K-theory.) By some of good cohomological properties of BO and BSp, we can tell the almost same thing as $\mathrm{Ad}^4\,\lambda$ and $\mathrm{Ad}^4\,\mu$ are homotopy equivalence, and we remove 'almost' later.

2. Main theorem

Theorem 2.1. *Let X, Y be of finite type CW-complex which are H-spaces having following properties,*
(1) $H^*(X;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1, w_2, w_3, \dots]$,
where $|w_i| = i$.
w_i's have following relation;

$$\begin{aligned} \mathrm{Sq}^1\, w_{2i} &\equiv w_{2i+1} && \mod J^2 \\ \mathrm{Sq}^2\, w_{2i} &\equiv w_{2i+2} && \mod J^2 \\ \mathrm{Sq}^{i-1}\, w_j &\equiv w_{2i-1} && \mod J^2 \\ \mathrm{Sq}^{2i-2}\, w_{2i} &\equiv w_{4i-2} && \mod J^2 \\ \mathrm{Sq}^{4i-3}\, w_{4i} &\equiv 0 && \mod J^2 \end{aligned}$$

(*where* $J = (w_1, w_2, \dots)$)
(2) $H^*(Y;\mathbb{Z}) = \mathbb{Z}[p_1, p_2, p_3, \dots]$, $|p_i| = 4i$.

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(3) *There exist maps*
$j : \mathbb{R}\mathrm{P}^\infty \to X, \quad j' : \mathbb{H}\mathrm{P}^\infty \to Y$
$\lambda : S^4 \wedge X \to Y, \quad \mu : S^4 \wedge Y \to X$
such that

(a) $(\lambda \circ (S^4 \wedge j))^* : H^{4*}(Y; \mathbb{Z}/2\mathbb{Z}) \to H^{4*}(S^4 \wedge \mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z})$
$(\mu \circ (S^4 \wedge j'))^* : H^*(X; \mathbb{Z}/2\mathbb{Z}) \to H^*(S^4 \wedge \mathbb{H}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z})$
are epic.

(b) $\mathrm{Ad}^4 \lambda : X \to \Omega^4 Y, \quad \mathrm{Ad}^4 \mu : Y \to \Omega^4 X$
are H-maps.

Then the maps $\widetilde{\mathrm{Ad}^4}\lambda : Y \to \Omega^4 X\langle 0\rangle$, $\widetilde{\mathrm{Ad}^4}\mu : X \to \Omega^4 Y\langle 0\rangle$ *are* $\bmod \mathcal{C}_2$-*homotopy equivalence as Serre's meaning.*

($\widetilde{\mathrm{Ad}^4}\lambda$ *and* $\widetilde{\mathrm{Ad}^4}\mu$ *are lifts of* $\mathrm{Ad}^4 \lambda$ *and* $\mathrm{Ad}^4 \mu$. $\mathcal{C}_2$ *is a class of all finite abelian groups whose orders are odd.*)

Proof. First I show $\widetilde{\mathrm{Ad}^4}\mu$ is $\bmod \mathcal{C}_2$ -homotopy equivalence. Setting k's as follows.

$$\begin{aligned} k &= \mu \circ (S^4 \wedge j') &&: S^4 \wedge \mathbb{H}\mathrm{P}^\infty \to X \\ k_1 &= \mathrm{Ad}\, \mu \circ (S^3 \wedge j') &&: S^3 \wedge \mathbb{H}\mathrm{P}^\infty \to \Omega X \\ k_2 &= \mathrm{Ad}^2 \mu \circ (S^2 \wedge j') &&: S^2 \wedge \mathbb{H}\mathrm{P}^\infty \to \Omega^2 X \\ k_3 &= \mathrm{Ad}^3 \mu \circ (S^1 \wedge j') &&: S^1 \wedge \mathbb{H}\mathrm{P}^\infty \to \Omega^3 X \\ k_4 &= \mathrm{Ad}^4 \mu \circ j' &&: \mathbb{H}\mathrm{P}^\infty \to \Omega^4 X \end{aligned}$$

Let $\alpha_m \in H^*(S^m; \mathbb{Z}/2\mathbb{Z})$, $\beta \in H^*(\mathbb{H}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z})$ be generators as ring and $\alpha_m \beta^n$ be the element coming to $\alpha_m \times \beta^n$ by the canonical map $H^*(S^m \wedge \mathbb{H}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \to H^*(S^m \times \mathbb{H}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z})$.

First of all, we have

$$H^*(X\langle 1\rangle; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w'_2, w'_3, w'_4, \dots],$$

where $w'_i = pr_1^* w_i$ $(i > 2)$.

Beginning to calculate of the spectral sequence of path fibration, we let σ be a suspension map and, with grance at relation of Sq^1 and Sq^{j-1}, have following:

$$H^*((\Omega X)\langle 0\rangle; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\sigma(w'_2), \sigma(w'_4), \sigma(w'_6), \dots]$$
$$\sigma(w'_{2^n(2i-1)-1}) = \sigma(w'_{2i})^{2^n}.$$

Then,

$$H^*((\Omega X)\langle 1\rangle; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[t_3, t_5, t_7, \dots],$$

where $t_{2i-1} = \sigma(w'_{2i})$, $\operatorname{Sq}^2 t_{2i-1} = t_{2i+1}$, $\operatorname{Sq}^{2i-2} t_{2i-1} = t_{4i-3}$, $\operatorname{Sq}^{2j-3} t_{2i-1} = 0$, and $\tilde{k_1}(t_{4i+3}) = \alpha_3\beta^i$.

Next considering the relation of Sq^2 and Sq^{2i-2} between t_{2i-1}'s, we have following:

$$H^*((\Omega^2 X)\langle 0\rangle; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[y_2, y_6, y_{10}, \dots],$$

where $y_i = \sigma(t_{i+1})$, $\operatorname{Sq}^{4i-3} y_{4i} = 0$ and $\tilde{k_2}^*(y_{4i+2}) = \alpha_2\beta^i$

Then,

$$H^*((\Omega^2 X)\langle 1\rangle; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[y_2, y_6, y_{10}, \dots],$$

where $\operatorname{Sq}^{4i-3} y_{4i} = 0$ and $\tilde{k_2}^*(y_{4i+2}) = \alpha_2\beta^i$.

Calculate the spectral sequence of path fibration in the same way,

$$H^*((\Omega^3 X)\langle 0\rangle; \mathbb{Z}/2\mathbb{Z}) = \bigwedge(x_1, x_5, x_9, \dots),$$

where $\sigma(y_{i+1}) = x_i$, $|x_i| = i$ and $\tilde{k_3}^*(x_{4i+1}) = \alpha_1\beta^i$.

Then we have the following.

$$H^*((\Omega^3 X)\langle 1\rangle; \mathbb{Z}/2\mathbb{Z}) = \bigwedge(x_5, x_9, x_{13}, \dots),$$

where $\tilde{k_3}^*(x_{4i+1}) = \alpha_1\beta^i$.

To calculate the spectral sequence of path fibration further, we take dual with the fact that x_i's are primitive.

$$H_*((\Omega^3 X)\langle 1\rangle; \mathbb{Z}/2\mathbb{Z}) = \bigwedge(\xi_5, \xi_9, \xi_{13}, \dots),$$

where $\xi_{4i+1} = (x_{4i+1})^*$, $\widetilde{k_3}_*((\alpha_1\beta^i)^*) = \xi_{4i+1}$.

Let τ be a transgression map. Now we have the following.

$$H_*((\Omega^4 X)\langle 0\rangle; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[q_4, q_8, q_{12}, \dots],$$

where $q_{4i} = \tau(\xi_{4i+1})$ and $\widetilde{k_4}_*((\beta^i)^*) = q_{4i}$.

As the image of a map $\widetilde{k_4}_*$ includes generators of $H^*((\Omega^4 X)\langle 0\rangle; \mathbb{Z}/2\mathbb{Z})$, so does $\widetilde{\operatorname{Ad}^4\mu}_*$. Then $\widetilde{\operatorname{Ad}^4\mu}_*$. is an isomorphism and, by mod $\mathcal{C}_2$-Whitehead's theorem, $\widetilde{\operatorname{Ad}^4\mu}$ is mod $\mathcal{C}_2$-homotopy equivalence.

I show $\widetilde{\mathrm{Ad}^4}\lambda$'s case next.

Setting h's as follows.

$$\begin{aligned} h &= \lambda \circ (S^4 \wedge j) &&: S^4 \wedge \mathbb{R}\mathrm{P}^\infty \to Y \\ h_1 &= \mathrm{Ad}\,\lambda \circ (S^3 \wedge j) &&: S^3 \wedge \mathbb{R}\mathrm{P}^\infty \to \Omega Y \\ h_2 &= \mathrm{Ad}^2\,\lambda \circ (S^2 \wedge j) &&: S^2 \wedge \mathbb{R}\mathrm{P}^\infty \to \Omega^2 Y \\ h_3 &= \mathrm{Ad}^3\,\lambda \circ (S^1 \wedge j) &&: S^1 \wedge \mathbb{R}\mathrm{P}^\infty \to \Omega^3 Y \\ h_4 &= \mathrm{Ad}^4\,\lambda \circ j &&: \mathbb{R}\mathrm{P}^\infty \to \Omega^4 Y \end{aligned}$$

Let $\alpha_m \in H^*(S^m; \mathbb{Z}/2\mathbb{Z})$, $\beta \in H^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z})$ be generators as ring and $\alpha_m\beta^n$ be the element coming to $\alpha_m \times \beta^n$ by a canonical map $H^*(S^m \wedge \mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \to H^*(S^m \times \mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z})$.

Calculate the spectral sequence of path fibration,

$$H^*(\Omega Y; \mathbb{Z}/2\mathbb{Z}) = \bigwedge(d_3, d_7, d_{11}, \dots),$$

where $d_{4i-1} = \sigma(p_{4i})$ and ${h_1}^*(d_{4i-1}) = \alpha_3\beta^{4(i-1)}$.

To calculate more, we have to take dual with the fact that x_i's are primitive. Then we have the following:

$$H_*(\Omega Y; \mathbb{Z}/2\mathbb{Z}) = \bigwedge(\delta_3, \delta_7, \delta_{11}, \dots),$$

where $(d_{4i-1})^* = \delta_{4i-1}$ and $h_{1*}((\alpha_3\beta^{4(i-1)})^*) = \delta_{4i-1}$.

Calculate the spectral sequence of path fibration,

$$H_*(\Omega^2 Y; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[c_2, c_6, c_{10}, \dots],$$

where, $c_{4i-2} = \tau(\delta_{4i-1})$, $h_{2*}((\alpha_2\beta^{4(i-1)})^*) = c_{4i-2}$.

Now that we know $Y_{(2)} \simeq ((\Omega^4 X)\langle 1\rangle)_{(2)}$, then we can use the homology operation in [2]. So, setting b's as follows:

$$b_{2^n} = {c_2}^{2^{(n-1)}},\ b_{2^n(4i-2)} = {c_{4i-2}}^{2^n},$$

we get the relations ${\mathrm{Sq}^2}_* b_{4i+2} = {b_{2i}}^2$ and

$$H_*(\Omega^2 Y; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[b_2, b_6, b_{10}, \dots].$$

Now we can calculate spectral sequence by using transgression and have relation between cohomology of $\Omega^3 Y$ and $S^1 \wedge \mathbb{R}\mathrm{P}^\infty$ with help of above Sq^2_* relation and that $\mathrm{Sq}^2_*\,\alpha_1\beta^{4i} = \alpha_1\beta^{4i-2}$.

$$H_*(\Omega^3 Y; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[a_1, a_3, a_5, \dots],$$

where $\tau(b_{i+1}) = a_i$ and $h_{3*}((\alpha_1\beta^{2i})^*) = a_{2i+1}$.

Then,

$$H_*(\Omega^3 Y\langle 1\rangle; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[{a_1}^2, a_3, a_5, \dots]$$

Again, using above homology operation, we have $\mathrm{Sq}^1_* \, a_{2^n(2i-1)-1} = a^n_{2i-1}$. Finally, in the same way and grance at the relation that $\mathrm{Sq}^1_* \, \beta^{2i} = \beta^{2i-1}$, we have following:

$$H_*(\Omega^4 Y\langle 0\rangle; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[v_1, v_2, v_3, v_4, \dots]$$

where $h_{4*}((\beta^i)^*) = v_i$.

Then checking the generators similarly, we see that $\widetilde{\mathrm{Ad}^4}\lambda$ is $\bmod\, \mathcal{C}_2$-homotopy equivalence. □

Next lemma shows that if there is a space Z which is related to X and Y by certain maps, we can tell X and Y are homotopy equivalent.

Lemma 2.2. *Suppose the following.*

(1) *X and Y satisfy the conditions in Theorem* 2.1 *and rank of $\pi_{4*}(Y) =$ rank of $\pi_{4*}(X)$.*

(2) *There exists a space Z whose homotopy groups are free.*

(3) *There exist maps as $\rho : \pi_*(Z) \to \pi_{*+4}(Z)$, which is isomorphism after tensoring $\mathbb{Z}[1/2]$, $c' : \pi_*(Y) \to \pi_*(Z)$, $c : \pi_*(X) \to \pi_*(Z)$, which are splitting and monic after tensoring $\mathbb{Z}[1/2]$. They satisfy following commutative diagrams.*

$$\begin{array}{ccc}
\pi_*(X) & \xrightarrow{(\widetilde{\mathrm{Ad}^4}\lambda)_*} & \pi_{*+4}(Y) \\
\Big\downarrow c & \circlearrowleft & \Big\downarrow c' \\
\pi_*(Z) & \xrightarrow{\rho} & \pi_{*+4}(Z)
\end{array}$$

$$\begin{array}{ccc}
\pi_*(Y) & \xrightarrow{(\widetilde{\mathrm{Ad}^4}\mu)_*} & \pi_{*+4}(X) \\
\Big\downarrow c' & \circlearrowleft & \Big\downarrow c \\
\pi_*(Z) & \xrightarrow{\rho} & \pi_{*+4}(Z)
\end{array}$$

Then $(\widetilde{\mathrm{Ad}^4}\lambda)_* : X \simeq (\Omega^4 Y)\langle 0\rangle$ and $(\widetilde{\mathrm{Ad}^4}\mu)_* : Y \simeq (\Omega^4 X)\langle 0\rangle$.

Proof. By the assumption, we see that $\pi_*(X)$ and $\pi_*(Y)$ are odd torsion free, and that $(\widetilde{\mathrm{Ad}^4}\lambda)_*$ and $(\widetilde{\mathrm{Ad}^4}\mu)_*$ are $\bmod\, \mathcal{C}_2$-homotopy equivalence. So we only need to check $(\widetilde{\mathrm{Ad}^4}\lambda)_*$ and $(\widetilde{\mathrm{Ad}^4}\mu)_*$ on the free parts of $\pi_*(X)$ and $\pi_*(Y)$.

$\pi_*(Y)$ has free parts only when $*$ is divided with 4, because of its cohomology. So we need to know $(\widetilde{\mathrm{Ad}^4}\lambda)_* : \pi_{4*}(X) \to \pi_{4*+4}(Y)$ and $(\widetilde{\mathrm{Ad}^4}\mu)_* : \pi_{4*}(Y) \to \pi_{4*+4}(X)$ are isomorphism

But it is easily verified by that $\pi_{4*}(Z)$ is free, rank of $\pi_{4*}(Y)$ = rank of $\pi_{4*}(X)$ and the commutative diagrams in the assumption. □

3. The case of BO and BSp

To apply theorem to BO and BSp, we have to construct maps λ, μ, j and j' for BO and BSp. To do that, we set notations of vector bundles.

$\underline{l_{\mathbb{R}}}$, $\underline{m_{\mathbb{C}}}$ and $\underline{n_{\mathbb{H}}}$ are rank l trivial real bundle, rank m trivial complex bundle and rank n trivial symplectic bundle.

Let ζ and η be the Hopf bundle of S^2 and S^4, and, η_2, η_∞, $\xi_{\mathrm{BO}(n)}$, $\xi_{\mathrm{BU}(n)}$ and $\xi_{\mathrm{BSp}(n)}$ be the universal bundles of $\mathbb{H}\mathrm{P}^2$, $\mathbb{H}\mathrm{P}^\infty$, $\mathrm{BO}(n)$, $\mathrm{BU}(n)$ and $\mathrm{BSp}(n)$, and virtual vector bundles ξ_{BO}, ξ_{BU}, ξ_{BSp} and $\xi_{\mathbb{R}\mathrm{P}^\infty}$ be $\lim_{n\to\infty}(\xi_{\mathrm{BO}(n)} - \underline{n_{\mathbb{R}}})$, $\lim_{n\to\infty}(\xi_{\mathrm{BU}(n)} - \underline{n_{\mathbb{C}}})$, $\lim_{n\to\infty}(\xi_{\mathrm{BSp}(n)} - \underline{n_{\mathbb{H}}})$ and $\xi_{\mathrm{BO}}|_{\mathbb{R}\mathrm{P}^\infty} = \xi_{\mathbb{R}\mathrm{P}^\infty}$

We consider a virtual vector bundle $(\eta - \underline{2_{\mathbb{C}}})\hat{\otimes}_{\mathbb{R}}\xi_{\mathrm{BO}}$ which is the element of $\widetilde{\mathrm{KSp}}(S^4 \wedge BO)$. We set $\lambda : S^4 \wedge \mathrm{BO} \to \mathrm{BSp}$ be a classifying map of the one.

Similarly we set $\mu' : \mathbb{H}\mathrm{P}^2 \wedge \mathrm{BSp} \to \mathrm{BU}$ be a classifying map of $(\eta_2 - \underline{2_{\mathbb{C}}})\hat{\otimes}_{\mathbb{C}}\xi_{\mathrm{BSp}} \in \widetilde{\mathrm{KU}}(\mathbb{H}\mathrm{P}^2 \wedge \mathrm{BSp})$. Since tensor product of two symplectic bundles has a real structure i.e. $(\eta_2 - \underline{2_{\mathbb{C}}})\hat{\otimes}_{\mathbb{C}}\xi_{\mathrm{BSp}} = \xi \otimes_{\mathbb{R}} \underline{1_{\mathbb{C}}}$, for certain virtual real vector bundle ξ on $\mathbb{H}\mathrm{P}^2 \wedge \mathrm{BSp}$, we can set $\mu : S^4 \wedge \mathrm{BSp} \to \mathrm{BO}$ be the classifying map of $\xi|_{S^4\wedge\mathrm{BSp}} \in \widetilde{\mathrm{KO}}(S^4 \wedge \mathrm{BSp})$.

Similarly we have $(\eta_2 - \underline{2_{\mathbb{C}}})\hat{\otimes}_{\mathbb{C}}(\eta_\infty - \underline{2_{\mathbb{C}}}) = \xi' \otimes_{\mathbb{R}} \underline{1_{\mathbb{C}}}$ for certain virtual real vector bundle ξ' on $\mathbb{H}\mathrm{P}^2 \wedge \mathbb{H}\mathrm{P}^\infty$.

Let $j : \mathbb{R}\mathrm{P}^\infty \to \mathrm{BO}$ and $j' : \mathbb{H}\mathrm{P}^\infty \to \mathrm{BSp}$ be inclusion.

It is easily seen as in [1] that $\mathrm{Ad}^4\,\lambda$ and $\mathrm{Ad}^4\,\mu$ are H-maps.

Lemma 3.1. $(\lambda\circ(S^4\wedge j))^* : H^{4*}(\mathrm{BSp};\mathbb{Z}/2\mathbb{Z}) \to H^{4*}(S^4\wedge\mathbb{R}\mathrm{P}^\infty);\mathbb{Z}/2\mathbb{Z})$ $(\mu\circ(S^4\wedge j'))^* : H^*(\mathrm{BO};\mathbb{Z}/2\mathbb{Z}) \to H^*(S^4\wedge\mathbb{H}\mathrm{P}^\infty;\mathbb{Z}/2\mathbb{Z})$ *are epic.*

Proof. Calculate total Chern class for λ.

$$\begin{aligned}
(\lambda\circ(S^4\wedge j))^*c(\xi_{\mathrm{BSp}}) &\equiv (S^4\wedge j)^*c((\eta - \underline{2_{\mathbb{C}}})\hat{\otimes}_{\mathbb{R}}\xi_{\mathrm{BO}})\\
&\equiv c((\eta - \underline{2_{\mathbb{C}}})\hat{\otimes}_{\mathbb{R}}(\xi_{\mathbb{R}\mathrm{P}^\infty} - \underline{1_{\mathbb{R}}}))\\
&\equiv c(\eta\hat{\otimes}_{\mathbb{R}}\xi_{\mathbb{R}\mathrm{P}^\infty})c(\xi_{\mathbb{R}\mathrm{P}^\infty}\hat{\otimes}_{\mathbb{R}}\underline{1_{\mathbb{C}}})^{-2}c(\eta)^{-1}\\
&\equiv (1+\alpha+\beta^2)(1+\beta)^{-2}(1+\alpha)^{-1}\\
&\equiv 1 + \sum_{i=1}^{\infty}\alpha\beta^{2i} \pmod 2
\end{aligned}$$

$(\alpha = c_2(\eta)\ \beta = c_1(\underline{1_c}\hat{\otimes}_{\mathbb{R}}\xi_{\mathbb{R}\mathrm{P}^\infty}) = w_1(\xi_{\mathbb{R}\mathrm{P}^\infty})^2$, where w_1 is a first Stiefel-Whitney class)

Then, $(\lambda \circ (S^4 \wedge j))^* : H^{4*}(\mathrm{BSp}; \mathbb{Z}/2\mathbb{Z}) \to H^{4*}(S^4 \wedge \mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z})$ is epic.

Calculate total Chern class for μ.

$$\begin{aligned}(\widetilde{\mu'} \circ (\mathbb{H}\mathrm{P}^2 \wedge j'))^* c(\xi_{\mathrm{BU}}) &\equiv c((\eta_2 - \underline{2_{\mathbb{C}}})\hat{\otimes}_{\mathbb{C}}(\eta_\infty - \underline{2_{\mathbb{C}}})) \\ &\equiv c(\xi' \otimes_{\mathbb{R}} \underline{1_{\mathbb{C}}}) \\ &\equiv w(\xi')^2 \qquad (\mathrm{mod}\, 2)\end{aligned}$$

On the other hand,

$$\begin{aligned}(\widetilde{\mu'} \circ (\mathbb{H}\mathrm{P}^2 \wedge j'))^* c(\xi_{\mathrm{BU}}) &\equiv c(\eta_2 - \underline{2_{\mathbb{C}}}\hat{\otimes}_{\mathbb{C}}\eta_\infty - \underline{2_{\mathbb{C}}}) \\ &\equiv c(\eta_2\hat{\otimes}_{\mathbb{C}}\eta_\infty)c(\eta_2)^{-2}c(\eta_\infty)^{-2} \\ &\equiv (1 + \alpha^{-2} + \beta^{-2})(1+\alpha)^{-2}(1+\beta)^{-2} \\ &\equiv \{1 + \alpha\sum_{i=1}^{\infty}\beta^i\}^2 \qquad (\mathrm{mod}\, 2)\end{aligned}$$

$(\alpha = c_2(\eta_2),\ \beta = c_2(\eta_\infty))$

Then we get $w(\xi') = 1 + \alpha\sum_{i=1}^{\infty}\beta^i + \alpha^2 f(\alpha, \beta)$. ($f(\alpha,\beta)$ is a formal power series of α and β)

Restricting to $S^4 \wedge \mathbb{H}\mathrm{P}^\infty$, we have the following.

$$(\mu \circ (S^4 \wedge j'))^* w(\xi_{\mathrm{BO}}) \equiv 1 + c_2(\eta)\sum_{i=1}^{\infty}\beta^i \qquad (\mathrm{mod}\, 2)$$

So, $(\mu \circ (S^4 \wedge j'))^* : H^*(\mathrm{BO}; \mathbb{Z}/2\mathbb{Z}) \to H^*(S^4 \wedge \mathbb{H}\mathrm{P}^\infty; \mathbb{Z}/2\mathbb{Z})$ is epic. □

Now that we only need to show the next lemma to prove Bott periodicity theorem.

Corollary 3.2 (Bott Periodicity Theorem)**.**

$$\mathrm{BSp} \simeq (\Omega^4\mathrm{BO})\langle 0\rangle,\ \mathrm{BO} \simeq (\Omega^4\mathrm{BSp})\langle 0\rangle$$

Proof. Recall that BU, BO and BSp are classifying spaces of cohomology theories $\widetilde{\mathrm{KU}}$, $\widetilde{\mathrm{KO}}$ and $\widetilde{\mathrm{KSp}}$, we can define maps ρ' and $\rho'' : S^2 \wedge \mathrm{BU} \to \mathrm{BU}$ to be the classifying maps of $(\zeta - \underline{1_{\mathbb{C}}})\hat{\otimes}_{\mathbb{C}}\xi_{\mathrm{BU}}$ and $(\bar{\zeta} - \underline{1_{\mathbb{C}}})\hat{\otimes}_{\mathbb{C}}\xi_{\mathrm{BU}}$ (see [1]), and, $r : \pi_*(\mathrm{BU}) \to \pi_*(\mathrm{BO})$, $c' : \pi_*(\overline{\mathrm{BSp}}) \to \pi_*(\mathrm{BU})$, $c : \pi_*(\mathrm{BO}) \to \pi_*(\mathrm{BU})$ and $q : \pi_*(\mathrm{BU}) \to \pi_*(\mathrm{BSp})$ to be the maps which induced from realization of complex bundle, complexification of symplectic one, complexification of real one and quaterniation of complex one.

We have $c \circ r = 2$ and $q \circ c' = 2$, then $c_* : \pi*(\mathrm{BO}) \otimes \mathbb{Z}[1/2] \to \pi*(\mathrm{BU}) \otimes \mathbb{Z}[1/2]$ and $c'_* : \pi*(\mathrm{BSp}) \otimes \mathbb{Z}[1/2] \to \pi*(\mathrm{BU}) \otimes \mathbb{Z}[1/2]$ are splitting and monic.

We set $\rho = (\widetilde{\mathrm{Ad}^2 \rho'})_* \circ (\widetilde{\mathrm{Ad}^2 \rho''})_* : \pi_*(\mathrm{BU}) \to \pi_{*+4}(\mathrm{BU})$. Then, by definition of ρ' and ρ'', and by the fact $(\zeta - \underline{1_{\mathbb{C}}}) \hat{\otimes}_{\mathbb{C}} (\bar{\zeta} - \underline{1_{\mathbb{C}}}) = \eta - \underline{1_{\mathbb{C}}}$, we also have commutative diagrams below.

$$
\begin{array}{ccc}
\pi_*(\mathrm{BO}) & \xrightarrow{(\widetilde{\mathrm{Ad}^4 \lambda})_*} & \pi_{*+4}(\mathrm{BSp}) \\
\downarrow c & \circlearrowleft & \downarrow c' \\
\pi_*(\mathrm{BU}) & \xrightarrow{\rho} & \pi_{*+4}(\mathrm{BU})
\end{array}
$$

$$
\begin{array}{ccc}
\pi_*(\mathrm{BSp}) & \xrightarrow{(\widetilde{\mathrm{Ad}^4 \mu})_*} & \pi_{*+4}(\mathrm{BO}) \\
\downarrow c' & \circlearrowleft & \downarrow c \\
\pi_*(\mathrm{BU}) & \xrightarrow{\rho} & \pi_{*+4}(\mathrm{BU})
\end{array}
$$

We already know that $\mathbb{Z}/2\mathbb{Z}$-coefficient cohomology rings and homotopy goups of BO, BU and BSp, and that ρ is isomorphism, so we can apply Lemma 2.2 and have periodicity. □

4. Characterization of BO

Let Y be a topological space and $Q(Y) = \lim_{n\to\infty} \Omega^n S^n Y$.

Let X be an infinite loop space. We consider a map $\Xi_X : Q(X) \to X$ which is defined as $\Xi_X = \lim_{n\to\infty} {\epsilon_n}^{-1} \circ \Omega^n (\mathrm{Ad}^n\, \epsilon_n)^{-1} : Q(X) \to X$, where $\epsilon_n : X \xrightarrow{\sim} \Omega^n B_n$.

We can see in [3] that there exists a map $\epsilon : BSp \to Q(\mathbb{H}\mathrm{P})$ s.t. $\Xi_{\mathrm{BSp}} \circ Q(i) \circ \epsilon \simeq id_{\mathrm{BSp}}$, where $i : \mathbb{H}\mathrm{P}^\infty \to \mathrm{BSp}$ is inclusion. And, by ϵ, we have $Q(\mathbb{H}\mathrm{P}^\infty) \simeq \mathrm{BSp} \times F$ and $\pi_*(F)$ is finite, so $H_*(F;\mathbb{Z})$ is finite.

We try to characterize BO through the characterization of BSp by means of homotopy equivalence above and by that BSp is an infinite loop space.

Theorem 4.1. *Suppose the following.*

(1) *Spaces X and Y satisfy the condition of Lemma* 2.2.

(2) *The map $j' : \mathbb{H}\mathrm{P}^\infty \to Y$ also satisfies that the image of $j'_* : H_*(\mathbb{H}\mathrm{P}^\infty;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ includes all algebra generators of $H_*(Y;\mathbb{Z})$.*

Then $\Xi_Y \circ Q(j) \circ \epsilon : \mathrm{BSp} \to Y$ is homotopy equivalence. And BO *is homotopy equivalent to X.*

Proof. We denote $i : \mathbb{H}\mathrm{P}^\infty \to Q(\mathbb{H}\mathrm{P}^\infty)$ be inclusion, so we have, by definition, that $\Xi_Y \circ Q(j) \circ i = j : \mathbb{H}\mathrm{P}^\infty \to Y$.

Then we can consider a commutative diagram below.

$$\begin{array}{ccccc}
\mathbb{H}\mathrm{P}^\infty & & \xrightarrow{\quad j \quad} & & Y \\
\downarrow{\scriptstyle i} & & & & \| \\
Q(\mathbb{H}\mathrm{P}^\infty) & \xrightarrow[Q(j)]{} & Q(Y) & \xrightarrow[\Xi_Y]{} & Y
\end{array}$$

And now we have a commutative diagram below.

$$\begin{array}{ccccc}
 & & H_*(\mathbb{H}\mathrm{P}^\infty;\mathbb{Z}) & \xrightarrow{j_*} & H_*(Y;\mathbb{Z}) \\
 & & \downarrow{\scriptstyle i_*} & & \| \\
H_*(\mathrm{BSp};\mathbb{Z}) & \xrightarrow{\epsilon_*} & H_*(Q(\mathbb{H}\mathrm{P}^\infty);\mathbb{Z}) & \xrightarrow{(\Xi_Y \circ Q(j))_*} & H_*(Y;\mathbb{Z}) \\
\| & & \downarrow{\scriptstyle proj} & & \| \\
H_*(\mathrm{BSp};\mathbb{Z}) & \xrightarrow[=]{\sim} & H_*(Q(\mathbb{H}\mathrm{P}^\infty);\mathbb{Z})/\,\mathrm{Tor} & \xrightarrow[\varphi]{} & H_*(Y;\mathbb{Z})
\end{array}$$

Since $H_*(Y;\mathbb{Z})$ is free and $\mathrm{Im}\ \varphi \supset \mathrm{Im}\ j_*$, in other words $\mathrm{Im}\ \varphi$ includes all algebra generators of $H_*(Y;\mathbb{Z})$, φ is isomorphism.

Then $(\Xi_Y \circ Q(j) \circ \epsilon)_* : H_*(\mathrm{BSp};\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ is isomorphism and, finally, we have $\Xi_Y \circ Q(j) \circ \epsilon : \mathrm{BSp} \to Y$ is homotopy equivalence. And this says that BO is homotopy equivalent to X. $\square$

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