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The mod 2 cohomology ring of the symmetric space EVI

By

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1. Introduction

The compact 1-connected irreducible symmetric spaces have been classified (E. Cartan, etc.). For classical cases, their cohomologies are well known (A. Borel, etc.). For exceptional cases, the integral cohomology rings of EI, EII, EIII, EIV, EVII, FI, FII and G are already determined ([6], [8], [11], [1], [12], [7], [3]). The remaining symmetric spaces $E\mathrm{V}, E\mathrm{VI}, E\mathrm{VIII}$ and EIX have 2-torsion, so their cohomologies are much more complicated. The purpose of this paper is to determine the mod 2 cohomology ring of EVI. Since EVI has only 2-torsion and the torsion elements of its integral cohomology are all of order 2 the additive structure of the integral cohomology can be completely determined by its mod 2 cohomology. As a homogeneous space, it is given by

$$E\mathrm{VI} = E_7/U_1, \quad U_1 = S^3 \cdot Spin(12), \quad S^3 \cap Spin(12) \cong \mathbb{Z}_2$$

where E_7 is the compact 1-connected simple Lie group of type E_7, U_1 is the identity component of the centralizer of an element $x_1 \in E_7$. Let C_1 be the centralizer of a suitable one dimensional torus containing x_1. Then

$$C_1 = T^1 \cdot Spin(12), \quad T^1 \cap Spin(12) \cong \mathbb{Z}_2$$

and we have a fibration:

$$S^2 \cong U_1/C_1 \longrightarrow E_7/C_1 \xrightarrow{p} E_7/U_1 = E\mathrm{VI}. \tag{1.1.1}$$

We consider the Gysin sequence associated with (1.1.1). In this case it is reduced to the following exact sequences since E_7/C_1 has no torsion and no odd dimensional part in its integral cohomology ([4]):

$$(*)_i \quad 0 \longrightarrow H^{2i-3}(E\mathrm{VI};A) \xrightarrow{h} H^{2i}(E\mathrm{VI};A) \xrightarrow{p^*} H^{2i}(E_7/C_1;A)$$
$$\xrightarrow{\theta} H^{2i-2}(E\mathrm{VI};A) \xrightarrow{h} H^{2i+1}(E\mathrm{VI};A) \longrightarrow 0$$

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where $\chi \in H^3(E\mathrm{VI}; A)$, $2\chi = 0$ and $A = \mathbb{Z}$ or $\mathbb{Z}_2$. The homomorphisms θ and h satisfy

$$\theta(p^*(x)y) = x\theta(y), \quad h(x) = \chi \cdot x.$$

On the other hand we determined the integral and mod 2 cohomology ring of E_7/C_1 ([10], [9]). Hence by considering the above exact sequences inductively, we will determine the mod 2 cohomology ring of EVI. The paper is organized as follows: In Section 2 we compute the invariant subalgebras of the Weyl groups in order to determine the rational cohomology ring of EVI. In Section 3 we discuss the integral and mod 2 cohomology of EVI in low degrees. In the final section, Section 4 we determine the mod 2 cohomology ring of EVI. Throughout this paper $\sigma_i(x_1, \dots, x_n)$ denotes the i-th elementary symmetric function in the variables $x_1, \dots, x_n$.

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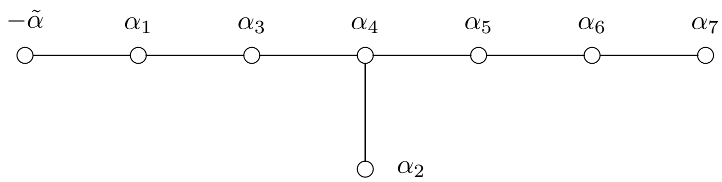
2. The rational cohomology ring of EVI

Let T be a maximal torus of E_7. According to [5] the completed Dynkin diagram of E_7 is



where α_i $(1 \leq i \leq 7)$ are the simple roots and $\tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ is the highest root. As usual we may regard each root as an element of $H^1(T; \mathbb{Z}) \xrightarrow{\sim} H^2(BT; \mathbb{Z})$.

Let C_1 be the centralizer of a one dimensional torus determined by $\alpha_i = 0$ $(i \neq 1)$ and U_1 the identity component of the centralizer of an element x_1 such that $\alpha_i(x_1) = 0$ for $i \neq 1$ and $\alpha_1(x_1) = 1/2$. Then the Weyl groups $W(\cdot)$ of E_7, U_1 and C_1 are given as follows:

$$\begin{aligned} &W(E_7) = \langle R_i \ (1 \leq i \leq 7) \rangle, \quad W(U_1) = \langle R_i \ (i \neq 1), \tilde{R} \rangle, \\ &W(C_1) = \langle R_i \ (i \neq 1) \rangle, \end{aligned}$$

where R_i (resp. $\tilde{R}$) denotes the reflection to the hyperplane $\alpha_i = 0$ (resp. $\tilde{\alpha} = 0$).

Note that ([7])

$$\begin{aligned} U_1 &= S^3 \cdot Spin(12), \quad S^3 \cap Spin(12) \cong \mathbb{Z}_2. \\ C_1 &= T^1 \cdot Spin(12), \quad T^1 \cap Spin(12) \cong \mathbb{Z}_2. \end{aligned}$$

Let $\{w_i\}_{1 \leq i \leq 7}$ be the fundamental weights corresponding to the system of the simple roots $\{\alpha_i\}_{1 \leq i \leq 7}$. We also regard each weight as an element of

$H^2(BT;\mathbb{Z})$ and then $\{w_i\}_{1\le i\le 7}$ forms a basis of $H^2(BT;\mathbb{Z})$. The action of R_i's and $\tilde{R}$ on $\{w_i\}_{1\le i\le 7}$ is given as follows:

$$R_i(w_i) = w_i - \sum_{j=1}^{7} \frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} w_j, \quad R_i(w_k) = w_k \quad \text{for} \quad k \ne i,$$

$$\tilde{R}(w_i) = w_i - m_i w_1 \quad \text{for} \quad \tilde{\alpha} = \sum_{i=1}^{7} m_i\alpha_i.$$

Following [12] we define

$$t_7 = w_7, \quad t_i = R_{i+1}(t_{i+1})\ (2 \le i \le 6), \quad t_1 = R_1(t_2),$$
$$c_i = \sigma_i(t_1, \dots, t_7), \quad t = w_2 = \frac{1}{3}c_1.$$

Then t and t_i's span $H^2(BT;\mathbb{Z})$ since each w_i is an integral linear combination of t and t_i's and we have the following isomorphism:

$$H^*(BT;\mathbb{Z}) = \mathbb{Z}[t_1, \dots, t_7, t]/(3t - c_1).$$

Furthermore the action of R_i's and $\tilde{R}$ on these elements is given by the following table:

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	$\tilde{R}$
t_1	t_2	$t - t_2 - t_3$						
t_2	t_1	$t - t_1 - t_3$	t_3					$t_1 + t_2 - t$
t_3		$t - t_1 - t_2$	t_2	t_4				$t_1 + t_3 - t$
t_4				t_3	t_5			$t_1 + t_4 - t$
t_5					t_4	t_6		$t_1 + t_5 - t$
t_6						t_5	t_7	$t_1 + t_6 - t$
t_7							t_6	$t_1 + t_7 - t$
t		$-t + t_4 + t_5 + t_6 + t_7$						$2t_1 - t$

where blanks indicate the trivial action.

Putting

$$t_0 = t - t_1 \quad \text{and} \quad \epsilon_i = t_{i+1} - \frac{1}{2}t_0\ (1 \le i \le 6),$$

we have

$$H^*(BT;\mathbb{Q}) = \mathbb{Q}[t_1, \dots, t_7] = \mathbb{Q}[t_0, \epsilon_1, \dots, \epsilon_6]$$

and the following table of the action:

	R_2	R_3	R_4	R_5	R_6	R_7	$\tilde{R}$
t_0							$-t_0$
ϵ_1	$-\epsilon_2$	ϵ_2					
ϵ_2	$-\epsilon_1$	ϵ_1	ϵ_3				
ϵ_3			ϵ_2	ϵ_4			
ϵ_4				ϵ_3	ϵ_5		
ϵ_5					ϵ_4	ϵ_6	
ϵ_6						ϵ_5	

From this table

Lemma 2.1. *The invariant subalgabras of the Weyl groups* $W(C_1), W(U_1)$ *are given as follows*:
(i) $H^*(BT;\mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4, p_5]$.
(ii) $H^*(BT;\mathbb{Q})^{W(U_1)} = \mathbb{Q}[t_0^2, p_1, p_2, e, p_3, p_4, p_5]$
where

$$p_i = \sigma_i(\epsilon_1^2, \cdots, \epsilon_6^2) \quad \textit{and} \quad e = \prod_{i=1}^{6} \epsilon_i.$$

Next as in [12] we put

$$x_i = 2t_i - t \ (1 \leq i \leq 7) \quad \text{and} \quad x_8 = t.$$

Then we have the following $W(E_7)$-invariant subset

$$S = \{x_i + x_j, -x_i - x_j \ (1 \leq i < j \leq 8)\} \subset H^2(BT;\mathbb{Q}).$$

Thus we have $W(E_7)$-invariant forms

$$I_n = \sum_{y \in S} y^n \in H^{2n}(BT;\mathbb{Q})^{W(E_7)}.$$

Consider the following elements ($J_i \in H^{2i}(BT;\mathbb{Q})$):

$$\begin{aligned}
J_2 &= c_2 - 4t^2,\\
J_6 &= c_3^2 + 8c_6 - 4c_5 t - 4c_3 t^3 + 4t^6,\\
J_8 &= 2c_4^2 - 3c_3c_5 + 12c_7 t - 3c_3c_4 t - 30c_6 t^2 + 24c_5 t^3 + 2c_4 t^4 + 2t^8,\\
J_{10} &= c_5^2 - 4c_3c_7 - 2c_4c_5 t + 2c_3c_5 t^2 + c_4^2 t^2 - 2c_3c_4 t^3 + 12c_7 t^3 - 8c_6 t^4 + 4c_4 t^6,\\
J_{12} &= -6t_0^8 u + 9t_0^4 u^2 + 2t_0^6 v - 12t_0^2 uv + u^3 + 3v^2,\\
J_{14} &= t_0^{14} - 6t_0^{10} u - 3t_0^6 u^2 + 4t_0^8 v - 6t_0^4 uv - 3u^2 v + 3t_0^2 v^2,\\
J_{18} &= -8t_0^{14} u + 24t_0^6 u^3 + 9t_0^2 u^4 - 8t_0^8 uv - 48t_0^4 u^2 v - 12u^3 v - 4t_0^6 v^2\\
&\quad + 24t_0^2 uv^2 - 8v^3,
\end{aligned}$$

where

$$t_0 = t - t_1, \quad u = \frac{1}{6}p_2 - \frac{13}{32}t_0^4, \quad v = e + \frac{3}{4}t_0^2 u - \frac{43}{64}t_0^6.$$

Then the following facts are proved ([12], [9]).

Lemma 2.2. *The invariant subalgebra of the Weyl group* $W(E_7)$ *is given as follows*:

$$H^*(BT;\mathbb{Q})^{W(E_7)} = \mathbb{Q}[I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}].$$

Lemma 2.3.
(i) $I_2 = -2^5 \cdot 3J_2$, $I_6 \equiv 2^8 \cdot 3^2 J_6 \mod \mathfrak{a}_6$,
$I_8 \equiv 2^{12} \cdot 5J_8 \mod \mathfrak{a}_8$, $I_{10} \equiv 2^{12} \cdot 3^2 \cdot 5 \cdot 7J_{10} \mod \mathfrak{a}_{10}$.
In $H^*(BT;\mathbb{Q})^{W(C_1)} = \mathbb{Q}[t_0, p_1, p_2, e, p_3, p_4, p_5]$ *we have*
(ii) $I_2 = 24(2p_1 + t_0^2)$, $I_6 = 2^8 \cdot 3^2 p_3 + 2^9 \cdot 3^2 \cdot 5e + decomp.$,
$I_8 = 2^{11} \cdot 3 \cdot 5p_4 + decomp.$, $I_{10} = 2^{12} \cdot 3^2 \cdot 5 \cdot 7p_5 + decomp.$
(iii) $I_{12} \equiv -2^{16} \cdot 3^4 \cdot 5J_{12} \mod \mathfrak{b}_{12}$, $I_{14} \equiv 2^{17} \cdot 3 \cdot 7 \cdot 11 \cdot 29J_{14} \mod \mathfrak{b}_{12}$,
$I_{18} \equiv 2^{20} \cdot 3^3 \cdot 1229J_{18} \mod \mathfrak{b}_{12}$
where decomp. means decomposable elements and $\mathfrak{a}_i$ (*resp.* $\mathfrak{b}_i$) *denotes the ideal of* $H^*(BT;\mathbb{Q})$ (*resp.* $H^*(BT;\mathbb{Q})^{W(C_1)}$) *generated by* I_j*'s for* $j < i, j \in \{2, 6, 8, 10, 12, 14, 18\}$.

Now we briefly review the classical results of A. Borel ([2]). Let G be a compact connected Lie group, U be a closed connected subgroup of G of maximal rank and T be a common maximal torus. Then both the rational cohomology spectral sequences for the fibrations

$$G/T \xrightarrow{\iota_0} BT \xrightarrow{\rho_0} BG, \quad G/U \xrightarrow{\iota} BU \xrightarrow{\rho} BG$$

collapse. In particular

$$\begin{aligned}
&\rho_0^* : H^*(BG;\mathbb{Q}) \to H^*(BT;\mathbb{Q}),\ \rho^* : H^*(BG;\mathbb{Q}) \to H^*(BU;\mathbb{Q}) \ \text{ injective,}\\
&\iota_0^* : H^*(BT;\mathbb{Q}) \to H^*(G/T;\mathbb{Q}),\ \iota^* : H^*(BU;\mathbb{Q}) \to H^*(G/U;\mathbb{Q}) \ \text{ surjective,}\\
&\ \text{and } \mathrm{Ker}\iota_0^* = (\rho_0^* H^+(BG;\mathbb{Q})),\ \mathrm{Ker}\iota^* = (\rho^* H^+(BG;\mathbb{Q})).
\end{aligned}$$

Furthermore $\mathrm{Im}\rho_0^*$ coincides with the invariant subalgebra $H^*(BT;\mathbb{Q})^{W(G)}$. Therefore we have the following description of $H^*(G/U;\mathbb{Q})$:

$$\begin{aligned}
H^*(G/U;\mathbb{Q}) \xleftarrow[\sim]{\iota^*} & H^*(BU;\mathbb{Q})/(\rho^* H^+(BG;\mathbb{Q}))\\
\cong & H^*(BT;\mathbb{Q})^{W(U)}/(H^+(BT;\mathbb{Q})^{W(G)}).
\end{aligned}$$

We apply this to the case $U = C_1$ and U_1. Then using Lemmas 2.1, 2.2 and 2.3 we have (For later use we replaced v by $v' = v - t_0^2 u$)

Lemma 2.4.
(i) $H^*(E_7/C_1;\mathbb{Q}) = \mathbb{Q}[t_0, u, v']/(J'_{12}, J'_{14}, J'_{18})$.
(ii) $H^*(E\mathrm{VI};\mathbb{Q}) = \mathbb{Q}[a, b, c]/(r_{12}, r_{14}, r_{18})$,
where a, b *and* c *are elements of* $H^*(E\mathrm{VI};\mathbb{Q})$ *determined by* $p^*(a) = t_0^2$, $p^*(b) =$

$2u$ *and* $p^*(c) = v'$,

$$\begin{aligned}
J'_{12} =& -4t_0^8u + 2t_0^6v' - 6t_0^2uv' + u^3 + 3v'^2,\\
J'_{14} =& t_0^{14} - 2t_0^{10}u - 6t_0^6u^2 + 4t_0^8v' - 3t_0^2u^3 - 3u^2v' + 3t_0^2v'^2,\\
J'_{18} =& -8t_0^{14}u - 8t_0^6u^3 - 3t_0^2u^4 - 16t_0^8uv' - 12t_0^{10}u^2 - 24t_0^4u^2v' - 12u^3v'\\
& - 4t_0^6v'^2 - 8v'^3,\\
r_{12} =& -2a^4b + 2a^3c - 3abc + \frac{1}{8}b^3 + 3c^2,\\
r_{14} =& a^7 - a^5b + 4a^4c - \frac{3}{2}a^3b^2 - \frac{3}{8}ab^3 + 3ac^2 - \frac{3}{4}b^2c,\\
r_{18} =& -4a^7b - 3a^5b^2 - 8a^4bc - a^3b^3 - 4a^3c^2 - 6a^2b^2c - \frac{3}{16}ab^4 - \frac{3}{2}b^3c - 8c^3.
\end{aligned}$$

Remark 2.5. As proved in the next section, a, b and c are all integral cohomology classes.

Furthermore we determined the integral cohomology ring of E_7/C_1 ([9], Theorem 5.7):

Theorem 2.6.

$$H^*(E_7/C_1;\mathbb{Z}) = \mathbb{Z}[t_0, u, v', w]/(\sigma'_9, \sigma'_{12}, \sigma'_{14}, \sigma'_{18})$$

where $\deg(t_0) = 2$, $\deg(u) = 8$, $\deg(v') = 12$, $\deg(w) = 18$ *and*

$$\begin{aligned}
\sigma'_9 &= 2w - t_0u^2,\\
\sigma'_{12} &= -4t_0^8u + 2t_0^6v' - 6t_0^2uv' + u^3 + 3v'^2,\\
\sigma'_{14} &= t_0^{14} - 2t_0^{10}u - 6t_0^6u^2 + 4t_0^8v' - 3t_0^2u^3 - 3u^2v' + 3t_0^2v'^2,\\
\sigma'_{18} &= -2t_0^{14}u - 2t_0^6u^3 - 3w^2 - 4t_0^8uv' - 3t_0^{10}u^2 - 6t_0^4u^2v' - 3u^3v' - t_0^6v'^2 - 2v'^3.
\end{aligned}$$

From this theorem (see also [10], Theorem 5.5) we have the following

Theorem 2.7.

$$H^*(E_7/C_1;\mathbb{Z}_2) = \mathbb{Z}_2[t_0, u, v', w]/(t_0u^2, u^3 + v'^2, t_0^{14} + u^2v', w^2 + v'^3).$$

Squaring operations on t_0, u, v', w *are given as follows*:

$$\begin{aligned}
&Sq^2(t_0) = t_0^2,\ Sq^2(u) = t_0u,\ Sq^4(u) = t_0^2u + v',\\
&Sq^2(v') = t_0^7 + t_0v',\ Sq^4(v') = t_0^8 + t_0^2v',\ Sq^8(v') = t_0^6u + t_0^4v' + t_0w + uv',\\
&Sq^2(w) = t_0^{10} + t_0^6u + uv',\ Sq^4(w) = t_0^{11} + t_0^7u,\\
&Sq^8(w) = t_0^{13} + t_0^9u + t_0^7v' + uw,\ Sq^{16}(w) = t_0^{13}u + t_0uv'^2 + u^2w.
\end{aligned}$$

Corollary 2.8. (i) *An additive basis of* $H^*(E_7/C_1;\mathbb{Z})$ *as a free module for degree* ≤ 20 *is given as follows*:

deg	0	2	4	6	8	10	12	14	16	18	20
	1	t_0	t_0^2	t_0^3	t_0^4	t_0^5	t_0^6	t_0^7	t_0^8	t_0^9	t_0^{10}
					u	t_0u	t_0^2u	t_0^3u	t_0^4u	t_0^5u	t_0^6u
							v'	t_0v'	t_0^2v'	t_0^3v'	t_0^4v'
									u^2	w	t_0w
											uv'

(ii) *An additive basis of* $H^*(E_7/C_1;\mathbb{Z}_2)$ *as a* $\mathbb{Z}_2$*-vector space is given as follows*:

$$\left\{\begin{array}{l} t_0^i,\ t_0^iu,\ t_0^iv',\ t_0^iw,\ t_0^iuv',\ t_0^iuw,\ t_0^iv'w,\ t_0^iuv'w\ (0\le i\le 13),\\ u^2,\ v'^2,\ u^2v',\ uv'^2,\ u^2w,\ v'^3,\ u^2v'^2,\ v'^2w,\ u^2v'w,\ uv'^2w,\ v'^4,\\ v'^3w,\ u^2v'^2w,\ v'^4w \end{array}\right\}.$$

3. The cohomology of EVI in low degrees

In this section we consider the integral and mod 2 cohomology of EVI in low degrees. As is mentioned in the introduction, we consider the Gysin sequence associated with the 2-sphere bundle $S^2\cong U_1/C_1\longrightarrow E_7/C_1\xrightarrow{p} E_7/U_1=E\mathrm{VI}$:

$$(*)_i\quad 0\longrightarrow H^{2i-3}(E\mathrm{VI};A)\xrightarrow{h}H^{2i}(E\mathrm{VI};A)\xrightarrow{p^*}H^{2i}(E_7/C_1;A)$$
$$\xrightarrow{\theta}H^{2i-2}(E\mathrm{VI};A)\xrightarrow{h}H^{2i+1}(E\mathrm{VI};A)\longrightarrow 0$$

where $A=\mathbb{Z}$ or $\mathbb{Z}_2$ and the homomorphisms θ and h satisfy

$$\theta(p^*(x)y)=x\theta(y),\quad h(x)=\chi\cdot x$$

for some $\chi\in H^3(E\mathrm{VI};A)$ such that $2\chi=0$. Since $H^{2i}(E_7/C_1;\mathbb{Z})$ is free, it follows from $(*)$ that

$$H^{\mathrm{odd}}(E\mathrm{VI};\mathbb{Z})=\chi\cdot H^{\mathrm{even}}(E\mathrm{VI};\mathbb{Z})\subset\mathrm{Im}\,h=\mathrm{Tor}H^*(E\mathrm{VI};\mathbb{Z})\tag{3.3.1}$$

and the latter is an elementary abelian 2-group. ($\mathrm{Tor}H^*(E\mathrm{VI};\mathbb{Z})$ means the torsion subgroup of $H^*(E\mathrm{VI};\mathbb{Z})$)

Since E_7 is 2-connected, $\pi_1(E\mathrm{VI})\cong\pi_0(U_1)=0,\pi_2(E\mathrm{VI})\cong\pi_1(U_1)\cong\mathbb{Z}_2$. Therefore

$$H_1(E\mathrm{VI};\mathbb{Z})=0,\quad H_2(E\mathrm{VI};\mathbb{Z})=\mathbb{Z}_2$$

and by the universal coefficient theorem we have

$$H^1(E\mathrm{VI};\mathbb{Z})=H^2(E\mathrm{VI};\mathbb{Z})=0,\quad H^3(E\mathrm{VI};\mathbb{Z})\neq 0.$$

Then by $(*)_1$:

$$0\longrightarrow\langle t_0\rangle\xrightarrow{\theta}\langle 1\rangle\xrightarrow{\chi\cdot}H^3(E\mathrm{VI};\mathbb{Z})\longrightarrow 0$$

we deduce

$$H^3(EVI;\mathbb{Z}) = \langle \chi \rangle \cong \mathbb{Z}_2, \quad \text{and} \quad \theta(t_0) = 2.$$

Here we change θ to $-\theta$ if it is necessary. Consider $(*)_1$ with mod 2 coefficient

$$0 \longrightarrow H^2(EVI;\mathbb{Z}_2) \xrightarrow{p^*} \langle t_0 \rangle \xrightarrow{\theta} \langle 1 \rangle \xrightarrow{y_3\cdot} H^3(EVI;\mathbb{Z}_2) \longrightarrow 0$$

where $y_3 = \chi \mod 2$. Since $\theta(t_0) = 2$ with integer coefficient, $\theta(t_0) \equiv 0 \mod 2$. Hence by the exactness there exists an element $y_2 \in H^2(EVI;\mathbb{Z}_2)$ such that $p^*(y_2) = t_0$ and we have

$$H^2(EVI;\mathbb{Z}_2) = \langle y_2 \rangle \quad \text{and} \quad H^3(EVI;\mathbb{Z}_2) = \langle y_3 \rangle.$$

Next consider $(*)_2$:

$$0 \longrightarrow H^4(EVI;\mathbb{Z}) \xrightarrow{p^*} \langle t_0^2 \rangle \xrightarrow{\theta} 0 \xrightarrow{\chi\cdot} H^5(EVI;\mathbb{Z}) \longrightarrow 0.$$

From this there exists an element $a \in H^4(EVI;\mathbb{Z})$ such that $p^*(a) = t_0^2$ and we have

$$H^4(EVI;\mathbb{Z}) = \langle a \rangle \cong \mathbb{Z} \quad \text{and} \quad H^5(EVI;\mathbb{Z}) = 0.$$

Considering with mod 2 coefficient

$$0 \longrightarrow H^4(EVI;\mathbb{Z}_2) \xrightarrow{p^*} \langle t_0^2 \rangle \xrightarrow{\theta} \langle y_2 \rangle \xrightarrow{y_3\cdot} H^5(EVI;\mathbb{Z}_2) \longrightarrow 0$$

we have $\theta(t_0^2) = \theta(p^*(y_2^2)) = 0$ and we deduce

$$H^4(EVI;\mathbb{Z}_2) = \langle y_2^2 \rangle \quad \text{and} \quad H^5(EVI;\mathbb{Z}_2) = \langle y_2 y_3 \rangle.$$

Note that $a \mod 2 = y_2^2$ since $p^*(a) = t_0^2$.

Next consider $(*)_3$:

$$0 \longrightarrow \langle \chi \rangle \xrightarrow{\chi\cdot} H^6(EVI;\mathbb{Z}) \xrightarrow{p^*} \langle t_0^3 \rangle \xrightarrow{\theta} \langle a \rangle \xrightarrow{\chi\cdot} H^7(EVI;\mathbb{Z}) \longrightarrow 0.$$

Since $\theta(t_0^3) = \theta(p^*(a)t_0) = a\theta(t_0) = 2a$, we deduce

$$H^6(EVI;\mathbb{Z}) = \langle \chi^2 \rangle \cong \mathbb{Z}_2, \quad \text{and} \quad H^7(EVI;\mathbb{Z}) = \langle a\chi \rangle \cong \mathbb{Z}_2.$$

Considering with mod 2 coefficient

$$0 \longrightarrow \langle y_3 \rangle \xrightarrow{y_3\cdot} H^6(EVI;\mathbb{Z}_2) \xrightarrow{p^*} \langle t_0^3 \rangle \xrightarrow{\theta} \langle y_2^2 \rangle \xrightarrow{y_3\cdot} H^7(EVI;\mathbb{Z}_2) \longrightarrow 0$$

we have $\theta(t_0^3) = \theta(p^*(y_2^3)) = 0$ and we deduce

$$H^6(EVI;\mathbb{Z}_2) = \langle y_2^3, y_3^2 \rangle \quad \text{and} \quad H^7(EVI;\mathbb{Z}_2) = \langle y_2^2 y_3 \rangle.$$

Next consider $(*)_4$:

$$0 \longrightarrow H^8(EVI;\mathbb{Z}) \xrightarrow{p^*} \langle t_0^4, u \rangle \xrightarrow{\theta} \langle \chi^2 \rangle \xrightarrow{\chi\cdot} H^9(EVI;\mathbb{Z}) \longrightarrow 0.$$

Then $\theta(t_0^4) = \theta(p^*(a^2)) = 0$. As to the image of u, there are two possibilities:

(i) $\theta(u) = \chi^2$,

(ii) $\theta(u) = 0$.

Lemma 3.1. (ii) *does not occur.*

Proof. If (ii) $\theta(u) = 0$ is true, $\theta(u) \equiv 0 \mod 2$. By the exactness there exists an element $y_8 \in H^8(E\mathrm{VI};\mathbb{Z}_2)$ such that $p^*(y_8) = u$. Then $\theta(v') = \theta(v + t_0^2 u) = \theta(Sq^4(u) + t_0^2 u) = \theta(p^*(Sq^4(y_8) + y_2^2 y_8)) = 0$. Hence there exists an element $y_{12} \in H^{12}(E\mathrm{VI};\mathbb{Z}_2)$ such that $p^*(y_{12}) = v'$. Applying Sq^8 on both sides, we have $p^*(Sq^8(y_{12})) = Sq^8(v') = t_0^6 u + t_0^4 v' + t_0 w + uv'$. Therefore by the exactness

$$\begin{aligned} 0 &= \theta(t_0^6 u) + \theta(t_0^4 v') + \theta(t_0 w) + \theta(uv') \\ &= \theta(p^*(y_2^6 y_8)) + \theta(p^*(y_2^4 y_{12})) + \theta(p^*(y_2) w) + \theta(p^*(y_8 y_{12})) \\ &= y_2\, \theta(w) \end{aligned}$$

and also $y_3\, \theta(w) = 0$. On the other hand since $p^*(y_8^3 + y_{12}^2) = u^3 + {v'}^2 = 0, p^*(y_2^{14} + y_8^2 y_{12}) = t_0^{14} + u^2 v' = 0$ by Theorem 2.7, we may put

$$y_8^3 + y_{12}^2 = y_3 \cdot f \quad \text{and} \quad y_2^{14} + y_8^2 y_{12} = y_3 \cdot g$$

for some elements $f, g \in H^*(E\mathrm{VI};\mathbb{Z}_2)$. Then using these relations

$$\begin{aligned} \theta({v'}^4 w) &= \theta(p^*(y_{12}^4) w) = y_{12}^4\, \theta(w) = y_{12}^2\, (y_8^3 + y_3 \cdot f)\, \theta(w) \\ &= y_{12}^2 y_8^3\, \theta(w) = y_{12} y_8\, (y_2^{14} + y_3 \cdot g)\, \theta(w) = 0. \end{aligned}$$

This contradicts the fact that $\theta : H^{66}(E_7/C_1;\mathbb{Z}_2) = \langle {v'}^4 w \rangle \longrightarrow H^{64}(E\mathrm{VI};\mathbb{Z}_2)$ is an isomorphism. □

Therefore (i) $\theta(u) = \chi^2$ is true. Then from $(*)_4$ there exists an element $b \in H^8(E\mathrm{VI};\mathbb{Z})$ such that $p^*(b) = 2u$ and we have

$$H^8(E\mathrm{VI};\mathbb{Z}) = \langle a^2, b \rangle \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad H^9(E\mathrm{VI};\mathbb{Z}) = 0,\ \chi^3 = 0.$$

Considering with mod 2 coefficient

$$0 \longrightarrow \langle y_2 y_3 \rangle \xrightarrow{y_3 \cdot} H^8(E\mathrm{VI};\mathbb{Z}_2) \xrightarrow{p^*} \langle t_0^4, u \rangle \xrightarrow{\theta} \langle y_2^3, y_3^2 \rangle \xrightarrow{y_3 \cdot} H^9(E\mathrm{VI};\mathbb{Z}_2) \longrightarrow 0$$

we have $\theta(t_0^4) = \theta(p^*(y_2^4)) = 0, \theta(u) = y_3^2$ and therefore

$$H^8(E\mathrm{VI};\mathbb{Z}_2) = \langle y_2^4, y_2 y_3^2 \rangle \quad \text{and} \quad H^9(E\mathrm{VI};\mathbb{Z}_2) = \langle y_2^3 y_3 \rangle$$

where $b \mod 2 = y_2 y_3^2$.

Next consider $(*)_5$:

$$0 \longrightarrow \langle a\chi \rangle \xrightarrow{\chi \cdot} H^{10}(E\mathrm{VI};\mathbb{Z}) \xrightarrow{p^*} \langle t_0^5, t_0 u \rangle \xrightarrow{\theta} \langle a^2, b \rangle \xrightarrow{\chi \cdot} H^{11}(E\mathrm{VI};\mathbb{Z}) \longrightarrow 0.$$

Then $\theta(t_0^5) = \theta(t_0 p^*(a^2)) = a^2 \theta(t_0) = 2a^2, 2\theta(t_0 u) = \theta(2 t_0 u) = \theta(t_0 p^*(b)) = b\theta(t_0) = 2b$ and therefore $\theta(t_0 u) = b$ since $H^8(E\mathrm{VI};\mathbb{Z}) = \langle a^2, b \rangle$ is free. Hence θ is injective and we have

$$H^{10}(E\mathrm{VI};\mathbb{Z}) = \langle a\chi^2 \rangle \cong \mathbb{Z}_2 \quad \text{and} \quad H^{11}(E\mathrm{VI};\mathbb{Z}) = \langle a^2 \chi \rangle \cong \mathbb{Z}_2,\ b\chi = 0.$$

Considering with mod 2 coefficient

$$0 \longrightarrow \langle y_2^2 y_3\rangle \xrightarrow{y_3\cdot} H^{10}(E\mathrm{VI};\mathbb{Z}_2) \xrightarrow{p^*} \langle t_0^5, t_0 u\rangle$$
$$\xrightarrow{\theta} \langle y_2^4, y_2 y_3^2\rangle \xrightarrow{y_3\cdot} H^{11}(E\mathrm{VI};\mathbb{Z}_2) \longrightarrow 0$$

we have $\theta(t_0^5) = \theta(p^*(y_2^5)) = 0, \theta(t_0 u) = \theta(p^*(y_2)u) = y_2\theta(u) = y_2 y_3^2$ and therefore

$$H^{10}(E\mathrm{VI};\mathbb{Z}_2) = \langle y_2^5, y_2^2 y_3^2\rangle \quad \text{and} \quad H^{11}(E\mathrm{VI};\mathbb{Z}_2) = \langle y_2^4 y_3\rangle.$$

Next consider $(*)_6$:

$$0 \longrightarrow H^{12}(E\mathrm{VI};\mathbb{Z}) \xrightarrow{p^*} \langle t_0^6, t_0^2 u, v'\rangle \xrightarrow{\theta} \langle a\chi^2\rangle \xrightarrow{\chi\cdot} H^{13}(E\mathrm{VI};\mathbb{Z}) \longrightarrow 0.$$

Then $\theta(t_0^6) = \theta(p^*(a^3)) = 0, \theta(t_0^2 u) = \theta(p^*(a)u) = a\theta(u) = a\chi^2$. As to the image of v', there are two possibilities:

(i) $\theta(v') = a\chi^2$,

(ii) $\theta(v') = 0$.

Now we assume the following lemma which will be proved at the end of this section.

Lemma 3.2. (i) *does not occur.*

Thererfore (ii) $\theta(v') = 0$ is true. Then from $(*)_6$ there exists an element $c \in H^{12}(E\mathrm{VI};\mathbb{Z})$ such that $p^*(c) = v'$ and we have

$$H^{12}(E\mathrm{VI};\mathbb{Z}) = \langle a^3, ab, c\rangle \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \quad \text{and} \quad H^{13}(E\mathrm{VI};\mathbb{Z}) = 0.$$

Considering with mod 2 coefficient

$$0 \longrightarrow \langle y_2^3 y_3\rangle \xrightarrow{y_3\cdot} H^{12}(E\mathrm{VI};\mathbb{Z}_2) \xrightarrow{p^*} \langle t_0^6, t_0^2 u, v'\rangle$$
$$\xrightarrow{\theta} \langle y_2^5, y_2^2 y_3^2\rangle \xrightarrow{y_3\cdot} H^{13}(E\mathrm{VI};\mathbb{Z}_2) \longrightarrow 0$$

we have $\theta(t_0^6) = \theta(p^*(y_2^6)) = 0, \theta(t_0^2 u) = \theta(p^*(y_2^2)u) = y_2^2\theta(u) = y_2^2 y_3^2, \theta(v') = 0$ and therefore

$$H^{12}(E\mathrm{VI};\mathbb{Z}_2) = \langle y_2^6, y_2^3 y_3^2, y_{12}\rangle \quad \text{and} \quad H^{13}(E\mathrm{VI};\mathbb{Z}_2) = \langle y_2^5 y_3\rangle$$

where $y_{12} = c \mod 2$.

We continue this argument up to degree ≤ 20.

Next consider $(*)_7$:

$$0 \longrightarrow \langle a^2\chi\rangle \xrightarrow{\chi\cdot} H^{14}(E\mathrm{VI};\mathbb{Z}) \xrightarrow{p^*} \langle t_0^7, t_0^3 u, t_0 v'\rangle$$
$$\xrightarrow{\theta} \langle a^3, ab, c\rangle \xrightarrow{\chi\cdot} H^{15}(E\mathrm{VI};\mathbb{Z}) \longrightarrow 0.$$

Then $\theta(t_0^7) = \theta(p^*(a^3)t_0) = a^3\theta(t_0) = 2a^3, \theta(t_0^3 u) = \theta(p^*(a)t_0 u) = a\theta(t_0 u) = ab, \theta(t_0 v') = \theta(p^*(c)t_0) = c\theta(t_0) = 2c$ and hence θ is injective and we have

$$H^{14}(E\mathrm{VI};\mathbb{Z}) = \langle a^2\chi^2\rangle \quad \text{and} \quad H^{15}(E\mathrm{VI};\mathbb{Z}) = \langle a^3\chi, c\chi\rangle.$$

Considering with mod 2 coefficient

$$0 \longrightarrow \langle y_2^4 y_3 \rangle \xrightarrow{y_3 \cdot} H^{14}(EVI;\mathbb{Z}_2) \xrightarrow{p^*} \langle t_0^7, t_0^3 u, t_0 v' \rangle \xrightarrow{\theta} \langle y_2^6, y_{12}, y_2^3 y_3^2 \rangle \xrightarrow{y_3 \cdot} H^{15}(EVI;\mathbb{Z}_2) \longrightarrow 0$$

we have $\theta(t_0^7) = \theta(p^*(y_2^7)) = 0, \theta(t_0^3 u) = \theta(p^*(y_2^3)u) = y_2^3\theta(u) = y_2^3 y_3^2, \theta(t_0 v') = \theta(p^*(y_2 y_{12})) = 0$ and therefore

$$H^{14}(EVI;\mathbb{Z}_2) = \langle y_2^7, y_2 y_{12}, y_2^4 y_3^2 \rangle \quad \text{and} \quad H^{15}(EVI;\mathbb{Z}_2) = \langle y_2^6 y_3, y_3 y_{12} \rangle.$$

Next consider $(*)_8$:

$$0 \longrightarrow H^{16}(EVI;\mathbb{Z}) \xrightarrow{p^*} \langle t_0^8, t_0^4 u, t_0^2 v', u^2 \rangle \xrightarrow{\theta} \langle a^2\chi^2 \rangle \xrightarrow{\chi \cdot} H^{17}(EVI;\mathbb{Z}) \longrightarrow 0.$$

Then $\theta(t_0^8) = \theta(p^*(a^4)) = 0, \theta(t_0^4 u) = \theta(p^*(a^2)u) = a^2\theta(u) = a^2\chi^2, \theta(t_0^2 v') = \theta(p^*(ac)) = 0$. As to the image of u^2, there are two possibilities:
(i) $\theta(u^2) = a^2\chi^2$,
(ii) $\theta(u^2) = 0$.

Lemma 3.3. (i) *does not occur.*

Proof. Consider

$$\theta : H^{18}(E_7/C_1;\mathbb{Z}) = \langle t_0^9, t_0^5 u, t_0^3 v', w \rangle \longrightarrow H^{16}(EVI;\mathbb{Z}).$$

Since $2w = t_0 u^2$ we have

$$4\theta(w) = \theta(4w) = \theta(2t_0 u^2) = \theta(p^*(b)t_0 u) = b\theta(t_0 u) = b^2.$$

Therefore if we put $\theta(w) = d$ then $b^2 = 4d$ and $4p^*(d) = 4u^2$. Thus $p^*(d) = u^2$ since $H^{16}(E_7/C_1;\mathbb{Z})$ is free. By the exactness we conclude $\theta(u^2) = 0$. □

Hence we have

$$H^{16}(EVI;\mathbb{Z}) = \langle a^4, a^2 b, ac, d \rangle,\ 4d = b^2 \quad \text{and} \quad H^{17}(EVI;\mathbb{Z}) = 0.$$

Considering with mod 2 coefficient

$$0 \longrightarrow \langle y_2^5 y_3 \rangle \xrightarrow{y_3 \cdot} H^{16}(EVI;\mathbb{Z}_2) \xrightarrow{p^*} \langle t_0^8, t_0^4 u, t_0^2 v', u^2 \rangle \xrightarrow{\theta} \langle y_2^7, y_2 y_{12}, y_2^4 y_3^2 \rangle \xrightarrow{y_3 \cdot} H^{17}(EVI;\mathbb{Z}_2) \longrightarrow 0$$

we have $\theta(t_0^8) = \theta(p^*(y_2^8)) = 0, \theta(t_0^4 u) = \theta(p^*(y_2^4)u) = y_2^4\theta(u) = y_2^4 y_3^2, \theta(t_0^2 v') = \theta(p^*(y_2^2 y_{12})) = 0, \theta(u^2) = 0$ and therefore

$$H^{16}(EVI;\mathbb{Z}_2) = \langle y_2^8, y_2^2 y_{12}, y_{16}, y_2^5 y_3^2 \rangle \quad \text{and} \quad H^{17}(EVI;\mathbb{Z}_2) = \langle y_2^7 y_3, y_2 y_3 y_{12} \rangle$$

where $y_{16} = d \mod 2$.

Next consider $(*)_9$:

$$0 \longrightarrow \langle a^3\chi, c\chi\rangle \xrightarrow{\chi\cdot} H^{18}(EVI;\mathbb{Z}) \xrightarrow{p^*} \langle t_0^9, t_0^5u, t_0^3v', w\rangle \xrightarrow{\theta} \langle a^4, a^2b, ac, d\rangle \xrightarrow{\chi\cdot} H^{19}(EVI;\mathbb{Z}) \longrightarrow 0.$$

Then $\theta(t_0^9) = \theta(p^*(a^4)t_0) = a^4\theta(t_0) = 2a^4, \theta(t_0^5u) = \theta(p^*(a^2)t_0u) = a^2\theta(t_0u)a^2b$, $\theta(t_0^3v') = \theta(p^*(ac)t_0) = ac\theta(t_0) = 2ac, \theta(w) = d$ and hence θ is injective and we have

$$H^{18}(EVI;\mathbb{Z}) = \langle a^3\chi^2, c\chi^2\rangle \quad \text{and} \quad H^{19}(EVI;\mathbb{Z}) = \langle a^4\chi, ac\chi\rangle,\ d\chi = 0.$$

Considering with mod 2 coefficient

$$0 \longrightarrow \langle y_2^6y_3, y_3y_{12}\rangle \xrightarrow{y_3\cdot} H^{18}(EVI;\mathbb{Z}_2) \xrightarrow{p^*} \langle t_0^9, t_0^5u, t_0^3v', w\rangle \xrightarrow{\theta} \langle y_2^8, y_2^2y_{12}, y_{16}, y_2^5y_3^2\rangle \xrightarrow{y_3\cdot} H^{19}(EVI;\mathbb{Z}_2) \longrightarrow 0$$

we have $\theta(t_0^9) = \theta(p^*(y_2^9)) = 0, \theta(t_0^5u) = \theta(p^*(y_2^5)u) = y_2^5\theta(u) = y_2^5y_3^2, \theta(t_0^3v') = \theta(p^*(y_2^3y_{12})) = 0, \theta(w) = y_{16}$. On the other hand $p^*(y_{12}) = v'$ implies $p^*(Sq^8(y_{12})) = Sq^8(v') = t_0^6u + t_0^4v' + t_0w + uv'$ and by the exactness we have

$$\begin{aligned}
0 &= \theta(t_0^6u) + \theta(t_0^4v') + \theta(t_0w) + \theta(uv') \\
&= \theta(p^*(y_2^6)u) + \theta(p^*(y_2^4y_{12})) + \theta(p^*(y_2)w) + \theta(p^*(y_{12})u) \\
&= y_2^6\theta(u) + y_2\theta(w) + y_{12}\theta(u) \\
&= y_2^6y_3^2 + y_2y_{16} + y_3^2y_{12}.
\end{aligned}$$

Therefore we deduce

$$\begin{aligned}
&H^{18}(EVI;\mathbb{Z}_2) = \langle y_2^9, y_2^3y_{12}, y_2^6y_3^2, y_3^2y_{12}\rangle, \quad y_2y_{16} = y_3^2y_{12} + y_2^6y_3^2. \\
&H^{19}(EVI;\mathbb{Z}_2) = \langle y_2^8y_3, y_2^2y_3y_{12}\rangle, \quad y_3y_{16} = 0.
\end{aligned}$$

Next consider $(*)_{10}$:

$$0 \longrightarrow H^{20}(EVI;\mathbb{Z}) \xrightarrow{p^*} \langle t_0^{10}, t_0^6u, t_0^4v', t_0w, uv'\rangle \xrightarrow{\theta} \langle a^3\chi^2, c\chi^2\rangle \xrightarrow{\chi\cdot} H^{21}(EVI;\mathbb{Z}) \longrightarrow 0.$$

Then $\theta(t_0^{10}) = \theta(p^*(a^5)) = 0, \theta(t_0^6u) = \theta(p^*(a^3)u) = a^3\theta(u) = a^3\chi^2, \theta(t_0^4v') = \theta(p^*(a^2c)) = 0, \theta(uv') = \theta(p^*(c)u) = c\theta(u) = c\chi^2$. Considering with mod 2 coefficient we have $\theta(t_0w) = \theta(p^*(y_2)w) = y_2\theta(w) = y_2y_{16} = y_3^2y_{12} + y_2^6y_3^2$. This implies $\theta(t_0w) = a^3\chi^2 + c\chi^2$ with integer coefficient. Therefore if we put $x = t_0w - t_0^6u - uv'$ we have $\theta(x) = 0$ and by the exactness there exists an element $e \in H^{20}(EVI;\mathbb{Z})$ such that $p^*(e) = x$. Then $p^*(2e) = 2x = 2t_0w - 2t_0^6u - 2uv' = p^*(ad - a^3b - bc)$ and we have $2e = ad - a^3b - bc$ since p^* is injective. Using x we have $H^{20}(E_7/C_1;\mathbb{Z}) = \langle t_0^{10}, t_0^6u, t_0^4v', x, uv'\rangle$ as a free module and we see easily

$$\begin{aligned}
&H^{20}(EVI;\mathbb{Z}) = \langle a^5, a^3b, a^2c, bc, e\rangle,\ 2e = ad - a^3b - bc. \\
&H^{21}(EVI;\mathbb{Z}) = 0.
\end{aligned}$$

Considering with mod 2 coefficient

$$0 \longrightarrow \langle y_2^7 y_3, y_2 y_3 y_{12}\rangle \xrightarrow{y_3\cdot} H^{20}(E\mathrm{VI};\mathbb{Z}_2) \xrightarrow{p^*} \langle t_0^{10}, t_0^6 u, t_0^4 v', x, uv'\rangle$$
$$\xrightarrow{\theta} \langle y_2^9, y_2^3 y_{12}, y_2^6 y_3^2, y_3^2 y_{12}\rangle \xrightarrow{y_3\cdot} H^{21}(E\mathrm{VI};\mathbb{Z}_2) \longrightarrow 0$$

we have $\theta(t_0^{10}) = \theta(p^*(y_2^{10})) = 0, \theta(t_0^6 u) = \theta(p^*(y_2^6)u) = y_2^6\theta(u) = y_2^6 y_3^2, \theta(t_0^4 v') = \theta(p^*(y_2^4 y_{12})) = 0, \theta(x) = 0, \theta(uv') = \theta(p^*(y_{12})u) = y_{12}\theta(u) = y_3^2 y_{12}$ and therefore

$$H^{20}(E\mathrm{VI};\mathbb{Z}_2) = \langle y_2^{10}, y_2^4 y_{12}, y_{20}, y_2^7 y_3^2, y_2 y_3^2 y_{12}\rangle.$$
$$H^{21}(E\mathrm{VI};\mathbb{Z}_2) = \langle y_2^9 y_3, y_2^3 y_3 y_{12}\rangle$$

where $y_{20} = e \mod 2$.

Thus we have determined $H^*(E\mathrm{VI};\mathbb{Z}), H^*(E\mathrm{VI};\mathbb{Z}_2)$ up to degree ≤ 20:

Lemma 3.4.
(i) $H^*(E\mathrm{VI};\mathbb{Z}) = \mathbb{Z}[\chi, a, b, c, d, e]/(2\chi, \chi^3, b\chi, 4d - b^2, d\chi, 2e - ad + bc + a^3 b)$,
(ii) $H^*(E\mathrm{VI};\mathbb{Z}_2) = \mathbb{Z}_2[y_2, y_3, y_{12}, y_{16}, y_{20}]/(y_3^3, y_2 y_{16} + y_3^2 y_{12} + y_2^6 y_3^2, y_3 y_{16})$ *for degree* ≤ 20.

We continue the computation with mod 2 coefficient up to degree ≤ 30.
Consider $(*)_{11}$:

$$0 \longrightarrow \langle y_2^8 y_3, y_2^2 y_3 y_{12}\rangle \xrightarrow{y_3\cdot} H^{22}(E\mathrm{VI};\mathbb{Z}_2) \xrightarrow{p^*} \langle t_0^{11}, t_0^7 u, t_0^5 v', t_0 x, t_0 uv'\rangle$$
$$\xrightarrow{\theta} \langle y_2^{10}, y_2^4 y_{12}, y_{20}, y_2^7 y_3^2, y_2 y_3^2 y_{12}\rangle \xrightarrow{y_3\cdot} H^{23}(E\mathrm{VI};\mathbb{Z}_2) \longrightarrow 0.$$

Then $\theta(t_0^{11}) = \theta(p^*(y_2^{11}) = 0, \theta(t_0^7 u) = \theta(p^*(y_2^7)u) = y_2^7\theta(u) = y_2^7 y_3^2, \theta(t_0^5 v') = \theta(p^*(y_2^5 y_{12})) = 0, \theta(t_0 x) = \theta(p^*(y_2 y_{20}) = 0, \theta(t_0 uv') = \theta(p^*(y_2 y_{12})u) = y_2 y_{12}\theta(u) = y_2 y_3^2 y_{12}$ and therefore

$$H^{22}(E\mathrm{VI};\mathbb{Z}_2) = \langle y_2^{11}, y_2^5 y_{12}, y_2 y_{20}, y_2^8 y_3^2, y_2^2 y_3^2 y_{12}\rangle.$$
$$H^{23}(E\mathrm{VI};\mathbb{Z}_2) = \langle y_2^{10} y_3, y_2^4 y_3 y_{12}, y_3 y_{20}\rangle.$$

Next consider $(*)_{12}$:

$$0 \longrightarrow \langle y_2^9 y_3, y_2^3 y_3 y_{12}\rangle \xrightarrow{y_3\cdot} H^{24}(E\mathrm{VI};\mathbb{Z}_2) \xrightarrow{p^*} \langle t_0^{12}, t_0^8 u, t_0^6 v', t_0^2 x, t_0^2 uv', v'^2\rangle$$
$$\xrightarrow{\theta} \langle y_2^{11}, y_2^5 y_{12}, y_2 y_{20}, y_2^8 y_3^2, y_2^2 y_3^2 y_{12}\rangle \xrightarrow{y_3\cdot} H^{25}(E\mathrm{VI};\mathbb{Z}_2) \longrightarrow 0.$$

Then $\theta(t_0^{12}) = \theta(p^*(y_2^{12})) = 0, \theta(t_0^8 u) = \theta(p^*(y_2^8)u) = y_2^8\theta(u) = y_2^8 y_3^2, \theta(t_0^6 v') = \theta(p^*(y_2^6 y_{12})) = 0, \theta(t_0^2 x) = \theta(p^*(y_2^2 y_{20})) = 0, \theta(t_0^2 uv') = \theta(p^*(y_2^2 y_{12})u) = y_2^2 y_{12}\theta(u) = y_2^2 y_3^2 y_{12}, \theta(v'^2) = \theta(p^*(y_{12}^2)) = 0$ and therefore

$$H^{24}(E\mathrm{VI};\mathbb{Z}_2) = \langle y_2^{12}, y_2^6 y_{12}, y_2^2 y_{20}, y_{12}^2, y_2^9 y_3^2, y_2^3 y_3^2 y_{12}\rangle.$$
$$H^{25}(E\mathrm{VI};\mathbb{Z}_2) = \langle y_2^{11} y_3, y_2^5 y_3 y_{12}, y_2 y_3 y_{20}\rangle.$$

Before considering $(*)_{13}$, we need to determine the action of the squaring operations on $y_2, y_3, y_{12}, y_{16}, y_{20}$.

Lemma 3.5.

$Sq^1(y_2) = y_3,\ Sq^1(y_3) = 0,\ Sq^1(y_{12}) = 0,\ Sq^1(y_{16}) = 0,\ Sq^1(y_{20}) = 0,$
$Sq^2(y_3) = y_2 y_3.$

Proof. Since $H^2(EVI;\mathbb{Z}) = 0, H^3(EVI;\mathbb{Z}) = \langle \chi \rangle \cong \mathbb{Z}_2, Sq^1(y_2) = \rho(\chi) = y_3$ by definition of Sq^1. Since $y_3, y_{12}, y_{16}, y_{20}$ are all mod 2 reductions of integral cohomology classes, Sq^1 on them are trivial. $Sq^1 Sq^2(y_3) = Sq^3(y_3) = y_3^2 \neq 0$ implies $Sq^2(y_3)$ does not vanish. As $H^5(EVI;\mathbb{Z}_2) = \langle y_2 y_3 \rangle$, $Sq^2(y_3) = y_2 y_3$. □

Lemma 3.6.

$$\begin{aligned} Sq^2(y_{12}) &= y_2^7 + y_2 y_{12} + y_2^4 y_3^2, \\ Sq^4(y_{12}) &= y_2^8 + y_2^2 y_{12} + \alpha' y_2^5 y_3^2, \\ Sq^8(y_{12}) &= y_{20} + y_2^4 y_{12} + \alpha'' y_2^7 y_3^2 + \beta'' y_2 y_3^2 y_{12} \end{aligned}$$

for some $\alpha', \alpha'', \beta'' \in \mathbb{Z}_2$.

Proof. Applying Sq^2 on both sides of $p^*(y_{12}) = v'$, we have $p^*(Sq^2(y_{12})) = t_0^7 + t_0 v' = p^*(y_2^7 + y_2 y_{12})$ from Theorem 2.7. Therefore in view of $(*)_7$, we may put

$$Sq^2(y_{12}) = y_2^7 + y_2 y_{12} + \alpha y_2^4 y_3^2$$

for some $\alpha \in \mathbb{Z}_2$. Applying Sq^2 on both sides, we have

$$(\alpha + 1) y_2^5 y_3^2 = 0 \quad \text{in } H^{16}(EVI;\mathbb{Z}_2).$$

Hence $\alpha = 1$ by Lemma 3.4 and we obtain the first assertion. Similarly we obtain the second and third assertions. □

Lemma 3.7.

$$\begin{aligned} Sq^2(y_{16}) &= 0, \\ Sq^4(y_{16}) &= y_2^7 y_3^2, \\ Sq^8(y_{16}) &= y_{12}^2 + \gamma'' y_2^9 y_3^2 + \delta'' y_2^3 y_3^2 y_{12} \end{aligned}$$

for some $\gamma'', \delta'' \in \mathbb{Z}_2$.

Proof. Applying Sq^2, Sq^4 on both sides of $p^*(y_{16}) = u^2$, we have $p^*(Sq^2(y_{16})) = 0, p^*(Sq^4(y_{16})) = 0$ from Theorem 2.7. Therefore in view of $(*)_9, (*)_{10}$, we may put

$$\begin{aligned} Sq^2(y_{16}) &= \gamma y_2^6 y_3^2 + \delta y_3^2 y_{12}, \\ Sq^4(y_{16}) &= \gamma' y_2^7 y_3^2 + \delta' y_2 y_3^2 y_{12} \end{aligned}$$

for some $\gamma, \delta, \gamma', \delta' \in \mathbb{Z}_2$. Now we apply Sq^2 on both sides of the relation $y_2 y_{16} = y_3^2 y_{12} + y_2^6 y_3^2$ and we have

$$\gamma y_2^7 y_3^2 + \delta y_2 y_3^2 y_{12} = 0 \quad \text{in } H^{20}(E\mathrm{VI}; \mathbb{Z}_2).$$

Hence $\gamma = \delta = 0$ by Lemma 3.4 and we obtain the first assertion. Furthermore applying Sq^4, we have

$$(\gamma' + 1)\, y_2^8 y_3^2 + \delta' y_2^2 y_3^2 y_{12} = 0 \quad \text{in } H^{22}(E\mathrm{VI}; \mathbb{Z}_2).$$

Hence $\gamma' = 1, \delta' = 0$ and we obtain the second assertion. The third assertion follows similarly. □

Similarly we can prove

Lemma 3.8.

$$\begin{aligned} Sq^2(y_{20}) &= y_2^{11} + y_2 y_{20} + \mu y_2^8 y_3^2 + \nu y_2^2 y_3^2 y_{12}, \\ Sq^4(y_{20}) &= y_{12}^2 + y_2^6 y_{12} + \mu' y_2^9 y_3^2 + \nu' y_2^3 y_3^2 y_{12}, \\ Sq^8(y_{20}) &= y_{12} y_{16} + y_2^8 y_{12} + \lambda'' y_2^{11} y_3^2 + \mu'' y_2^5 y_3^2 y_{12} + \nu'' y_2 y_3^2 y_{20} \end{aligned}$$

for some $\mu, \nu, \mu', \nu', \lambda'', \mu'', \nu'' \in \mathbb{Z}_2$.

Now we apply Sq^8 on both sides of the relation $y_2 y_{16} = y_3^2 y_{12} + y_2^6 y_3^2$. Then using Lemmas 3.6 and 3.7 we have

Lemma 3.9. *There exists a relation of the form*

$$y_2 y_{12}^2 = y_3^2 y_{20} + \gamma'' y_2^{10} y_3^2 + \delta'' y_2^4 y_3^2 y_{12} \tag{3.3.2}$$

where γ'', δ'' *are as in Lemma* 3.7.

Moreover applying Sq^1 on both sides of (3.3.2), we have

Lemma 3.10. *There exists a relation of the form*

$$y_3 y_{12}^2 = 0. \tag{3.3.3}$$

Now consider $(*)_{13}$:

$$0 \longrightarrow \langle y_2^{10} y_3, y_2^4 y_3 y_{12}, y_3 y_{20} \rangle \xrightarrow{y_3 \cdot} H^{26}(E\mathrm{VI}; \mathbb{Z}_2) \xrightarrow{p^*} \langle t_0^{13}, t_0^9 u, t_0^7 v', t_0^3 x, t_0^3 u v', u w \rangle$$
$$\xrightarrow{\theta} \langle y_2^{12}, y_2^6 y_{12}, y_2^2 y_{20}, y_{12}^2, y_2^9 y_3^2, y_2^3 y_3^2 y_{12} \rangle \xrightarrow{y_3 \cdot} H^{27}(E\mathrm{VI}; \mathbb{Z}_2) \longrightarrow 0.$$

Then $\theta(t_0^{13}) = \theta(p^*(y_2^{13})) = 0, \theta(t_0^9 u) = \theta(p^*(y_2^9) u) = y_2^9 \theta(u) = y_2^9 y_3^2, \theta(t_0^7 v') = \theta(p^*(y_2^7 y_{12})) = 0, \theta(t_0^3 x) = \theta(p^*(y_2^3 y_{20})) = 0, \theta(t_0^3 u v') = \theta(p^*(y_2^3 y_{12}) u) = y_2^3 y_{12} \times \theta(u) = y_2^3 y_3^2 y_{12}$. As to the image of uw

Lemma 3.11.

$$\theta(uw) = y_{12}^2 + y_2^3 y_3^2 y_{12}.$$

Proof. Since $2w = t_0u^2$ with integer coefficient

$$\begin{aligned}2uw = t_0u^3 =& t_0(4t_0^8u - 2t_0^6v' + 6t_0^2uv' - 3{v'}^2)\\ =& 4t_0^9u - 2t_0^7v' + 6t_0^3uv' - 3t_0{v'}^2\end{aligned}$$

by Theorem 2.6. Then

$$\begin{aligned}2\theta(uw) &= 4\theta(t_0^9u) - 2\theta(t_0^7v') + 6\theta(t_0^3uv') - 3\theta(t_0{v'}^2)\\ &= 4a^4b - 4a^3c + 6abc - 6c^2.\end{aligned}$$

Therefore

$$\theta(uw) = 2a^4b - 2a^3c + 3abc - 3c^2$$

since $H^{24}(E\mathrm{VI};\mathbb{Z})$ is free. Applying the mod 2 reduction ρ, we have the required result. □

Therefore we have

$$\begin{aligned}H^{26}(E\mathrm{VI};\mathbb{Z}_2) &= \langle y_2^{13}, y_2^7y_{12}, y_2^3y_{20}, y_2^{10}y_3^2, y_2^4y_3^2y_{12}, y_3^2y_{20}\rangle,\\ y_2y_{12}^2 &= y_3^2y_{20} + \gamma''y_2^{10}y_3^2 + \delta''y_2^4y_3^2y_{12}.\\ H^{27}(E\mathrm{VI};\mathbb{Z}_2) &= \langle y_2^{12}y_3, y_2^6y_3y_{12}, y_2^2y_3y_{20}\rangle, \quad y_3y_{12}^2 = 0.\end{aligned}$$

Before considering $(*)_{14}$, we need a lemma.

Lemma 3.12. *There exists a relation of the form*

$$y_2^{14} = y_{12}y_{16} + y_2^{11}y_3^2 + y_2^5y_3^2y_{12}. \tag{3.3.4}$$

Proof. By Lemma 2.4 there exists a relation

$$r_{14} = a^7 - a^5b + 4a^4c - \frac{3}{2}a^3b^2 - \frac{3}{8}ab^3 + 3ac^2 - \frac{3}{4}b^2c = 0 \tag{3.3.5}$$

in $H^{28}(E\mathrm{VI};\mathbb{Q})$. Substituting $b^2 = 4d$, $b^3 = 4bd = 16a^4b - 16a^3c + 24abc - 24c^2$, $2e = ad - bc - a^3b$ into (3.3.5), we have

$$a^7 - 13a^5b + 10a^4c - 15a^2bc - 12a^2e + 12ac^2 - 3cd = 0 \tag{3.3.6}$$

in $H^{28}(E\mathrm{VI};\mathbb{Z})$ since a, b, c, d, e are all integral cohomology classes. Therefore applying the mod 2 reduction ρ to (3.3.6) we have the required result. □

Now consider $(*)_{14}$:

$$0 \longrightarrow \langle y_2^{11}y_3, y_2^5y_3y_{12}, y_2y_3y_{20}\rangle \xrightarrow{y_3\cdot} H^{28}(E\mathrm{VI};\mathbb{Z}_2) \xrightarrow{p^*} \langle t_0^{10}u, t_0^8v', t_0^4x, t_0^4uv',$$
$$t_0uw, u^2v'\rangle \xrightarrow{\theta} \langle y_2^{13}, y_2^7y_{12}, y_2^3y_{20}, y_2^{10}y_3^2, y_2^4y_3^2y_{12}, y_3^2y_{20}\rangle \xrightarrow{y_3\cdot} H^{29}(E\mathrm{VI};\mathbb{Z}_2) \longrightarrow 0.$$

Then $\theta(t_0^{10}u) = \theta(p^*(y_2^{10})u) = y_2^{10}\theta(u) = y_2^{10}y_3^2, \theta(t_0^8v') = \theta(p^*(y_2^8y_{12})) = 0, \theta(t_0^4x) = \theta(p^*(y_2^4y_{20})) = 0, \theta(t_0^4uv') = \theta(p^*(y_2^4y_{12})u) = y_2^4y_{12}\theta(u) = y_2^4y_3^2y_{12}$, $\theta(u^2v') = \theta(p^*(y_{12}y_{16})) = 0$ and

$$\begin{aligned}\theta(t_0uw) = \theta(p^*(y_2)uw) = y_2\theta(uw) &= y_2y_{12}^2 + y_2^4y_3^2y_{12}\\ &= y_3^2y_{20} + \gamma''y_2^{10}y_3^2 + (\delta''+1)y_2^4y_3^2y_{12}.\end{aligned}$$

Therefore we deduce

$$\begin{aligned}H^{28}(E\mathrm{VI};\mathbb{Z}_2) &= \langle y_2^8y_{12}, y_2^4y_{20}, y_{12}y_{16}, y_2^{11}y_3^2, y_2^5y_3^2y_{12}, y_2y_3^2y_{20}\rangle,\\ y_2^{14} &= y_{12}y_{16} + y_2^{11}y_3^2 + y_2^5y_3^2y_{12}.\\ H^{29}(E\mathrm{VI};\mathbb{Z}_2) &= \langle y_2^{13}y_3, y_2^7y_3y_{12}, y_2^3y_3y_{20}\rangle.\end{aligned}$$

Next consider $(*)_{15}$:

$$0 \longrightarrow \langle y_2^{12}y_3, y_2^6y_3y_{12}, y_2^2y_3y_{20}\rangle \xrightarrow{y_3\cdot} H^{30}(E\mathrm{VI};\mathbb{Z}_2) \xrightarrow{p^*} \langle t_0^{11}u, t_0^9v', t_0^5x, t_0^5uv', t_0^2uw, v'w\rangle \xrightarrow{\theta} \langle y_2^8y_{12}, y_2^4y_{20}, y_{12}y_{16}, y_2^{11}y_3^2, y_2^5y_3^2y_{12}, y_2y_3^2y_{20}\rangle \xrightarrow{y_3\cdot} H^{31}(E\mathrm{VI};\mathbb{Z}_2) \longrightarrow 0.$$

Then $\theta(t_0^{11}u) = \theta(p^*(y_2^{11})u) = y_2^{11}\theta(u) = y_2^{11}y_3^2, \theta(t_0^9v') = \theta(p^*(y_2^9y_{12})) = 0, \theta(t_0^5x) = \theta(p^*(y_2^5y_{20})) = 0, \theta(t_0^5uv') = \theta(p^*(y_2^5y_{12})u) = y_2^5y_{12}\theta(u) = y_2^5y_3^2y_{12}$, $\theta(t_0^2uw) = \theta(p^*(y_2^2)uw) = y_2^2\theta(uw) = y_2y_3^2y_{20} + \gamma''y_2^{11}y_3^2 + (\delta''+1)y_2^5y_3^2y_{12}, \theta(v'w) = \theta(p^*(y_{12})w) = y_{12}\theta(w) = y_{12}y_{16}$ and therefore

$$\begin{aligned}H^{30}(E\mathrm{VI};\mathbb{Z}_2) &= \langle y_2^9y_{12}, y_2^5y_{20}, y_2^{12}y_3^2, y_2^6y_3^2y_{12}, y_2^2y_3^2y_{20}\rangle.\\ H^{31}(E\mathrm{VI};\mathbb{Z}_2) &= \langle y_2^8y_3y_{12}, y_2^4y_3y_{20}\rangle.\end{aligned}$$

From these results we can determine γ'', δ'' as follows: First we apply Sq^2 on both sides of (3.3.2). Then we have

$$(\gamma'' + \delta'' + 1)\, y_2^{11}y_3^2 = 0 \quad \text{in } H^{28}(E\mathrm{VI};\mathbb{Z}_2).$$

Hence $\delta'' = \gamma'' + 1$. Furthermore we apply Sq^4 on both sides of (3.3.2) $(\delta'' = \gamma'' + 1)$. Then using Lemma 3.12 and (3.3.3) we have

$$\gamma''y_2^6y_3^2y_{12} = 0 \quad \text{in } H^{30}(E\mathrm{VI};\mathbb{Z}_2).$$

Hence $\gamma'' = 0$. Thus there exists a relation of the form

$$y_2y_{12}^2 = y_3^2y_{20} + y_2^4y_3^2y_{12}. \tag{3.3.7}$$

Lemma 3.13. *Moreover there exists relations of the form*:
(i) $y_{12}^3 = y_{16}y_{20} + y_2^5y_3^2y_{20}$,
(ii) $y_{12}y_{16}^2 = y_2^{13}y_3^2y_{12}$,
(iii) $y_{16}^2y_{20} = y_2^{13}y_3^2y_{20}$.

Proof. Applying Sq^8 to (3.3.4), the first assertion follows. Since $p^*(y_2^{13}y_{20}) = t_0^{13}(t_0w + t_0^6u + uv') = u^2v'w + t_0^{13}uv'$,

$$0 = \theta p^*(y_2^{13}y_{20}) = \theta(u^2v'w) + \theta(t_0^{13}uv') = y_{12}y_{16}^2 + y_2^{13}y_3^2y_{12}$$

by the exactness and the second assertion follows. Applying Sq^8 on both sides of $y_{12}y_{16}^2 = y_2^{13}y_3^2y_{12}$, the last assertion follows. $\square$

Proof of Lemma 3.2. If $\theta(v') = a\chi^2$ is true, $\theta(v) = \theta(v' + t_0^2u) = a\chi^2 + a\chi^2 = 0$. Hence there exists an element $c' \in H^{12}(E\mathrm{VI};\mathbb{Z})$ such that $p^*(c') = v$. Then we can discuss $(*)_7 \sim (*)_{15}$ in the same way as above and we have elements d, e' of $H^*(E\mathrm{VI};\mathbb{Z})$ such that

$$p^*(d) = u^2,\ p^*(e') = x' = t_0w + uv.$$

Putting $y'_{12} = c' \mod 2,\ y_{16} = d \mod 2,\ y'_{20} = e' \mod 2$ we obtain

(i) $y_3^3 = 0,\ y_2y_{16} = y_3^2y'_{12}, y_3y_{16} = 0,\ y_2{y'_{12}}^2 = y_3^2y'_{20} + \gamma''y_2^{10}y_3^2 + \delta''y_2^4y_3^2y'_{12},\ y_3{y'_{12}}^2 = 0,\ y_2^{14} + y'_{12}y_{16} + y_2^2{y'_{12}}^2 + y_2^{11}y_3^2 = 0.$

(ii) $Sq^1(y'_{12}) = 0,\ Sq^2(y'_{12}) = y_2^7 + y_2y'_{12} + y_2^4y_3^2,$
$Sq^4(y'_{12}) = y_2^8 + \alpha'y_2^5y_3^2,\ Sq^8(y'_{12}) = y'_{20} + \alpha''y_2^7y_3^2 + \beta''y_2y_3^2y'_{12},$
$Sq^1(y_{16}) = 0,\ Sq^2(y_{16}) = y_2^6y_3^2,\ Sq^4(y_{16}) = y_2y_3^2y'_{12},$
$Sq^8(y_{16}) = {y'_{12}}^2 + \gamma''y_2^9y_3^2 + \delta''y_2^3y_3^2y'_{12}$

for some $\alpha', \alpha'', \beta'', \gamma'', \delta'' \in \mathbb{Z}_2$. Now we apply Sq^4 on both sides of $y_2^{14} + y'_{12}y_{16} + y_2^2{y'_{12}}^2 + y_2^{11}y_3^2 = 0$. Then using above results we obtain $y_2^{13}y_3^2 = 0$. On the other hand by $(*)_{14}$ we see that $y_2^{13}y_3 \neq 0$. Since $H^{29}(E\mathrm{VI};\mathbb{Z}_2) \xrightarrow{y_3\cdot} H^{32}(E\mathrm{VI};\mathbb{Z}_2)$ is injective we have $y_2^{13}y_3^2 \neq 0$. This is a contradiction. $\square$

4. The mod 2 cohomology ring of EVI

In this section we determine the mod 2 cohomology ring of EVI.

From Lemma 3.4 we have elements $y_i \in H^i(E\mathrm{VI};\mathbb{Z}_2)$ $(i = 2, 3, 12, 16, 20)$ such that

(4.a)(i) $p^*(y_2) = t_0,\ p^*(y_3) = 0,\ p^*(y_{12}) = v',\ p^*(y_{16}) = u^2,$
$p^*(y_{20}) = x = t_0w + t_0^6u + uv'.$

(ii) $\theta(u) = y_3^2,\ \theta(w) = y_{16},\ \theta(uw) = y_{12}^2 + y_2^3y_3^2y_{12}.$

(iii) $y_3^3 = 0,\ y_2y_{16} = y_3^2y_{12} + y_2^6y_3^2,\ y_3y_{16} = 0,\ y_2y_{12}^2 = y_3^2y_{20} + y_2^4y_3^2y_{12},$
$y_3y_{12}^2 = 0,\ y_2^{14} = y_{12}y_{16} + y_2^{11}y_3^2 + y_2^5y_3^2y_{12},\ y_{12}^3 = y_{16}y_{20} + y_2^5y_3^2y_{20},$
$y_{12}y_{16}^2 = y_2^{13}y_3^2y_{12},\ y_{16}^2y_{20} = y_2^{13}y_3^2y_{20}.$

We define the graded $\mathbb{Z}_2$-vector spaces as follows $(\deg(y_j) = j)$:

$$B_0^* = \langle y_{16},\ y_{12}^2,\ y_{16}^2,\ y_{12}y_{16},\ y_{12}^2y_{16},\ y_{16}y_{20},\ y_{12}y_{16}y_{20}\rangle,$$
$$B_1^* = \langle y_2^i,\ y_2^iy_{12},\ y_2^iy_{20},\ y_2^iy_{12}y_{20}\ (0 \le i \le 13)\rangle,$$
$$B_2^* = \langle y_2^iy_3^2,\ y_2^iy_3^2y_{12},\ y_2^iy_3^2y_{20},\ y_2^iy_3^2y_{12}y_{20}\ (0 \le i \le 13)\rangle,$$
$$B^* = B_0^* \oplus B_1^* \oplus B_2^*,$$
$$C^* = \langle y_2^iy_3,\ y_2^iy_3y_{12},\ y_2^iy_3y_{20},\ y_2^iy_3y_{12}y_{20}\ (0 \le i \le 13)\rangle.$$

Moreover define the homomorphisms

$$\begin{aligned}
&h : C^* \longrightarrow B^* \quad \text{by} \ \ h(\xi) = y_3 \cdot \xi, \quad \xi \in C^*,\\
&h' : B^* \longrightarrow C^* \quad \text{by} \ \ h'(B_0^*) = 0,\ h'(B_2^*) = 0,\ h'(\xi) = y_3 \cdot \xi, \quad \xi \in B_1^*,\\
&p^* : B^* \longrightarrow H^*(E_7/C_1; \mathbb{Z}_2) \quad \text{by}\\
&p^*(y_2) = t_0,\ p^*(y_3) = 0,\ p^*(y_{12}) = v',\ p^*(y_{16}) = u^2,\\
&p^*(y_{20}) = x = t_0 w + t_0^6 u + uv' \text{ and the multiplicativity } p^*(\xi\eta) = p^*(\xi)p^*(\eta).
\end{aligned}$$

For each monomial basis of Corollary 2.8 define

$\theta : H^*(E_7/C_1; \mathbb{Z}_2) \longrightarrow B^*$ by

(4.b)

$$\theta(t_0^i) = 0 \ (0 \le i \le 13), \quad \theta(t_0^i u) = y_2^i y_3^2 \ (0 \le i \le 13),$$

$$\theta(t_0^i v') = 0 \ (0 \le i \le 13), \quad \theta(t_0^i w) = \begin{cases} y_{16} & i = 0 \\ y_2^{i-1} y_3^2 y_{12} + y_2^{i+5} y_3^2 & 1 \le i \le 8 \\ y_2^{i-1} y_3^2 y_{12} & 9 \le i \le 13 \end{cases},$$

$$\theta(t_0^i uv') = y_2^i y_3^2 y_{12} \ (0 \le i \le 13), \quad \theta(t_0^i uw) = \begin{cases} y_{12}^2 + y_2^3 y_3^2 y_{12} & i = 0 \\ y_2^{i-1} y_3^2 y_{20} & 1 \le i \le 13 \end{cases},$$

$$\theta(t_0^i v' w) = \begin{cases} y_{12} y_{16} & i = 0 \\ y_2^{i+5} y_3^2 y_{12} & 1 \le i \le 8 \\ 0 & 9 \le i \le 13 \end{cases},$$

$$\theta(t_0^i uv' w) = \begin{cases} y_{16} y_{20} + y_2^5 y_3^2 y_{20} & i = 0 \\ y_2^{i-1} y_3^2 y_{12} y_{20} & 1 \le i \le 13 \end{cases},$$

$$\begin{aligned}
&\theta(u^2) = 0, \quad \theta(v'^2) = 0, \quad \theta(u^2 v') = 0, \quad \theta(uv'^2) = 0, \quad \theta(u^2 w) = y_{16}^2,\\
&\theta(v'^3) = 0, \quad \theta(u^2 v'^2) = 0, \quad \theta(v'^2 w) = y_{12}^2 y_{16}, \quad \theta(u^2 v' w) = y_2^{13} y_3^2 y_{12},\\
&\theta(v'^4) = 0, \quad \theta(uv'^2 w) = y_{12} y_{16} y_{20} + y_2^5 y_3^2 y_{12} y_{20}, \quad \theta(v'^3 w) = y_2^{13} y_3^2 y_{20},\\
&\theta(u^2 v'^2 w) = 0, \quad \theta(v'^4 w) = y_2^{13} y_3^2 y_{12} y_{20}.
\end{aligned}$$

Then

Lemma 4.1. *For each n, the following sequece is exact*:

$$0 \longrightarrow C^{2n-3} \xrightarrow{h} B^{2n} \xrightarrow{p^*} H^{2n}(E_7/C_1; \mathbb{Z}_2) \xrightarrow{\theta} B^{2n-2} \xrightarrow{h'} C^{2n+1} \longrightarrow 0.$$

Proof. By the definition of $h : C^* \longrightarrow B^*$, $h' : B^* \longrightarrow C^*$, we see easily that h is injective, h' is surjective and $\operatorname{Im} h = B_2^*$, $\operatorname{Ker} h' = B_0^* \oplus B_2^*$. On the other hand by the definition of θ, it is verified directly that $\operatorname{Im}\theta = B_0^* \oplus B_2^*$

and therefore $\operatorname{Im}\theta = \operatorname{Ker} h'$ and $\operatorname{Ker}\theta$ has a basis

$$t_0^i \ (0 \le i \le 13), \quad t_0^i v' \ (0 \le i \le 13), \quad \begin{cases} t_0^i v'w + t_0^{i+5} uv' & 1 \le i \le 8 \\ t_0^i v'w & 9 \le i \le 13 \end{cases},$$

$$\begin{cases} t_0^i w + t_0^{i+5} u + t_0^{i-1} uv' & 1 \le i \le 8 \\ t_0^i w + t_0^{i-1} uv' & 9 \le i \le 13 \end{cases}, \quad u^2 v' w + t_0^{13} uv',$$

$$u^2,\ v'^2,\ u^2 v',\ uv'^2,\ v'^3,\ u^2 v'^2,\ v'^4,\ u^2 v'^2 w.$$

Then considering the image of $B_0^* \oplus B_1^*$ under p^*, we see that $B_0^* \oplus B_1^*$ is mapped isomorphically onto $\operatorname{Ker}\theta$. Thus the exactness of the sequence is proved. □

Theorem 4.2. *An additive basis of $H^*(E\mathrm{VI}; \mathbb{Z}_2)$ as a $\mathbb{Z}_2$-vector space is given as follows*:

$$\left\{ \begin{array}{l} y_2^i,\ y_2^i y_{12},\ y_2^i y_{20},\ y_2^i y_{12} y_{20}, \\ y_2^i y_3,\ y_2^i y_3 y_{12},\ y_2^i y_3 y_{20},\ y_2^i y_3 y_{12} y_{20}, \\ y_2^i y_3^2,\ y_2^i y_3^2 y_{12},\ y_2^i y_3^2 y_{20},\ y_2^i y_3^2 y_{12} y_{20} \ (0 \le i \le 13), \\ y_{16},\ y_{12}^2,\ y_{16}^2,\ y_{12} y_{16},\ y_{12}^2 y_{16},\ y_{16} y_{20},\ y_{12} y_{16} y_{20} \end{array} \right\}.$$

Proof. We prove that the natural maps

$$f_n : B^{2n} \longrightarrow H^{2n}(E\mathrm{VI}; \mathbb{Z}_2), \quad g_n : C^{2n+1} \longrightarrow H^{2n+1}(E\mathrm{VI}; \mathbb{Z}_2)$$

are isomorphisms by induction on n. In view of Lemma 4.1 and $(*)_n$, it is sufficient to prove that the formulae for (4.b) is still valid for $\theta : H^{2n}(E_7/C_1; \mathbb{Z}_2) \longrightarrow H^{2n-2}(E\mathrm{VI}; \mathbb{Z}_2)$ under the inductive hypothesis on $H^{2n-2}(E\mathrm{VI}; \mathbb{Z}_2)$. This can be done using (4.a) and the property $\theta(p^*(x)y) = x\theta(y)$. □

In order to determine the ring structure of $H^*(E\mathrm{VI}; \mathbb{Z}_2)$, we consider another relations between $y_2, y_3, y_{12}, y_{16}, y_{20}$.

Lemma 4.3. *There exists relations of the form*
(i) $y_{20}^2 = y_{12}^2 y_{16} + y_2^{11} y_3^2 y_{12}$,
(ii) $y_{12}^2 y_{20} = y_2^{13} y_3^2 y_{12} + y_2^3 y_3^2 y_{12} y_{20}$,
(iii) $y_{16}^3 = y_{12} y_{16} y_{20} + y_2^5 y_3^2 y_{12} y_{20}$.

Proof. Since $p^*(y_{20}^2 + y_{12}^2 y_{16}) = 0$, we may put

$$y_{20}^2 = y_{12}^2 y_{16} + p y_2^{11} y_3^2 y_{12} + q y_2^7 y_3^2 y_{20} + r y_2 y_3^2 y_{12} y_{20} \tag{4.4.1}$$

for some $p, q, r \in \mathbb{Z}_2$. First we apply Sq^2 on both sides of (4.4.1). Then

$$0 = r(y_2^8 y_3^2 y_{20} + y_2^{12} y_3^2 y_{12} + y_2^2 y_3^2 y_{12} y_{20}) \quad \text{in } H^{42}(E\mathrm{VI}; \mathbb{Z}_2).$$

Hence $r = 0$ by Theorem 4.2. Next we apply Sq^4 on both sides of (4.4.1) $(r = 0)$. Then using Lemma 3.13 we have

$$(p + q + 1)\ y_2^{13} y_3^2 y_{12} = 0 \quad \text{in } H^{44}(E\mathrm{VI}; \mathbb{Z}_2).$$

Hence $q = p + 1$. Furthermore we apply Sq^8, we have

$$(p+1)\, y_2^{11} y_3^2 y_{20} = 0 \quad \text{in } H^{48}(E\mathrm{VI}; \mathbb{Z}_2).$$

Hence $p = 1$ and the first assertion follows. Since $p^*(y_{12}^2 y_{20}) = 0$, we may put

$$y_{12}^2 y_{20} = p' y_2^{13} y_3^2 y_{12} + q' y_2^9 y_3^2 y_{20} + r' y_2^3 y_3^2 y_{12} y_{20} \tag{4.4.2}$$

for some $p', q', r' \in \mathbb{Z}_2$. Multiplying by y_2 on both sides of (4.4.2), we obtain

$$y_2^4 y_3^2 y_{12} y_{20} = q' y_2^{10} y_3^2 y_{20} + r' y_2^4 y_3^2 y_{12} y_{20} \quad \text{in } H^{46}(E\mathrm{VI}; \mathbb{Z}_2).$$

Hence $q' = 0, r' = 1$. Furthremore multiplying by y_{20}, we obtain

$$y_2^{13} y_3^2 y_{12} y_{20} = p' y_2^{13} y_3^2 y_{12} y_{20} \quad \text{in } H^{64}(E\mathrm{VI}; \mathbb{Z}_2).$$

Hence $p' = 1$ and the second assertion follows. Similarly since $p^*(y_{16}^3 + y_{12} y_{16} y_{20}) = 0$, we may put

$$y_{16}^3 = y_{12} y_{16} y_{20} + p'' y_2^5 y_3^2 y_{12} y_{20} + q'' y_2^{11} y_3^2 y_{20} \tag{4.4.3}$$

for some $p'', q'' \in \mathbb{Z}_2$. Multiplying by y_2 on both sides of (4.4.3), we obtain

$$0 = (p''+1) y_2^6 y_3^2 y_{12} y_{20} + q'' y_2^{12} y_3^2 y_{20} \quad \text{in } H^{50}(E\mathrm{VI}; \mathbb{Z}_2).$$

Hence $p'' = 1, q'' = 0$ and the last assertion follows. □

Theorem 4.4. *The mod 2 cohomology ring of EVI is given as follows*:

$$H^*(E\mathrm{VI}; \mathbb{Z}_2) = \mathbb{Z}_2[y_2, y_3, y_{12}, y_{16}, y_{20}]/J$$

for the ideal

$$J = \left(\begin{array}{l} y_3^3,\ y_2 y_{16} + y_3^2 y_{12} + y_2^6 y_3^2,\ y_3 y_{16},\ y_2 y_{12}^2 + y_3^2 y_{20} + y_2^4 y_3^2 y_{12}, \\ y_3 y_{12}^2,\ y_2^{14} + y_{12} y_{16} + y_2^{11} y_3^2 + y_2^5 y_3^2 y_{12},\ y_{12}^3 + y_{16} y_{20} + y_2^5 y_3^2 y_{20}, \\ y_{20}^2 + y_{12}^2 y_{16} + y_2^{11} y_3^2 y_{12},\ y_{12}^2 y_{20} + y_2^{13} y_3^2 y_{12} + y_2^3 y_3^2 y_{12} y_{20}, \\ y_{12} y_{16}^2 + y_2^{13} y_3^2 y_{12},\ y_{16}^3 + y_{12} y_{16} y_{20} + y_2^5 y_3^2 y_{12} y_{20},\ y_{16}^2 y_{20} + y_2^{13} y_3^2 y_{20} \end{array} \right).$$

Proof. By the previous arguments we see that J vanishes in $H^*(E\mathrm{VI}; \mathbb{Z}_2)$. By use of the relations in J, we see that every monomial in $y_2, y_3, y_{12}, y_{16}, y_{20}$ is a linear combination of the basis in Theorem 4.2. Thus Theorem 4.4 is established. □

Finally we comment the additive structure of $H^*(E\mathrm{VI}; \mathbb{Z})$. Using Lemma 3.5 and Theorem 4.2 we see that

$$\operatorname{Im} Sq^1 = \left\langle \begin{array}{l} y_2^{2i} y_3,\ y_2^{2i} y_3 y_{12},\ y_2^{2i} y_3 y_{20},\ y_2^{2i} y_3 y_{12} y_{20}, \\ y_2^{2i} y_3^2,\ y_2^{2i} y_3^2 y_{12},\ y_2^{2i} y_3^2 y_{20},\ y_2^{2i} y_3^2 y_{12} y_{20}\ (0 \le i \le 6) \end{array} \right\rangle$$

as a $\mathbb{Z}_2$-vector space. Because Sq^1 is the mod 2 Bockstein homomorphism and $\mathrm{Tor} H^*(E\mathrm{VI}; \mathbb{Z})$ consists of elements of order 2 we deduce

Proposition 4.5. *The mod 2 reduction* $\rho : H^*(E\mathrm{VI};\mathbb{Z}) \longrightarrow H^*(E\mathrm{VI};\mathbb{Z}_2)$ *maps* $\mathrm{Tor}\, H^*(E\mathrm{VI};\mathbb{Z})$ *isomorphically onto* $\mathrm{Im}\, Sq^1$.

Using this proposition and the results of $H^*(E\mathrm{VI};\mathbb{Q})$ the additive structure of $H^*(E\mathrm{VI};\mathbb{Z})$ can be completely determined.

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