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The integral cohomology ring of E_7/T

By

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1. Introduction

Let G be a compact connected Lie group and T be its maximal torus. The quotient space G/T is called a flag manifold and plays an important role in topology, representation theory, etc. Since G has a finite covering group which is a direct product of a torus and compact 1-connected simple Lie groups, G/T is homeomorphic to a direct product of quotients of compact 1-connected simple Lie groups by maximal tori. On the other hand each factor has no torsion according to [3]. Therefore in order to determine the integral cohomology ring of G/T, it suffices to consider the case when G is 1-connected simple by the Künneth formula. For $G = SU(n), Sp(n), Spin(n), G_2, F_4$ and E_6, the integral cohomology ring of G/T is known ([1], [4], [9]). The purpose of this paper is to determine the integral cohomology ring of E_7/T. The method used here is quite similar to that in [9], [10].

The paper is organized as follows: In Section 2 we discuss the action of the Weyl groups on $H^*(BT;\mathbb{Q})$ and compute the invariant subalgebras of the Weyl groups. The rational cohomology rings of $E_7/C_1, C_1 = T^1 \cdot Spin(12)$ and E_7/T are determined in Section 3. Section 4 is a preparation for Section 5. In the final section, Section 5 we determine the integral cohomology rings of E_7/C_1 and E_7/T.

Throughout this paper $H^*(\cdot)$ denotes the integral cohomology ring and $\sigma_i(x_1, \cdots, x_n)$ denotes the i-th elementary symmetric function in the variables $x_1, \cdots, x_n$.

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2. The rational invariant subalgebras of the Weyl groups

Let T be a maximal torus of E_7. According to [5] the Dynkin diagram of E_7 is

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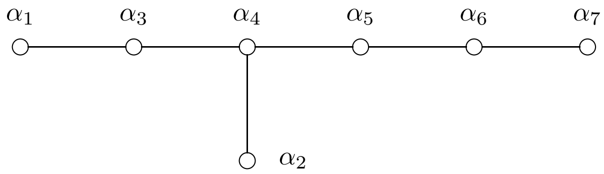

where α_i's are the simple roots. As usual we may regard each root as an element of $H^1(T) \xrightarrow{\sim} H^2(BT)$.

Let C_1 be the centralizer of the one dimensional torus determined by $\alpha_i = 0$ $(i \neq 1)$. Then the Weyl groups $W(\cdot)$ of E_7, C_1 are given as follows:

$$W(E_7) = \langle R_i \ (1 \leq i \leq 7) \rangle \ , \quad W(C_1) = \langle R_i \ (i \neq 1) \rangle \ ,$$

where R_i denotes the reflection to the hyperplane defined by $\alpha_i = 0$.

Note that ([6])

$$C_1 = T^1 \cdot Spin(12) \ , \quad T^1 \cap Spin(12) \cong \mathbb{Z}_2 \ .$$

Let $\{w_i\}_{1 \leq i \leq 7}$ be the fundamental weights of E_7 corresponding to the system of the simple roots $\{\alpha_i\}_{1 \leq i \leq 7}$. We also regard each weight as an element of $H^2(BT)$ and then $\{w_i\}_{1 \leq i \leq 7}$ forms a basis of $H^2(BT)$. The action of R_i's on $\{w_i\}_{1 \leq i \leq 7}$ is given as follows:

$$R_i(w_i) = w_i - \sum_{j=1}^{7} \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} w_j \quad \text{and} \quad R_i(w_k) = w_k \quad \text{for} \quad k \neq i \ .$$

Following [10] we define

$$\begin{aligned} &t_7 = w_7 \ , \quad t_i = R_{i+1}(t_{i+1}) \ \ (2 \leq i \leq 6) \ , \quad t_1 = R_1(t_2) \ , \\ &c_i = \sigma_i(t_1, \cdots, t_7) \ , \quad t = \frac{1}{3} c_1 \ . \end{aligned}$$

Then t and t_i $(1 \leq i \leq 7)$ span $H^2(BT)$ since each w_i is an integral linear combination of t and t_i $(1 \leq i \leq 7)$ and we have the following isomorphism:

$$H^*(BT) = \mathbb{Z}[t_1, \cdots, t_7, t]/(3t - c_1) \ .$$

Furthermore the action of R_i's on these elements is given by the following table:

	R_1	R_2	R_3	R_4	R_5	R_6	R_7
t_1	t_2	$t - t_2 - t_3$					
t_2	t_1	$t - t_1 - t_3$	t_3				
t_3		$t - t_1 - t_2$	t_2	t_4			
t_4				t_3	t_5		
t_5					t_4	t_6	
t_6						t_5	t_7
t_7							t_6
t		$-t + t_4 + t_5 + t_6 + t_7$					

where blanks indicate the trivial action.

Putting

$$t_0 = t - t_1 \quad \text{and} \quad \epsilon_i = t_{i+1} - \frac{1}{2}t_0 \ \ (1 \leq i \leq 6)\,,$$

we have

$$H^*(BT;\mathbb{Q}) = \mathbb{Q}[t_1, \cdots, t_7] = \mathbb{Q}[t_0, \epsilon_1, \epsilon_2, \cdots, \epsilon_6]$$

and the following table of the action:

	R_2	R_3	R_4	R_5	R_6	R_7
ϵ_1	$-\epsilon_2$	ϵ_2				
ϵ_2	$-\epsilon_1$	ϵ_1	ϵ_3			
ϵ_3			ϵ_2	ϵ_4		
ϵ_4				ϵ_3	ϵ_5	
ϵ_5					ϵ_4	ϵ_6
ϵ_6						ϵ_5
t_0						

From this table

Lemma 2.1. *The invariant subalgebra of $W(C_1)$ is given as follows*:

$$H^*(BT;\mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4, p_5]\,,$$

where

$$p_i = \sigma_i(\epsilon_1^2, \cdots, \epsilon_6^2) \quad \text{and} \quad e = \prod_{i=1}^{6} \epsilon_i\,.$$

We can compute p_i's in the following way: Put

$$b_i = \sigma_i(\epsilon_1, \cdots, \epsilon_6)$$

so that

$$\begin{aligned}\sum_{i\geq 0}(-1)^i p_i &= \prod_{j=1}^{6}(1-\epsilon_j^2) = \prod_{j=1}^{6}(1+\epsilon_j)\prod_{j=1}^{6}(1-\epsilon_j)\\ &= \left(\sum_{k\geq 0} b_k\right)\left(\sum_{l\geq 0}(-1)^l b_l\right) = \sum_{k,l\geq 0}(-1)^l b_k b_l\,.\end{aligned}$$

Therefore

$$p_i = \sum_{k+l=2i}(-1)^{l+i} b_k b_l\,.$$

More precisely

$$p_1 = b_1^2 - 2b_2\,, \quad p_2 = b_2^2 - 2b_1b_3 + 2b_4\,, \quad p_3 = b_3^2 - 2b_2b_4 + 2b_1b_5 - 2b_6\,,$$
$$p_4 = b_4^2 - 2b_3b_5 + 2b_2b_6\,, \quad p_5 = b_5^2 - 2b_4b_6\,, \quad p_6 = b_6^2\,.$$

On the other hand since

$$\begin{aligned}\left(1-\frac{1}{2}t_0+t_1\right)\sum_{n=0}^{6} b_n &= \left(1-\frac{1}{2}t_0+t_1\right)\prod_{i=1}^{6}(1+\epsilon_i)\\ &= \left(1-\frac{1}{2}t_0+t_1\right)\prod_{i=1}^{6}\left(1+t_{i+1}-\frac{1}{2}t_0\right) = \prod_{i=1}^{7}\left(1-\frac{1}{2}t_0+t_i\right)\\ &= \sum_{i=0}^{7}\left(1-\frac{1}{2}t_0\right)^{7-i} c_i\,,\end{aligned}$$

we have

$$b_n + \left(-\frac{1}{2}t_0+t_1\right)b_{n-1} = \sum_{i=0}^{n}\binom{7-i}{n-i}\left(-\frac{1}{2}t_0\right)^{n-i} c_i \quad (1 \le n \le 6)$$

and then

$$\begin{aligned}&b_1 = 2t_1\,, \quad b_2 = c_2 - 2t_1^2 - 8t_1t_0 - \frac{15}{4}t_0^2\,,\\ &b_3 = c_3 - (t_1+2t_0)c_2 + 2t_1^3 + 7t_1^2t_0 + 11t_1t_0^2 + 5t_0^3\,,\\ &b_4 = c_4 - \left(t_1+\frac{3}{2}t_0\right)c_3 + \left(t_1^2+\frac{3}{2}t_1t_0+\frac{3}{2}t_0^2\right)c_2 - 2t_1^4 - 6t_1^3t_0 - \frac{15}{2}t_1^2t_0^2\\ &\qquad - 7t_1t_0^3 - \frac{45}{16}t_0^4\,,\\ &b_5 = c_5 - (t_1+t_0)c_4 + \left(t_1^2+t_1t_0+\frac{3}{4}t_0^2\right)c_3 - \left(t_1^3+t_1^2t_0+\frac{3}{4}t_1t_0^2+\frac{1}{2}t_0^3\right)c_2\\ &\qquad + 2t_1^5 + 5t_1^4t_0 + \frac{9}{2}t_1^3t_0^2 + \frac{13}{4}t_1^2t_0^3 + \frac{17}{8}t_1t_0^4 + \frac{3}{4}t_0^5\,,\\ &b_6 = c_6 - \left(t_1+\frac{1}{2}t_0\right)c_5 + \left(t_1^2+\frac{1}{2}t_1t_0+\frac{1}{4}t_0^2\right)c_4 - \left(t_1^3+\frac{1}{2}t_1^2t_0+\frac{1}{4}t_1t_0^2\right.\\ &\qquad \left.+\frac{1}{8}t_0^3\right)c_3 + \left(t_1^4+\frac{1}{2}t_1^3t_0+\frac{1}{4}t_1^2t_0^2+\frac{1}{8}t_1t_0^3+\frac{1}{16}t_0^4\right)c_2 - 2t_1^6 - 4t_1^5t_0\\ &\qquad - 2t_1^4t_0^2 - t_1^3t_0^3 - \frac{1}{2}t_1^2t_0^4 - \frac{1}{4}t_1t_0^5 - \frac{5}{64}t_0^6\,.\end{aligned}$$

By these relations we can compute p_i's.

Next we put

$$x_i = 2t_i - t\ (1 \le i \le 7) \quad \text{and} \quad x_8 = t\,.$$

Then we have the following $W(E_7)$-invariant subset

$$S = \{x_i + x_j, -x_i - x_j \ (1 \le i < j \le 8)\} \subset H^2(BT;\mathbb{Q}) \,.$$

Thus we have $W(E_7)$-invariant forms

$$I_n = \sum_{y\in S} y^n \in H^{2n}(BT;\mathbb{Q})^{W(E_7)} \,.$$

As in [10] Section 2 I_n is computed by the formula:

$$I_n = (16 - 2^n)s_n + \sum_{0<i<n} \binom{n}{i} s_i s_{n-i} \quad \text{for} \ \ n \quad \text{even},$$

where $s_n = x_1^n + \cdots + x_8^n$ and s_n is written as a polynomial in d_i's, $d_i = \sigma_i(x_1, \cdots, x_8)$ by use of the Newton formula:

$$s_n = \sum_{1\le i<n} (-1)^{i-1} s_{n-i} d_i + (-1)^{n-1} n d_n \quad (d_n = 0 \quad \text{for} \quad n > 8) \,.$$

Moreover we rewrite d_i in terms of t and c_i's by the formulae:

$$\begin{aligned} d_i &= e_i + t e_{i-1} && (1 \le i \le 8) \,, \\ e_n &= \sum_{i=0}^{n} (-1)^{n-i} 2^i \binom{7-i}{n-i} c_i t^{n-i} && (1 \le n \le 7) \,, \end{aligned}$$

where $e_i = \sigma_i(x_1, \cdots, x_7)$. Therefore I_n can be written as a polynomial in t and c_i's. Then the next lemma is proved in [10] Lemma 2.1.

Lemma 2.2. *The invariant subalgebra of $W(E_7)$ is given as follows*:

$$H^*(BT;\mathbb{Q})^{W(E_7)} = \mathbb{Q}[I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}] \,.$$

Consider the following elements ($J_i \in H^{2i}(BT;\mathbb{Q})$):

$$\begin{aligned} J_2 &= c_2 - 4t^2, \\ J_6 &= c_3^2 + 8c_6 - 4c_5 t - 4c_3 t^3 + 4t^6, \\ J_8 &= 2c_4^2 - 3c_3c_5 + 12c_7 t - 3c_3c_4 t - 30c_6 t^2 + 24c_5 t^3 + 2c_4 t^4 + 2t^8, \\ J_{10} &= c_5^2 - 4c_3c_7 - 2c_4c_5 t + 2c_3c_5 t^2 + c_4^2 t^2 - 2c_3c_4 t^3 + 12c_7 t^3 - 8c_6 t^4 + 4c_4 t^6, \\ J_{12} &= -6t_0^8 u + 9t_0^4 u^2 + 2t_0^6 v - 12t_0^2 uv + u^3 + 3v^2, \\ J_{14} &= t_0^{14} - 6t_0^{10} u - 3t_0^6 u^2 + 4t_0^8 v - 6t_0^4 uv - 3u^2 v + 3t_0^2 v^2, \\ J_{18} &= -8t_0^{14} u + 24t_0^6 u^3 + 9t_0^2 u^4 - 8t_0^8 uv - 48t_0^4 u^2 v - 12u^3 v - 4t_0^6 v^2 \\ &\quad + 24t_0^2 uv^2 - 8v^3, \end{aligned}$$

where

$$t_0 = t - t_1 \,, \quad u = \frac{1}{6}p_2 - \frac{13}{32}t_0^4 \,, \quad v = e + \frac{3}{4}t_0^2 u - \frac{43}{64}t_0^6 \,.$$

Put

$$A = H^*(BT;\mathbb{Q})^{\langle R_3,\cdots,R_7\rangle}\,.$$

A is a subalgebra of $H^*(BT;\mathbb{Q})$ containing $H^*(BT;\mathbb{Q})^{W(C_1)}$. Denote by

$$\mathfrak{a}_i \subset A \quad (\text{resp. } \mathfrak{b}_i \subset H^*(BT;\mathbb{Q})^{W(C_1)})$$

the ideal of A (resp. of $H^*(BT;\mathbb{Q})^{W(C_1)}$) generated by I_j's for $j < i, \quad j \in \{2,6,8,10,12,14,18\}$. Then we have the following

Lemma 2.3.

(i) $I_2 = -2^5 \cdot 3J_2\,, \qquad I_6 \equiv 2^8 \cdot 3^2 J_6 \mod \mathfrak{a}_6\,,$
$I_8 \equiv 2^{12} \cdot 5J_8 \mod \mathfrak{a}_8\,, \qquad I_{10} \equiv 2^{12}\cdot 3^2 \cdot 5 \cdot 7J_{10} \mod \mathfrak{a}_{10}\,.$

In $H^*(BT;\mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0,p_1,p_2,e,p_3,p_4,p_5]$ *we have*

(ii) $I_2 = 24(2p_1 + t_0^2)\,, \qquad I_6 = 2^8 \cdot 3^2 p_3 + 2^9 \cdot 3^2 \cdot 5e + \textit{decomp.}\,,$
$I_8 = 2^{11}\cdot 3 \cdot 5p_4 + \textit{decomp.}\,, \qquad I_{10} = 2^{12}\cdot 3^2\cdot 5\cdot 7p_5 + \textit{decomp.}\,.$

(iii) $I_{12} \equiv -2^{16}\cdot 3^4 \cdot 5J_{12} \mod \mathfrak{b}_{12}\,,$
$I_{14} \equiv 2^{17}\cdot 3\cdot 7\cdot 11\cdot 29J_{14} \mod \mathfrak{b}_{12}\,,$
$I_{18} \equiv 2^{20}\cdot 3^3 \cdot 1229J_{18} \mod \mathfrak{b}_{12}\,,$

where decomp. means decomposable elements.

Proof. (i) Using the previous notations it is verified directly that

$$\begin{aligned} I_2 &= -24d_2\,,\\ I_6 &\equiv 36(d_3^2 + 8d_6) \mod \mathfrak{a}_6\,,\\ I_8 &\equiv 80(2d_4^2 - 3d_3d_5 + 24d_8) \mod \mathfrak{a}_8\,,\\ I_{10} &\equiv 1260(d_5^2 - 4d_3d_7) \mod \mathfrak{a}_{10}\,. \end{aligned}$$

Rewriting d_i's in terms of t and c_i's we have the required results.

(ii) Since $I_{10} \in H^*(BT;\mathbb{Q})^{W(E_7)} \subset H^*(BT;\mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0,p_1,p_2,e,p_3,p_4,p_5]$ we may put

$$I_{10} = \alpha\, p_5 + \text{decomp.} \quad \text{for some} \quad \alpha \in \mathbb{Q}\,.$$

Take the following values of variables; $t_0 = 0, \epsilon_i = \zeta^i$ for $i = 1,2,3,4,5$ and $\epsilon_6 = 0$ where $\zeta = \exp(2\pi\sqrt{-1}/10)$. Then we have easily that $p_1 = p_2 = e = p_3 = p_4 = 0,\ p_5 = 1$ and

$$\begin{aligned} &x_1 = t = \frac{1}{2}\,(\zeta + \zeta^2 + \zeta^3 + \zeta^4 - 1)\,,\ x_i = 2\zeta^{i-1} - t \ \ (2 \le i \le 6)\,,\\ &x_7 = -t\,,\ x_8 = t\,. \end{aligned}$$

Then

$$S = \left\{ \begin{array}{l} 2\zeta^i \ (0 \le i \le 9),\ 2\zeta^i \ (0 \le i \le 9),\ 2\zeta^i \ (0 \le i \le 9)\ , 0, 0, 0, 0, 0, 0, \\ 2\zeta^i(1+\zeta^2)\ (0 \le i \le 9), 2\zeta^i(1+\zeta^4)\ (0 \le i \le 9) \end{array} \right\}.$$

Therefore

$$\begin{aligned} \alpha = \sum_{y \in S} y^{10} &= 2^{11} \cdot 5\{3 + (1+\zeta^2)^{10} + (1+\zeta^4)^{10}\} \\ &= 2^{11} \cdot 5(3 + 123) = 2^{12} \cdot 3^2 \cdot 5 \cdot 7\,. \end{aligned}$$

For I_8, take $t_0 = 0$, $\epsilon_i = \zeta^i$ $(1 \le i \le 4)$, $\epsilon_5 = \epsilon_6 = 0$ for $\zeta = \exp(2\pi\sqrt{-1}/8)$. For I_6, take $t_0 = 0$, $\epsilon_i = \zeta^i$ $(1 \le i \le 3)$, $\epsilon_4 = \epsilon_5 = \epsilon_6 = 0$ for $\zeta = \exp(2\pi\sqrt{-1}/6)$ and furthermore $t_0 = 0$, $\epsilon_i = \zeta^i$ $(1 \le i \le 6)$ for $\zeta = \exp(2\pi \sqrt{-1}/6)$. I_2 is computed directly.

(iii) First note that $H^*(BT;\mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4, p_5] = \mathbb{Q}[t_0, I_2, u, v, I_6, I_8, I_{10}]$ by (ii).

Since $I_{18} \in H^*(BT;\mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4, p_5] = \mathbb{Q}[t_0, I_2, u, v, I_6, I_8, I_{10}]$ we may put

$$(*) \quad \begin{aligned} I_{18} \equiv 2^{22} \cdot 3^4 \cdot 1229(&\lambda_1 t_0^{18} + \lambda_2 t_0^{14} u + \lambda_3 t_0^{10} u^2 + \lambda_4 t_0^6 u^3 + \lambda_5 t_0^2 u^4 + \lambda_6 t_0^{12} v \\ &+ \lambda_7 t_0^8 uv + \lambda_8 t_0^4 u^2 v + \lambda_9 u^3 v + \lambda_{10} t_0^6 v^2 + \lambda_{11} t_0^2 uv^2 + \lambda_{12} v^3) \mod \mathfrak{b}_{12} \end{aligned}$$

for some $\lambda_i \in \mathbb{Q}$.

Here we assume the following lemma which will be proved later.

Lemma 2.4.

$$\begin{aligned} A/(t, \mathfrak{a}_{12}) &= A/(t, I_2, I_6, I_8, I_{10}) \\ &= \mathbb{Q}[t_0, c_3, c_4, c_5, c_6]/\Bigg(c_3^2 + 8c_6, c_4^2 - \frac{3}{2}c_3c_5\ , \\ &\qquad c_5^2 + 4t_0c_3c_6 + 4t_0^2c_3c_5 + 4t_0^3c_3c_4 - 32t_0^4c_6 + 4t_0^7c_3\Bigg)\,. \end{aligned}$$

In particular (i) *the following relations hold in* $A/(t, \mathfrak{a}_{12})$:

$$\begin{aligned} &c_3^2 = -8c_6\,, \quad c_4^2 = \frac{3}{2}c_3c_5\,, \\ &c_5^2 = -4t_0c_3c_6 - 4t_0^2c_3c_5 - 4t_0^3c_3c_4 + 32t_0^4c_6 - 4t_0^7c_3\,. \end{aligned}$$

(ii) $A/(t, \mathfrak{a}_{12})$ *has a basis* $\{t_0^i c_3^j c_4^k c_5^l c_6^m \ (0 \le i, m, 0 \le j, k, l \le 1)\}$ *as a* $\mathbb{Q}$*-vector space.*

We consider the relation $(*)$ in $A/(t, \mathfrak{a}_{12})$. Then

$$I_{18} \equiv 2^{22} \cdot 3^4 \cdot 1229 \Big(-\frac{1}{18}c_3c_4c_5c_6 - \frac{2}{3}c_6^3 + t_0c_5c_6^2 - \frac{2}{9}t_0^2c_4c_6^2 - 4t_0^3c_3c_6^2 + \frac{5}{9}t_0^3c_4c_5c_6 - \frac{11}{3}t_0^4c_3c_5c_6 - \frac{32}{9}t_0^5c_3c_4c_6 + \frac{32}{3}t_0^6c_6^2 + \frac{1}{9}t_0^6c_3c_4c_5 - \frac{77}{3}t_0^7c_5c_6 - \frac{244}{9}t_0^8c_4c_6 - \frac{76}{3}t_0^9c_3c_6 - \frac{4}{9}t_0^9c_4c_5 - \frac{2}{3}t_0^{10}c_3c_5 + \frac{4}{9}t_0^{11}c_3c_4 - \frac{64}{3}t_0^{12}c_6 - \frac{2}{9}t_0^{14}c_4 \Big) .$$

On the other hand since

$$b_1 \equiv -2t_0 \mod (t, I_2) , \quad b_2 \equiv \frac{9}{4}t_0^2 \mod (t, I_2) ,$$
$$b_3 \equiv c_3 - t_0^3 \mod (t, I_2) , \quad b_4 \equiv c_4 - \frac{1}{2}t_0c_3 + \frac{11}{16}t_0^4 \mod (t, I_2) ,$$
$$b_5 \equiv c_5 + \frac{3}{4}t_0^2c_3 + \frac{3}{8}t_0^5 \mod (t, I_2) ,$$
$$b_6 = e \equiv c_6 + \frac{1}{2}t_0c_5 + \frac{3}{4}t_0^2c_4 + \frac{5}{8}t_0^3c_3 + \frac{43}{64}t_0^6 \mod (t, I_2) ,$$

we have

$$u = \frac{1}{6}p_2 - \frac{13}{32}t_0^4 \equiv \frac{1}{3}c_4 + \frac{1}{2}t_0c_3 \mod (t, I_2) ,$$
$$v = e + \frac{3}{4}t_0^2u - \frac{43}{64}t_0^6 \equiv c_6 + \frac{1}{2}t_0c_5 + t_0^2c_4 + t_0^3c_3 \mod (t, I_2) .$$

Therefore in $A/(t, \mathfrak{a}_{12})$ we have

$$t_0^{18}, \quad t_0^{14}u \equiv \frac{1}{3}t_0^{14}c_4 + \frac{1}{2}t_0^{15}c_3 , \quad t_0^{10}u^2 \equiv \frac{1}{6}t_0^{10}c_3c_5 + \frac{1}{3}t_0^{11}c_3c_4 - 2t_0^{12}c_6 ,$$
$$t_0^6u^3 \equiv \frac{1}{18}t_0^6c_3c_4c_5 - 2t_0^7c_5c_6 - 2t_0^8c_4c_6 - t_0^9c_3c_6 ,$$
$$t_0^2u^4 \equiv \frac{8}{9}t_0^3c_3c_6^2 - \frac{8}{9}t_0^3c_4c_5c_6 - \frac{10}{9}t_0^4c_3c_5c_6 - \frac{4}{9}t_0^5c_3c_4c_6 - \frac{28}{9}t_0^6c_6^2 + \frac{8}{9}t_0^9c_3c_6 ,$$
$$t_0^{12}v \equiv t_0^{12}c_6 + \frac{1}{2}t_0^{13}c_5 + t_0^{14}c_4 + t_0^{15}c_3 ,$$
$$t_0^8uv \equiv \frac{1}{3}t_0^8c_4c_6 + \frac{1}{6}t_0^9c_4c_5 + \frac{1}{2}t_0^9c_3c_6 + \frac{3}{4}t_0^{10}c_3c_5 + \frac{5}{6}t_0^{11}c_3c_4 - 4t_0^{12}c_6 ,$$
$$t_0^4u^2v \equiv \frac{1}{6}t_0^4c_3c_5c_6 + \frac{1}{3}t_0^5c_3c_4c_6 + \frac{2}{3}t_0^6c_6^2 + \frac{1}{3}t_0^6c_3c_4c_5 - \frac{11}{3}t_0^7c_5c_6 - 2t_0^8c_4c_6 + \frac{2}{3}t_0^9c_3c_6 + \frac{8}{3}t_0^{12}c_6 ,$$

$$
\begin{aligned}
u^3v \equiv{} & \frac{1}{18}c_3c_4c_5c_6 - 2t_0c_5c_6^2 - \frac{10}{9}t_0^2c_4c_6^2 + \frac{17}{3}t_0^3c_3c_6^2 - \frac{23}{9}t_0^3c_4c_5c_6 \\
& + \frac{5}{2}t_0^4c_3c_5c_6 + \frac{41}{9}t_0^5c_3c_4c_6 - \frac{136}{3}t_0^6c_6^2 + \frac{8}{9}t_0^8c_4c_6 + \frac{20}{3}t_0^9c_3c_6 , \\
t_0^6v^2 \equiv{} & t_0^6c_6^2 + t_0^7c_5c_6 + 2t_0^8c_4c_6 + t_0^9c_3c_6 + t_0^9c_4c_5 + \frac{3}{2}t_0^{10}c_3c_5 + t_0^{11}c_3c_4 - t_0^{15}c_3 , \\
t_0^2uv^2 \equiv{} & \frac{1}{3}t_0^2c_4c_6^2 + \frac{1}{2}t_0^3c_3c_6^2 + \frac{1}{3}t_0^3c_4c_5c_6 + \frac{3}{2}t_0^4c_3c_5c_6 + \frac{4}{3}t_0^5c_3c_4c_6 + 12t_0^6c_6^2 \\
& + t_0^6c_3c_4c_5 + 6t_0^7c_5c_6 + 12t_0^8c_4c_6 + 16t_0^9c_3c_6 - \frac{1}{3}t_0^{11}c_3c_4 + 20t_0^{12}c_6 , \\
v^3 \equiv{} & c_6^3 + \frac{3}{2}t_0c_5c_6^2 + 3t_0^2c_4c_6^2 + 3t_0^3c_4c_5c_6 + 4t_0^4c_3c_5c_6 + 80t_0^6c_6^2 + t_0^6c_3c_4c_5 \\
& + 72t_0^7c_5c_6 + 80t_0^8c_4c_6 + 69t_0^9c_3c_6 - \frac{1}{2}t_0^{10}c_3c_5 - 3t_0^{11}c_3c_4 + 80t_0^{12}c_6 .
\end{aligned}
$$

Using Lemma 2.4, as the solution of $(*)$ we obtain

$$
\begin{aligned}
& \lambda_1 = 0 , \quad \lambda_2 = -\frac{2}{3} , \quad \lambda_3 = 0 , \quad \lambda_4 = 2 , \quad \lambda_5 = \frac{3}{4} , \quad \lambda_6 = 0 , \\
& \lambda_7 = -\frac{2}{3} , \quad \lambda_8 = -4 , \quad \lambda_9 = -1 , \quad \lambda_{10} = -\frac{1}{3} , \quad \lambda_{11} = 2 , \quad \lambda_{12} = -\frac{2}{3} .
\end{aligned}
$$

Thus

$$
\begin{aligned}
I_{18} \equiv{} & 2^{22}\cdot 3^4 \cdot 1229 \Big(-\frac{2}{3}t_0^{14}u + 2t_0^6u^3 + \frac{3}{4}t_0^2u^4 - \frac{2}{3}t_0^8uv - 4t_0^4u^2v - u^3v \\
& - \frac{1}{3}t_0^6v^2 + 2t_0^2uv^2 - \frac{2}{3}v^3 \Big) \\
\equiv{} & 2^{20}\cdot 3^3 \cdot 1229\, (-8t_0^{14}u + 24t_0^6u^3 + 9t_0^2u^4 - 8t_0^8uv - 48t_0^4u^2v - 12u^3v \\
& - 4t_0^6v^2 + 24t_0^2uv^2 - 8v^3) \\
\equiv{} & 2^{20}\cdot 3^3 \cdot 1229 J_{18} \qquad \bmod \mathfrak{b}_{12} .
\end{aligned}
$$

Similar direct computations give the required results for I_{12}, I_{14} .

Proof of Lemma 2.4. Put

$$\tilde{c}_i = \sigma_i(t_2, \cdots, t_7) .$$

Since

$$\sum_{n=0}^{7} c_n = \prod_{i=1}^{7}(1+t_i) = (1+t_1)\prod_{i=2}^{7}(1+t_i) = (1+t_1)\sum_{n=0}^{6}\tilde{c}_n ,$$

we have

$$c_n = \tilde{c}_n + t_1\tilde{c}_{n-1} \quad (0 \le n \le 7) .$$

Conversely

$$\tilde{c}_n = c_n - t_1c_{n-1} + t_1^2c_{n-2} - \cdots + (-1)^n t_1^n \quad (0 \le n \le 6) .$$

In particular the following relation holds:

$$c_7 = t_1 c_6 - t_1^2 c_5 + \cdots + t_1^7 .$$

Therefore by the previous table we see that

$$\begin{aligned} A &= H^*(BT;\mathbb{Q})^{\langle R_3,\cdots,R_7\rangle} \\ &= \mathbb{Q}[t_1,\tilde{c}_1,\tilde{c}_2,\cdots,\tilde{c}_6] \\ &= \mathbb{Q}[t_1,c_1,c_2,\cdots,c_7]/(c_7 - t_1c_6 + t_1^2c_5 - \cdots - t_1^7) \\ &= \mathbb{Q}[t_1,c_1,c_2,\cdots,c_6] . \end{aligned}$$

On the other hand since

$$\begin{aligned} c_7 &= t_1c_6 - t_1^2c_5 + t_1^3c_4 - t_1^4c_3 + t_1^5c_2 - t_1^6c_1 + t_1^7 \\ &\equiv -t_0c_6 - t_0^2c_5 - t_0^3c_4 - t_0^4c_3 - t_0^5c_2 - t_0^7 \qquad \bmod (t) \\ &\equiv -t_0c_6 - t_0^2c_5 - t_0^3c_4 - t_0^4c_3 - t_0^7 \qquad \bmod (t, J_2) , \end{aligned}$$

we have

$$\begin{aligned} J_2 &\equiv c_2 \quad \bmod (t) , \\ J_6 &\equiv c_3^2 + 8c_6 \quad \bmod (t) , \\ J_8 &\equiv 2c_4^2 - 3c_3c_5 \quad \bmod (t) , \\ J_{10} &\equiv c_5^2 - 4c_3c_7 \quad \bmod (t) \\ &\equiv c_5^2 + 4t_0c_3c_6 + 4t_0^2c_3c_5 + 4t_0^3c_3c_4 - 32t_0^4c_6 + 4t_0^7c_3 \quad \bmod (t, J_2, J_6) . \end{aligned}$$

Therefore

$$\begin{aligned} A/(t,\mathfrak{a}_{12}) &= A/(t,J_2,J_6,J_8,J_{10}) \\ &= \mathbb{Q}[t_1,c_1,c_2,c_3,c_4,c_5,c_6]/\Big(t, c_2, c_3^2 + 8c_6, c_4^2 - \frac{3}{2}c_3c_5, \\ &\qquad c_5^2 + 4t_0c_3c_6 + 4t_0^2c_3c_5 + 4t_0^3c_3c_4 - 32t_0^4c_6 + 4t_0^7c_3\Big) \\ &= \mathbb{Q}[t_0,c_3,c_4,c_5,c_6]/\Big(c_3^2 + 8c_6, c_4^2 - \frac{3}{2}c_3c_5, \\ &\qquad c_5^2 + 4t_0c_3c_6 + 4t_0^2c_3c_5 + 4t_0^3c_3c_4 - 32t_0^4c_6 + 4t_0^7c_3\Big) . \end{aligned}$$

□

Consequently Lemma 2.3 is established. □

3. The rational cohomology rings of E_7/T and E_7/C_1

Let G be a compact connected Lie group and T be a maximal torus of G. According to Borel [1] rational cohomology spectral sequence for the fibration

$$G/T \xrightarrow{\iota_0} BT \xrightarrow{\rho_0} BG$$

collapses. In particular

$$\begin{aligned}&\rho_0^* : H^*(BG;\mathbb{Q}) \longrightarrow H^*(BT;\mathbb{Q}) \text{ is injective,}\\ &\iota_0^* : H^*(BT;\mathbb{Q}) \longrightarrow H^*(G/T;\mathbb{Q}) \text{ is surjective}\\ &\text{and} \quad \operatorname{Ker}\iota_0^* = (\rho_0^* H^+(BG;\mathbb{Q}))\,,\end{aligned}$$

where $H^+(\cdot) = \bigoplus_{i>0} H^i(\cdot)$ and (A) denotes an ideal generated by A. Furthermore the image of ρ_0^* coincides with the subalgebra of $H^*(BT;\mathbb{Q})$ which consists of the elements invariant under the action of the Weyl group $W(G)$. Thus

$$\begin{aligned}H^*(BG;\mathbb{Q}) &\xrightarrow[\sim]{\rho_0^*} H^*(BT;\mathbb{Q})^{W(G)},\\ H^*(G/T;\mathbb{Q}) &\xleftarrow[\sim]{\iota_0^*} H^*(BT;\mathbb{Q})/(\rho_0^* H^+(BG;\mathbb{Q}))\\ &\quad = H^*(BT;\mathbb{Q})/(H^+(BT;\mathbb{Q})^{W(G)})\,.\end{aligned}$$

Let U be a closed connected subgroup of G of maximal rank and consider the fibration

$$G/U \xrightarrow{\iota} BU \xrightarrow{\rho} BG\,.$$

Since $H^*(G/U;\mathbb{Q})$ has vanishing odd dimensional part by Borel [1] again, rational cohomology spectral sequence for this fibration also collapses. In particular

$$\begin{aligned}H^*(G/U;\mathbb{Q}) &\xleftarrow[\sim]{\iota^*} H^*(BU;\mathbb{Q})/(\rho^* H^+(BG;\mathbb{Q}))\\ &\quad \cong H^*(BT;\mathbb{Q})^{W(U)}/(H^+(BT;\mathbb{Q})^{W(G)})\end{aligned}$$

and the homomorphism

$$p^* : H^*(G/U;\mathbb{Q}) \longrightarrow H^*(G/T;\mathbb{Q})$$

induced by the projection $p : G/T \longrightarrow G/U$ is equivalent to the natural inclusion

$$H^*(BT;\mathbb{Q})^{W(U)} \longrightarrow H^*(BT;\mathbb{Q})\,.$$

Apply these results to the fibrations

(3.1) $$E_7/T \xrightarrow{\iota_0} BT \xrightarrow{\rho_0} BE_7\,,$$

(3.2) $$E_7/C_1 \xrightarrow{\iota} BC_1 \xrightarrow{\rho} BE_7\,.$$

Then since $(I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}) = (J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18})$ as ideals by Lemma 2.3 (i), (iii) we have

$$\begin{aligned}H^*(E_7/T;\mathbb{Q}) &= H^*(BT;\mathbb{Q})/(H^+(BT;\mathbb{Q})^{W(E_7)})\\ &= \mathbb{Q}[t_1,\cdots,t_7]/(I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18})\\ &= \mathbb{Q}[t_1,\cdots,t_7]/(J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18})\,.\end{aligned}$$

Since $H^*(BT;\mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4, p_5] = \mathbb{Q}[t_0, I_2, u, v, I_6, I_8, I_{10}]$ by Lemma 2.3 (ii) we have

$$\begin{aligned} H^*(E_7/C_1;\mathbb{Q}) &= H^*(BT;\mathbb{Q})^{W(C_1)}/(H^+(BT;\mathbb{Q})^{W(E_7)}) \\ &= \mathbb{Q}[t_0, I_2, u, v, I_6, I_8, I_{10}]/(I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}) \\ &= \mathbb{Q}[t_0, u, v]/(I_{12}, I_{14}, I_{18}) \\ &= \mathbb{Q}[t_0, u, v]/(J_{12}, J_{14}, J_{18}) \qquad \text{by Lemma 2.3 (iii).} \end{aligned}$$

Thus we have the following

Lemma 3.1.

(i) $H^*(E_7/T;\mathbb{Q}) = \mathbb{Q}[t_1, \cdots, t_7]/(J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18})$.

(ii) $H^*(E_7/C_1;\mathbb{Q}) = \mathbb{Q}[t_0, u, v]/(J_{12}, J_{14}, J_{18})$.

4. The mod p cohomology ring of E_7/C_1

The purpose of this section is to prove the following

Proposition 4.1. *$H^*(E_7/C_1)$ is generated as a ring by elements of degree ≤ 18.*

Proof. Since E_7/C_1 has no torsion it is sufficient to prove the mod p case of the proposition for each prime p.

For $p \geq 5$; since E_7 and $C_1 = T^1 \cdot Spin(12)$ have no p- torsion the mod p spectral sequence for the fibration (3.2) collapses ([1]). Therefore the mod p version of Lemma 3.1(ii) is valid and the result follows.

For $p = 3$; Since $C_1 = T^1 \cdot Spin(12)$ has no 3-torsion $H^*(BC_1;\mathbb{Z}_3)$ is a polynomial ring generated by elements of even degree ([1]). Therefore the analogous arguments to the proof of [8] Theorem 2.1 can be applied to the fibration

$$E_7 \xrightarrow{\pi} E_7/C_1 \xrightarrow{\iota} BC_1 .$$

Then $H^*(E_7/C_1;\mathbb{Z}_3)$ is generated by elements of degree ≤ 12 and the result follows.

For $p = 2$; according to [7]

$$H^*(E_7/C_1;\mathbb{Z}_2) = \mathbb{Z}_2[t_0, u, v, w]/(t_0u^2, u^3 + v^2, t_0^{14} + u^2v, w^2 + v^3) ,$$

where deg $t_0 = 2$, deg $u = 8$, deg $v = 12$, deg $w = 18$. Therefore the result follows. □

5. The integral cohomology rings of E_7/C_1 and E_7/T

Consider the fibration

$$C_1/T \xrightarrow{i} E_7/T \xrightarrow{p} E_7/C_1 .$$

Since $H^*(E_7/C_1)$ and $H^*(C_1/T)$ are torsion free and have vanishing odd dimensional part by Bott [3] the following sequence

$$\mathbb{Z} \longrightarrow H^*(E_7/C_1) \xrightarrow{p^*} H^*(E_7/T) \xrightarrow{i^*} H^*(C_1/T) \longrightarrow \mathbb{Z}$$

is co-exact; that is

$$p^* \text{ is injective, } i^* \text{ is surjective}$$
$$\text{and} \quad \operatorname{Ker} i^* = (p^* H^+(E_7/C_1)) .$$

Note that p^* is a split monomorphism so that $\operatorname{Im} p^*$ is a direct summand of $H^*(E_7/T)$.

Therefore we will know about the generators of $H^*(E_7/C_1)$ by considering $\operatorname{Ker} i^*$. In order to investigate $\operatorname{Ker} i^*$ we will determine $H^*(E_7/T)$ up to degree ≤ 20.

Lemma 5.1.

$$\begin{aligned} H^*(E_7/T) =& \mathbb{Z}[t_1, \cdots, t_7, t, \gamma_3, \gamma_4, \gamma_5, \gamma_9] \\ & /(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{10}) \quad \textit{for degree } \leq 20 , \end{aligned}$$

where $t_1, \cdots, t_7, t \in H^2$ as in Section 2, $\gamma_i \in H^{2i}$ $(i = 3, 4, 5, 9)$ and

$$\begin{aligned} &\rho_1 = c_1 - 3t , \quad \rho_2 = c_2 - 4t^2, \quad \rho_3 = c_3 - 2\gamma_3 , \quad \rho_4 = c_4 + 2t^4 - 3\gamma_4 , \\ &\rho_5 = c_5 - 3t\gamma_4 + 2t^2\gamma_3 - 2\gamma_5 , \quad \rho_6 = \gamma_3^2 + 2c_6 - 2t\gamma_5 - 3t^2\gamma_4 + t^6, \\ &\rho_8 = 3\gamma_4^2 - 2\gamma_3\gamma_5 + 2tc_7 - 6t\gamma_3\gamma_4 - 9t^2c_6 + 12t^3\gamma_5 + 15t^4\gamma_4 - 6t^5\gamma_3 - t^8, \\ &\rho_9 = 2c_6\gamma_3 + t^2c_7 - 3t^3c_6 - 2\gamma_9 , \quad \rho_{10} = \gamma_5^2 - 2c_7\gamma_3 + 3t^3c_7 . \end{aligned}$$

Proof. The most part of the lemma is proved in [10] Theorem 4.1. The only part to prove is to determine the relation ρ_{10}, but this follows immediately by definition of ρ_{10} and Lemma 2.3 (i). □

Remark 5.2. Our γ_9 is slightly different from that in [10]

On the other hand since C_1/T is homeomorphic to $SO(12)/T^6, T^6$ the canonical maximal torus of $SO(12)$, we have ([9] Corollary 2.2)

Proposition 5.3.

$$\begin{aligned} H^*(C_1/T) &\cong H^*(SO(12)/T^6) \\ &= \mathbb{Z}[t'_1, \cdots, t'_6, e_2, e_6, e_{10}]/(r_1, r_2, r_3, r_4, r_5, r_6, r'_6, r'_8, r'_{10}) , \end{aligned}$$

where $\{t'_i\}_{1 \leq i \leq 6}$ is the canonical basis of $H^2(SO(12)/T^6)$ determined by T^6, $e_{2i} \in H^{2i} (i = 1, 3, 5)$ are elements such that $2e_{2i} = c'_i$ for $c'_i = \sigma_i(t'_1, \cdots, t'_6)$ and

$$\begin{aligned} &r_1 = c'_1 - 2e_2 , \quad r_2 = c'_2 - 2e_2^2 , \quad r_3 = c'_3 - 2e_6 , \quad r_4 = c'_4 + 2e_2^4 - 4e_2e_6 , \\ &r_5 = c'_5 - 2e_{10} , \quad r_6 = c'_6 , \quad r'_6 = -e_6^2 - 2e_2e_{10} - 2e_2^6 + 4e_2^3e_6 , \\ &r'_8 = e_2^8 - 4e_2^5e_6 + 4e_2^2e_6^2 - 2e_6e_{10} , \quad r'_{10} = -e_{10}^2 . \end{aligned}$$

Next we consider the homomorphism $i^* : H^*(E_7/T) \longrightarrow H^*(C_1/T) \cong H^*(SO(12)/T^6)$. By comparing the Dynkin diagram of C_1 with that of $SO(12)$ ([2]) we see easily that

$$i^*(t_1) = e_2 , \quad i^*(t_i) = t'_{8-i} \ (2 \le i \le 7) .$$

Therefore

$$\sum_{n=0}^{7} i^*(c_n) = \prod_{i=1}^{7}(1 + i^*(t_i)) = (1 + e_2)\prod_{i=1}^{6}(1 + t'_i) = (1 + e_2)\sum_{n=0}^{6} c'_n$$

and we have

$$i^*(c_n) = c'_n + e_2 c'_{n-1} \ (0 \le n \le 7) .$$

In particular

$$3i^*(t) = i^*(3t) = i^*(c_1) = c'_1 + e_2 = 3e_2 .$$

Thus

$$i^*(t) = e_2$$

since $H^*(SO(12)/T^6)$ is torsion free. Moreover we can compute i^*-images of $\gamma_3, \gamma_4, \gamma_5$ and γ_9 from the relations ρ_3, ρ_4, ρ_5 and ρ_9. Thus we have the following

Lemma 5.4. *The homomorphism* $i^* : H^*(E_7/T) \longrightarrow H^*(C_1/T) \cong H^*(SO(12)/T^6)$ *is given as follows*:

$$\begin{aligned} &i^*(t_1) = e_2 , \quad i^*(t_i) = t'_{8-i} \ (2 \le i \le 7) , \quad i^*(t) = e_2 , \\ &i^*(\gamma_3) = e_6 + e_2^3 , \quad i^*(\gamma_4) = 2e_2e_6 , \quad i^*(\gamma_5) = e_{10} , \\ &i^*(\gamma_9) = 2e_2e_6e_{10} - e_2^4 e_{10} . \end{aligned}$$

Lemma 5.5. *Kernel of the homomorphism* $i^* : H^*(E_7/T) \longrightarrow H^*(C_1/T) \cong H^*(SO(12)/T^6)$ *is an ideal generated by* $t_0 = t - t_1, \gamma_4 - 2t_1\gamma_3 + 2t_1^4, c_6 - 2t_1\gamma_5, \gamma_9 - 2t_1\gamma_3\gamma_5 + 3t_1^4\gamma_5$.

Proof. Put

$$I = (t_0, \gamma_4 - 2t_1\gamma_3 + 2t_1^4, c_6 - 2t_1\gamma_5, \gamma_9 - 2t_1\gamma_3\gamma_5 + 3t_1^4\gamma_5) \subset H^*(E_7/T) .$$

By Lemma 5.4 we see easily that the ideal I is contained in $\operatorname{Ker} i^*$. Therefore i^* induces a map

$$H^*(E_7/T)/I \longrightarrow H^*(C_1/T) \cong H^*(SO(12)/T^6) .$$

Since $\rho_9 = 2c_6\gamma_3 + t^2c_7 - 3t^3c_6 - 2\gamma_9 \in I$ we have from Lemma 5.1

$$\begin{aligned} H^*(E_7/T)/I =&\mathbb{Z}[t_1, \cdots, t_7, t, \gamma_3, \gamma_4, \gamma_5, \gamma_9] \\ &/(t - t_1, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \gamma_4 - 2t_1\gamma_3 + 2t_1^4, c_6 - 2t_1\gamma_5\ , \\ &\gamma_9 - 2t_1\gamma_3\gamma_4 + 3t_1^4\gamma_5, \rho_6, \rho_8, \rho_{10}) \\ =&\mathbb{Z}[t_1, \cdots, t_7, \gamma_3, \gamma_5] \\ &/(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, c_6 - 2t_1\gamma_5, \rho_6, \rho_8, \rho_{10}) \quad \text{for degree } \leq 20\,. \end{aligned}$$

On the other hand by Lemma 5.4 it is verified directly that

$$\begin{aligned} &i^*(\rho_i) = r_i\ (1 \leq i \leq 5)\,, \quad i^*(c_6 - 2t_1\gamma_5) = r_6\,, \quad i^*(\rho_6) = -r_6'\,, \\ &i^*(\rho_8) = r_8'\,, \quad i^*(\rho_{10}) = -r_{10}'\,. \end{aligned}$$

Therefore this map is an isomorphism for degree ≤ 20. Thus

$$\operatorname{Ker} i^* = I \quad \text{for degree } \leq 20\,.$$

Since $\operatorname{Ker} i^*$ is generated by elements of degree ≤ 18 from Proposition 4.1 the above equality holds without restriction on degree. □

From this lemma we see that $H^*(E_7/C_1)$ is generated by some four elements $\tilde{t_0} \in H^2, \tilde{u} \in H^8, \tilde{v} \in H^{12}$ and $\tilde{w} \in H^{18}$ such that

$$(t_0, \gamma_4 - 2t_1\gamma_3 + 2t_1^4, c_6 - 2t_1\gamma_5, \gamma_9 - 2t_1\gamma_3\gamma_5 + 3t_1^4\gamma_5) = (\tilde{t_0}, \tilde{u}, \tilde{v}, \tilde{w})$$

as ideals. So we must identify these generators.

Remark 5.6. As is well known $W(E_7)$ acts on $H^*(E_7/T)$, so dose $W(C_1)$ as the subgroup of $W(E_7)$ and the image of $p^* : H^*(E_7/C_1) \longrightarrow H^*(E_7/T)$ is contained in the invariant subalgebra $H^*(E_7/T)^{W(C_1)}$. In this case, as is proved in [6] Proposition 3.2 Imp^* coincides with $H^*(E_7/T)^{W(C_1)}$ and we can identify $H^*(E_7/C_1)$ with $H^*(E_7/T)^{W(C_1)}$. Therefore finding the generators $\tilde{t_0}, \tilde{u}, \tilde{v}, \tilde{w}$ is equivalent to finding $W(C_1)$-invariant elements including $t_0, \gamma_4 - 2t_1\gamma_3 + 2t_1^4, c_6 - 2t_1\gamma_5, \gamma_9 - 2t_1\gamma_3\gamma_5 + 3t_1^4\gamma_5$ respectively.

Hereafter we may identify $H^*(E_7/C_1)$ with $\operatorname{Im} p^*$ and regard it as a subalgebra of $H^*(E_7/T)$.

First note that in $H^*(BT; \mathbb{Q})$

$$p_2 \equiv 2c_4 - 3(2t_1 + t_0)c_3 + 16t_1^4 + 52t_1^3t_0 + 66t_1^2t_0^2 + 34t_1t_0^3 + \frac{103}{16}t_0^4 \mod (I_2)\,,$$

$$\begin{aligned} e \equiv\ & c_6 - \left(t_1 + \frac{1}{2}t_0\right)c_5 + \left(t_1^2 + \frac{1}{2}t_1t_0 + \frac{1}{4}t_0^2\right)c_4 \\ &- \left(t_1^3 + \frac{1}{2}t_1^2t_0 + \frac{1}{4}t_1t_0^2 + \frac{1}{8}t_0^3\right)c_3 + 2t_1^6 + 6t_1^5t_0 + 7t_1^4t_0^2 + \frac{7}{2}t_1^3t_0^3 + \frac{7}{4}t_1^2t_0^4 \\ &+ \frac{3}{4}t_1t_0^5 + \frac{11}{64}t_0^6 \mod (I_2)\,. \end{aligned}$$

On the other hand in $H^*(E_7/T) \hookrightarrow H^*(E_7/T;\mathbb{Q})$ we have

$$\begin{aligned} c_3 &= 2\gamma_3 , \\ c_4 &= 3\gamma_4 - 2t^4 = 3\gamma_4 - 2t_1^4 - 8t_1^3t_0 - 12t_1^2t_0^2 - 8t_1t_0^3 - 2t_0^4 , \\ c_5 &= 2\gamma_5 + 3t\gamma_4 - 2t^2\gamma_3 = 2\gamma_5 + 3(t_1+t_0)\gamma_4 - 2(t_1^2 + 2t_1t_0 + t_0^2)\gamma_3 . \end{aligned}$$

Therefore in $H^*(E_7/T;\mathbb{Q})$

$$\begin{aligned} p_2 &= 6\gamma_4 - 6(2t_1+t_0)\gamma_3 + 12t_1^4 + 36t_1^3t_0 + 42t_1^2t_0^2 + 18t_1t_0^3 + \frac{39}{16}t_0^4 , \\ e &= c_6 - (2t_1+t_0)\gamma_5 - \left(3t_1t_0 + \frac{3}{4}t_0^2\right)\gamma_4 + \left(4t_1^2t_0 + \frac{7}{2}t_1t_0^2 + \frac{3}{4}t_0^3\right)\gamma_3 \\ &\quad - 3t_1^5t_0 - \frac{19}{2}t_1^4t_0^2 - \frac{25}{2}t_1^3t_0^3 - \frac{29}{4}t_1^2t_0^4 - \frac{9}{4}t_1t_0^5 - \frac{21}{64}t_0^6 . \end{aligned}$$

Now let us determine our generators $\tilde{t_0}, \tilde{u}, \tilde{v}$ and $\tilde{w}$. Obviously we can take $t_0 = t - t_1$ as our generator $\tilde{t_0}$. By Lemma 3.1 (ii) we may write

$$\tilde{u} = \alpha p_2 + \beta t_0^4 \quad \text{in } H^*(E_7/C_1;\mathbb{Q})$$

for some $\alpha, \beta \in \mathbb{Q}$. On the other hand by Lemma 5.5 we may write

$$\tilde{u} = \gamma_4 - 2t_1\gamma_3 + 2t_1^4 + f \quad \text{in } \operatorname{Im} p^* \subset H^*(E_7/T)$$

for some $f \in H^8(E_7/T) \cap (t_0)$. Hence in $H^*(E_7/T;\mathbb{Q})$

$$\begin{aligned} &\gamma_4 - 2t_1\gamma_3 + 2t_1^4 + f = \alpha p_2 + \beta t_0^4 \\ &\quad = \alpha\left\{6\gamma_4 - 6(2t_1+t_0)\gamma_3 + 12t_1^4 + 36t_1^3t_0 + 42t_1^2t_0^2 + 18t_1t_0^3 + \frac{39}{16}t_0^4\right\} + \beta t_0^4 \\ &\quad = 6\alpha(\gamma_4 - 2t_1\gamma_3 + 2t_1^4) - 6\alpha t_0\gamma_3 + 36\alpha t_1^3t_0 + 42\alpha t_1^2t_0^2 + 18\alpha t_1t_0^3 \\ &\qquad + \left(\frac{39}{16}\alpha + \beta\right)t_0^5 \end{aligned}$$

and we may take

$$\alpha = \frac{1}{6}, \quad \beta = -\frac{13}{32} \quad \text{and} \quad f = -t_0\gamma_3 + 6t_1^3t_0 + 7t_1^2t_0^2 + 3t_1t_0^3 .$$

Thus

$$(5.1) \qquad u = \frac{1}{6}p_2 - \frac{13}{32}t_0^4 = \gamma_4 - (2t_1+t_0)\gamma_3 + 2t_1^4 + 6t_1^3t_0 + 7t_1^2t_0^2 + 3t_1t_0^3$$

can be chosen as our generator $\tilde{u}$.

Similarly we may write

$$\begin{aligned} \tilde{v} &= \alpha e + \beta t_0^2 u + \gamma t_0^6 \quad \text{in } H^*(E_7/C_1;\mathbb{Q}) \\ &= c_6 - 2t_1\gamma_5 + g \quad \text{in } \operatorname{Im} p^* \subset H^*(E_7/T) \end{aligned}$$

for some $\alpha, \beta, \gamma \in \mathbb{Q}$ and some $g \in H^{12}(E_7/T) \cap (t_0, u)$. Hence

$$\begin{aligned} c_6 - 2t_1\gamma_5 + g &= \alpha(c_6 - 2t_1\gamma_5) - \alpha t_0\gamma_5 + \left\{-3\alpha t_1 t_0 + \left(-\frac{3}{4}\alpha + \beta\right) t_0^2\right\}\gamma_4 \\ &+ \left\{4\alpha t_1^2 t_0 + \left(\frac{7}{2}\alpha - 2\beta\right) t_1 t_0^2 + \left(\frac{3}{4}\alpha - \beta\right) t_0^3\right\}\gamma_3 - 3\alpha t_1^5 t_0 \\ &+ \left(-\frac{19}{2}\alpha + 2\beta\right) t_1^4 t_0^2 + \left(-\frac{25}{2}\alpha + 6\beta\right) t_1^3 t_0^3 + \left(-\frac{29}{4}\alpha + 7\beta\right) t_1^2 t_0^4 \\ &+ \left(-\frac{9}{4}\alpha + 3\beta\right) t_1 t_0^5 + \left(-\frac{21}{64}\alpha + \gamma\right) t_0^6 \end{aligned}$$

and we may take

$$\begin{aligned} &\alpha = 1\,, \quad \beta = \frac{3}{4}\,, \quad \gamma = -\frac{43}{64}\,, \\ &g = -t_0\gamma_5 - 3t_1t_0\gamma_4 + (4t_1^2t_0 + 2t_1t_0^2)\gamma_3 - 3t_1^5t_0 - 8t_1^4t_0^2 - 8t_1^3t_0^3 - 2t_1^2t_0^4 - t_0^6 \\ & = -t_0\gamma_5 - 3t_1t_0u + (-2t_1^2t_0 - t_1t_0^2)\gamma_3 + 3t_1^5t_0 + 10t_1^4t_0^2 + 13t_1^3t_0^3 + 7t_1^2t_0^4 - t_0^6\,. \end{aligned}$$

Thus

$$\begin{aligned} v &= e + \frac{3}{4}t_0^2u - \frac{43}{64}t_0^6 \\ &= c_6 - (2t_1 + t_0)\gamma_5 - 3t_1t_0\gamma_4 + (4t_1^2t_0 + 2t_1t_0^2)\gamma_3 - 3t_1^5t_0 - 8t_1^4t_0^2 - 8t_1^3t_0^3 \\ &\quad - 2t_1^2t_0^4 - t_0^6 \end{aligned}$$

can be chosen as our generator $\tilde{v}$.

Next consider the element

$$w = \frac{1}{2}t_0u^2 \quad \text{in} \quad H^*(E_7/C_1;\mathbb{Q})\,.$$

Then direct computation yields

$$\begin{aligned} w = \gamma_9 &- c_6\gamma_3 + (t_1^3 + 3t_1^2t_0 + 8t_1t_0^2 + 5t_0^3)c_6 + (t_1^4 + 2t_1^3t_0 - 7t_1^2t_0^2 - 13t_1t_0^3 \\ &- 5t_0^4)\gamma_5 + t_0\gamma_3\gamma_5 - t_0\gamma_4^2 + (t_1t_0 + 2t_0^2)\gamma_3\gamma_4 + (2t_1^4t_0 - 14t_1^2t_0^3 - 18t_1t_0^4 \\ &- 6t_0^5)\gamma_4 + (-3t_1^5t_0 - 8t_1^4t_0^2 + 14t_1^2t_0^4 + 12t_1t_0^5 + 3t_0^6)\gamma_3 + 2t_1^8t_0 + 12t_1^7t_0^2 \\ &+ 28t_1^6t_0^3 + 32t_1^5t_0^4 + 16t_1^4t_0^5 - 3t_1^2t_0^7 - t_1t_0^8\,. \end{aligned}$$

Here we used (5.1) and the relations ρ_6, ρ_8, ρ_9. This shows that w is contained in $H^*(E_7/C_1)$. Then

$$w \equiv \gamma_9 - 2t_1\gamma_3\gamma_5 + 3t_1^4\gamma_5 \qquad \text{mod } (t_0, u, v)\,.$$

Therefore

$$w = \frac{1}{2}t_0u^2$$

can be chosen as our generator $\tilde{w}$. Using w

$$\begin{aligned} J_{18} =& -8t_0^{14}u + 24t_0^6u^3 + 36w^2 - 8t_0^8uv - 48t_0^4u^2v - 12u^3v - 4t_0^6v^2 \\ &+ 24t_0^2uv^2 - 8v^3 \\ =&4(-2t_0^{14} + 6t_0^6u^3 + 9w^2 - 2t_0^8uv - 12t_0^4u^2v - 3u^3v - t_0^6v^2 + 6t_0^2uv^2 - 2v^3)\,. \end{aligned}$$

Therefore in view of Lemma 3.1 (ii) we have the following

Theorem 5.7.

$$H^*(E_7/C_1) = \mathbb{Z}[t_0, u, v, w]/(\sigma_9, \sigma_{12}, \sigma_{14}, \sigma_{18})\,,$$

where $\deg t_0 = 2$*,* $\deg u = 8$*,* $\deg v = 12$*,* $\deg w = 18$ *and*

$$\begin{aligned} \sigma_9 &= 2w - t_0u^2, \\ \sigma_{12} &= -6t_0^8u + 9t_0^4u^2 + 2t_0^6v - 12t_0^2uv + u^3 + 3v^2, \\ \sigma_{14} &= t_0^{14} - 6t_0^{10}u - 3t_0^6u^2 + 4t_0^8v - 6t_0^4uv - 3u^2v + 3t_0^2v^2, \\ \sigma_{18} &= -2t_0^{14}u + 6t_0^6u^3 + 9w^2 - 2t_0^8uv - 12t_0^4u^2v - 3u^3v - t_0^6v^2 + 6t_0^2uv^2 - 2v^3. \end{aligned}$$

Remark 5.8. (i) We have chosen our t_0, u, v, w so that their mod 2 reductions coincide with t_0, u, v, w in [7].

(ii) Our γ_5, γ_9 are slightly different from those in [7]. If we denote those in [7] by γ_5', γ_9', the following relations hold:

$$\begin{aligned} \gamma_5 &\equiv \gamma_5' + (t_1 + t_0)c_4 + t_1^5 + t_1^4t_0 + t_1t_0^4 + t_0^5 \qquad \mathrm{mod}\ 2\,, \\ \gamma_9 &\equiv \gamma_9' + c_4\gamma_5' + (t_1^4 + t_0^4)\gamma_5' \qquad \mathrm{mod}\ 2\,. \end{aligned}$$

Next consider the integral cohomology ring of E_7/T. General description of $H^*(E_7/T)$ is given in [8] Proposition 3.2. We need only to determine the relations ρ_{12}, ρ_{14} and ρ_{18}. As remarked earlier $\mathrm{Im}\, p^*$ is a direct summand of $H^*(E_7/T)$ so that σ_{12}, σ_{14} and σ_{18} are not divisible in $H^*(E_7/T)$. Hence we can take σ_{12}, σ_{14} and σ_{18} as our relations ρ_{12}, ρ_{14} and ρ_{18} respectively. Thus we have the following

Theorem 5.9.

$$\begin{aligned} H^*(E_7/T) =&\mathbb{Z}[t_1, \cdots, t_7, t, \gamma_3, \gamma_4, \gamma_5, \gamma_9] \\ &/(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{18})\,, \end{aligned}$$

where $t_1, \cdots, t_7, t \in H^2$ *as in Section 2,* $\gamma_i \in H^{2i}$ $(i = 3, 4, 5, 9)$ *and*

$$\begin{aligned} &\rho_1 = c_1 - 3t\,, \quad \rho_2 = c_2 - 4t^2\,, \quad \rho_3 = c_3 - 2\gamma_3\,, \quad \rho_4 = c_4 + 2t^4 - 3\gamma_4\,, \\ &\rho_5 = c_5 - 3t\gamma_4 + 2t^2\gamma_3 - 2\gamma_5\,, \quad \rho_6 = \gamma_3^2 + 2c_6 - 2t\gamma_5 - 3t^2\gamma_4 + t^6\,, \\ &\rho_8 = 3\gamma_4^2 - 2\gamma_3\gamma_5 + 2tc_7 - 6t\gamma_3\gamma_4 - 9t^2c_6 + 12t^3\gamma_5 + 15t^4\gamma_4 - 6t^5\gamma_3 - t^8\,, \\ &\rho_9 = 2c_6\gamma_3 + t^2c_7 - 3t^3c_6 - 2\gamma_9\,, \quad \rho_{10} = \gamma_5^2 - 2c_7\gamma_3 + 3t^3c_7\,, \\ &\rho_{12} = -6t_0^8u + 9t_0^4u^2 + 2t_0^6v - 12t_0^2uv + u^3 + 3v^2\,, \\ &\rho_{14} = t_0^{14} - 6t_0^{10}u - 3t_0^6u^2 + 4t_0^8v - 6t_0^4uv - 3u^2v + 3t_0^2v^2\,, \\ &\rho_{18} = -2t_0^{14}u + 6t_0^6u^3 + 9w^2 - 2t_0^8uv - 12t_0^4u^2v - 3u^3v - t_0^6v^2 + 6t_0^2uv^2 - 2v^3 \end{aligned}$$

for

$$\begin{aligned}
t_0 &= t - t_1 \,, \\
u &= \gamma_4 - (2t_1 + t_0)\gamma_3 + 2t_1^4 + 6t_1^3 t_0 + 7t_1^2 t_0^2 + 3t_1 t_0^3 \,, \\
v &= c_6 - (2t_1 + t_0)\gamma_5 - 3t_1 t_0 \gamma_4 + (4t_1^2 t_0 + 2t_1 t_0^2)\gamma_3 - 3t_1^5 t_0 - 8t_1^4 t_0^2 - 8t_1^3 t_0^3 \\
&\quad - 2t_1^2 t_0^4 - t_0^6 \,, \\
w &= \frac{1}{2} t_0 u^2 \\
&= \gamma_9 - c_6\gamma_3 + (t_1^3 + 3t_1^2 t_0 + 8t_1 t_0^2 + 5t_0^3)c_6 + (t_1^4 + 2t_1^3 t_0 - 7t_1^2 t_0^2 - 13t_1 t_0^3 \\
&\quad - 5t_0^4)\gamma_5 + t_0\gamma_3\gamma_5 - t_0\gamma_4^2 + (t_1 t_0 + 2t_0^2)\gamma_3\gamma_4 + (2t_1^4 t_0 - 14t_1^2 t_0^3 - 18t_1 t_0^4 \\
&\quad - 6t_0^5)\gamma_4 + (-3t_1^5 t_0 - 8t_1^4 t_0^2 + 14t_1^2 t_0^4 + 12t_1 t_0^5 + 3t_0^6)\gamma_3 + 2t_1^8 t_0 + 12t_1^7 t_0^2 \\
&\quad + 28t_1^6 t_0^3 + 32t_1^5 t_0^4 + 16t_1^4 t_0^5 - 3t_1^2 t_0^7 - t_1 t_0^8 \,.
\end{aligned}$$

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