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3-graded decompositions of exceptional Lie algebras $\mathfrak{g}$ and group realizations of $\mathfrak{g}_{ev}, \mathfrak{g}_0$ and $\mathfrak{g}_{ed}$ Part I, $G = G_2, F_4, E_6$

By

Ichiro Yokota

The ν-graded decomposition of simple Lie algebras $\mathfrak{g}$, $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k, [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, has been studied by many mathematicians. Firstly the case of $\nu = 1$ was studied by S. Kobayashi–T. Nagano [4]. The case of $\nu = 2$, S. Kaneyuki [3] classified and determined the types of subalgebras $\mathfrak{g}_{ev}, \mathfrak{g}_0$ of $\mathfrak{g}$ and in the exceptional case, S. Gomyo [1] gave explicit realization of each $\mathfrak{g}_k$, I. Yokota [8], [9], [10] gave group realization of $\mathfrak{g}_{ev}, \mathfrak{g}_0$. Now, recently M. Hara [2] classified the 3-graded decomposition of simple Lie algebras $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

and determined the types of subalgebras $\mathfrak{g}_{ev} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2, \mathfrak{g}_0$ and $\mathfrak{g}_{ed} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_3$ of $\mathfrak{g}$. The following table is the results of $\mathfrak{g}_{ev}, \mathfrak{g}_0, \mathfrak{g}_{ed}$ for the exceptional Lie algebras $\mathfrak{g}$ of type G_2, F_4 and E_6.

$\mathfrak{g}$	$\dim \mathfrak{g}_1, \dim \mathfrak{g}_2, \dim \mathfrak{g}_3$ $\mathfrak{g}_0$	$\mathfrak{g}_{ev}$ $\mathfrak{g}_{ed}$
${\mathfrak{g}_2}^C$	2, 1, 2 $C \oplus \mathfrak{sl}(2, C)$	$\mathfrak{sl}(2, C) \oplus \mathfrak{sl}(2, C)$ $\mathfrak{sl}(3, C)$
$\mathfrak{g}_{2(2)}$	2, 1, 2 $\boldsymbol{R} \oplus \mathfrak{sl}(2, \boldsymbol{R})$	$\mathfrak{sl}(2, \boldsymbol{R}) \oplus \mathfrak{sl}(2, \boldsymbol{R})$ $\mathfrak{sl}(3, \boldsymbol{R})$
${\mathfrak{f}_4}^C$	12, 6, 2 $C \oplus \mathfrak{sl}(2, C) \oplus \mathfrak{sl}(3, C)$	$\mathfrak{sl}(2, C) \oplus \mathfrak{sp}(3, C)$ $\mathfrak{sl}(3, C) \oplus \mathfrak{sl}(3, C)$
$\mathfrak{f}_{4(4)}$	12, 6, 2 $\boldsymbol{R} \oplus \mathfrak{sl}(2, \boldsymbol{R}) \oplus \mathfrak{sl}(3, \boldsymbol{R})$	$\mathfrak{sl}(2, \boldsymbol{R}) \oplus \mathfrak{sp}(3, \boldsymbol{R})$ $\mathfrak{sl}(3, \boldsymbol{R}) \oplus \mathfrak{sl}(3, \boldsymbol{R})$
${\mathfrak{e}_6}^C$	18, 9, 2 $C \oplus \mathfrak{sl}(2, C) \oplus \mathfrak{sl}(3, C) \oplus \mathfrak{sl}(3, C)$	$\mathfrak{sl}(2, C) \oplus \mathfrak{sl}(6, C)$ $\mathfrak{sl}(3, C) \oplus \mathfrak{sl}(3, C) \oplus \mathfrak{sl}(3, C)$
$\mathfrak{e}_{6(6)}$	18, 9, 2 $\boldsymbol{R} \oplus \mathfrak{sl}(2, \boldsymbol{R}) \oplus \mathfrak{sl}(3, \boldsymbol{R}) \oplus \mathfrak{sl}(3, \boldsymbol{R})$	$\mathfrak{sl}(2, \boldsymbol{R}) \oplus \mathfrak{sl}(6, \boldsymbol{R})$ $\mathfrak{sl}(3, \boldsymbol{R}) \oplus \mathfrak{sl}(3, \boldsymbol{R}) \oplus \mathfrak{sl}(3, \boldsymbol{R})$

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$\mathfrak{e}_{6(2)}$	$18, \quad 9, \quad 2$	$\mathfrak{sl}(2,\boldsymbol{R}) \oplus \mathfrak{su}(3,3)$
	$\boldsymbol{R} \oplus \mathfrak{sl}(2,\boldsymbol{R}) \oplus \mathfrak{sl}(3,C)$	$\mathfrak{sl}(3,\boldsymbol{R}) \oplus \mathfrak{sl}(3,C)$
${\mathfrak{e}_6}^C$	$16, \quad 9, \quad 4$	$\mathfrak{sl}(2,C) \oplus \mathfrak{sl}(6,C)$
	$C \oplus C \oplus \mathfrak{sl}(2,C) \oplus \mathfrak{sl}(4,C)$	$C \oplus \mathfrak{sl}(2,C) \oplus \mathfrak{sl}(5,C)$
$\mathfrak{e}_{6(6)}$	$16, \quad 9, \quad 4$	$\mathfrak{sl}(2,\boldsymbol{R}) \oplus \mathfrak{sl}(6,\boldsymbol{R})$
	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{sl}(2,\boldsymbol{R}) \oplus \mathfrak{sl}(4,\boldsymbol{R})$	$\boldsymbol{R} \oplus \mathfrak{sl}(2,\boldsymbol{R}) \oplus \mathfrak{sl}(5,\boldsymbol{R})$
${\mathfrak{e}_6}^C$	$15, \quad 10, \quad 1$	$C \oplus \mathfrak{so}(10,C)$
	$C \oplus C \oplus \mathfrak{sl}(5,C)$	$C \oplus \mathfrak{sl}(2,C) \oplus \mathfrak{sl}(5,C)$
$\mathfrak{e}_{6(6)}$	$15, \quad 10, \quad 1$	$\boldsymbol{R} \oplus \mathfrak{so}(5,5)$
	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{sl}(5,\boldsymbol{R})$	$\boldsymbol{R} \oplus \mathfrak{sl}(2,\boldsymbol{R}) \oplus \mathfrak{sl}(5,\boldsymbol{R})$
${\mathfrak{e}_6}^C$	$11, \quad 10, \quad 5$	$C \oplus \mathfrak{so}(10,C)$
	$C \oplus C \oplus \mathfrak{sl}(5,C)$	$C \oplus \mathfrak{sl}(6,C)$
$\mathfrak{e}_{6(6)}$	$11, \quad 10, \quad 5$	$\boldsymbol{R} \oplus \mathfrak{so}(5,5)$
	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{sl}(5,\boldsymbol{R})$	$\boldsymbol{R} \oplus \mathfrak{sl}(6,\boldsymbol{R})$
${\mathfrak{e}_6}^C$	$8, \quad 8, \quad 8$	$C \oplus \mathfrak{so}(10,C)$
	$C \oplus C \oplus \mathfrak{so}(8,C)$	$C \oplus \mathfrak{so}(10,C)$
$\mathfrak{e}_{6(6)}$	$8, \quad 8, \quad 8$	$\boldsymbol{R} \oplus \mathfrak{so}(5,5)$
	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{so}(4,4)$	$\boldsymbol{R} \oplus \mathfrak{so}(5,5)$
$\mathfrak{e}_{6(-26)}$	$8, \quad 8, \quad 8$	$\boldsymbol{R} \oplus \mathfrak{so}(1,9)$
	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \mathfrak{so}(8)$	$\boldsymbol{R} \oplus \mathfrak{so}(1,9)$

Now, for the exceptional Lie groups G of type G_2, F_4 and E_6, we realize the subgroups G_{ev}, G_0, G_{ed} of G corresponding to the subalgebras $\mathfrak{g}_{ev}, \mathfrak{g}_0, \mathfrak{g}_{ed}$ of $\mathfrak{g} = \mathrm{Lie}G$. Our results are as follows.

$G \quad \dim\mathfrak{g}_1, \dim\mathfrak{g}_2, \dim\mathfrak{g}_3$	G_{ev}
G_0	G_{ed}
${G_2}^C \quad 2, \quad 1, \quad 2$	$(Sp(1,C) \times Sp(1,C))/\boldsymbol{Z}_2$
$(Sp(1,C) \times C^*)/\boldsymbol{Z}_2$	$SL(3,C)$
$G_{2(2)} \quad 2, \quad 1, \quad 2$	$(Sp(1,\boldsymbol{R}) \times Sp(1,\boldsymbol{R}))/\boldsymbol{Z}_2 \times 2$
$(Sp(1,\boldsymbol{R}) \times \boldsymbol{R}^+) \times 2$	$SL(3,\boldsymbol{R})$
${F_4}^C \quad 12, \quad 6, \quad 2$	$(Sp(1,C) \times Sp(3,C))/\boldsymbol{Z}_2$
$(Sp(1,C) \times C^* \times SL(3,C))/\boldsymbol{Z}_6$	$(SL(3,C) \times SL(3,C))/\boldsymbol{Z}_3$
$F_{4(4)} \quad 12, \quad 6, \quad 2$	$(Sp(1,\boldsymbol{R}) \times Sp(3,\boldsymbol{R}))/\boldsymbol{Z}_2 \times 2$
$(Sp(1,\boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(3,\boldsymbol{R})) \times 2$	$(SL(3,\boldsymbol{R}) \times SL(3,\boldsymbol{R})) \times 3$
${E_6}^C \quad 18, \quad 9, \quad 2$	$(Sp(1,C) \times SL(6,C))/\boldsymbol{Z}_2$
$(Sp(1,C) \times C^* \times SL(3,C) \times SL(3,C))/\boldsymbol{Z}_6$	$(SL(3,C) \times SL(3,C) \times SL(3,C))/\boldsymbol{Z}_3$
$E_{6(6)} \quad 18, \quad 9, \quad 2$	$(Sp(1,\boldsymbol{R}) \times SL(6,\boldsymbol{R}))/\boldsymbol{Z}_2 \times 2$
$(Sp(1,\boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(3,\boldsymbol{R}) \times SL(3,\boldsymbol{R})) \times 2$	$(SL(3,\boldsymbol{R}) \times SL(3,\boldsymbol{R}) \times SL(3,\boldsymbol{R})) \times 3$
$E_{6(2)} \quad 18, \quad 9, \quad 2$	$(Sp(1,\boldsymbol{R}) \times SU(3,3))/\boldsymbol{Z}_2 \times 2$
$(Sp(1,\boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(3,C)) \times 2$	$SL(3,\boldsymbol{R}) \times SL(3,C)$
${E_6}^C \quad 16, \quad 9, \quad 4$	$(Sp(1,C) \times SL(6,C))/\boldsymbol{Z}_2$
$(C^* \times C^* \times SL(2,C) \times SL(4,C))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$	$(Sp(1,C) \times C^* \times SL(5,C))/\boldsymbol{Z}_2$

$E_{6(6)}$ 16, 9, 4	$(Sp(1,\boldsymbol{R}) \times SL(6,\boldsymbol{R}))/\boldsymbol{Z}_2 \times 2$
$(\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times SL(2,\boldsymbol{R}) \times SL(4,\boldsymbol{R})) \times 2$	$(Sp(1,\boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(5,\boldsymbol{R})) \times 2$
${E_6}^C$ 15, 10, 1	$(C^* \times Spin(10,C))/\boldsymbol{Z}_4$
$(C^* \times C^* \times SL(5,C))/\boldsymbol{Z}_2$	$(Sp(1,C) \times C^* \times SL(5,C))/\boldsymbol{Z}_2$
$E_{6(6)}$ 15, 10, 1	$(\boldsymbol{R}^+ \times spin(5,5)) \times 2$
$(\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times SL(5,\boldsymbol{R})) \times 2$	$(Sp(1,\boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(5,\boldsymbol{R})) \times 2$
${E_6}^C$ 11, 10, 5	$(C^* \times Spin(10,C))/\boldsymbol{Z}_4$
$(C^* \times C^* \times SL(5,C))/\boldsymbol{Z}_2$	$(C^* \times SL(6,C))/\boldsymbol{Z}_2$
$E_{6(6)}$ 11, 10, 5	$(\boldsymbol{R}^+ \times spin(5,5)) \times 2$
$(\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times SL(5,\boldsymbol{R})) \times 2$	$(\boldsymbol{R}^+ \times SL(6,\boldsymbol{R})) \times 2$
${E_6}^C$ 8, 8, 8	$(C^* \times Spin(10,C))/\boldsymbol{Z}_4$
$(C^* \times C^* \times Spin(8,C))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_4)$	$(C^* \times Spin(10,C))/\boldsymbol{Z}_4$
$E_{6(6)}$ 8, 8, 8	$(\boldsymbol{R}^+ \times spin(5,5)) \times 2$
$(\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times spin(4,4)) \times 2^2$	$(\boldsymbol{R}^+ \times spin(5,5)) \times 2$
$E_{6(-26)}$ 8, 8, 8	$\boldsymbol{R}^+ \times Spin(9,1)$
$(\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times Spin(8)) \times 2^2$	$\boldsymbol{R}^+ \times Spin(9,1)$

1. Group G_2

1.1. Lie groups of type G_2 and some subgroups of ${G_2}^C$

We use the same notations and definitions as in [8]. For example,
the Cayley algebra $\mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H}e_4$ and algebras $\boldsymbol{C}', \boldsymbol{H}'$,
the groups ${G_2}^C = \{\alpha \in \mathrm{Iso}_C(\mathfrak{C}^C) \,|\, \alpha(xy) = (\alpha x)(\alpha y)\}, G_2$ and $G_{2(2)}$,
the involutive automorphisms $\gamma, \gamma_1, \gamma_2$ of G_2 and $G_{2(2)} = ({G_2}^C)^{\tau\gamma_1}$,
the Lie algebra $\mathfrak{so}(8) = \mathfrak{so}(\mathfrak{C})$ of the group $SO(8) = SO(\mathfrak{C})$, elements G_{kl} of $\mathfrak{so}(8)$ and the Lie algebra ${\mathfrak{g}_2}^C$ of the group ${G_2}^C$,
group isomorphisms $Sp(n, \boldsymbol{H}^C) \cong Sp(n, C), SU(n, \boldsymbol{C}^C) \cong SL(n, C)$, $SU(n, \boldsymbol{C}') \cong SL(n, \boldsymbol{R}), U(1, \boldsymbol{C}^C) \cong C^*, U(1, \boldsymbol{C}') \cong \boldsymbol{R}^*$ etc.

We shall review and add some notations and definitions. The Cayley algebra $\mathfrak{C}$ naturally contains the field $\boldsymbol{C}$ of complex numbers as $\boldsymbol{C} = \{x + ye_1 \,|\, x, y \in \boldsymbol{R}\}$. Now, to an element

$$x = a + m_1e_2 + m_2e_4 + m_3e_6, \quad a, m_1, m_2, m_3 \in \boldsymbol{C}$$

of $\mathfrak{C}$, we associate an element

$$a + \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

of the algebra $\boldsymbol{C} \oplus \boldsymbol{C}^3$ with the multiplication

$$(a + \boldsymbol{m})(b + \boldsymbol{n}) = (ab - \langle \boldsymbol{m}, \boldsymbol{n} \rangle) + (a\boldsymbol{n} + \overline{b}\boldsymbol{m} - \overline{\boldsymbol{m} \times \boldsymbol{n}}),$$

where $\langle \boldsymbol{m}, \boldsymbol{n} \rangle = {}^t\boldsymbol{m}\overline{\boldsymbol{n}}$ and $\boldsymbol{m} \times \boldsymbol{n}$ is the exterior product of $\boldsymbol{m}, \boldsymbol{n}$. Note that $\boldsymbol{C} \oplus \boldsymbol{C}^3$ is a left $\boldsymbol{C}$-module. Hereafter we identify $\mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H}e_4$ and $\boldsymbol{C} \oplus \boldsymbol{C}^3$.

We define $\varphi : Sp(1) \times Sp(1) \to G_2$ and $\psi : SU(3) \to G_2$ by

$$\varphi(p,q)(m+ne_4) = qm\overline{q} + (pn\overline{q})e_4, \qquad m+ne_4 \in \boldsymbol{H} \oplus \boldsymbol{H}e_4 = \mathfrak{C},$$
$$\psi(P)(a+\boldsymbol{m}) = a + P\boldsymbol{m}, \qquad a+\boldsymbol{m} \in \boldsymbol{C} \oplus \boldsymbol{C}^3 = \mathfrak{C},$$

respectively. Then for the induced mappings $\varphi_* : \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \to \mathfrak{g}_2$ of φ and $\psi_* : \mathfrak{su}(3) \to \mathfrak{g}_2$ of ψ, we have

$$\varphi_*(e_1, 0) = -G_{45} + G_{67}, \qquad \varphi_*(0, e_1) = -2G_{23} + G_{45} + G_{67},$$
$$\psi_*(\mathrm{diag}(e_1, -e_1, 0)) = -G_{23} + G_{45}, \quad \psi_*(\mathrm{diag}(0, e_1, -e_1)) = -G_{45} + G_{67}.$$

Now, we define $\boldsymbol{R}$-linear transformations γ, δ_4 and w_3 of $\mathfrak{C}$ by

$$\gamma = \varphi(1, -1), \quad \delta_4 = \varphi(1, -e_1), \quad w_3 = \psi(\mathrm{diag}(\omega_1, \omega_1, \omega_1)),$$

where $\omega_1 = -(1/2) + (\sqrt{3}/2)e_1 \in \boldsymbol{C} \subset \boldsymbol{H} \subset \mathfrak{C}$. The explicit forms of γ, δ_4 and w_3 are

$$\left.\begin{array}{l} \gamma(m+ne_4) = m - ne_4, \\ \delta_4(m+ne_4) = -e_1 m e_1 + (ne_1)e_4, \end{array}\right. \quad m+ne_4 \in \boldsymbol{H} \oplus \boldsymbol{H}e_4,$$
$$\left.\begin{array}{l} w_3(a+\boldsymbol{m}) = a + \omega_1\boldsymbol{m}, \\ \delta_4(a+\boldsymbol{m}) = a + D_4\boldsymbol{m}, \end{array}\right. \quad a+\boldsymbol{m} \in \boldsymbol{C} \oplus \boldsymbol{C}^3,$$

where $D_4 = \mathrm{diag}(-1, e_1, e_1) \in SU(3)$. Then $\gamma, \delta_4, w_3 \in G_2 \subset {G_2}^C$ and $\gamma^2 = 1, {\delta_4}^4 = 1, {w_3}^3 = 1$.

1.2. Subgroups of type ${\boldsymbol{C}_1}^C \oplus {\boldsymbol{C}_1}^C, {\boldsymbol{C}_1}^C \oplus \boldsymbol{C}$ and ${\boldsymbol{A}_2}^C$ of ${\boldsymbol{G}_2}^C$

In the Lie algebra ${\mathfrak{g}_2}^C = \mathrm{Lie}{G_2}^C$, let

$$Z = i(-2G_{23} + G_{45} + G_{67}).$$

Theorem 1.1. *The* 3*-graded decomposition of* $\mathfrak{g}_{2(2)} = ({\mathfrak{g}_2}^C)^{\tau\gamma_1}$ (*or* ${\mathfrak{g}_2}^C$),

$$\mathfrak{g}_{2(2)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\mathrm{ad}Z, Z = i(-2G_{23} + G_{45} + G_{67})$, *is given by*

$$\mathfrak{g}_0 = \{i(2G_{23} - G_{45} - G_{67}), i(G_{45} - G_{67}), G_{46} + G_{57}, i(G_{47} - G_{56})\}\, 4$$
$$\mathfrak{g}_{-1} = \left\{\begin{array}{l} (2G_{15} + G_{26} - G_{37}) - i(2G_{14} + G_{27} + G_{36}), \\ (2G_{17} - G_{24} + G_{35}) - i(2G_{16} - G_{25} - G_{34}) \end{array}\right\} \ 2$$
$$\mathfrak{g}_{-2} = \{(-2G_{13} + G_{46} - G_{57}) - i(2G_{12} - G_{47} - G_{56})\} 1$$
$$\mathfrak{g}_{-3} = \{(G_{24} + G_{35}) + i(G_{25} - G_{34}), (G_{26} + G_{37}) + i(G_{27} - G_{36})\} 2$$
$$\mathfrak{g}_1 = \tau(\mathfrak{g}_{-1})\tau, \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau, \quad \mathfrak{g}_3 = \tau(\mathfrak{g}_{-3})\tau.$$

Proof. We can prove this theorem in a way similar to [8] Theorem 1.6, using [8] Lemmas 1.2 and 1.5. □

Since $iZ = 2G_{23} - G_{45} - G_{67} = \varphi_*(0, -e_1) = \psi_*(\mathrm{diag}(-2e_1, e_1, e_1))$, we have

$$z_2 = \exp\frac{2\pi i}{2}Z = \gamma, \quad z_4 = \exp\frac{2\pi i}{4}Z = \delta_4, \quad z_3 = \exp\frac{2\pi i}{3}Z = w_3.$$

Now, since $({\mathfrak{g}_2}^C)_{ev} = ({\mathfrak{g}_2}^C)^{z_2}, ({\mathfrak{g}_2}^C)_0 = ({\mathfrak{g}_2}^C)^{z_4}, ({\mathfrak{g}_2}^C)_{ed} = ({\mathfrak{g}_2}^C)^{z_3}$, we shall determine the group structures of

$$({G_2}^C)_{ev} = ({G_2}^C)^{z_2}, \quad ({G_2}^C)_0 = ({G_2}^C)^{z_4}, \quad ({G_2}^C)_{ed} = ({G_2}^C)^{z_3}.$$

Theorem 1.2. (1) $({G_2}^C)_{ev} \cong (Sp(1, C) \times Sp(1, C))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1, 1), (-1, -1)\}$.
(2) $({G_2}^C)_0 \cong (Sp(1, C) \times C^*)/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1, 1), (-1, -1)\}$.
(3) $({G_2}^C)_{ed} \cong SL(3, C)$.

Proof. (1) We define $\varphi : Sp(1, \boldsymbol{H}^C) \times Sp(1, \boldsymbol{H}^C) \to ({G_2}^C)_{ev} = ({G_2}^C)^{z_2} = ({G_2}^C)^{\gamma}$ by

$$\varphi(p, q)(m + ne) = qm\overline{q} + (pn\overline{q})e_4, \quad m + ne \in \boldsymbol{H}^C \oplus \boldsymbol{H}^C e_4 = \mathfrak{C}^C.$$

Then φ is well-defined, is a homomorphism and $\mathrm{Ker}\varphi = \boldsymbol{Z}_2$. Since $({G_2}^C)^{\gamma}$ is connected and $\dim_C(\mathfrak{sp}(1, \boldsymbol{H}^C) \oplus \mathfrak{sp}(1, \boldsymbol{H}^C)) = 3 + 3 = 6 = 4 + 1 \times 2 = \dim_C(({\mathfrak{g}_2}^C)_{ev})$ (Theorem 1.1), φ is onto. Therefore $({G_2}^C)_{ev} \cong (Sp(1, \boldsymbol{H}^C) \times Sp(1, \boldsymbol{H}^C))/\boldsymbol{Z}_2 \cong (Sp(1, C) \times Sp(1, C))/\boldsymbol{Z}_2$.
(2) The restriction mapping $\varphi : Sp(1, \boldsymbol{H}^C) \times U(1, \boldsymbol{C}^C) \to ({G_2}^C)_0 = ({G_2}^C)^{z_4} = ({G_2}^C)^{\delta_4}$ of φ of (1) above is well-defined and $\mathrm{Ker}\varphi = \boldsymbol{Z}_2$. Since $({G_2}^C)^{\delta_4}$ is connected and $\dim_C(\mathfrak{sp}(1, \boldsymbol{H}^C) \oplus \mathfrak{u}(1, \boldsymbol{C}^C)) = 3 + 1 = 4 = \dim_C \times (({\mathfrak{g}_2}^C)_0)$ (Theorem 1.1), φ is onto. Therefore $({G_2}^C)_0 \cong (Sp(1, \boldsymbol{H}^C) \times U(1, \boldsymbol{C}^C))/\boldsymbol{Z}_2 \cong (Sp(1, C) \times C^*)/\boldsymbol{Z}_2$.
(3) We define $\psi : SU(3, \boldsymbol{C}^C) \to ({G_2}^C)_{ed} = ({G_2}^C)^{z_3} = ({G_2}^C)^{w_3}$ by

$$\psi(P)(a + \boldsymbol{m}) = a + P\boldsymbol{m}, \quad a + \boldsymbol{m} \in \boldsymbol{C}^C \oplus (\boldsymbol{C}^C)^3 = \mathfrak{C}^C.$$

Then ψ is well-defined, is a homomorphism and one-to-one. Since $({G_2}^C)^{w_3}$ is connected and $\dim_C(\mathfrak{su}(3, \boldsymbol{C}^C)) = 8 = 4 + 2 \times 2 = \dim_C(({\mathfrak{g}_2}^C)_{ed})$ (Theorem 1.1), ψ is onto. Therefore $({G_2}^C)_{ed} \cong SU(3, \boldsymbol{C}^C) \cong SL(3, C)$. □

1.2.1. Subgroups of type $\boldsymbol{C}_{1(1)} \oplus \boldsymbol{C}_{1(1)}, \boldsymbol{C}_{1(1)} \oplus \boldsymbol{R}$ and $\boldsymbol{A}_{2(2)}$ of $\boldsymbol{G}_{2(2)}$

We use the same notations as in 1.2. Since $(\mathfrak{g}_{2(2)})_{ev} = ({\mathfrak{g}_2}^C)^{z_2} \cap ({\mathfrak{g}_2}^C)^{\tau\gamma_1}$, $(\mathfrak{g}_{2(2)})_0 = ({\mathfrak{g}_2}^C)^{z_4} \cap ({\mathfrak{g}_2}^C)^{\tau\gamma_1}, (\mathfrak{g}_{2(2)})_{ed} = ({\mathfrak{g}_2}^C)^{z_3} \cap ({\mathfrak{g}_2}^C)^{\tau\gamma_1}$, we shall determine the group structures of

$$(G_{2(2)})_{ev} = ({G_2}^C)^{z_2} \cap ({G_2}^C)^{\tau\gamma_1}, \quad (G_{2(2)})_0 = ({G_2}^C)^{z_4} \cap ({G_2}^C)^{\tau\gamma_1},$$
$$(G_{2(2)})_{ed} = ({G_2}^C)^{z_3} \cap ({G_2}^C)^{\tau\gamma_1}.$$

Theorem 1.3. (1) $(G_{2(2)})_{ev} \cong (Sp(1,\boldsymbol{R}) \times Sp(1,\boldsymbol{R}))/\boldsymbol{Z}_2 \times \{1,\gamma_2\}, \boldsymbol{Z}_2 = \{(1,1),(-1,-1)\}$.

(2) $(G_{2(2)})_0 \cong (Sp(1,\boldsymbol{R}) \times \boldsymbol{R}^+) \times \{1,\gamma_2\}$.

(3) $(G_{2(2)})_{ed} \cong SL(3,\boldsymbol{R})$.

Proof. (1) For $\alpha \in (G_{2(2)})_{ev} \subset ({G_2}^C)^\gamma$, there exist $p, q \in Sp(1,\boldsymbol{H}^C)$ such that $\alpha = \varphi(p,q)$ (Theorem 1.2 (1)). From $\gamma_1\tau\alpha\tau\gamma_1 = \alpha$, we have $\varphi(\gamma_1\tau p, \gamma_1\tau q) = \varphi(p,q)$ ([8], Lemma 1.8 (2)). Hence

$$\gamma_1\tau p = p,\ \gamma_1\tau q = q \quad \text{or} \quad \gamma_1\tau p = -p,\ \gamma_1\tau q = -q.$$

In the former case, $p, q \in Sp(1,\boldsymbol{H}')$. Hence the group of the former case is $(Sp(1,\boldsymbol{H}') \times Sp(1,\boldsymbol{H}'))/\boldsymbol{Z}_2 \cong (Sp(1,\boldsymbol{R}) \times Sp(1,\boldsymbol{R}))/\boldsymbol{Z}_2$. In the latter case, $p = q = e_1$ satisfies these conditions and $\varphi(e_1,e_1) = \gamma_2$ ([8], Lemma 1.8 (1)). Therefore $(G_{2(2)})_{ev} \cong (Sp(1,\boldsymbol{R}) \times Sp(1,\boldsymbol{R}))/\boldsymbol{Z}_2 \times \{1,\gamma_2\}$.

(2) For $\alpha \in (G_{2(2)})_0 \subset ({G_2}^C)^{\delta_4}$, there exist $p \in Sp(1,\boldsymbol{H}^C)$ and $a \in U(1,\boldsymbol{C}^C)$ such that $\alpha = \varphi(p,a)$ (Theorem 1.2 (2)). From $\gamma_1\tau\alpha\tau\gamma_1 = \alpha$, we have $\varphi(\gamma_1\tau p,\ \gamma_1\tau a) = \varphi(p,a)$ ([8], Lemma 1.8 (2)). Hence

$$\gamma_1\tau p = p, \gamma_1\tau a = a \quad \text{or} \quad \gamma_1\tau p = -p, \gamma_1\tau a = -a.$$

In the former case, $p \in Sp(1,\boldsymbol{H}'), a \in U(1,\boldsymbol{C}')$, hence the group of the former case is $(Sp(1,\boldsymbol{H}') \times U(1,\boldsymbol{C}'))/\boldsymbol{Z}_2 \cong (Sp(1,\boldsymbol{R}) \times \boldsymbol{R}^*)/\boldsymbol{Z}_2(\boldsymbol{Z}_2 = \{(1,1),(-1,-1)\}) \cong Sp(1,\boldsymbol{R}) \times \boldsymbol{R}^+$. In the latter case, $p = a = e_1$ satisfies these conditions and $\varphi(e_1,e_1) = \gamma_2$ ([8], Lemma 1.8 (1)). Therefore $(G_{2(2)})_0 \cong (Sp(1,\boldsymbol{R}) \times \boldsymbol{R}^+) \times \{1,\gamma_2\}$.

(3) For $\alpha \in (G_{2(2)})_{ed} \subset ({G_2}^C)^{w_3}$, there exists $P \in SU(3,\boldsymbol{C}^C)$ such that $\alpha = \psi(P)$ (Theorem 1.2 (3)). Using $\tau\psi(P)\tau = \psi(\tau P)$ and $\gamma_1\psi(P)\gamma_1 = \psi(\overline{P})$, from $\gamma_1\tau\alpha\tau\gamma_1 = \alpha$, we have $\psi(\tau\overline{P}) = \psi(P)$. Hence $\tau\overline{P} = P$, that is, $P \in SU(3,\boldsymbol{C}')$. Therefore $(G_{2(2)})_{ed} \cong SU(3,\boldsymbol{C}') \cong SL(3,\boldsymbol{R})$. □

2. Group F_4

2.1. Lie groups of type $\boldsymbol{F}_4$ and some subgroups of ${\boldsymbol{F}_4}^C$

We use the same notations and definitions as in [8]. For example,

the Jordan algebras $\mathfrak{J} = \mathfrak{J}(3,\mathfrak{C}), \mathfrak{J}(3,\boldsymbol{H}), \mathfrak{J}(3,\boldsymbol{C})$ with the Jordan multiplication $X \circ Y$, the inner product (X,Y) and the Freudenthal multiplication $X \times Y$ and elements $E_k, F_k(x)$ of $\mathfrak{J}^C$,

the groups ${F_4}^C = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,|\, \alpha(X \times Y) = \alpha X \times \alpha Y\}, F_4$ and $F_{4(4)}$,

the involutive automorphisms $\gamma_1, \gamma_2, \sigma$ of F_4 and $F_{4(4)} = ({F_4}^C)^{\tau\gamma_1}$,

the Lie algebras ${\mathfrak{f}_4}^C, \mathfrak{f}_4, \mathfrak{f}_{4(4)}$ and elements $\tilde{A}_k(a)$ of ${\mathfrak{f}_4}^C$,

the principle of triality and the identification $D_1 \in \mathfrak{so}(8) \leftrightarrow \delta(D_1,D_2,D_3) \in \mathfrak{f}_4$, etc.

We shall review and add some notations and definitions. To an element

$X = \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \in \mathfrak{J}(3, \mathfrak{C})$, we associate an element

$$\begin{pmatrix} \xi_1 & m_3 & \overline{m_2} \\ \overline{m_3} & \xi_2 & m_1 \\ m_2 & \overline{m_1} & \xi_3 \end{pmatrix} + (n_1, n_2, n_3), \quad x_k = m_k + n_k e_4 \in \boldsymbol{H} \oplus \boldsymbol{H} e_4 = \mathfrak{C}$$

of the algebra $\mathfrak{J}(3, \boldsymbol{H}) \oplus \boldsymbol{H}^3$ with the multiplication

$$(M_1 + \boldsymbol{n}_1) \times (M_2 + \boldsymbol{n}_2) = \Big(M_1 \times M_2 - \frac{1}{2}(\boldsymbol{n}_1{}^* \boldsymbol{n}_2 + \boldsymbol{n}_2{}^* \boldsymbol{n}_1)\Big) - \frac{1}{2}(\boldsymbol{n}_1 M_2 + \boldsymbol{n}_2 M_1).$$

The $\boldsymbol{R}$-linear transformations γ and δ_4 of $\mathfrak{C}$ are extended to the $\boldsymbol{R}$-linear transformations of $\mathfrak{J}(3, \mathfrak{C})$ as

$$\begin{array}{l} \gamma(M + \boldsymbol{n}) = M - \boldsymbol{n}, \\ \delta_4(M + \boldsymbol{n}) = -D_{e_1} M D_{e_1} + \boldsymbol{n} D_{e_1}, \end{array} \quad M + \boldsymbol{n} \in \mathfrak{J}(3, \boldsymbol{H}) \oplus \boldsymbol{H}^3 = \mathfrak{J}(3, \mathfrak{C}),$$

where $D_{e_1} = \mathrm{diag}(e_1, e_1, e_1) \in Sp(3)$. Furthermore, to an element $\begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix}$ $\in \mathfrak{J}(3, \mathfrak{C})$, we associate an element

$$\begin{pmatrix} \xi_1 & a_3 & \overline{a_2} \\ \overline{a_3} & \xi_2 & a_1 \\ a_2 & \overline{a_1} & \xi_3 \end{pmatrix} + (\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3), \quad x_k = a_k + \boldsymbol{m}_k \in \boldsymbol{C} \oplus \boldsymbol{C}^3 = \mathfrak{C}$$

of the algebra $\mathfrak{J}(3, \boldsymbol{C}) \oplus M(3, \boldsymbol{C})$ with the multiplication and the inner product

$$\begin{aligned} (X + M) \times (Y + N) &= \Big(X \times Y - \frac{1}{2}(M^* N + N^* M)\Big) \\ &\quad - \frac{1}{2}(MY + NX - \overline{M \times N}), \\ (X + M, Y + N) &= (X, Y) + \mathrm{tr}(M^* N + N^* M), \end{aligned}$$

where for $M = (\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3), N = (\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) \in M(3, \boldsymbol{C})$, $M \times N \in M(3, \boldsymbol{C})$ is defined as

$$M \times N = \begin{pmatrix} \boldsymbol{m}_2 \times \boldsymbol{n}_3 & \boldsymbol{m}_3 \times \boldsymbol{n}_1 & \boldsymbol{m}_1 \times \boldsymbol{n}_2 \\ + & + & + \\ \boldsymbol{n}_2 \times \boldsymbol{m}_3 & \boldsymbol{n}_3 \times \boldsymbol{m}_1 & \boldsymbol{n}_1 \times \boldsymbol{m}_2 \end{pmatrix}.$$

The $\boldsymbol{R}$-linear transformations δ_4 and w_3 of $\mathfrak{C}$ are extended to the $\boldsymbol{R}$-linear transformations of $\mathfrak{J}(3, \mathfrak{C})$ as

$$\begin{array}{l} \delta_4(X + M) = X + D_4 M, \\ w_3(X + M) = X + \omega_1 M, \end{array} \quad X + M \in \mathfrak{J}(3, \boldsymbol{C}) \oplus M(3, \boldsymbol{C}) = \mathfrak{J}(3, \mathfrak{C}).$$

2.2. Subgroups of type ${\boldsymbol{C}_1}^C \oplus {\boldsymbol{C}_3}^C, {\boldsymbol{C}_1}^C \oplus \boldsymbol{C} \oplus {\boldsymbol{A}_2}^C$ and ${\boldsymbol{A}_2}^C \oplus {\boldsymbol{A}_2}^C$ of ${\boldsymbol{F}_4}^C$

In the Lie algebra ${\mathfrak{f}_4}^C$, let

$$Z = i(-2G_{23} + G_{45} + G_{67}).$$

Theorem 2.1. *The* 3*-graded decomposition of* $\mathfrak{f}_{4(4)} = ({\mathfrak{f}_4}^C)^{\tau\gamma_1}$ *(or* ${\mathfrak{f}_4}^C$*),*

$$\mathfrak{f}_{4(4)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\mathrm{ad}Z, Z = i(-2G_{23} + G_{45} + G_{67})$*, is given by*

$$\mathfrak{g}_0 = \left\{ \begin{array}{lllll} G_{46} + G_{57}, & i(G_{47} - G_{56}), & \tilde{A}_1(1), & \tilde{A}_2(1), & \tilde{A}_3(1), \\ iG_{01}, iG_{23}, iG_{45}, iG_{67}, & & i\tilde{A}_1(e_1), & i\tilde{A}_2(e_1), & i\tilde{A}_3(e_1) \end{array} \right\} \; 12$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} (2G_{15} + G_{26} - G_{37}) - i(2G_{14} + G_{27} + G_{36}), \\ (2G_{17} - G_{24} + G_{35}) - i(2G_{16} - G_{25} - G_{34}), \\ G_{04} + iG_{05}, G_{06} + iG_{07}, iG_{14} - G_{15}, iG_{16} - G_{17}, \\ \tilde{A}_1(e_4 + ie_5), \tilde{A}_2(e_4 + ie_5), \tilde{A}_3(e_4 + ie_5), \\ \tilde{A}_1(e_6 + ie_7), \tilde{A}_2(e_6 + ie_7), \tilde{A}_3(e_6 + ie_7) \end{array} \right\} \; 12$$

$$\mathfrak{g}_{-2} = \left\{ \begin{array}{ll} G_{02} - iG_{03}, & (-2G_{13} + G_{46} - G_{57}) - i(2G_{12} - G_{47} - G_{56}), \\ iG_{12} + G_{13}, & \tilde{A}_1(e_2 - ie_3), \tilde{A}_2(e_2 - ie_3), \tilde{A}_3(e_2 - ie_3) \end{array} \right\} \; 6$$

$$\mathfrak{g}_{-3} = \{(G_{24} + G_{35}) + i(G_{25} - G_{34}), (G_{26} + G_{37}) + i(G_{27} - G_{36})\} \; 2$$

$$\mathfrak{g}_1 = \tau(\mathfrak{g}_{-1})\tau, \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau, \quad \mathfrak{g}_3 = \tau(\mathfrak{g}_{-3})\tau.$$

Proof. Note that for $D_1 = -2G_{23} + G_{45} + G_{67} \in \mathfrak{so}(8)$ we have also $D_2 = D_3 = -2G_{23} + G_{45} + G_{67}$. We can then prove this theorem in a way similar to Theorem 1.1, using [8] Lemmas 1.5 and 2.3. □

As is shown in ${G_2}^C$, we have

$$z_2 = \exp\frac{2\pi i}{2}Z = \gamma, \quad z_4 = \exp\frac{2\pi i}{4}Z = \delta_4, \quad z_3 = \exp\frac{2\pi i}{3}Z = w_3.$$

Now, since $({\mathfrak{f}_4}^C)_{ev} = ({\mathfrak{f}_4}^C)^{z_2}, ({\mathfrak{f}_4}^C)_0 = ({\mathfrak{f}_4}^C)^{z_4}, ({\mathfrak{f}_4}^C)_{ed} = ({\mathfrak{f}_4}^C)^{z_3}$, we shall determine the group structures of

$$({F_4}^C)_{ev} = ({F_4}^C)^{z_2}, \quad ({F_4}^C)_0 = ({F_4}^C)^{z_4}, \quad ({F_4}^C)_{ed} = ({F_4}^C)^{z_3}.$$

Theorem 2.2. (1) $({F_4}^C)_{ev} \cong (Sp(1,C) \times Sp(3,C))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(1,E), (-1,-E)\}$.

(2) $({F_4}^C)_0 \cong (Sp(1,C) \times C^* \times SL(3,C))/\boldsymbol{Z}_6$, $\boldsymbol{Z}_6 = \{(1,1,E), (1,\omega,\omega^2 E), (1,\omega^2 E,\omega E), (-1,-1,E), (-1,-\omega,\omega^2 E), (-1,-\omega^2,\omega E)\}$.

(3) $({F_4}^C)_{ed} \cong (SL(3,C) \times SL(3,C))/\boldsymbol{Z}_3$, $\boldsymbol{Z}_3 = \{(1,E), (\omega E,\omega E), (\omega^2 E, \omega^2 E)\}$.

Where $\omega = -(1/2) + (\sqrt{3}/2)i \in C$.

Proof. (1) We define $\varphi : Sp(1, \boldsymbol{H}^C) \times Sp(3, \boldsymbol{H}^C) \to ({F_4}^C)_{ev} = ({F_4}^C)^{z_2} = ({F_4}^C)^{\gamma}$ by

$$\varphi(p, A)(M + \boldsymbol{n}) = AMA^* + p\boldsymbol{n}A^*, \quad M + \boldsymbol{n} \in \mathfrak{J}(3, \boldsymbol{H}^C) \oplus (\boldsymbol{H}^C)^3 = \mathfrak{J}(3, \mathfrak{C}^C).$$

We can then prove this in a way similar to Theorem 1.2 (1).

(2) Using the restriction mapping $\varphi : Sp(1, \boldsymbol{H}^C) \times U(3, \boldsymbol{C}^C) \to ({F_4}^C)_0 = ({F_4}^C)^{z_4} = ({F_4}^C)^{\delta_4}$ of φ, in a way similar to (1) above, we have $({F_4}^C)_0 \cong (Sp(1, \boldsymbol{H}^C) \times U(3, \boldsymbol{C}^C))/\boldsymbol{Z}_2(\boldsymbol{Z}_2 = \{(1, E), (-1, -E)\}) \cong (Sp(1, \boldsymbol{H}^C) \times U(1, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_3)$ $(\boldsymbol{Z}_3 = \{(1, 1, E), (1, \omega_1, {\omega_1}^2 E), (1, {\omega_1}^2, \omega_1 E)\}) \cong (Sp(1, C) \times C^* \times SL(3, C))/\boldsymbol{Z}_6$. (Note that under the isomorphism $f : SL(3, C) \to SU(3, \boldsymbol{C}^C)$, ω is translated to ${\omega_1}^2$).

(3) We define $\psi : SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C) \to ({F_4}^C)_{ed} = ({F_4}^C)^{z_3} = ({F_4}^C)^{w_3}$ by

$$\psi(P, A)(X + M) = AXA^* + PMA^*, \; X + M \in \mathfrak{J}(3, \boldsymbol{C}^C) \oplus M(3, \boldsymbol{C}^C) = \mathfrak{J}(3, \mathfrak{C}^C).$$

Then ψ is well-defined ([5]), is a homomorphism and $\mathrm{Ker}\psi = \boldsymbol{Z}_3$. Since $({F_4}^C)^{w_3}$ is connected and $\dim_C(\mathfrak{su}(3, \boldsymbol{C}^C) \oplus \mathfrak{su}(3, \boldsymbol{C}^C)) = 8 + 8 = 16 = 12 + 2 \times 2 = \dim_C(({\mathfrak{f}_4}^C)_{ed})$ (Theorem 2.1), ψ is onto. Therefore $({F_4}^C)_{ed} \cong (SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C))/\boldsymbol{Z}_3 \cong (SL(3, C) \times SL(3, C))/\boldsymbol{Z}_3$. □

2.2.1. Subgroups of type $\boldsymbol{C}_{1(1)} \oplus \boldsymbol{C}_{3(3)}, \boldsymbol{C}_{1(1)} \oplus \boldsymbol{R} \oplus \boldsymbol{A}_{2(2)}$ and $\boldsymbol{A}_{2(2)} \oplus \boldsymbol{A}_{2(2)}$ of $\boldsymbol{F}_{4(4)}$

We use the same notations as in **2.2**. Since $(\mathfrak{f}_{4(4)})_{ev} = ({\mathfrak{f}_4}^C)^{z_2} \cap ({\mathfrak{f}_4}^C)^{\tau\gamma_1}$, $(\mathfrak{f}_{4(4)})_0 = ({\mathfrak{f}_4}^C)^{z_4} \cap ({\mathfrak{f}_4}^C)^{\tau\gamma_1}, (\mathfrak{f}_{4(4)})_{ed} = ({\mathfrak{f}_4}^C)^{z_3} \cap ({\mathfrak{f}_4}^C)^{\tau\gamma_1}$, we shall determine the group structures of

$$(F_{4(4)})_{ev} = ({F_4}^C)^{z_2} \cap ({F_4}^C)^{\tau\gamma_1}, \quad (F_{4(4)})_0 = ({F_4}^C)^{z_4} \cap ({F_4}^C)^{\tau\gamma_1},$$
$$(F_{4(4)})_{ed} = ({F_4}^C)^{z_3} \cap ({F_4}^C)^{\tau\gamma_1}.$$

Theorem 2.3. (1) $(F_{4(4)})_{ev} \cong (Sp(1, \boldsymbol{R}) \times Sp(3, \boldsymbol{R}))/\boldsymbol{Z}_2 \times \{1, \gamma_2\}$, $\boldsymbol{Z}_2 = \{(1, E), (-1, -E)\}$.

(2) $(F_{4(4)})_0 \cong (Sp(1, \boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(3, \boldsymbol{R})) \times \{1, \gamma_2\}$.

(3) $(F_{4(4)})_{ed} \cong (SL(3, \boldsymbol{R}) \times SL(3, \boldsymbol{R})) \times \{1, \omega 1, \omega^2 1\}$.

Proof. (1) and (2) are proved from Theorem 2.2 in a way similar to Theorem 1.3 (1) and (2), using [8] Lemma 2.6.

(3) For $\alpha \in (F_{4(4)})_{ed} \subset ({F_4}^C)^{w_3}$, there exist $P, A \in SU(3, \boldsymbol{C}^C)$ such that $\alpha = \psi(P, A)$ (Theorem 2.2 (3)). Using $\gamma_1\tau\psi(P, A)\tau\gamma_1 = \psi(\tau\overline{P}, \tau\overline{A})$, from $\gamma_1\tau\alpha\tau\gamma_1 = \alpha$ we have $\psi(\tau\overline{P}, \tau\overline{A}) = \psi(P, A)$. Hence

$$\begin{cases} \tau\overline{P} = P \\ \tau\overline{A} = A \end{cases}, \quad \begin{cases} \tau\overline{P} = \omega P \\ \tau\overline{A} = \omega A \end{cases}, \text{ or } \quad \begin{cases} \tau\overline{P} = \omega^2 P \\ \tau\overline{A} = \omega^2 A \end{cases}.$$

In the first case, $P, A \in SU(3, \boldsymbol{C}') \cong SL(3, \boldsymbol{R})$, so the group of the first case is $SL(3, \boldsymbol{R}) \times SL(3, \boldsymbol{R})$. In the last two cases, $P = A = \omega E$ (resp. $P = A = \omega^2 E$) satisfies the conditions and $\psi(\omega E, \omega E) = \omega^2 1$ (resp. $\psi(\omega^2 E, \omega^2 E) = \omega 1$). Therefore $(F_{4(4)})_{ed} \cong (SL(3, \boldsymbol{R}) \times SL(3, \boldsymbol{R})) \times \{1, \omega 1, \omega^2 1\}$. □

3. Group E_6

3.1. Lie groups of type $\boldsymbol{E}_6$ and some subgroups of $\boldsymbol{E}_6{}^C$

We use the same notations and definitions as in [8]. For example,

the groups $E_6{}^C = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,|\, \det\alpha X = \det X\} = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,|\, {}^t\alpha^{-1} (X \times Y) = \alpha X \times \alpha Y\}, E_6, E_{6(6)}, E_{6(2)}$ and $E_{6(-26)}$,

the involutive automorphisms $\gamma, \gamma_1, \sigma, \sigma', \lambda, \tau_1$ of the group E_6 and $E_{6(6)} = (E_6{}^C)^{\tau\gamma_1}$, $E_{6(2)} = (E_6{}^C)^{\lambda\tau\gamma_1}$, $E_{6(-26)} = (E_6{}^C)^{\tau_1}$,

the Lie algebra $\mathfrak{e}_6{}^C$ of the group $E_6{}^C$ and elements $\tilde{F}_k(a)$ of $\mathfrak{e}_6{}^C$ etc.

We shall review and add some notations and definitions. Let $k : M(3, \boldsymbol{H}^C) \to MJ(6, \boldsymbol{C}^C) = \{P \in M(6, \boldsymbol{C}^C) | JP = \overline{P}J\}$ (resp. $k : (\boldsymbol{H}^C)^3 \to MJ(2, 6, \boldsymbol{C}^C) = \{P \in M(2, 6, \boldsymbol{C}^C) | JP = \overline{P}J\}$) be the C-linear isomorphism defined by

$$k\Big((a + be_2)\Big) = \left(\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}\right), \quad a, b \in \boldsymbol{C}^C,$$

and we denote the inverse k^{-1} of k by h. We define $\varphi_1 : Sp(1, \boldsymbol{H}^C) \times SU^*(6, \boldsymbol{C}^C) \to (E_6{}^C)^\gamma$ by

$$\varphi_1(p, A)(M + \boldsymbol{n}) = (hA)M(hA)^* + p\boldsymbol{n}(hA)^{-1},$$
$$M + \boldsymbol{n} \in \mathfrak{J}(3, \boldsymbol{H}^C) \oplus (\boldsymbol{H}^C)^3 = \mathfrak{J}^C.$$

Then φ_1 is well-defined, is a homomorphism and $\mathrm{Ker}\varphi_1 = \{(1, E), (-1 - E)\} = \boldsymbol{Z}_2$. Since $(E_6{}^C)^\gamma$ is connected and $\dim_C(\mathfrak{sp}(1, \boldsymbol{H}^C) \oplus \mathfrak{su}^*(6, \boldsymbol{C}^C)) = 3 + 35 = 38 = 36 + 2 = \dim_C((\mathfrak{e}_6{}^C)^\gamma)$ (see Theorems 3.1 and 3.2 (1)), φ_1 is onto. Therefore $(E_6{}^C)^\gamma \cong (Sp(1, \boldsymbol{H}^C) \times SU^*(6, \boldsymbol{C}^C))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1, E), (-1, -E)\}$ ([6], Proposition 3.5.4). Furthermore, note that the mapping $f : SL(6, C) \to SU^*(6, \boldsymbol{C}^C)$, $f(A) = \varepsilon A - \overline{\varepsilon} JAJ$, where $\varepsilon = (1/2)(1 + ie_1)$, gives an isomorphism, and we define $\varphi : Sp(1, \boldsymbol{H}^C) \times SL(6, C) \to (E_6{}^C)^\gamma$ by $\varphi(p, A) = \varphi_1(p, f(A))$, then we have also an isomorphism

$$(E_6{}^C)^\gamma \cong (Sp(1, \boldsymbol{H}^C) \times SL(6, C))/\boldsymbol{Z}_2.$$

Then for the induced mapping $\varphi_* : \mathfrak{sp}(1, \boldsymbol{H}^C) \oplus \mathfrak{sl}(6, C) \to \mathfrak{e}_6{}^C$ of φ, we have

$$\varphi_*(e_1, \mathrm{diag}(0, 0, 0, 0, 0, 0)) = -G_{45} + G_{67}$$
$$\varphi_*(0, \mathrm{diag}(i, -i, 0, 0, 0, 0)) = -G_{45} - G_{67}$$
$$\varphi_*(0, \mathrm{diag}(0, i, -i, 0, 0, 0)) = -\tfrac{1}{2}G_{01} - \tfrac{1}{2}G_{23} + \tfrac{1}{2}G_{45} + \tfrac{1}{2}G_{67} + i(E_1 - E_2)^\sim$$
$$\varphi_*(0, \mathrm{diag}(0, 0, i, -i, 0, 0)) = G_{01} + G_{23}$$
$$\varphi_*(0, \mathrm{diag}(0, 0, 0, i, -i, 0)) = -G_{23} + i(E_2 - E_3)^\sim$$
$$\varphi_*(0, \mathrm{diag}(0, 0, 0, 0, i, -i)) = -G_{01} + G_{23}.$$

From the facts above, we have also

$$\begin{aligned} G_{01} &= \varphi_*(0, \mathrm{diag}(0,0,i/2,-i/2,-i/2,i/2)) \\ G_{23} &= \varphi_*(0, \mathrm{diag}(0,0,i/2,-i/2,i/2,-i/2)) \\ G_{45} &= \varphi_*(-e_1/2, \mathrm{diag}(-i/2,i/2,0,0,0,0)) \\ G_{67} &= \varphi_*(e_1/2, \mathrm{diag}(-i/2,i/2,0,0,0,0)) \end{aligned}$$

$$\begin{aligned} i(E_1 - E_2) &= \varphi_*(0, \mathrm{diag}(i/2,i/2,-i/2,-i/2,0,0)) \\ i(E_2 - E_3) &= \varphi_*(0, \mathrm{diag}(0,0,i/2,i/2,-i/2,-i/2)). \end{aligned}$$

For $\theta \in C, \theta \neq 0$ and $a \in \mathfrak{C}^C, a\overline{a} = 1$, we define C-linear transformations $\phi(\theta)$ and $D(a)$ of $\mathfrak{J}^C$ by

$$\phi(\theta)\begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \theta^4\xi_1 & \theta x_3 & \theta\overline{x_2} \\ \theta\overline{x_3} & \theta^{-2}\xi_2 & \theta^{-2}x_1 \\ \theta x_2 & \theta^{-2}\overline{x_1} & \theta^{-2}\xi_3, \end{pmatrix},$$

$$D(a)\begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & x_3 a & \overline{a x_2} \\ \overline{x_3 a} & \xi_2 & \overline{a} x_1 \overline{a} \\ a x_2 & a\overline{x_1} a & \xi_3 \end{pmatrix},$$

respectively. Then $\phi(\theta), D(a) \in {E_6}^C$. Usually we denote $\sigma = D(-1)$.

The mapping $\varphi : Sp(1, \boldsymbol{H}^C) \times SL(6, C) \to {E_6}^C$ has the following properties.

$$\begin{aligned} &\varphi(1, \mathrm{diag}(\omega,\omega,\omega,\omega,\omega,\omega)) = \omega^2 1, \\ &\varphi(1, \mathrm{diag}(-1,-1,-1,-1,-1,-1)) = \gamma, \\ &\varphi(e_1, \mathrm{diag}(-i,i,-i,i,-i,i)) = \gamma_2, \\ &\varphi(1, \mathrm{diag}(i,-i,i,-i,i,-i)) = \delta_4, \\ &\varphi(1, \mathrm{diag}(1,1,-1,-1,-1,-1)) = \sigma, \\ &\varphi(1, \mathrm{diag}(1,1,i,-i,-i,i)) = D(e_1), \\ &\varphi(1, \mathrm{diag}(\omega,\omega^2,\omega,\omega^2,\omega,\omega^2)) = w_3. \end{aligned}$$

3.2. Subgroups of type ${\boldsymbol{C}_1}^C \oplus {\boldsymbol{A}_5}^C, {\boldsymbol{C}_1}^C \oplus \boldsymbol{C} \oplus {\boldsymbol{A}_2}^C \oplus {\boldsymbol{A}_2}^C$ and ${\boldsymbol{A}_2}^C \oplus {\boldsymbol{A}_2}^C \oplus {\boldsymbol{A}_2}^C$ of ${\boldsymbol{E}_6}^C$

In the Lie algebra ${\mathfrak{e}_6}^C$, let

$$Z = i(-2G_{23} + G_{45} + G_{67}).$$

Theorem 3.1. *The* 3*-graded decomposition of* $\mathfrak{e}_{6(6)} = ({\mathfrak{e}_6}^C)^{\tau\gamma_1}$ *(or* ${\mathfrak{e}_6}^C$*),*

$$\mathfrak{e}_{6(6)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\mathrm{ad}Z, Z = i(-2G_{23} + G_{45} + G_{67})$, *is given by*

$$\mathfrak{g}_0 = \left\{ \begin{array}{llll} G_{46} + G_{57},\ i(G_{47} - G_{56}), & \tilde{A}_1(1), & \tilde{A}_2(1), & \tilde{A}_3(1), \\ iG_{01},\ iG_{23},\ iG_{45},\ iG_{67}, & i\tilde{A}_1(e_1), & i\tilde{A}_2(e_1) & i\tilde{A}_3(e_1), \\ (E_1 - E_2)^\sim, (E_2 - E_3)^\sim, & \tilde{F}_1(1), & \tilde{F}_2(1), & \tilde{F}_3(1), \\ & i\tilde{F}_1(e_1), & i\tilde{F}_2(e_1), & i\tilde{F}_3(e_1) \end{array} \right\} \ 20$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} (2G_{15} + G_{26} - G_{37}) - i(2G_{14} + G_{27} + G_{36}), \\ (2G_{17} - G_{24} + G_{35}) - i(2G_{16} - G_{25} - G_{34}), \\ G_{04} + iG_{05},\ G_{06} + iG_{07},\ iG_{14} - G_{15},\ iG_{16} - G_{17}, \\ \tilde{A}_1(e_4 + ie_5),\ \tilde{A}_1(e_6 + ie_7),\ \tilde{F}_1(e_4 + ie_5),\ \tilde{F}_1(e_6 + ie_7), \\ \tilde{A}_2(e_4 + ie_5),\ \tilde{A}_2(e_6 + ie_7),\ \tilde{F}_2(e_4 + ie_5),\ \tilde{F}_2(e_6 + ie_7), \\ \tilde{A}_3(e_4 + ie_5),\ \tilde{A}_3(e_6 + ie_7),\ \tilde{F}_3(e_4 + ie_5),\ \tilde{F}_3(e_6 + ie_7) \end{array} \right\} \ 18$$

$$\mathfrak{g}_{-2} = \left\{ \begin{array}{llll} (-2G_{13} + G_{46} - G_{57}) - i(2G_{12} - G_{47} - G_{56}), & & & \\ G_{02} - iG_{03}, & \tilde{A}_1(e_2 - ie_3), & \tilde{A}_2(e_2 - ie_3), & \tilde{A}_3(e_2 - ie_3), \\ iG_{12} + G_{13}, & \tilde{F}_1(e_2 - ie_3), & \tilde{F}_2(e_2 - ie_3), & \tilde{F}_3(e_2 - ie_3) \end{array} \right\} \ 9$$

$$\mathfrak{g}_{-3} = \{(G_{24} + G_{35}) + i(G_{25} - G_{34}), (G_{26} + G_{37}) + i(G_{27} - G_{36})\} \ 2$$

$$\mathfrak{g}_1 = \tau(\mathfrak{g}_{-1})\tau, \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau, \quad \mathfrak{g}_3 = \tau(\mathfrak{g}_{-3})\tau.$$

Proof. We can prove this theorem in a way similar to Theorem 2.1, using [8] Lemma 3.3. □

Since $iZ = 2G_{23} - G_{45} - G_{67} = \varphi_*(0, \mathrm{diag}(i, -i, i, -i, i, -i))$, we have

$$\begin{aligned} z_2 &= \exp \frac{2\pi i}{2} Z = \varphi(1, \mathrm{diag}(-1, -1, -1, -1, -1, -1)) = \gamma, \\ z_4 &= \exp \frac{2\pi i}{4} Z = \varphi(1, \mathrm{diag}(i, -i, i, -i, i, -i)) = \delta_4, \\ z_3 &= \exp \frac{2\pi i}{3} Z = \varphi(1, \mathrm{diag}(\omega, \omega^2, \omega, \omega^2, \omega, \omega^2)) = w_3. \end{aligned}$$

Now, we shall determine the group structures of

$$({E_6}^C)_{ev} = ({E_6}^C)^{z_2}, \quad ({E_6}^C)_0 = ({E_6}^C)^{z_4}, \quad ({E_6}^C)_{ed} = ({E_6}^C)^{z_3}.$$

Theorem 3.2. (1) $({E_6}^C)_{ev} \cong (Sp(1, C) \times SL(6, C))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(1, E), (-1, -E)\}$.

(2) $({E_6}^C)_0 \cong (Sp(1, C) \times C^* \times SL(3, C) \times SL(3, C))/\boldsymbol{Z}_6$, $\boldsymbol{Z}_6 = \{(1, 1, E, E), (1, \omega, \omega^2 E, \omega E), (1, \omega^2, \omega E, \omega^2 E), (-1, -1, E, E), (-1, -\omega, \omega^2 E, \omega E), (-1, -\omega^2, \omega E, \omega^2 E)\}$.

(3) $({E_6}^C)_{ed} \cong (SL(3, C) \times SL(3, C) \times SL(3, C))/\boldsymbol{Z}_3$, $\boldsymbol{Z}_3 = \{(E, E, E), (\omega E, \omega E, \omega E), (\omega^2 E, \omega^2 E, \omega^2 E)\}$.

Proof. (1) $({E_6}^C)_{ev} = ({E_6}^C)^{z_2} = ({E_6}^C)^{\gamma} \cong (Sp(1, C) \times SL(6, C))/\boldsymbol{Z}_2$ is already shown.

(2) Since z_4 is conjugate to

$$z_4{}' = \varphi(1, \mathrm{diag}(i, i, i, -i, -i, -i))$$

under the adjoint action of $SL(6, \boldsymbol{R}) \subset ({E_6}^C)^{\tau\gamma_1}$, we use $z_4{}'$ instead of z_4. Now, using the restriction mapping $\varphi : Sp(1, \boldsymbol{H}^C) \times S(GL(3, C) \times GL(3, C)) \to ({E_6}^C)_0 = ({E_6}^C)^{z_4{}'}$ of φ, in a way similar to Theorem 1.2, we have $({E_6}^C)_0 \cong (Sp(1, \boldsymbol{H}^C) \times S(GL(3, C) \times GL(3, C)))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(1, E), (-1, -E)\}$. Since $h : C^* \times SL(3, C) \times SL(3, C) \to S(GL(3, C) \times GL(3, C)), h(z, A_1, A_2) = \begin{pmatrix} zA_1 & 0 \\ 0 & z^{-1}A_2 \end{pmatrix}$ induces an isomorphism $S(GL(3, C) \times GL(3, C)) \cong (C^* \times SL(3, C) \times SL(3, C))/\boldsymbol{Z}_3, \boldsymbol{Z}_3 = \{(1, E, E), (\omega, \omega^2 E, \omega E), (\omega^2, \omega E, \omega^2 E)\}$, we have $({E_6}^C)_0 \cong (Sp(1, C) \times C^* \times SL(3,\ C) \times SL(3, C))/\boldsymbol{Z}_6$.

(3) We define $\psi : SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C) \to ({E_6}^C)_{ed} = ({E_6}^C)^{z_3} = ({E_6}^C)^{w_3}$ by

$$\psi(P, A, B)(X + M) = h(A, B)Xh(A, B)^* + PM\tau h(A, B)^*,$$
$$X + M \in \mathfrak{J}(3, \boldsymbol{C}^C) \oplus M(3, \boldsymbol{C}^C) = \mathfrak{J}^C,$$

where $h : M(3, \boldsymbol{C}^C) \times M(3, \boldsymbol{C}^C) \to M(3, \boldsymbol{C}^C)$ is the mapping defined by

$$h(A, B) = \varepsilon A + \overline{\varepsilon} B, \quad \varepsilon = \frac{1 + ie_1}{2}.$$

Using

$${}^t h(A, B)^{-1} = \tau h(A, B) = h(\tau B, \tau A),$$

we can verify that ψ is well-defined ([5]), ψ is a homomorphism and $\mathrm{Ker}\psi = \boldsymbol{Z}_3$. Since $({E_6}^C)^{w_3}$ is connected and $\dim_C(\mathfrak{su}(3, \boldsymbol{C}^C) \oplus \mathfrak{su}(3, \boldsymbol{C}^C) \oplus \mathfrak{su}(3, \boldsymbol{C}^C)) = 8 + 8 + 8 = 24 = 20 + 2 \times 2 = \dim_C(({\mathfrak{e}_6}^C)_{ed})$ (Theorem 3.1), ψ is onto. Therefore $({E_6}^C)_{ed} \cong (SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C) \times SU(3, \boldsymbol{C}^C))/\boldsymbol{Z}_3 \cong (SL(3, C) \times SL(3, C) \times SL(3, C))/\boldsymbol{Z}_3$. □

3.2.1. Subgroups of type $\boldsymbol{C}_{1(1)} \oplus \boldsymbol{A}_{5(5)}, \boldsymbol{C}_{1(1)} \oplus \boldsymbol{R} \oplus \boldsymbol{A}_{2(2)} \oplus \boldsymbol{A}_{2(2)}$ and $\boldsymbol{A}_{2(2)} \oplus \boldsymbol{A}_{2(2)} \oplus \boldsymbol{A}_{2(2)}$ of $\boldsymbol{E}_{6(6)}$

Using the same notations as in 3.2, we shall determine the group structures of

$$(E_{6(6)})_{ev} = ({E_6}^C)^{z_2} \cap ({E_6}^C)^{\tau\gamma_1}, \quad (E_{6(6)})_0 = ({E_6}^C)^{z_4{}'} \cap ({E_6}^C)^{\tau\gamma_1},$$
$$(E_{6(6)})_{ed} = ({E_6}^C)^{z_3} \cap ({E_6}^C)^{\tau\gamma_1}.$$

Theorem 3.3. (1) $(E_{6(6)})_{\mathrm{ev}} \cong (Sp(1, \boldsymbol{R}) \times SL(6, \boldsymbol{R}))/\boldsymbol{Z}_2 \times \{1, \gamma_2\}$, $\boldsymbol{Z}_2 = \{(1, E), (-1, -E)\}$.

(2) $(E_{6(6)})_0 \cong (Sp(1, \boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(3, \boldsymbol{R}) \times SL(3, \boldsymbol{R})) \times \{1, \gamma_2\}$.

(3) $(E_{6(6)})_{ed} \cong (SL(3, \boldsymbol{R}) \times SL(3, \boldsymbol{R}) \times SL(3, \boldsymbol{R})) \times \{1, \omega 1, \omega^2 1\}$.

Proof. (1) and (2) Using $\gamma_1 \tau \varphi(p, A) \tau \gamma_1 = \varphi(\gamma_1 \tau p, \tau A)$, $p \in Sp(1, \boldsymbol{H}^C)$, $A \in SL(6, C)$ ([8], Lemma 3.6) and $\varphi(e_1, -iI) = \gamma_2$ in Theorem 3.2, we can prove this in a way similar to Theorem 1.3 (1) and (2).

(3) Using $\gamma_1\tau\psi(P,A,B)\tau\gamma_1 = \psi(\tau\overline{P},\tau\overline{A},\tau\overline{B}),\ P,A,B \in SU(3,\boldsymbol{C}^C)$, we can prove this in a way similar to Theorem 2.3 (3). □

3.2.2. Subgroups of type $\boldsymbol{C}_{1(1)} \oplus \boldsymbol{A}_{5(1)}, \boldsymbol{C}_{1(1)} \oplus \boldsymbol{R} \oplus {\boldsymbol{A}_2}^C$ and $\boldsymbol{A}_{2(2)} \oplus {\boldsymbol{A}_2}^C$ of $\boldsymbol{E}_{6(2)}$

Theorem 3.4. *The* 3*-graded decomposition of* $\mathfrak{e}_{6(2)} = ({\mathfrak{e}_6}^C)^{\lambda\tau\gamma_1}$,

$$\mathfrak{e}_{6(2)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\mathrm{ad}Z, Z = i(-2G_{23} + G_{45} + G_{67})$, *is given by exchanging*

$$F_k(a) \to iF_k(a), \qquad iF_k(a) \to F_k(a),$$
$$(E_1 - E_2)^\sim \to i(E_1 - E_2)^\sim, \qquad (E_2 - E_3)^\sim \to i(E_2 - E_3)^\sim,$$

in the table of Theorem 3.1.

Proof. We can prove this theorem in a way similar to Theorem 3.1, using [8] Lemma 3.8. □

Using the same notations as in **3.2**, we shall determine the group structures of

$$(E_{6(2)})_{ev} = ({E_6}^C)^{z_2} \cap ({E_6}^C)^{\lambda\tau\gamma_1}, \quad (E_{6(2)})_0 = ({E_6}^C)^{z_4} \cap ({E_6}^C)^{\lambda\tau\gamma_1},$$
$$(E_{6(6)})_{ed} = ({E_6}^C)^{z_3} \cap ({E_6}^C)^{\lambda\tau\gamma_1},$$

Theorem 3.5. (1) $(E_{6(2)})_{ev} \cong (Sp(1,\boldsymbol{R}) \times SU(3,3))/\boldsymbol{Z}_2 \times \{1,\gamma_2\}$, $\boldsymbol{Z}_2 = \{(1,E),(-1,-E)\}$.
(2) $(E_{6(2)})_0 \cong (Sp(1,\boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(3,C)) \times \{1,\gamma_2\}$.
(3) $(E_{6(2)})_{ed} \cong SL(3,\boldsymbol{R}) \times SL(3,C)$.

Proof. (1) For $\alpha \in (E_{6(2)})_{ev} \subset ({E_6}^C)^\gamma$, there exist $p \in Sp(1,\boldsymbol{H}^C)$ and $A \in SL(6,C)$ such that $\alpha = \varphi(p,A)$ (Theorem 3.2 (1)). From $\gamma_1\tau{}^t\alpha^{-1}\tau\gamma_1 = \alpha$, we have $\varphi(\gamma_1\tau p, -J^t(\tau A)^{-1}J) = \varphi(p,A)$ ([8], Lemmas 3.6 (2) and 3.10). Hence

$$\gamma_1\tau p = p,\ -J^t(\tau A)^{-1}J = A \quad \text{or} \quad \gamma_1\tau p = -p,\ -J^t(\tau A)^{-1}J = -A.$$

In the former case, $p \in Sp(1,\boldsymbol{H}') \cong Sp(1,\boldsymbol{R})$ and the group $\{A \in SL(6,C) \,|\, -J^t(\tau A)^{-1}J = A\}$ is $\{A \in SL(6,C) \,|\, {}^t(\tau A)JA = J\} \cong \{A \in SL(6,C) \,|\, {}^t(\tau A)IA = I\} \cong SU(3,3)$. Hence the group of the former case is $(Sp(1,\boldsymbol{R}) \times SU(3,3))/\boldsymbol{Z}_2$. In the latter case, $p = e_1$ and $A = -iI$ satisfy these conditions and $\varphi(e_1,-iI) = \gamma_2$. Therefore $(E_{6(2)})_{ev} \cong (Sp(1,\boldsymbol{R}) \times SU(3,3))/\boldsymbol{Z}_2 \times \{1,\gamma_2\}$.

(2) For $\alpha \in (E_{6(2)})_0 \subset ({E_6}^C)^{\delta_4}$, there exist $p \in Sp(1,\boldsymbol{H}^C)$ and $A \in SL(6,C), \delta A\,\delta^{-1} = A$ ($\delta = \mathrm{diag}(i,-i,i,-i,i,-i) = iI$) such that $\alpha = \varphi(p,A)$ (Theorem 3.2 (2)). From $\gamma_1\tau{}^t\alpha^{-1}\tau\gamma_1 = \alpha$, we have $\varphi(\gamma_1\tau p, -J^t(\tau A)^{-1}J) = \varphi(p,A)$ as in (1) above. Hence

$$\gamma_1\tau p = p, -J^t(\tau A)^{-1}J = A \quad \text{or} \quad \gamma_1\tau p = -p, -J^t(\tau A)^{-1}J = -A.$$

In the former case, $p \in Sp(1, \boldsymbol{H}') \cong Sp(1, \boldsymbol{R})$ and

$$
\begin{aligned}
G &= \{A \in SL(6, C) | \delta A \delta^{-1} = A, {}^t(\tau A)JA = J\} \\
&\cong \left\{ A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in SL(6, C), A_1, A_2 \in GL(3, C) \right. \\
&\qquad \left| \begin{pmatrix} {}^t(\tau A_1) & 0 \\ 0 & {}^t(\tau A_2) \end{pmatrix} \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \right\} \\
&= \left\{ A \in SL(6, C) \,\middle|\, A = \begin{pmatrix} A_1 & 0 \\ 0 & {}^t(\tau A_1)^{-1} \end{pmatrix},\ A_1 \in GL(3, C) \right\}.
\end{aligned}
$$

From $\det A = 1$, we have $\det A_1(\tau(\det A_1)^{-1}) = 1$, so $\det A_1 \in \boldsymbol{R}$, hence $G \cong \{A_1 \in GL(3, C) \,|\, \det A_1 \in \boldsymbol{R}\} \cong \boldsymbol{R}^* \times SL(3, C)$. Thus the group of the former case is $(Sp(1, \boldsymbol{H}') \times \boldsymbol{R}^* \times SL(3, C))/\boldsymbol{Z}_2\,(\boldsymbol{Z}_2 = \{(1, 1, E), (-1, -1, E)\}) \cong Sp(1, \boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(3, C)$. In the latter case, $p = e_1, A = -iI$ satisfy these conditions and $\varphi(e_1, -iI) = \gamma_2$. Therefore $(E_{6(2)})_{ed} \cong (Sp(1, \boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(3, C)) \times \{1, \gamma_2\}$.

(3) For $\alpha \in (E_{6(2)})_{ed} \subset ({E_6}^C)^{w_3}$, there exist $P, A, B \in SU(3, \boldsymbol{C}^C)$ such that $\alpha = \psi(P, A, B)$ (Theorem 3.2 (3)). Since ${}^t\psi(P, A, B)^{-1} = \psi(P, \tau B, \tau A)$ (Theorem 3.2 (3)), we have $\gamma_1 \tau {}^t\psi(P, A, B)^{-1} \tau\gamma_1 = \psi(\tau\gamma_1 P, \overline{B}, \overline{A})$. From $\gamma_1 \tau {}^t\alpha^{-1} \tau\gamma_1 = \alpha$, we have $\psi(\tau\gamma_1 P, \overline{B}, \overline{A}) = \psi(P, A, B)$. Hence

$$
\begin{cases} \tau\gamma_1 P = P \\ \quad B = \overline{A} \end{cases}, \quad \begin{cases} \tau\gamma_1 P = \omega P \\ \quad A = \omega\overline{B} \\ \quad B = \omega\overline{A} \end{cases}, \text{ or } \begin{cases} \tau\gamma_1 P = \omega^2 P \\ \quad A = \omega^2\overline{B} \\ \quad B = \omega^2\overline{A} \end{cases}.
$$

In the first case, $P \in SL(3, \boldsymbol{C}') \cong SL(3, \boldsymbol{R})$ and $A \in SL(3, \boldsymbol{C}^C) \cong SL(3, C)$. The last two cases are false. Therefore $(E_{6(2)})_{ed} \cong SL(3, \boldsymbol{R}) \times SL(3, C)$. □

3.3. Subgroups of type ${\boldsymbol{C}_1}^C \oplus {\boldsymbol{A}_5}^C, \boldsymbol{C} \oplus \boldsymbol{C} \oplus {\boldsymbol{A}_1}^C \oplus {\boldsymbol{A}_3}^C$ and ${\boldsymbol{C}_1}^C \oplus \boldsymbol{C} \oplus {\boldsymbol{A}_4}^C$ of ${\boldsymbol{E}_6}^C$

In the Lie algebra ${\mathfrak{e}_6}^C$, let

$$
Z = i(G_{45} - G_{67}) + \frac{4}{3}(2E_1 - E_2 - E_3)^\sim.
$$

Theorem 3.6. *The* 3*-graded decomposition of* $\mathfrak{e}_{6(6)} = ({\mathfrak{e}_6}^C)^{\tau\gamma_1}$ (*or* ${\mathfrak{e}_6}^C$),

$$
\mathfrak{e}_{6(6)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3
$$

with respect to $\mathrm{ad}Z, Z = i(G_{45} - G_{67}) + (4/3)(2E_1 - E_2 - E_3)^\sim$, *is given by*

$$\mathfrak{g}_0 = \left\{\begin{array}{l} iG_{01},\ G_{02},\ iG_{03},\ iG_{12},\ G_{13},\ iG_{23},\ (E_1-E_2)^\sim, \\ iG_{45},\ iG_{67},\ G_{46}-G_{57},\ i(G_{47}+G_{56}),\ (E_2-E_3)^\sim, \\ \tilde{A}_1(1),\ i\tilde{A}_1(e_1), \tilde{A}_1(e_2),\ i\tilde{A}_1(e_3), \\ \tilde{F}_1(1),\ i\tilde{F}_1(e_1),\ \tilde{F}_1(e_2),\ i\tilde{F}_1(e_3) \end{array}\right\} 20$$

$$\mathfrak{g}_{-1} = \left\{\begin{array}{l} G_{04}+iG_{05},\ G_{06}-iG_{07},\ iG_{14}-G_{15},\ iG_{16}+G_{17}, \\ G_{24}+iG_{25},\ G_{26}-iG_{27},\ iG_{34}-G_{35},\ iG_{36}+G_{37}, \\ \tilde{A}_1(e_4+ie_5),\ \tilde{A}_1(e_6-ie_7),\ \tilde{F}_1(e_4+ie_5),\ \tilde{F}_1(e_6-ie_7), \\ \tilde{A}_2(e_4+ie_5)-\tilde{F}_2(e_4+ie_5),\ \tilde{A}_2(e_6+ie_7)+\tilde{F}_2(e_6+ie_7), \\ \tilde{A}_3(e_4-ie_5)-\tilde{F}_3(e_4-ie_5),\ \tilde{A}_3(e_6+ie_7)-\tilde{F}_3(e_6+ie_7) \end{array}\right\} 16$$

$$\mathfrak{g}_{-2} = \left\{\begin{array}{lll} (G_{46}+G_{57})-i(G_{47}-G_{56}), & & \\ \tilde{A}_2(1)+\tilde{F}_2(1), & i\tilde{A}_2(e_1)+i\tilde{F}_2(e_1), & \tilde{A}_2(e_2)+\tilde{F}_2(e_2), \\ i\tilde{A}_2(e_3)+i\tilde{F}_2(e_3), & \tilde{A}_3(1)-\tilde{F}_3(1), & i\tilde{A}_3(e_1)-i\tilde{F}_3(e_1), \\ \tilde{A}_3(e_2)-\tilde{F}_3(e_2), & i\tilde{A}_3(e_3)-i\tilde{F}_3(e_3) & \end{array}\right\} 9$$

$$\mathfrak{g}_{-3} = \left\{\begin{array}{l} \tilde{A}_2(e_4+ie_5)+\tilde{F}_2(e_4+ie_5),\ \tilde{A}_2(e_6-ie_7)+\tilde{F}_2(e_6-ie_7), \\ \tilde{A}_3(e_4+ie_5)-\tilde{F}_3(e_4+ie_5),\ \tilde{A}_3(e_6-ie_7)-\tilde{F}_3(e_6-ie_7) \end{array}\right\} 4$$

$$\mathfrak{g}_1 = \tau(\lambda(\mathfrak{g}_{-1}))\tau, \quad \mathfrak{g}_2 = \tau(\lambda(\mathfrak{g}_{-2}))\tau, \quad \mathfrak{g}_3 = \tau(\lambda(\mathfrak{g}_{-3}))\tau.$$

Proof. Note that for $D_1 = G_{45} - G_{67} \in \mathfrak{so}(8)$ we have also $D_2 = D_3 = G_{45} - G_{67}$. Then we can prove this theorem in the similar way to [8] Theorem 3.9, using [8] Lemmas 2.3, 3.3 and 3.17. □

Since $iZ = (-G_{45}+G_{67})+(4/3)i(2E_1-E_2-E_3)^\sim = \varphi_*(e_1, \mathrm{diag}(4i/3, 4i/3, -2i/3, -2i/3, -2i/3, -2i/3))$, we have

$$\begin{aligned} z_2 &= \exp\frac{2\pi i}{2}Z = \varphi(-1, \mathrm{diag}(\omega^2,\omega^2,\omega^2,\omega^2,\omega^2,\omega^2)) = \omega\gamma, \\ z_4 &= \exp\frac{2\pi i}{4}Z = \varphi(e_1, \mathrm{diag}(\omega,\omega,-\omega,-\omega,-\omega,-\omega)) \\ &= \omega^2\varphi(e_1, \mathrm{diag}(1,1,-1,-1,-1,-1)), \\ \Big(z_3 &= \exp\frac{2\pi i}{3}Z = \varphi(\omega_1, \mathrm{diag}(\nu^4,\nu^4,\nu^{-2},\nu^{-2},\nu^{-2},\nu^{-2}))\Big), \nu = e^{2\pi i/9}. \end{aligned}$$

Since $Z' = i(-G_{45} - G_{67}) + (4/3)(2E_1 - E_2 - E_3)^\sim$ is conjugate to $Z = i(G_{45} - G_{67}) + (4/3)(2E_1 - E_2 - E_3)^\sim$ under the adjoint action of $({E_6}^C)^{\tau\gamma_1}$, (in fact, for $\delta = \exp(\pi G_{24}) \in F_4 \cap ({E_6}^C)^{\tau\gamma_1}$, we have $\delta^{-1}Z'\delta = Z$), we use the following ${z_3}'$ instead of z_3. Since $iZ' = (G_{45}+G_{67})+(4/3)i(2E_1-E_2-E_3)^\sim = \varphi_*(0, \mathrm{diag}(i/3, 7i/3, -2i/3, -2i/3, -2i/3, -2i/3))$, we have

$${z_3}' = \exp\frac{2\pi i}{3}Z' = \varphi(1, \mathrm{diag}(\nu, \nu^{-2},\nu^{-2},\nu^{-2},\nu^{-2},\nu^{-2})).$$

Now, we shall determine the group structures of

$$({E_6}^C)_{ev} = ({E_6}^C)^{z_2}, \quad ({E_6}^C)_0 = ({E_6}^C)^{z_4}, \quad ({E_6}^C)_{ed} = ({E_6}^C)^{{z_3}'}.$$

Theorem 3.7. (1) $({E_6}^C)_{ev} \cong (Sp(1,C) \times SL(6,C))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(1,E), (-1, -E)\}$.

(2) $({E_6}^C)_0 \cong (C^* \times C^* \times SL(2,C) \times SL(4,C))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2), \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 = \{(1,1,E,\ E),(1,-1,-E,-E),(-1,1,-E,-E),(-1,-1,E,E)\}$.

(3) $({E_6}^C)_{ed} \cong (Sp(1,C) \times C^* \times SL(5,C))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1,1,E),(-1,-1,-E)\}$.

Proof. (1) $({E_6}^C)_{ev} = ({E_6}^C)^{z_2} = ({E_6}^C)^{\omega\gamma} = ({E_6}^C)^{\gamma}$ (since $\omega 1$ is a central element of ${E_6}^C) \cong (Sp(1,C) \times SL(6,C))/\boldsymbol{Z}_2$ (Theorem 3.2 (1)).

(2) Using the restriction mapping $\varphi : U(1,\boldsymbol{C}^C) \times S(GL(2,C) \times GL(4,C)) \to ({E_6}^C)_0 = ({E_6}^C)^{z_4}$ of φ, we can prove this in a similar way to Theorem 3.2 (2).

(3) Using the restriction mapping $\varphi : Sp(1,\boldsymbol{H}^C) \times S(GL(1,C) \times GL(5,C)) \to ({E_6}^C)_{ed} = ({E_6}^C)^{{z_3}'}$ of φ, we can prove this in a way similar to (2) above. □

3.3.1. Subgroups of type $\boldsymbol{C}_{1(1)} \oplus \boldsymbol{A}_{5(5)}, \boldsymbol{R} \oplus \boldsymbol{R} \oplus \boldsymbol{A}_{1(1)} \oplus \boldsymbol{A}_{3(3)}$ and $\boldsymbol{C}_{1(1)} \oplus \boldsymbol{R} \oplus \boldsymbol{A}_{4(4)}$ of $\boldsymbol{E}_{6(6)}$

Using the same notations as in 3.3, we shall determine the group structures of

$$(E_{6(6)})_{ev} = ({E_6}^C)^{z_2} \cap ({E_6}^C)^{\tau\gamma_1}, \quad (E_{6(6)})_0 = ({E_6}^C)^{z_4} \cap ({E_6}^C)^{\tau\gamma_1},$$
$$(E_{6(6)})_{ed} = ({E_6}^C)^{{z_3}'} \cap ({E_6}^C)^{\tau\gamma_1}.$$

Theorem 3.8. (1) $(E_{6(6)})_{ev} \cong (Sp(1,\boldsymbol{R}) \times SL(6,\boldsymbol{R}))/\boldsymbol{Z}_2 \times \{1,\gamma_2\}$, $\boldsymbol{Z}_2 = \{(1,E),(-1,-E)\}$.

(2) $(E_{6(6)})_0 \cong (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times SL(2,\boldsymbol{R}) \times SL(4,\boldsymbol{R})) \times \{1,\gamma_2\}$.

(3) $(E_{6(6)})_{ed} \cong (Sp(1,\boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(5,\boldsymbol{R})) \times \{1,\gamma_2\}$.

Proof. We can prove this theorem from Theorem 3.7, in a way similar to Theorem 3.3 (1) and (2), using [8] Lemma 3.6. □

3.4. Subgroups of type $\boldsymbol{C} \oplus {\boldsymbol{D}_5}^C, \boldsymbol{C} \oplus \boldsymbol{C} \oplus {\boldsymbol{A}_4}^C$ and ${\boldsymbol{C}_1}^C \oplus \boldsymbol{C} \oplus {\boldsymbol{A}_4}^C$ of ${\boldsymbol{E}_6}^C$

In the Lie algebra ${\mathfrak{e}_6}^C$, let

$$Z = i(-G_{45} + 2G_{67}) + \frac{1}{3}(2E_1 - E_2 - E_3)^{\sim}.$$

Theorem 3.9. *The* 3*-graded decomposition of* $\mathfrak{e}_{6(6)} = ({\mathfrak{e}_6}^C)^{\tau\gamma_1}$ (*or* ${\mathfrak{e}_6}^C$),

$$\mathfrak{e}_{6(6)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\mathrm{ad}Z, Z = i(-G_{45} + 2G_{67}) + (1/3)(2E_1 - E_2 - E_3)^{\sim}$, *is given by*

$$\mathfrak{g}_0 = \left\{ \begin{array}{l} iG_{01},\, G_{02},\, iG_{03},\, iG_{12},\, G_{13},\, iG_{23},\, iG_{45},\, iG_{67}, \\ \tilde{A}_1(1), \quad i\tilde{A}_1(e_1), \quad \tilde{A}_1(e_2), \quad i\tilde{A}_1(e_3), \quad (E_1 - E_2)^\sim, \\ \tilde{F}_1(1), \quad i\tilde{F}_1(e_1), \quad \tilde{F}_1(e_2), \quad i\tilde{F}_1(e_3), \quad (E_2 - E_3)^\sim, \\ \tilde{A}_2(1+ie_1) - \tilde{F}_2(1+ie_1), \qquad \tilde{A}_2(1-ie_1) + \tilde{F}_2(1-ie_1), \\ \tilde{A}_2(e_2+ie_3) + \tilde{F}_2(e_2+ie_3), \quad \tilde{A}_2(e_2-ie_3) - \tilde{F}_2(e_2-ie_3), \\ \tilde{A}_3(1+ie_1) - \tilde{F}_3(1+ie_1), \qquad \tilde{A}_3(1-ie_1) + \tilde{F}_3(1-ie_1), \\ \tilde{A}_3(e_2+ie_3) - \tilde{F}_3(e_2+ie_3), \quad \tilde{A}_3(e_2-ie_3) + \tilde{F}_3(e_2-ie_3) \end{array} \right\} 26$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} G_{04} - iG_{05},\, iG_{14} + G_{15},\, G_{24} - iG_{25}, iG_{34} + G_{35}, \\ (G_{46} - G_{57}) + i(G_{47} + G_{56}), \quad \tilde{A}_1(e_4 - ie_5),\, \tilde{F}_1(e_4 - ie_5), \\ \tilde{A}_2(1+ie_1) + \tilde{F}_2(1+ie_1), \qquad \tilde{A}_2(e_2-ie_3) + \tilde{F}_2(e_2-ie_3), \\ \tilde{A}_2(e_4-ie_5) - \tilde{F}_2(e_4-ie_5), \quad \tilde{A}_2(e_6+ie_7) - \tilde{F}_2(e_6+ie_7), \\ \tilde{A}_3(1-ie_1) - \tilde{F}_3(1-ie_1), \qquad \tilde{A}_3(e_2-ie_3) - \tilde{F}_3(e_2-ie_3), \\ \tilde{A}_3(e_4-ie_5) + \tilde{F}_3(e_4-ie_5), \quad \tilde{A}_3(e_6-ie_7) - \tilde{F}_3(e_6-ie_7) \end{array} \right\} 15$$

$$\mathfrak{g}_{-2} = \left\{ \begin{array}{l} G_{06} + iG_{07},\, iG_{16} - G_{17},\, G_{26} + iG_{27},\, iG_{36} - G_{37}, \\ \tilde{A}_1(e_6 + ie_7),\, \tilde{F}_1(e_6 + ie_7), \\ \tilde{A}_2(e_4-ie_5) + \tilde{F}_2(e_4-ie_5),\, \tilde{A}_2(e_6+ie_7) + \tilde{F}_2(e_6+ie_7), \\ \tilde{A}_3(e_4-ie_5) - \tilde{F}_3(e_4-ie_5),\, \tilde{A}_3(e_6+ie_7) - \tilde{F}_3(e_6+ie_7) \end{array} \right\} 10$$

$$\mathfrak{g}_{-3} = \{(G_{46} + G_{57}) + i(G_{47} - G_{56})\}\ 1$$

$$\mathfrak{g}_1 = \tau(\lambda(\mathfrak{g}_{-1}))\tau, \quad \mathfrak{g}_2 = \tau(\lambda(\mathfrak{g}_{-2}))\tau, \quad \mathfrak{g}_3 = \tau(\lambda(\mathfrak{g}_{-3}))\tau.$$

Proof. Note that for $D_1 = -G_{45} + 2G_{67} \in \mathfrak{so}(8)$ we have

$$D_2 = \frac{1}{2}G_{01} - \frac{1}{2}G_{23} - \frac{3}{2}G_{45} + \frac{3}{2}G_{67}, \quad D_3 = -\frac{1}{2}G_{01} - \frac{1}{2}G_{23} - \frac{3}{2}G_{45} + \frac{3}{2}G_{67}.$$

We can then prove this theorem in a way similar to Theorem 3.6. □

Since $iZ = (G_{45} - 2G_{67}) + (1/3)i(2E_1 - E_2 - E_3)^\sim = \varphi_*(-3e_1/2, \mathrm{diag}(5i/6, -i/6, -i/6, -i/6, -i/6, -i/6))$, we have

$$\left(z_2 = \exp\frac{2\pi i}{2}Z. = \varphi(e_1, \mathrm{diag}(\theta^5, \theta^{-1}, \theta^{-1}, \theta^{-1}, \theta^{-1}, \theta^{-1}, \theta^{-1})), \theta = e^{\pi i/6}\right),$$

$$z_4 = \exp\frac{2\pi i}{4}Z = \varphi(-\delta, \mathrm{diag}(\mu^5, \mu^{-1}, \mu^{-1}, \mu^{-1}, \mu^{-1}, \mu^{-1})), \left\{ \begin{array}{l} \delta = e^{\pi e_1/4} \\ \mu = e^{\pi i/12}, \end{array} \right.$$

$$z_3 = \exp\frac{2\pi i}{3}Z = \varphi(-1, \mathrm{diag}(\kappa^5, \kappa^{-1}, \kappa^{-1}, \kappa^{-1}, \kappa^{-1}, \kappa^{-1})), \kappa = e^{\pi i/9}.$$

Since $Z' = i(G_{01} - 2G_{23}) + (1/3)(2E_1 - E_2 - E_3)^\sim$ is conjugate to $Z = i(G_{45} - 2G_{67}) + (1/3)(2E_1 - E_2 - E_3)^\sim$ under the adjoint action of $({E_6}^C)^{\tau\gamma_1}$, (in fact, for $\delta = \exp((\pi/2)(G_{04} + G_{15} + G_{26} + G_{37})) \in F_4 \cap ({E_6}^C)^{\tau\gamma_1}$, we have $\delta^{-1}Z'\delta = Z$), we consider the following ${z_2}'$, moreover ${z_2}''$ instead of z_2. Since $iZ' = (-G_{01} + 2G_{23}) + (1/3)i(2E_1 - E_2 - E_3)^\sim = \varphi_*(0, \mathrm{diag}(i/3, i/3, i/3, -2i/3, 4i/3, -5i/3))$, we have

$$\begin{aligned} {z_2}' &= \exp\frac{2\pi i}{2}Z' = \varphi(1, \mathrm{diag}(-\omega^2, -\omega^2, -\omega^2, \omega^2, \omega^2, -\omega^2)) \\ &= \omega\varphi(1, \mathrm{diag}(-1, -1, -1, 1, 1, -1)) \end{aligned}$$

which is conjugate to

$$z_2{}'' = \omega\varphi(1, \mathrm{diag}(1, 1, -1, -1, -1, -1)) = \omega\sigma$$

under the adjoint action of $SL(6, \boldsymbol{R}) \subset ({E_6}^C)^{\tau\gamma_1}$.

Now, we shall determine the group structures of

$$({E_6}^C)_{ev} = ({E_6}^C)^{z_2{}''}, \quad ({E_6}^C)_0 = ({E_6}^C)^{z_4}, \quad ({E_6}^C)_{ed} = ({E_6}^C)^{z_3}.$$

Theorem 3.10. (1) $({E_6}^C)_{ev} \cong (C^* \times Spin(10, C))/\boldsymbol{Z}_4$, $\boldsymbol{Z}_4 = \{(1, 1), (-1, \sigma), (i, \phi(-i)), (-i, \phi(i))\}$.

(2) $({E_6}^C)_0 \cong (C^* \times C^* \times SL(5, C))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1, 1, E), (-1, -1, -E)\}$.

(3) $({E_6}^C)_{ed} \cong (Sp(1, C) \times C^* \times SL(5, C))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1, 1, E), (-1, -1, -E)\}$.

Proof. (1) Let $Spin(10, C) = ({E_6}^C)_{E_1}$ ([8], Proposition 3.22 (2)). We define $\phi : U(1, \boldsymbol{C}^C) \times Spin(10, C) \to ({E_6}^C)_{ev} = ({E_6}^C)^{z_2{}''} = ({E_6}^C)^{\omega\sigma} = ({E_6}^C)^{\sigma}$ by

$$\phi(\theta, \beta) = \phi(\theta)\beta.$$

Then ϕ is well-defined, is a homomorphism and $\mathrm{Ker}\phi = \boldsymbol{Z}_4$. Since $({E_6}^C)^{\sigma}$ is connected and $\dim_C(\mathfrak{u}(1, \boldsymbol{C}^C) \oplus \mathfrak{so}(10, C)) = 1 + 45 = 46 = 26 + 10 \times 2 = \dim_C(({\mathfrak{e}_6}^C)_{ev})$ (Theorem 3.9), ϕ is onto. Therefore $({E_6}^C)_{ev} \cong (U(1, \boldsymbol{C}^C) \times Spin(10, C))/\boldsymbol{Z}_4 \cong (C^* \times Spin(10, C))/\boldsymbol{Z}_4$.

(2) Using the restriction mapping $\varphi : U(1, \boldsymbol{C}^C) \times S(GL(1, C) \times GL(5, C)) \to ({E_6}^C)^{z_4}$ of φ, we can prove this in a similar way to Theorems 3.2 (2) and 3.7 (2).

(3) is proved in a way similar to Theorem 3.7 (3). □

3.4.1. Subgroups of type $\boldsymbol{R} \oplus \boldsymbol{D}_{5(5)}, \boldsymbol{R} \oplus \boldsymbol{R} \oplus \boldsymbol{A}_{4(4)}$ and $\boldsymbol{C}_{1(1)} \oplus \boldsymbol{R} \oplus \boldsymbol{A}_{4(4)}$ of $\boldsymbol{E}_{6(6)}$

Using the same notations as in 3.4, we shall determine the group structures of

$$(E_{6(6)})_{ev} = ({E_6}^C)^{z_2{}''} \cap ({E_6}^C)^{\tau\gamma_1}, \quad (E_{6(6)})_0 = ({E_6}^C)^{z_4} \cap ({E_6}^C)^{\tau\gamma_1},$$
$$(E_{6(6)})_{ed} = ({E_6}^C)^{z_3} \cap ({E_6}^C)^{\tau\gamma_1}.$$

We define $\rho \in E_6 \subset {E_6}^C$ by

$$\rho = \varphi(1, \mathrm{diag}(1, -1, 1, -1, 1, 1)).$$

Theorem 3.11. (1) $(E_{6(6)})_{ev} \cong (\boldsymbol{R}^+ \times spin(5, 5)) \times \{1, \rho\}$.

(2) $(E_{6(6)})_0 \cong (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times SL(5, \boldsymbol{R})) \times \{1, \gamma_2\}$.

(3) $(E_{6(6)})_{ed} \cong (Sp(1, \boldsymbol{R}) \times \boldsymbol{R}^+ \times SL(5, \boldsymbol{R})) \times \{1, \gamma_2\}$.

Proof. (1) $(E_{6(6)})_{ev} = ({E_6}^C)^{\sigma} \cap ({E_6}^C)^{\tau\gamma_1} \cong (\boldsymbol{R}^+ \times spin(5, 5)) \times \{1, \rho\}$ ([8], Theorem 3.25 (1)).

(2) is proved from Theorem 3.10, in a way similar to Theorem 3.8 (2).
(3) is as same as Theorem 3.8 (3). □

3.5. Subgroups of type $C \oplus D_5{}^C, C \oplus C \oplus A_4{}^C$ and $C \oplus A_5{}^C$ of $E_6{}^C$

In the Lie algebra $\mathfrak{e}_6{}^C$, let

$$Z = i(G_{45} + 2G_{67}) + (2E_1 - E_2 - E_3)^\sim.$$

Theorem 3.12. *The* 3*-graded decomposition of* $\mathfrak{e}_{6(6)} = (\mathfrak{e}_6{}^C)^{\tau\gamma_1}$ (*or* $\mathfrak{e}_6{}^C$),

$$\mathfrak{e}_{6(6)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\mathrm{ad}Z, Z = i(G_{45} + 2G_{67}) + (2E_1 - E_2 - E_3)^\sim$, *is given by*

$$\mathfrak{g}_0 = \left\{ \begin{array}{l} iG_{01},\, G_{02},\, iG_{03},\, iG_{12},\, G_{13},\, iG_{23},\, iG_{45},\, iG_{67}, \\ \tilde{A}_1(1), \quad i\tilde{A}_1(e_1), \quad \tilde{A}_1(e_2), \quad i\tilde{A}_1(e_3), \quad (E_1 - E_2)^\sim, \\ \tilde{F}_1(1), \quad i\tilde{F}_1(e_1), \quad \tilde{F}_1(e_2), \quad i\tilde{F}_1(e_3), \quad (E_2 - E_3)^\sim, \\ \tilde{A}_2(1 + ie_1) - \tilde{F}_2(1 + ie_1), \qquad \tilde{A}_2(1 - ie_1) + \tilde{F}_2(1 - ie_1), \\ \tilde{A}_2(e_2 + ie_3) + \tilde{F}_2(e_2 + ie_3), \quad \tilde{A}_2(e_2 - ie_3) - \tilde{F}_2(e_2 - ie_3), \\ \tilde{A}_3(1 + ie_1) - \tilde{F}_3(1 + ie_1), \qquad \tilde{A}_3(1 - ie_1) + \tilde{F}_3(1 - ie_1), \\ \tilde{A}_3(e_2 + ie_3) - \tilde{F}_3(e_2 + ie_3), \quad \tilde{A}_3(e_2 - ie_3) + \tilde{F}_3(e_2 - ie_3) \end{array} \right\} 26$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} G_{04} + iG_{05},\, iG_{14} - G_{15},\, G_{24} + iG_{25}, iG_{34} - G_{35}, \\ (G_{46} + G_{57}) - i(G_{47} - G_{56}),\, \tilde{A}_1(e_4 + ie_5),\, \tilde{F}_1(e_4 + ie_5), \\ \tilde{A}_2(e_4 + ie_5) + \tilde{F}_2(e_4 + ie_5),\, \tilde{A}_2(e_6 - ie_7) + \tilde{F}_2(e_6 - ie_7), \\ \tilde{A}_3(e_4 + ie_5) - \tilde{F}_3(e_4 + ie_5),\, \tilde{A}_3(e_6 - ie_7) - \tilde{F}_3(e_6 - ie_7) \end{array} \right\} 11$$

$$\mathfrak{g}_{-2} = \left\{ \begin{array}{l} G_{06} + iG_{07},\, iG_{16} - G_{17},\, G_{26} + iG_{27},\, iG_{36} - G_{37}, \\ \tilde{A}_1(e_6 + ie_7),\, \tilde{F}_1(e_6 + ie_7), \\ \tilde{A}_2(e_4 - ie_5) + \tilde{F}_2(e_4 - ie_5),\, \tilde{A}_2(e_6 + ie_7) + \tilde{F}_2(e_6 + ie_7), \\ \tilde{A}_3(e_4 - ie_5) - \tilde{F}_3(e_4 - ie_5),\, \tilde{A}_3(e_6 + ie_7) - \tilde{F}_3(e_6 + ie_7) \end{array} \right\} 10$$

$$\mathfrak{g}_{-3} = \left\{ \begin{array}{l} (G_{46} - G_{57}) + i(G_{47} + G_{56}), \\ \tilde{A}_2(1 + ie_1) + \tilde{F}_2(1 + ie_1), \quad \tilde{A}_2(e_2 - ie_3) + \tilde{F}_2(e_2 - ie_3), \\ \tilde{A}_3(1 - ie_1) - \tilde{F}_3(1 - ie_1), \quad \tilde{A}_3(e_2 - ie_3) - \tilde{F}_3(e_2 - ie_3) \end{array} \right\} 5$$

$$\mathfrak{g}_1 = \tau(\lambda(\mathfrak{g}_{-1}))\tau, \quad \mathfrak{g}_2 = \tau(\lambda(\mathfrak{g}_{-2}))\tau, \quad \mathfrak{g}_3 = \tau(\lambda(\mathfrak{g}_{-3}))\tau.$$

Proof. Note that for $D_1 = G_{45} + 2G_{67} \in \mathfrak{so}(8)$ we have

$$D_2 = \frac{3}{2}G_{01} - \frac{3}{2}G_{23} - \frac{1}{2}G_{45} + \frac{1}{2}G_{67}, \quad D_3 = -\frac{3}{2}G_{01} - \frac{3}{2}G_{23} - \frac{1}{2}G_{45} + \frac{1}{2}G_{67}.$$

We can then prove this theorem in a way similar to Theorem 3.9, using [8] Lemmas 2.14, 3.3 and 3.29. □

Since $iZ = (-G_{45}-2G_{67})+i(2E_1-E_2-E_3)^{\sim} = \varphi_*(-e_1/2, \mathrm{diag}(5i/2, -i/2, -i/2, -i/2, -i/2, -i/2))$, we have

$$\Big(z_2 = \exp\frac{2\pi i}{2}Z = \varphi(-e_1, \mathrm{diag}(i, -i, -i, -i, -i, -i))\Big),$$
$$z_4 = \exp\frac{2\pi i}{4}Z = \varphi(\delta^{-1}, \mathrm{diag}(d^5, d^{-1}, d^{-1}, d^{-1}, d^{-1}, d^{-1})), \begin{cases} \delta = e^{\pi e_1/4} \\ d = e^{\pi i/4}, \end{cases}$$
$$z_3 = \exp\frac{2\pi i}{3}Z = \varphi(-\omega_1, \mathrm{diag}(-\omega, -\omega, -\omega, -\omega, -\omega, -\omega)) = \omega^2\varphi(\omega_1, (1,1,1,1,1,1)).$$

Since $Z' = i(G_{01}+2G_{23}) + (2E_1-E_2-E_3)^{\sim}$ is conjugate to $i(G_{45}+2G_{67}) + (2E_1-E_2-E_3)^{\sim}$ under the adjoint action of $F_4 \cap ({E_6}^C)^{\tau\gamma_1}$ (see **3.4**), we consider the following ${z_2}'$, moreover ${z_2}''$ instead of z_2. Since $iZ' = (-G_{01} - 2G_{23}) + i(2E_1-E_2-E_3)^{\sim} = \varphi_*(0, \mathrm{diag}(i, i, -2i, i, -i, 0))$, we have

$${z_2}' = \exp\frac{2\pi i}{2}Z' = \varphi(1, \mathrm{diag}(-1,-1,1,-1,-1,1)),$$

which is conjugate to

$${z_2}'' = \varphi(1, \mathrm{diag}(1,1,-1,-1,-1,-1)) = \sigma$$

under the adjoint action of $SL(6, \boldsymbol{R}) \subset ({E_6}^C)^{\tau\gamma_1}$.

Now, we shall determine the group structures of

$$({E_6}^C)_{ev} = ({E_6}^C)^{{z_2}''}, \quad ({E_6}^C)_0 = ({E_6}^C)^{z_4}, \quad ({E_6}^C)_{ed} = ({E_6}^C)^{z_3}.$$

Theorem 3.13. (1) $({E_6}^C)_{ev} \cong (C^* \times Spin(10, C))/\boldsymbol{Z}_4$, $\boldsymbol{Z}_4 = \{(1,1), (-1,\sigma), (i, \phi(-i)), (-i, \phi(i))\}$.

(2) $({E_6}^C)_0 \cong (C^* \times C^* \times SL(5, C))/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2 = \{(1,1,E), (-1,-1,-E)\}$.

(3) $({E_6}^C)_{ed} \cong (C^* \times SL(6, C))/\boldsymbol{Z}_2, \boldsymbol{Z}_2 = \{(1,1), (-1,-E)\}$.

Proof. (1) and (2) are proved in a way similar to Theorem 3.10 (1) and (2).

(3) Using the restriction mapping $\varphi : U(1, \boldsymbol{C}^C) \times SL(6, C) \to ({E_6}^C)^{z_3}$ of φ, we can prove this in a way similar to Theorem 3.7 (2). □

3.5.1. Subgroups of type $\boldsymbol{R} \oplus \boldsymbol{D}_{5(5)}, \boldsymbol{R} \oplus \boldsymbol{R} \oplus \boldsymbol{A}_{4(4)}$ and $\boldsymbol{R} \oplus \boldsymbol{A}_{5(5)}$ of $\boldsymbol{E}_{6(6)}$

Using the same notations as in **3.5**, we shall determine the group structures of

$$(E_{6(6)})_{ev} = ({E_6}^C)^{{z_2}''} \cap ({E_6}^C)^{\tau\gamma_1}, \quad (E_{6(6)})_0 = ({E_6}^C)^{z_4} \cap ({E_6}^C)^{\tau\gamma_1},$$
$$(E_{6(6)})_{ed} = ({E_6}^C)^{z_3} \cap ({E_6}^C)^{\tau\gamma_1}.$$

Theorem 3.14. (1) $(E_{6(6)})_{ev} \cong (\boldsymbol{R}^+ \times spin(5,5)) \times \{1, \rho\}$.
(2) $(E_{6(6)})_0 \cong (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times SL(5, \boldsymbol{R})) \times \{1, \gamma_2\}$.
(3) $(E_{6(6)})_{ed} \cong (\boldsymbol{R}^+ \times SL(6, \boldsymbol{R})) \times \{1, \gamma_2\}$.

Proof. (1) is as same as Theorem 3.11 (1).
(2) is as same as Theorem 3.11 (2).
(3) is proved from Theorem 3.13 in a way similar to Theorem 3.8 (2). □

3.6. Subgroups of type $\boldsymbol{C} \oplus \boldsymbol{D}_5{}^C, \boldsymbol{C} \oplus \boldsymbol{C} \oplus \boldsymbol{D}_4{}^C$ and $\boldsymbol{C} \oplus \boldsymbol{D}_5{}^C$ of $\boldsymbol{E}_6{}^C$

In the Lie algebra $\mathfrak{e}_6{}^C$, let

$$Z = 2iG_{01} + \frac{4}{3}(2E_1 - E_2 - E_3)^\sim.$$

Theorem 3.15. *The 3-graded decomposition of* $\mathfrak{e}_{6(6)} = (\mathfrak{e}_6{}^C)^{\tau\gamma_1}$ *(or* $\mathfrak{e}_6{}^C$*),*

$$\mathfrak{e}_{6(6)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\mathrm{ad}Z, Z = 2iG_{01} + (4/3)(2E_1 - E_2 - E_3)^\sim$, *is given by*

$$\mathfrak{g}_0 = \left\{ \begin{array}{l} iG_{01},\ iG_{23},\ G_{24},\ iG_{25},\ G_{26},\ iG_{27},\ iG_{34},\ G_{35},\ (E_1 - E_2)^\sim, \\ iG_{36},\ G_{37},\ iG_{45},\ G_{46},\ iG_{47},\ iG_{56},\ G_{57},\ iG_{67},\ (E_2 - E_3)^\sim, \\ \tilde{A}_1(e_2),\quad i\tilde{A}_1(e_3),\quad \tilde{A}_1(e_4),\quad i\tilde{A}_1(e_5),\quad \tilde{A}_1(e_6),\quad i\tilde{A}_1(e_7), \\ \tilde{F}_1(e_2),\quad i\tilde{F}_1(e_3),\quad \tilde{F}_1(e_4),\quad i\tilde{F}_1(e_5),\quad \tilde{F}_1(e_6),\quad i\tilde{F}_1(e_7) \end{array} \right\} \ 30$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{ll} \tilde{A}_2(1 + ie_1) + \tilde{F}_2(1 + ie_1), & \tilde{A}_2(e_2 + ie_3) + \tilde{F}_2(e_2 + ie_3), \\ \tilde{A}_2(e_4 + ie_5) + \tilde{F}_2(e_4 + ie_5), & \tilde{A}_2(e_6 + ie_7) + \tilde{F}_2(e_6 + ie_7), \\ \tilde{A}_3(1 + ie_1) - \tilde{F}_3(1 + ie_1), & \tilde{A}_3(e_2 - ie_3) - \tilde{F}_3(e_2 - ie_3), \\ \tilde{A}_3(e_4 - ie_5) - \tilde{F}_3(e_4 - ie_5), & \tilde{A}_3(e_6 - ie_7) - \tilde{F}_3(e_6 - ie_7) \end{array} \right\} \ 8$$

$$\mathfrak{g}_{-2} = \left\{ \begin{array}{ll} G_{02} - iG_{12},\ iG_{03} + G_{13},\ G_{04} - iG_{14}, & \tilde{A}_1(1 - ie_1), \\ iG_{05} + G_{15},\ G_{06} - iG_{16},\ iG_{07} + G_{17}, & \tilde{F}_1(1 - ie_1) \end{array} \right\} \ 8$$

$$\mathfrak{g}_{-3} = \left\{ \begin{array}{ll} \tilde{A}_2(1 - ie_1) + \tilde{F}_2(1 - ie_1), & \tilde{A}_2(e_2 - ie_3) + \tilde{F}_2(e_2 - ie_3), \\ \tilde{A}_2(e_4 - ie_5) + \tilde{F}_2(e_4 - ie_5), & \tilde{A}_2(e_6 - ie_7) + \tilde{F}_2(e_6 - ie_7), \\ \tilde{A}_3(1 - ie_1) - \tilde{F}_3(1 - ie_1), & \tilde{A}_3(e_2 + ie_3) - \tilde{F}_3(e_2 + ie_3), \\ \tilde{A}_3(e_4 + ie_5) - \tilde{F}_3(e_4 + ie_5), & \tilde{A}_3(e_6 + ie_7) - \tilde{F}_3(e_6 + ie_7) \end{array} \right\} \ 8$$

$$\mathfrak{g}_1 = \tau(\lambda(\mathfrak{g}_{-1}))\tau, \quad \mathfrak{g}_2 = \tau(\lambda(\mathfrak{g}_{-2}))\tau, \quad \mathfrak{g}_3 = \tau(\lambda(\mathfrak{g}_{-3}))\tau.$$

Proof. Note that for $D_1 = 2G_{01} \in \mathfrak{so}(8)$ we have

$$D_2 = -G_{01} - G_{23} - G_{45} - G_{67}, \quad D_3 = -G_{01} + G_{23} + G_{45} + G_{67}.$$

We can then prove this theorem in a way similar to [8] Theorem 3.21, using [8] Lemmas 3.3 and 3.17. □

Since $iZ = -2G_{01} + (4/3)i(2E_1 - E_2 - E_3)^\sim = \varphi_*(0, \mathrm{diag}(4i/3, 4i/3, -5i/3, i/3, i/3, -5i/3)$, we have

$$z_2 = \exp\frac{2\pi i}{2}Z = \varphi(1, \mathrm{diag}(\omega^2, \omega^2, -\omega^2, -\omega^2, -\omega^2, -\omega^2)) = \omega\sigma,$$
$$z_4 = \exp\frac{2\pi i}{4}Z = \varphi(1, \mathrm{diag}(\omega, \omega, i\omega, -i\omega, -i\omega, i\omega)) = \omega^2 D(e_1),$$
$$\left(z_3 = \exp\frac{2\pi i}{3}Z = \varphi(1, \mathrm{diag}(\nu^4, \nu^4, \nu^4, \nu, \nu, \nu^4))\right),\ \nu = e^{2\pi i/9}.$$

z_3 is conjugate to

$$z_3{}' = \varphi(1, \mathrm{diag}(\nu, \nu, \nu^4, \nu^4, \nu^4, \nu^4))$$

under the adjoint action of $SL(6, \boldsymbol{R}) \subset ({E_6}^C)^{\tau\gamma_1}$. The explicit form of $z_3{}'$ is

$$z_3{}' \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \nu^2\xi_1 & \nu^5 x_3 & \nu^5\overline{x_2} \\ \nu^5\overline{x_3} & \nu^8\xi_2 & \nu^8 x_1 \\ \nu^5 x_2 & \nu^8\overline{x_1} & \nu^8\xi_3 \end{pmatrix}.$$

Now, we shall determine the group structures of

$$({E_6}^C)_{ev} = ({E_6}^C)^{z_2}, \quad ({E_6}^C)_0 = ({E_6}^C)^{z_4}, \quad ({E_6}^C)_{ed} = ({E_6}^C)^{z_3{}'}.$$

Theorem 3.16. (1) $({E_6}^C)_{ev} \cong (C^* \times Spin(10, C))/\boldsymbol{Z}_4$, $\boldsymbol{Z}_4 = \{(1, 1), (-1, \sigma), (i, \phi(-i)), (-i, \phi(i))\}$.

(2) $({E_6}^C)_0 \cong (C^* \times C^* \times Spin(8, C))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_4), \boldsymbol{Z}_2 = \{(1, 1, 1), (1, -1, \sigma)\}, \boldsymbol{Z}_4 = \{(1, 1, 1), (-1, -1, 1), (i, e_1, \phi(-i)D(-e_1)), (-i, -e_1, \phi(i)D(e_1))\}$.

(3) $({E_6}^C)_{ed} \cong (C^* \times Spin(10, C))/\boldsymbol{Z}_4$, $\boldsymbol{Z}_4 = \{(1, 1), (-1, \sigma), (i, \phi(-i)), (-i, \phi(i))\}$.

Proof. (1) $({E_6}^C)_{ev} = ({E_6}^C)^{z_2} = ({E_6}^C)^{\omega\sigma} = ({E_6}^C)^{\sigma} \cong (C^* \times Spin(10, C))/\boldsymbol{Z}_4$ (Theorem 3.10 (1)).

(2) Let $Spin(8, C) = ({E_6}^C)_{E_1, F_1(1), F_1(e_1)}$ ([8], Proposition 3.22 (1)). We define $\phi : C^* \times U(1, \boldsymbol{C}^C) \times Spin(8, C) \to ({E_6}^C)_0 = ({E_6}^C)^{z_4} = ({E_6}^C)^{\omega D(e_1)} = ({E_6}^C)^{D(e_1)}$ by

$$\phi(\theta, a, \beta) = \phi(\theta)D(a)\beta.$$

Then ϕ is well-defined, is a homomorphism and $\mathrm{Ker}\phi = \boldsymbol{Z}_2 \times \boldsymbol{Z}_4$. Since $({E_6}^C)^{D(e_1)}$ is connected and $\dim_C(C \oplus \mathfrak{u}(1, \boldsymbol{C}^C) \oplus \mathfrak{so}(8, C)) = 1 + 1 + 28 = 30 = \dim_C(({\mathfrak{e}_6}^C)_0)$ (Theorem 3.15), ϕ is onto. Therefore $({E_6}^C)_0 \cong (C^* \times U(1, \boldsymbol{C}^C) \times Spin(8, C))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_4) \cong (C^* \times C^* \times Spin(8, C))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_4)$.

(3) C-vector subspaces

$$\{\xi_1 E_1 \,|\, \xi_1 \in C\}, \quad \{F_2(x_2) + F_3(x_3) \,|\, x_2, x_3 \in \mathfrak{C}^C\},$$
$$\{\xi_2 E_2 + \xi_3 E_3 + F_1(x_1) \,|\, \xi_2, \xi_3 \in C, x_1 \in \mathfrak{C}^C\}$$

of $\mathfrak{J}^C$ are invariant under the action of the group $({E_6}^C)^{z_3{}'}$. In particular, $\alpha \in ({E_6}^C)^{z_3{}'}$ commutes with σ. Hence we have $({E_6}^C)^{z_3{}'} \subset ({E_6}^C)^{\sigma}$. Conversely, since $({E_6}^C)^{z_3{}'}, ({E_6}^C)^{\sigma}$ are connected and $\dim_C(({\mathfrak{e}_6}^C)^{z_3{}'}) = 30 + 8 \times 2$

(Theorem 3.15) $= 46 = 1 + 45 = \dim_C(C \oplus \mathfrak{so}(10, C)) = \dim_C((\mathfrak{e}_6{}^C)^\sigma)$, we have $(E_6{}^C)^{z_3{}'} = (E_6{}^C)^\sigma$. Therefore $(E_6{}^C)_{ed} = (E_6{}^C)^\sigma \cong (C^* \times Spin(10, C))/\boldsymbol{Z}_4$ (Theorem 3.10 (1)). □

3.6.1. Subgroups of type $\boldsymbol{R} \oplus \boldsymbol{D}_{5(5)}, \boldsymbol{R} \oplus \boldsymbol{R} \oplus \boldsymbol{D}_{4(4)}$ and $\boldsymbol{R} \oplus \boldsymbol{D}_{5(5)}$ of $\boldsymbol{E}_{6(6)}$

Using the same notations as in **3.6**, we shall determine the group structures of

$$(E_{6(6)})_{ev} = (E_6{}^C)^{z_2} \cap (E_6{}^C)^{\tau\gamma_1}, \quad (E_{6(6)})_0 = (E_6{}^C)^{z_4} \cap (E_6{}^C)^{\tau\gamma_1},$$
$$(E_{6(6)})_{ed} = (E_6{}^C)^{z_3{}'} \cap (E_6{}^C)^{\tau\gamma_1}.$$

Theorem 3.17. (1) $(E_{6(6)})_{ev} \cong (\boldsymbol{R}^+ \times spin(5, 5)) \times \{1, \rho\}$.
(2) $(E_{6(6)})_0 \cong (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times spin(4, 4)) \times (\{1, \sigma'\} \times \{1, \rho\})$.
(3) $(E_{6(6)})_{ed} \cong (\boldsymbol{R}^+ \times spin(5, 5)) \times \{1, \rho\}$.

Proof. (1) and (3) are as same as Theorem 3.11 (1).
(2) is found in [8] Theorem 3.25 (2). □

3.6.2. Subgroups of type $\boldsymbol{R} \oplus \boldsymbol{D}_{5(-27)}$, $\boldsymbol{R} \oplus \boldsymbol{R} \oplus \boldsymbol{D}_{4(-28)}$ and $\boldsymbol{R} \oplus \boldsymbol{D}_{5(-27)}$ of $E_{6(-26)}$

Let $\tau_1 = \delta_1{}^{-1}\tau\delta_1, \delta_1 = \exp(\pi/2)i\tilde{F}_1(1)$ ([8], 3.4.4) and we use the fact that $E_{6(-26)} = (E_6{}^C)^{\tau_1}$.

Theorem 3.18. *The* 3-*graded decomposition of* $\mathfrak{e}_{6(-26)} = (\mathfrak{e}_6{}^C)^{\tau_1}$,

$$\mathfrak{e}_{6(6)} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

with respect to $\mathrm{ad}Z, Z = 2iG_{01} + (4/3)(2E_1 - E_2 - E_3)^\sim$, *is given by*

$$\mathfrak{g}_0 = \left\{ \begin{array}{l} iG_{01}, G_{23}, G_{24}, G_{25}, G_{26}, G_{27}, G_{34}, G_{35}, (2E_1 - E_2 - E_3)^\sim, \\ G_{36}, G_{37}, G_{45}, G_{46}, G_{47}, G_{56}, G_{57}, G_{67}, i(E_2 - E_3)^\sim, \\ \tilde{A}_1(e_2), \quad \tilde{A}_1(e_3), \quad \tilde{A}_1(e_4), \quad \tilde{A}_1(e_5), \quad \tilde{A}_1(e_6), \quad \tilde{A}_1(e_7), \\ i\tilde{F}_1(e_2), \quad i\tilde{F}_1(e_3), \quad i\tilde{F}_1(e_4), \quad i\tilde{F}_1(e_5), \quad i\tilde{F}_1(e_6), \quad i\tilde{F}_1(e_7) \end{array} \right\} 30$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} (\tilde{A}_2(1 + ie_1) + \tilde{F}_2(1 + ie_1)) + i(\tilde{A}_3(1 + ie_1) - \tilde{F}_3(1 + ie_1)), \\ i(\tilde{A}_2(1 + ie_1) + \tilde{F}_2(1 + ie_1)) + (\tilde{A}_3(1 + ie_1) - \tilde{F}_3(1 + ie_1)), \\ (\tilde{A}_2(e_2 + ie_3) + \tilde{F}_2(e_2 + ie_3)) - i(\tilde{A}_3(e_2 - ie_3) - \tilde{F}_3(e_2 - ie_3)), \\ i(\tilde{A}_2(e_2 + ie_3) + \tilde{F}_2(e_2 + ie_3)) - (\tilde{A}_3(e_2 - ie_3) - \tilde{F}_3(e_2 - ie_3)), \\ (\tilde{A}_2(e_4 + ie_5) + \tilde{F}_2(e_4 + ie_5)) - i(\tilde{A}_3(e_4 - ie_5) - \tilde{F}_3(e_4 - ie_5)), \\ i(\tilde{A}_2(e_4 + ie_5) + \tilde{F}_2(e_4 + ie_5)) - (\tilde{A}_3(e_4 - ie_5) - \tilde{F}_3(e_4 - ie_5)), \\ (\tilde{A}_2(e_6 + ie_7) + \tilde{F}_2(e_6 + ie_7)) - i(\tilde{A}_3(e_6 - ie_7) - \tilde{F}_3(e_6 - ie_7)), \\ i(\tilde{A}_2(e_6 + ie_7) + \tilde{F}_2(e_6 + ie_7)) - (\tilde{A}_3(e_6 - ie_7) - \tilde{F}_3(e_6 - ie_7)) \end{array} \right\} 8$$

$$\mathfrak{g}_{-2} = \left\{ \begin{array}{ll} iG_{02} + G_{12}, iG_{03} + G_{13}, iG_{04} + G_{14} & i\tilde{A}_1(1 - ie_1), \\ iG_{05} + G_{15}, iG_{06} + G_{16}, iG_{07} + G_{17} & \tilde{F}_1(1 - ie_1) \end{array} \right\} 8$$

$$\mathfrak{g}_{-3} = \left\{ \begin{array}{l} (\tilde{A}_2(1-ie_1)+\tilde{F}_2(1-ie_1))+i(\tilde{A}_3(1-ie_1)-\tilde{F}_3(1-ie_1)), \\ i(\tilde{A}_2(1-ie_1)+\tilde{F}_2(1-ie_1))+(\tilde{A}_3(1-ie_1)-\tilde{F}_3(1-ie_1)), \\ (\tilde{A}_2(e_2-ie_3)+\tilde{F}_2(e_2-ie_3))-i(\tilde{A}_3(e_2+ie_3)-\tilde{F}_3(e_2+ie_3)), \\ i(\tilde{A}_2(e_2-ie_3)+\tilde{F}_2(e_2-ie_3))-(\tilde{A}_3(e_2+ie_3)-\tilde{F}_3(e_2+ie_3)), \\ (\tilde{A}_2(e_4-ie_5)+\tilde{F}_2(e_4-ie_5))-i(\tilde{A}_3(e_4+ie_5)-\tilde{F}_3(e_4+ie_5)), \\ i(\tilde{A}_2(e_4+ie_5)+\tilde{F}_2(e_4-ie_5))-(\tilde{A}_3(e_4+ie_5)-\tilde{F}_3(e_4+ie_5)), \\ (\tilde{A}_2(e_6-ie_7)+\tilde{F}_2(e_6-ie_7))-i(\tilde{A}_3(e_6+ie_7)-\tilde{F}_3(e_6+ie_7)), \\ i(\tilde{A}_2(e_6-ie_7)+\tilde{F}_2(e_6-ie_7))-(\tilde{A}_3(e_6+ie_7)-\tilde{F}_3(e_6+ie_7)) \end{array} \right\}_8$$

$$\mathfrak{g}_1 = \tau(\lambda(\mathfrak{g}_{-1}))\tau, \quad \mathfrak{g}_2 = \tau(\lambda(\mathfrak{g}_{-2}))\tau, \quad \mathfrak{g}_3 = \tau(\lambda(\mathfrak{g}_{-3}))\tau.$$

Proof. We can prove this theorem in a way similar to [8] Theorem 3.35, using [8] Lemma 3.34. □

Using the same notations as in 3.6, we shall determine the group structures of

$$(E_{6(-26)})_{ev} = ({E_6}^C)^{z_2} \cap ({E_6}^C)^{\tau_1}, \quad (E_{6(-26)})_0 = ({E_6}^C)^{z_4} \cap ({E_6}^C)^{\tau_1},$$
$$(E_{6(-26)})_{ed} = ({E_6}^C)^{z_3} \cap ({E_6}^C)^{\tau_1}.$$

Theorem 3.19. (1) $(E_{6(-26)})_{ev} \cong \boldsymbol{R}^+ \times Spin(9,1))$.
(2) $(E_{6(-26)})_0 \cong (\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times Spin(8)) \times (\{1, \sigma'\} \times \{1, \rho\})$.
(3) $(E_{6(-26)})_{ed} \cong \boldsymbol{R}^+ \times Spin(9,1)$.

Proof. (1) and (3) are found in [8] Theorem 3.37 (1).
(2) is found in [8] Theorem 3.37 (2). □

339-5, Okada-matsuoka
Matsumoto 390-0312 Japan

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