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The invariance of analytic assembly maps under the groupoid equivalence

By

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Introduction

The original motivation for the work of Baum and Connes ([1], [2]) was to construct a geometric or topological version $K^*(M,G)$ of the K-theory group $K_*(C_r^*(M \rtimes G))$, where $C_r^*(M \rtimes G)$ is the reduced C^*-algebra associated to the Lie group action of G on a manifold M. $K^*(M,G)$ is much easier to calculate than $K_*(C_r^*(M \rtimes G))$ since there are geometric and topological tools available for the calculation of $K^*(M,G)$. The cocycles of $K^*(M,G)$ are triples (Z,σ,f), where Z is a proper smooth G-manifold, $f : Z \to M$ is a G-equivariant smooth submersion, and σ is a G-equivariant symbol along the fibers of f. The (reduced) analytic assembly map $\mu_r : K^*(M,G) \to K_*(C_r^*(M \rtimes G))$ is defined as follows: on each fiber the symbol σ gives an elliptic operator D_x, and $\mu_r(Z,\sigma,f)$ is the index of the family (D_x). It is conjectured by P. Baum and A. Connes that this map is an isomorphism.

It has many important implications in topology and analysis. For instance, the rational injectivity of μ_r implies the Novikov conjecture on the homotopy invariance of higher signatures ([11]), and the Gromov-Lawson-Rosenberg conjecture on manifolds admitting metrics of positive scalar curvature ([17]). The surjectivity of μ_r implies the generalized Kadison conjecture on the nonexistence of projections in $C_r^*(\Gamma)$ where Γ is a torsion-free discrete group.

In [6], A. Connes sketched the construction of the analytic assembly map for a general smooth groupoid $\mathcal{G}$,

$$K_{top}^*(\mathcal{G}) \xrightarrow{\mu_{\mathcal{G}}} K_*(C^*(\mathcal{G})).$$

Then he conjectured that the composition of $\mu_{\mathcal{G}}$ with the canonical map from $K_*(C^*(\mathcal{G}))$ to $K_*(C_r^*(\mathcal{G}))$, which will be called the reduced analytic assembly map, is an isomorphism. This conjecture will be called the *Baum-Connes conjecture for $\mathcal{G}$*.

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In this paper, we explicitly construct analytic assembly maps for general smooth groupoids and then prove that they are invariant under the groupoid equivalence. Since C^*-algebras of two equivalent groupoids have the same K-theory, this result is a strong evidence for the Baum-Connes conjecture for general smooth groupoids.

1. Some basic facts on groupoids

The contents in this section are well known. In order to fix the notation, we have collected them here.

Definition 1.1. A *groupoid* consists of a set $\mathcal{G}$, a subset $\mathcal{G}^{(0)} \subset \mathcal{G}$, two maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$, and a law of composition $\cdot : \mathcal{G}^{(2)} \to \mathcal{G}$, where

$$\mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G} | s(\gamma_1) = r(\gamma_2)\},$$

satisfying the following:

$s(\gamma_1 \cdot \gamma_2) = s(\gamma_2)$, $r(\gamma_1 \cdot \gamma_2) = r(\gamma_1)$ for any $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$,

1. $s(x) = r(x) = x$ for any $x \in \mathcal{G}^{(0)}$,
2. $\gamma \cdot s(\gamma) = \gamma$, $r(\gamma) \cdot \gamma = \gamma$ for any $\gamma \in \mathcal{G}$,
3. $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$ for $(\gamma_1, \gamma_2), (\gamma_2, \gamma_3) \in \mathcal{G}^{(2)}$, and
4. each $\gamma \in \mathcal{G}$ has a two-sided inverse γ^{-1}, with

$$\gamma \cdot \gamma^{-1} = r(\gamma), \quad \gamma^{-1} \cdot \gamma = s(\gamma).$$

We may regard a groupoid $\mathcal{G}$ as a small category where every morphism is an isomorphism. Indeed, we take $\mathcal{G}^{(0)}$ as the collection of objects. Then the collection of morphisms from x to y consists of γ's with $s(\gamma) = x$ and $r(\gamma) = y$. A *subgroupoid* can be defined as a subcategory of $\mathcal{G}$. A subgroupoid is called a *full subgroupoid* if it is a full subcategory of $\mathcal{G}$. A subgroupoid $\mathcal{G}'$ of $\mathcal{G}$ is called a *component subgroupoid* if it satisfies the following: if γ is any element in $\mathcal{G}$, and if $x = s(\gamma)$ lies in $\mathcal{G}'$, then $y = r(\gamma)$ also lies in $\mathcal{G}'$.

Definition 1.2. A *smooth groupoid* is a groupoid $\mathcal{G}$ with differential structures on $\mathcal{G}$ and $\mathcal{G}^{(0)}$ in which the maps r, s are submersions, and the inclusion $\mathcal{G}^{(0)} \to \mathcal{G}$ is smooth as well as the composition $\mathcal{G}^{(2)} \to \mathcal{G}$. We allow $\mathcal{G}$ to be a manifold with boundary. In this case, we require that the boundary is a full component subgroupoid of $\mathcal{G}$.

Example 1.3. (1) Any manifold P is a smooth groupoid, where the set of units is all of P. It has no composition structure except the trivial compositions, $x \cdot x = x$ for $x \in P$. This is called a trivial groupoid.

(2) A Lie group G is a smooth groupoid, where $G^{(0)} = \{e\}$, $s(g) = r(g) = e$ and the composition is the group multiplications. Here e denotes the identity of G. This is the opposite case to the above example.

(3) Assume that a Lie group G acts on a manifold M from the right. We take $\mathcal{G} = M \times G$, $\mathcal{G}^{(0)} = M \times \{e\}$, and $r(x, g) = x$, $s(x, g) = xg$. The composition is given by

$$(x, g_1) \cdot (xg_1, g_2) = (x, g_1 g_2).$$

This is a smooth groupoid, which will be denoted by $M \rtimes G$.

(4) Let $\{\mathcal{G}_\alpha\}_{\alpha\in I}$ be a collection of groupoids indexed by I. Then the disjoint union $\mathcal{G} = \cup_{\alpha\in I}\mathcal{G}_\alpha$ is a groupoid. Note that $\mathcal{G}_\alpha$ is a full component subgroupoid of $\mathcal{G}$. When each $\mathcal{G}_\alpha$, $\mathcal{G}$ and I are smooth, and the canonical map $p : \mathcal{G} \to I$ is a submersion, $\mathcal{G}$ is called a *smooth groupoid of parameterized groupoids over* I. In particular, the total space E of a smooth vector bundle $p : E \to M$ is a smooth groupoid of parameterized abelian groups over M.

(5) Let M be a manifold. $\mathcal{G} = M \times M$ is a smooth groupoid, where $\mathcal{G}^{(0)}$ is the diagonal identified with M, $r(x, y) = x$, $s(x, y) = y$, and $(x, y) \cdot (y, z) = (x, z)$.

Let $\Omega^{1/2}$ is the line bundle over a smooth groupoid $\mathcal{G}$ whose fiber at $\gamma \in \mathcal{G}$, $r(\gamma) = x$, $s(\gamma) = y$, is the linear space of maps

$$\rho : \Lambda^k T_\gamma(\mathcal{G}^x) \otimes \Lambda^k T_\gamma(\mathcal{G}_y) \to \mathbb{C}$$

such that $\rho(\lambda v) = |\lambda|^{1/2}\rho(v)$ for $\lambda \in \mathbb{R}$. Here

$$\mathcal{G}^x = \{\gamma \in \mathcal{G} \mid r(\gamma) = x\}, \quad \mathcal{G}_y = \{\gamma \in \mathcal{G} \mid s(\gamma) = y\},$$

and $k = \dim(\mathcal{G}^x) = \dim(\mathcal{G}_y)$. We consider the linear space $C_c^\infty(\mathcal{G}, \Omega^{1/2})$ of compactly supported smooth sections of $\Omega^{1/2}$, We define a convolution product and a $*$-operation on $C_c^\infty(\mathcal{G}, \Omega^{1/2})$: for $f, g \in C_c^\infty(\mathcal{G}, \Omega^{1/2})$,

$$(f * g)(\gamma) = \int_{\gamma_1\cdot\gamma_2=\gamma} f(\gamma_1)g(\gamma_2),$$
$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

They define a $*$-algebra structure on $C_c^\infty(\mathcal{G}, \Omega^{1/2})$. To obtain a C^*-algebra, we need to take a completion of $C_c^\infty(\mathcal{G}, \Omega^{1/2})$. Usually, it is completed in two ways; the maximal C^*-algebra $C^*(\mathcal{G})$ and the reduced C^*-algebra $C_r^*(\mathcal{G})$. We omit it and refer to [6].

Definition 1.4. Let $\mathcal{G}$ be a smooth groupoid. Regarding $\mathcal{G}$ as a small category, we define a right $\mathcal{G}$-action on a smooth manifold P as a contravariant functor F from $\mathcal{G}$ to the category $\mathcal{M}$ of smooth manifolds and smooth maps satisfying the following three properties.

1. Let P_x denote $F(x)$, for $x \in \mathcal{G}^{(0)}$. P_x's are submanifolds of P and they form a partition of P.
2. The map $\sigma : P \to \mathcal{G}^{(0)}$, given by $\sigma(p) = x$, when $p \in P_x$, is a submersion.
3. The map

$$P \times_{\mathcal{G}^{(0)}} \mathcal{G} \to P$$
$$(p, \gamma) \mapsto F(\gamma)(p)$$

is smooth, where $P \times_{\mathcal{G}^{(0)}} \mathcal{G}$ is the fibered product, that is $P \times_{\mathcal{G}^{(0)}} \mathcal{G} = \{(p, \gamma) \in P \times \mathcal{G} \ : \ \sigma(p) = r(\gamma)\}$.

P will be called a $\mathcal{G}$-manifold. We abbreviate $F(\gamma)(p)$ by $p \cdot \gamma$, or simply by $p\gamma$.

Remark. (1) Similarly, we may define a left $\mathcal{G}$-action. It is defined as a covariant functor from $\mathcal{G}$ to $\mathcal{M}$. It is obvious how to modify the above definition.

(2) $P \times_{\mathcal{G}^{(0)}} \mathcal{G}$ has a natural groupoid structure. We put $(P \times_{\mathcal{G}^{(0)}} \mathcal{G})^{(0)} = P$,[*1] $r(p,\gamma) = p$, $s(p,\gamma) = p \cdot \gamma$, and $(p,\gamma) \cdot (p \cdot \gamma, \gamma') = (p, \gamma\gamma')$. We denote this groupoid by $P \rtimes \mathcal{G}$.

Example 1.5. (1) For any smooth groupoid $\mathcal{G}$ there is a natural action of $\mathcal{G}$ on $\mathcal{G}^{(0)}$. The functor F sends $\gamma : x \to y$ to the trivial map $\{y\} \to \{x\}$.

(2) A smooth groupoid $\mathcal{G}$ acts on itself by the groupoid multiplication.

(3) Let $h : E_1 \to E_2$ be a vector bundle map over a manifold M. Then the groupoid E_1 acts on E_2. We let $F(x) = E_{2,x}$ and $v_1 \in E_{1,x}$ acts on $E_{2,x}$ as follows: $v_2 \cdot v_1 = v_2 + h(v_1)$, for any $v_2 \in E_{2,x}$.

(4) A group Γ acts on V. Let $E\Gamma$ be the universal Γ-bundle. Then the groupoid $V \rtimes \Gamma$ acts on $V \times E\Gamma$ freely and properly as follows: let $\sigma(v,x) = v \in V = (V \rtimes \Gamma)^{(0)}$ and $(v,x) \cdot (v,g) := (vg, xg)$. The quotient space of $V \times E\Gamma$ by this groupoid action is $V \times_\Gamma E\Gamma$. $V \times E\Gamma \to V \times_\Gamma E\Gamma$ is the universal $V \rtimes \Gamma$-bundle.

(5) For a $\mathcal{G}$-manifold P we obtain another $\mathcal{G}$-manifold $T_{\mathcal{G}}P$ by replacing each P_x, $x \in \mathcal{G}^{(0)}$, by its tangent bundle $T(P_x)$. The total space $T_{\mathcal{G}}P$ is the kernel of the map $d\sigma$. γ acts as the differential from $T(P_y)$ to $T(P_x)$, where $x = s(\gamma)$, $y = r(\gamma)$.

Definition 1.6. (1) A $\mathcal{G}$-manifold P is said to be *proper* if the following map is proper:

$$\begin{aligned} P \times_{\mathcal{G}^{(0)}} \mathcal{G} &\to P \times P \\ (p,\gamma) &\to (p, p \cdot \gamma). \end{aligned}$$

(2) A smooth groupoid $\mathcal{G}$ is *proper* if the following map is proper:

$$\begin{aligned} &\mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)} \\ &\gamma \to (r(\gamma), s(\gamma)). \end{aligned}$$

Remark. Note that a $\mathcal{G}$-manifold P is proper if and only if $P \rtimes \mathcal{G}$ is proper.

Definition 1.7 ([14])**.** Let $\mathcal{G}$ and $\mathcal{H}$ be smooth groupoids. They are said to be (smoothly) equivalent if there exists a manifold Z such that

1. $\mathcal{G}$ has a free and proper left action on Z with $\rho : Z \to \mathcal{G}^{(0)}$,
2. $\mathcal{H}$ has a free and proper right action on Z with $\sigma : Z \to \mathcal{H}^{(0)}$,
3. the $\mathcal{G}$ and $\mathcal{H}$ actions commute,
4. the map ρ induces a diffeomorphism of $Z/\mathcal{H}$ onto $\mathcal{G}^{(0)}$, and
5. the map σ induces a diffeomorphism of $\mathcal{G}\backslash Z$ onto $\mathcal{H}^{(0)}$.

Z is said to be a $(\mathcal{G},\mathcal{H})$-equivalence.

[*1] We identify p with $(p, \sigma(p))$.

Remark. (1) Indeed, this is an equivalence relation. If Z is a $(\mathcal{G}, \mathcal{H})$-equivalence and Y is a $(\mathcal{H}, \mathcal{K})$-equivalence, then a $(\mathcal{G}, \mathcal{K})$-equivalence is given by the quotient of $Z \times_{\mathcal{H}^{(0)}} Y$ obtained by the diagonal action of $\mathcal{H}$.

(2) G is naturally isomorphic to $(Z \times_\sigma Z)/\mathcal{H}$ where

$$(Z \times_\sigma Z) = \{(z_1, z_2) \in Z \times Z : \sigma(z_1) = \sigma(z_2)\}.$$

For any $[z_1, z_2] \in (Z \times_\sigma Z)/\mathcal{H}$, there is a unique $\gamma \in \mathcal{G}$ such that $\gamma \cdot z_1 = z_2$. (Note that $\sigma(z_1) = \sigma(z_2)$.) The correspondence $[z_1, z_2] \mapsto \gamma$ is the desired isomorphism between $(Z \times_\sigma Z)/\mathcal{H}$ and $\mathcal{G}$.

Example 1.8. (1) Let $\mathcal{G}_\mathcal{F}$ be the holonomy groupoid of a foliated space $(M, \mathcal{F})$ ([20]), and $T \subset M$ be a complete transversal, that is, a transversal which meets every leaf (but T need not be connected). Then $\mathcal{G}_T^T = \{\gamma \in \mathcal{G}_\mathcal{F} : r(\gamma), s(\gamma) \in T\}$ is an etale (or discrete) groupoid which is equivalent to $\mathcal{G}_\mathcal{F}$. We take $\mathcal{G}_T = \{\gamma : s(\gamma) \in T\}$ as a $(\mathcal{G}_\mathcal{F}, \mathcal{G}_T^T)$-equivalence.

(2) Let $\mathcal{G}$ be a transitive groupoid. Then for any unit $x \in \mathcal{G}^{(0)}$, the Lie group $H = \mathcal{G}_x^x$ is equivalent to $\mathcal{G}$. $\mathcal{G}_x$ is a $(\mathcal{G}, H)$-equivalence. Here $\mathcal{G}_x^x = \mathcal{G}^x \cap \mathcal{G}_x$.

(3) Suppose that two Lie groups H and K act freely and properly on a manifold M and assume that their actions commute. The manifold M/H (respectively, M/K) carries a K (respectively, H) action. With these two actions M is a $(M/K \rtimes H, M/H \rtimes K)$-equivalence.

(4) If a smooth groupoid $\mathcal{G}$ acts on P freely and properly, then the trivial groupoid $P/\mathcal{G}$ is equivalent to $P \rtimes \mathcal{G}$ with P a $(P/\mathcal{G}, P \rtimes \mathcal{G})$-equivalence. $P/\mathcal{G}$ acts on P trivially and the right action of $P \rtimes \mathcal{G}$ on P is given by $p \cdot (p, \gamma) = p \cdot \gamma$.

(5) For a proper groupoid $\mathcal{G}$, consider the subgroupoid $\widehat{\mathcal{G}}$ which is the inverse image of the diagonal under the map

$$\mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}.$$

It is easy to see that $\mathcal{G}$ is equivalent to $\widehat{\mathcal{G}}$. Since $\mathcal{G}$ is proper, its automorphism groups $\mathcal{G}_x^x$ are compact. So $\widehat{\mathcal{G}}$ is a parameterized compact groups. Hence we conclude that a proper groupoid is equivalent to a groupoid which is a parameterized compact groups. In particular, for a proper $\mathcal{G}$-manifold P, the groupoid $P \rtimes \mathcal{G}$ is equivalent to a parameterized compact groups.

For equivalent groupoids $\mathcal{G}$ and $\mathcal{H}$ with a $(\mathcal{G}, \mathcal{H})$-equivalence Z as in Definition 1.7, we define a left $C_c(\mathcal{G})$-action and a right $C_c(\mathcal{H})$-action on $C_c(Z)$ as follows: for $f \in C_c(\mathcal{G})$, $g \in C_c(\mathcal{H})$ and $\varphi \in C_c(Z)$,

$$(f \cdot \varphi)(z) = \int_{\mathcal{G}^{\rho(z)}} f(\gamma)\varphi(\gamma^{-1} \cdot z),$$
$$(\varphi \cdot g)(z) = \int_{\mathcal{H}^{\sigma(z)}} \varphi(z \cdot \delta) g(\delta^{-1}).$$

Then $f \cdot \varphi$ and $\varphi \cdot g$ are in $C_c(Z)$ ([14]).

Now we define a $C_c(\mathcal{H})$-valued inner-product on $C_c(Z)$:

$$\langle \varphi, \psi \rangle_{C_c(\mathcal{H})}(\delta) = \int_{\mathcal{G}^{\rho(z)}} \overline{\varphi(\gamma^{-1} \cdot z)} \psi(\gamma^{-1} \cdot z \cdot \delta),$$

where $\sigma(z) = r(\delta)$. Note that the definition is independent of the choice of z with $\sigma(z) = r(\delta)$.

Similarly, we define a $C_c(\mathcal{G})$-valued inner product:

$$\langle \varphi, \psi \rangle_{C_c(\mathcal{G})}(\gamma) = \int_{\mathcal{H}^{\sigma(z)}} \varphi(\gamma^{-1} \cdot z \cdot \delta)\overline{\psi(z \cdot \delta)},$$

where $\rho(z) = r(\gamma)$.

Then the following identities are straightforward to prove:

$$\begin{aligned} f \cdot \langle \varphi, \psi \rangle_{C_c(\mathcal{G})} &= \langle f \cdot \varphi, \psi \rangle_{C_c(\mathcal{G})}, \\ \langle \varphi, \psi \rangle_{C_c(\mathcal{H})} \cdot g &= \langle \varphi, \psi \cdot g \rangle_{C_c(\mathcal{H})}, \\ (f_1 * f_2) \cdot \varphi &= f_1 \cdot (f_2 \cdot \varphi), \\ \varphi \cdot (g_1 * g_2) &= (\varphi \cdot g_1) \cdot g_2, \\ \langle \varphi, \psi \rangle^*_{C_c(\mathcal{G})} &= \langle \psi, \varphi \rangle_{C_c(\mathcal{G})}, \\ \langle \varphi, \psi \rangle^*_{C_c(\mathcal{H})} &= \langle \psi, \varphi \rangle_{C_c(\mathcal{H})}. \end{aligned}$$

The following is the main theorem of [14].

Theorem 1.9. *The $C_c(\mathcal{G}) - C_c(\mathcal{H})$-bimodule $C_c(Z)$ defined above can be naturally completed into a $C^*(\mathcal{G}) - C^*(\mathcal{H})$-equivalence bimodule $\mathcal{E}$. That is, $C^*(\mathcal{G})$ and $C^*(\mathcal{H})$ are Morita equivalent.*

For those who are not familiar with Morita equivalence, we refer to [16].

Remark. (1) So the $C^*(\mathcal{G})-C^*(\mathcal{H})$-equivalence bimodule $\mathcal{E}$ defines an invertible element $[\mathcal{E}] \in KK(C^*(\mathcal{G}), C^*(\mathcal{H}))$. Hence we have an isomorphism

$$K_*(C^*(\mathcal{G})) \xrightarrow{\cdot \otimes [\mathcal{E}]} K_*(C^*(\mathcal{H})),$$

where $\cdot \otimes [\mathcal{E}]$ denotes the Kasparov product by $[\mathcal{E}]$. This isomorphism is called *the isomorphism induced by the $(\mathcal{G}, \mathcal{H})$-equivalence Z.*

(2) The above theorem still holds when we take the reduced C^*-algebras. That is, the $C_c(\mathcal{G})$-$C_c(\mathcal{H})$-bimodule $C_c(Z)$ is naturally completed into a $C_r^*(\mathcal{G})$-$C_r^*(\mathcal{H})$-equivalence bimodule $\mathcal{E}_r$. We have the following commutative diagram:

$$\begin{array}{ccc} K_*(C^*(\mathcal{G})) & \xrightarrow{(r_{\mathcal{G}})_*} & K_*(C_r^*(\mathcal{G})) \\ {\scriptstyle \cdot\otimes[\mathcal{E}]}\downarrow & & \downarrow{\scriptstyle \cdot\otimes[\mathcal{E}_r]} \\ K_*(C^*(\mathcal{H})) & \xrightarrow[(r_{\mathcal{H}})_*]{} & K_*(C_r^*(\mathcal{H})) \end{array}$$

where $(r_{\mathcal{G}})_*$ and $(r_{\mathcal{H}})_*$ are the induced maps from the canonical surjections $r_{\mathcal{G}} : C^*(\mathcal{G}) \to C_r^*(\mathcal{G})$ and $r_{\mathcal{H}} : C^*(\mathcal{H}) \to C_r^*(\mathcal{H})$, respectively.

Definition 1.10. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be smooth groupoids. A *strong deformation* from $\mathcal{G}_1$ to $\mathcal{G}_2$ is given by another smooth groupoid of parameterized groupoids over $[0, 1)$ (see (4) of Example 1.3) whose restriction to $(0, 1)$ is $\mathcal{G}_2 \times (0, 1)$ and restriction to 0 is $\mathcal{G}_1$.

Remark. A strong deformation $\mathcal{G}$ from $\mathcal{G}_1$ to $\mathcal{G}_2$ defines a continuous field of C^*-algebras over $[0,1)$, where the restriction to $(0,1)$ is a constant field with fiber $C^*(\mathcal{G}_2)$, and the fiber over 0 is $C^*(\mathcal{G}_1)$. Hence $\mathcal{G}$ defines an E-theory element in $E(C^*(\mathcal{G}_1), C^*(\mathcal{G}_2))$. Recall that a cycle of $E(A,B)$ is an asymptotic homomorphism from $\mathcal{K}\otimes A$ to $\mathcal{K}\otimes B$, where $\mathcal{K}$ is the C^*-algebra of compact operators on a separable Hilbert space. An asymptotic homomorphism from a C^*-algebra A to another C^*-algebra B is a family $\{\varphi_t\}_{t\in(1,\infty)}$ of maps from A to B, satisfying the following two conditions:

1. For any $a \in A$, the map $t \to \varphi_t(a)$ is norm continuous.
2. For any a, $b \in A$, $\lambda \in \mathbb{C}$, we have

$$\lim_{t\to\infty}(\varphi_t(a)+\lambda\varphi_t(b)-\varphi_t(a+\lambda b))=0,$$
$$\lim_{t\to\infty}(\varphi_t(ab)-\varphi_t(a)\varphi_t(b))=0,$$
$$\lim_{t\to\infty}(\varphi_t(a^*)-\varphi_t(a)^*)=0.$$

If we are given a continuous field $(A(t),\Gamma)$ of C^*-algebras over the interval $[0,1)$ whose fiber at 0 is $A(0)=A$, and whose restriction to $(0,1)$ is the constant field with fiber $A(t)=B$ for $t\in(0,1)$, then we obtain an asymptotic homomorphism from A to B: for any $a\in A=A(0)$, choose a continuous section $\sigma_a\in\Gamma$, and define $\varphi_t(a)=\sigma_a(1/t)$. For more details about E-theory, see [7], or [9].

2. Construction of analytic assembly maps for general groupoids

Definition 2.1 (Semi-direct products)**.** A smooth groupoid $\mathcal{G}$ acts on another smooth groupoid $\mathcal{H}$ with $\tau:\mathcal{H}\to\mathcal{G}^{(0)}$. Assume that this action satisfies the following conditions.

1. For each $x\in\mathcal{G}^{(0)}$, $\tau^{-1}(x)$ is a full component subgroupoid of $\mathcal{G}$. Hence, if δ and $\delta'\in\mathcal{H}$ are composable and one of $\delta\cdot\gamma$ and $\delta'\cdot\gamma$ is defined, then the other one as well as $(\delta\delta')\cdot\gamma$ are defined.

2. For $\delta,\delta'\in\tau^{-1}(x)$, $\gamma\in\mathcal{G}$ with $r(\gamma)=x$, where δ and δ' are composable, then $\delta\cdot\gamma$ and $\delta'\cdot\gamma$ are also composable with the equality

$$(\delta\delta')\cdot\gamma=(\delta\cdot\gamma)(\delta'\cdot\gamma).$$

Then we define $\mathcal{H}\overrightarrow{\rtimes}\mathcal{G}$, *the semi-direct product of* $\mathcal{H}$ *by the action of* $\mathcal{G}$ as follows. As a manifold, $\mathcal{H}\overrightarrow{\rtimes}\mathcal{G}$ is $\mathcal{H}\times_{\mathcal{G}^{(0)}}\mathcal{G}=\{(\delta,\gamma)\colon\ \tau(\delta)=r(\gamma)\ \}$. (δ_1,γ_1) and (δ_2,γ_2) are composable if and only if γ_1, γ_2 are composable and $\delta_2=\delta_2'\cdot\gamma_1$ with δ_1 and δ_2' composable. Their composition is given by

$$(\delta_1,\gamma_1)(\delta_2'\cdot\gamma_1,\gamma_2)=(\delta_1\delta_2',\gamma_1\gamma_2).$$

The other maps are given by $s(\delta,\gamma)=(s(\delta)\cdot\gamma,s(\gamma))$, $r(\delta,\gamma)=(r(\delta),r(\gamma))$, and $(\delta,\gamma)^{-1}=(\delta^{-1}\cdot\gamma,\gamma^{-1})$. The units are $\mathcal{H}^{(0)}$, identifying $(u,\tau(u))$ with u.

Remark. (1) Note that the groupoid $P\rtimes\mathcal{G}$ is an example of a semi-direct product, where P is regarded as a trivial groupoid.

(2) When a group G acts on another group H as group homomorphisms, then $H \overrightarrow{\rtimes} G$ is the usual semi-direct product. But, in taking $H \rtimes G$, we completely forget the group structure of H.

In [6], A. Connes constructed the tangent groupoid $\mathcal{G}_M$ for a Riemannian manifold M, which is the union of TM and $(M \times M) \times (0,1)$. This is a strong deformation from TM to $M \times M$. Its induced map from $K_*(C^*(TM)) = K^*(T^*M)$ to $K_*(\mathcal{K}) = \mathbb{Z}$ is the Atiyah-Singer index map. For details, see [5], [6]. We generalize the tangent groupoid for the $\mathcal{G}$-equivariant case. Before we do so, we need to make some observations. For a proper $\mathcal{G}$-manifold P, the action of $\mathcal{G}$ on $T_{\mathcal{G}}P$ satisfies the conditions of the above definition. Hence we may take the semi-direct product $T_{\mathcal{G}}P \overrightarrow{\rtimes} \mathcal{G}$. We have an isomorphism between $C^*(T_{\mathcal{G}}P \overrightarrow{\rtimes} \mathcal{G})$ and $C^*(T^*_{\mathcal{G}}P \rtimes \mathcal{G})$, where $\gamma \in \mathcal{G}$ acts on $T^*_{\mathcal{G}}P$ as the inverse of the codifferential. The proof is essentially the same as that of the following simple fact: for a group G acting on an abelian group H as homomorphisms, we have $C^*(H \overrightarrow{\rtimes} G) \cong C^*(\widehat{H} \rtimes G) = C_0(\widehat{H}) \rtimes G$, which is due to the Fourier transform. Here $\widehat{H}$ denotes the dual group of H.

Definition 2.2. Let P be a proper $\mathcal{G}$-manifold with a submersion $\sigma : P \to \mathcal{G}^{(0)}$. We put a $\mathcal{G}$-invariant metric on $T_{\mathcal{G}}P$. We can do so because the $\mathcal{G}$-action is proper. We denote the tangent groupoid of $P_x = \sigma^{-1}(x)$ by $\mathcal{G}_{P_x}$. Then let $\mathcal{G}'_P$ be the smooth groupoid obtained by putting together $\mathcal{G}_{P_x}$'s. That is, $\mathcal{G}'_P$ is the closure of $(P \times_\sigma P) \times (0,1)$ in the usual tangent groupoid $\mathcal{G}_P$ of P, where

$$P \times_\sigma P = \{(p_1,p_2) \in P \times P : \sigma(p_1) = \sigma(p_2)\}.$$

$P \times_\sigma P$ is a groupoid with $s(p_1,p_2) = p_2$, $(p_1,p_2)(p_2,p_3) = (p_1,p_3)$. As a set, $\mathcal{G}'_P$ is equal to $T_{\mathcal{G}}P \cup [(P \times_\sigma P) \times (0,1)]$. The $\mathcal{G}$-action on P induces another $\mathcal{G}$-action on $\mathcal{G}'_P$ which satisfies the conditions in Definition 2.1. Therefore, we can take the semi-direct product $\mathcal{G}'_P \overrightarrow{\rtimes} \mathcal{G}$, which is a strong deformation from $T_{\mathcal{G}}P \overrightarrow{\rtimes} \mathcal{G}$ to $(P \times_\sigma P) \overrightarrow{\rtimes} \mathcal{G}$. But it is easy to show that $(P \times_\sigma P) \overrightarrow{\rtimes} \mathcal{G}$ is equivalent to $\mathcal{G}$. Indeed we take $P \times_{\mathcal{G}^{(0)}} \mathcal{G}$ as an equivalence between these two groupoids. We define

$$\rho' : P \times_{\mathcal{G}^{(0)}} \mathcal{G} \to ((P \times_\sigma P) \overrightarrow{\rtimes} \mathcal{G})^{(0)} = P,$$
$$\sigma' : P \times_{\mathcal{G}^{(0)}} \mathcal{G} \to \mathcal{G}^{(0)},$$

as $\rho'(p,\gamma) = ((p,p), r(\gamma))$ (identified with p) and $\sigma'(p,\gamma) = s(\gamma)$. We define

$$((p_1,p_2),\gamma') \cdot (p,\gamma) := (p_1, \gamma'\gamma)$$

whenever $p_2\gamma' = p$ and $s(\gamma') = r(\gamma)$. Also we define

$$(p,\gamma) \cdot \gamma' := (p, \gamma\gamma')$$

whenever $s(\gamma) = r(\gamma')$. Then $P \times_{\mathcal{G}^{(0)}} \mathcal{G}$ is a $((P \times_\sigma P) \overrightarrow{\rtimes} \mathcal{G}, \mathcal{G})$-equivalence. Hence we obtain an element of $E(C^*(T^*_{\mathcal{G}}P \rtimes \mathcal{G}), C^*(\mathcal{G}))$. We denote this E-theory element by $\mathrm{Ind}^{\mathcal{G}}_P$. The induced map

$$\mathrm{Ind}^{\mathcal{G}}_P : K_*(C^*(T^*_{\mathcal{G}}P \rtimes \mathcal{G})) \to K_*(C^*(\mathcal{G}))$$

is called the $\mathcal{G}$-equivariant index map determined by the $\mathcal{G}$-manifold P. (Note that we are using the same notation for the induced group homomorphism.) Since we have chosen a $\mathcal{G}$-invariant metric on $T_{\mathcal{G}}P$, we may identify $T^*_{\mathcal{G}}P$ with $T_{\mathcal{G}}P$. Hence we regard $\mathrm{Ind}^{\mathcal{G}}_P$ as an element in $E(C^*(T_{\mathcal{G}}P \rtimes \mathcal{G}), C^*(\mathcal{G}))$. So we drop $*$ in the induced map: hence

$$\mathrm{Ind}^{\mathcal{G}}_P : K_*(C^*(T_{\mathcal{G}}P \rtimes \mathcal{G})) \to K_*(C^*(\mathcal{G})).$$

Remark. Any submersion $f : P \to M$ is a M-manifold if M is regarded as a trivial groupoid. If $p : E \to M$ is a vector bundle, then $\mathrm{Ind}^M_E = g!$, where g is the projection from the total space of $\ker(dp)^*$ to M. Note that $\ker(dp)^*$ is a vector bundle over M, that is, $E \oplus E^*$. Hence Ind^M_E induces the Thom isomorphism between $K^*(E \oplus E^*)$ and $K^*(M)$.

Definition 2.3. Let $\tau_V : V \to \mathcal{G}^{(0)}$ and $\tau_W : W \to \mathcal{G}^{(0)}$ be proper right $\mathcal{G}$-manifolds. Then a smooth map $f : V \to W$ is said to be $\mathcal{G}$-equivariant if $\tau_W(f(v)) = \tau_V(v)$ and $f(v \cdot \gamma) = f(v) \cdot \gamma$.

For a $\mathcal{G}$-equivariant map $f : P_1 \to P_2$, we construct a $\mathcal{G}$-equivariant version of $(df)!$.

Let $h : V \to W$ be a $\mathcal{G}$-equivariant map. Then each $0 \le \varepsilon \le 1$ gives us a groupoid $h^*(T_{\mathcal{G}}W) \rtimes_\varepsilon T_{\mathcal{G}}V$ where multiplication is given by

$$(\eta, \xi) \cdot (\eta', \xi') = (\eta, \xi + \xi') \quad \text{if} \quad \eta + \varepsilon(dh)(\xi) = \eta',$$

that is, it is the groupoid which comes from the groupoid action of $T_{\mathcal{G}}V$ on $h^*(T_{\mathcal{G}}W)$ given by $\eta \cdot \xi = \eta + \varepsilon(dh)(\xi)$. (It is clear when and only when $\eta \cdot \xi$ is defined.) The family $(h^*(T_{\mathcal{G}}W) \rtimes_\varepsilon T_{\mathcal{G}}V)_{0 \le \varepsilon < 1}$ form a groupoid $\mathcal{R}^{\mathcal{G}}_h$, where the fibers over $(0,1)$ are all isomorphic to as a set $\mathcal{R}^{\mathcal{G}}_h$ is $(T_{\mathcal{G}}V \oplus h^*(T_{\mathcal{G}}W)) \times [0,1)$. There is the canonical action of $\mathcal{G}$ on $\mathcal{R}^{\mathcal{G}}_h$, which satisfies the conditions in Definition 2.1. Hence we may take the semi-direct product $\mathcal{R}^{\mathcal{G}}_h \overrightarrow{\rtimes} \mathcal{G}$, which is a strong deformation from $[h^*(T_{\mathcal{G}}W) \rtimes_0 T_{\mathcal{G}}V] \overrightarrow{\rtimes} \mathcal{G}$ to $[h^*(T_{\mathcal{G}}W) \rtimes_1 T_{\mathcal{G}}V] \overrightarrow{\rtimes} \mathcal{G}$. Note that to $C^*([T^*_{\mathcal{G}}V \oplus h^*(T_{\mathcal{G}}W)] \rtimes \mathcal{G})$. So $\mathcal{R}^{\mathcal{G}}_h \overrightarrow{\rtimes} \mathcal{G}$ gives us an element

$$\delta_h \in E\left(C^*([T^*_{\mathcal{G}}V \oplus h^*(T_{\mathcal{G}}W)] \rtimes \mathcal{G}), C^*([h^*(T_{\mathcal{G}}W) \rtimes_1 T_{\mathcal{G}}V] \overrightarrow{\rtimes} \mathcal{G})\right).$$

For a smooth map $f : M \to N$, A. Connes constructed $\mathcal{G}(f)$ which is a strong deformation from $f^*(TN) \rtimes_1 TM$ to $N \times (M \times M)$.[*2] For each $x \in \mathcal{G}^{(0)}$, we have a map $h_x : V_x \to W_x$, where $V_x = \tau_V^{-1}(x)$ and $W_x = \tau_W^{-1}(x)$. We put $\mathcal{G}(h_x)$ together. More explicitly, forgetting $\mathcal{G}$-manifold structure for a moment, we take the groupoid $\mathcal{G}(h)$ constructed in the non-equivariant case. Then $\mathcal{Q}^{\mathcal{G}}_h$ is the closure of $W \times_{\mathcal{G}^{(0)}} (V \times_{\tau_V} V) \times (0,1)$ in $\mathcal{G}(h)$, where

$$W \times_{\mathcal{G}^{(0)}} (V \times_{\tau_V} V) = \{(w, v_1, v_2) : \tau_W(w) = \tau_V(v_1) = \tau_V(v_2)\}.$$

We have a canonical action of $\mathcal{G}$ on $\mathcal{Q}^{\mathcal{G}}_h$, which satisfies the conditions in Definition 1.2. Hence we take the semi-direct product $\mathcal{Q}^{\mathcal{G}}_h \overrightarrow{\rtimes} \mathcal{G}$, which is a strong

[*2]See p. 108 of [6].

deformation from $[h^*(T_{\mathcal{G}}W) \rtimes_1 T_{\mathcal{G}}V]\overrightarrow{\rtimes}\mathcal{G}$ to $[W \times_{\mathcal{G}^{(0)}} (V \times_{\tau_V} V)]\overrightarrow{\rtimes}\mathcal{G}$. Since $[W \times_{\mathcal{G}^{(0)}} (V \times_{\tau_V} V)]\overrightarrow{\rtimes}\mathcal{G}$ is equivalent[*3] to $W \rtimes \mathcal{G}$, we have an E-theory element

$$\pi_h \in E(C^*([h^*(T_{\mathcal{G}}W) \rtimes_1 T_{\mathcal{G}}V]\overrightarrow{\rtimes}\mathcal{G}), C^*(W \rtimes \mathcal{G})).$$

Composing π_h with δ_h, we obtain

$$[h_{pr}!]_{\mathcal{G}} \in E\left(C^*([T^*_{\mathcal{G}}V \oplus h^*(T_{\mathcal{G}}W)] \rtimes \mathcal{G}), C^*(W \rtimes \mathcal{G})\right).$$

When $h = df$ for some $\mathcal{G}$-equivariant map $f : P_1 \to P_2$ (hence $V = T_{\mathcal{G}}P_1$ and $W = T_{\mathcal{G}}P_2$), then the bundle $T^*_{\mathcal{G}}V \oplus h^*(T_{\mathcal{G}}W)$ over V is equal to $F \oplus F$ for some $\mathcal{G}$-equivariant bundle F. (Note that the bundle $T^*_{\mathcal{G}}V \oplus h^*(T_{\mathcal{G}}W)$ has $\mathcal{G}$-invariant complex structure.) F becomes a right $V \rtimes \mathcal{G}$-manifold with submersion $F \to V$. Hence we have

$$\mathrm{Ind}_F^{V\rtimes\mathcal{G}} \in E(C^*((F \oplus F) \rtimes \mathcal{G}), C^*(V \rtimes \mathcal{G})),$$

which induces the "$\mathcal{G}$-equivariant Thom isomorphism".

Definition 2.4. We define

$$[df!]_{\mathcal{G}} = [df_{pr}!]_{\mathcal{G}} \circ (\mathrm{Ind}_F^{V\rtimes\mathcal{G}})^{-1} \in E(C^*(T_{\mathcal{G}}P_1 \rtimes \mathcal{G}), C^*(T_{\mathcal{G}}P_2 \rtimes \mathcal{G})).$$

Definition 2.5 ([6])**.** Let $\mathcal{G}$ be a smooth groupoid. Then a geometric cycle for $\mathcal{G}$ is given by a proper $\mathcal{G}$-manifold P and an element $y \in K_*(C^*(T_{\mathcal{G}}P \rtimes \mathcal{G}))$. Two geometric cycles are equivalent if there exists a proper $\mathcal{G}$-manifold P and $\mathcal{G}$-equivariant maps $f_j : P_j \to P$ such that $[df_1!]_{\mathcal{G}}(y_1) = [df_2!]_{\mathcal{G}}(y_2)$. Then we define $K^*_{top}(\mathcal{G})$ as the set of geometric cycles modulo the above equivalence relation and call it the group of topological $\mathcal{G}$-indices.

Now we define the analytic assembly map for a smooth groupoid $\mathcal{G}$. Let (P, y) be a geometric cycle, $y \in K_*(C^*(T_{\mathcal{G}}P \rtimes \mathcal{G}))$. Then $\mu_{\mathcal{G}}(y)$ is defined as $\mathrm{Ind}_P^{\mathcal{G}}(y)$. This gives us a well-defined map

$$\mu_{\mathcal{G}} : K^*_{top}(\mathcal{G}) \to K_*(C^*(\mathcal{G})),$$

which is called the analytic assembly map for the smooth groupoid $\mathcal{G}$. Well-definedness follows from the functoriality of shrink maps and the fact that $\mathrm{Ind}_P^{\mathcal{G}} = [d\sigma!]_{\mathcal{G}}$, where $\sigma : P \to \mathcal{G}^{(0)}$ is the submersion required in the right $\mathcal{G}$-action. By composing $\mu_{\mathcal{G}}$ with the natural homomorphism

$$K_*(C^*(\mathcal{G})) \to K_*(C^*_r(\mathcal{G})),$$

we obtain another map, called the *reduced analytic assembly map for* $\mathcal{G}$,

$$\mu_{\mathcal{G},r} : K^*_{top}(\mathcal{G}) \to K_*(C^*_r(\mathcal{G})).$$

A. Connes conjectured that $\mu_{\mathcal{G},r}$ is an isomorphism. This conjecture was proved for some cases: for instance, the cases of fundamental groups of negatively

[*3] $W \times_{\mathcal{G}^{(0)}} V \times_{\mathcal{G}^{(0)}} \mathcal{G}$ is an equivalence between them.

curved compact Riemannian manifolds ([12]), discrete subgroups of $SO(n,1)$ or $SU(n,1)$ ([3]), free groups ([15], [8]), connected linear reductive groups ([19]), p-adic $GL(n)$ ([4]), and the case of foliations whose holonomy groupoid is Hausdorff and amenable ([18]).

3. Analytic assembly maps and the groupoid equivalence

Note that the collection of proper right $\mathcal{G}$-manifolds form a category $\mathcal{M}(\mathcal{G})$, where morphisms are $\mathcal{G}$-equivariant maps. We proceed to show that equivalent groupoids possess equivalent categories.

Suppose that $\mathcal{G}$ and $\mathcal{H}$ are equivalent as in Definition 1.7. Let $\tau : P \to \mathcal{G}^{(0)}$ be a proper right $\mathcal{G}$-manifold. We let $P \times_{\mathcal{G}^{(0)}} Z = \{(p,z) : \tau(p) = \rho(z)\}$. We have a free and proper left action of $\mathcal{G}$ on $P \times_{\mathcal{G}^{(0)}} Z$ given by $\gamma \cdot (p,z) = (p\gamma^{-1}, \gamma z)$. We define a right $\mathcal{H}$-action on the smooth manifold $\mathcal{G}\backslash(P \times_{\mathcal{G}^{(0)}} Z)$ as follows: define a submersion

$$\tau' : \mathcal{G}\backslash(P \times_{\mathcal{G}^{(0)}} Z) \to \mathcal{H}^{(0)}, \quad [p,z] \mapsto \sigma(z),$$

and $[p,z] \cdot \delta = [p, z \cdot \delta]$ for $\delta \in \mathcal{H}$. This action is proper if P is a proper $\mathcal{G}$-manifold. For any morphism $f : P_1 \to P_2$ in $\mathcal{M}(\mathcal{G})$, we define

$$\hat{f} : \mathcal{G}\backslash(P_1 \times_{\mathcal{G}^{(0)}} Z) \to \mathcal{G}\backslash(P_2 \times_{\mathcal{G}^{(0)}} Z)$$

by $\hat{f}([p_1,z]) = [f(p_1), z]$.

Theorem 3.1. *For an object P and a morphism $f : P_1 \to P_2$ in $\mathcal{M}(\mathcal{G})$ we define $\Phi(P) = \mathcal{G}\backslash(P \times_{\mathcal{G}^{(0)}} Z)$ and $\Phi(f) = \hat{f}$. Then $\Phi(f)$ is in $\mathcal{M}(\mathcal{H})$ and Φ is a functor from $\mathcal{M}(\mathcal{G})$ to $\mathcal{M}(\mathcal{H})$.*

Proof. Clearly $\Phi(f)$ is smooth and we have

$$\hat{f}([p_1,z] \cdot \delta) = \hat{f}([p_1, z \cdot \delta]) = [f(p_1), z \cdot \delta] = [f(p_1), z] \cdot \delta = f([p,z]) \cdot \delta.$$

So $\hat{f}$ is in $\mathcal{M}(\mathcal{H})$. It is also clear that $\Phi(f_1 \circ f_2) = \Phi(f_1) \circ \Phi(f_2)$. □

Similarly we define a functor

$$\Psi : \mathcal{M}(\mathcal{H}) \to \mathcal{M}(\mathcal{G})$$

by $\Psi(Q) = (Q \times_{\mathcal{H}^{(0)}} Z)/\mathcal{H}$ for an object Q in $\mathcal{M}(\mathcal{H})$. We denote $(\Psi \circ \Phi)(P)$ and $(\Psi \circ \Phi)(f)$ simply by $\widehat{\widehat{P}}$ and $\widehat{\widehat{f}}$ respectively. We define a map

$$\Delta_P : \widehat{\widehat{P}} \to P$$

by $[[p_1,z_1],z_2] \mapsto p_1 \cdot \gamma$ where γ is the unique element such that $z_1 = \gamma \cdot z_2$. Then it is easy to check that Δ_P is an isomorphism in $\mathcal{M}(\mathcal{G})$. For $f : P_1 \to P_2$

in $\mathcal{M}(\mathcal{G})$, we have the following commutative diagram:

$$\begin{array}{ccc} P_1 & \xrightarrow{f} & P_2 \\ {\scriptstyle \Delta_{P_1}}\uparrow & & \uparrow{\scriptstyle \Delta_{P_2}} \\ \widehat{\widehat{P_1}} & \xrightarrow{\widehat{\widehat{f}}} & \widehat{\widehat{P_2}} \,. \end{array}$$

Hence, we have the following theorem.

Theorem 3.2. *For equivalent groupoids $\mathcal{G}$ and $\mathcal{H}$, we have equivalent categories $\mathcal{M}(\mathcal{G})$ and $\mathcal{M}(\mathcal{H})$.*

Theorem 3.3. *Suppose that $\mathcal{G}$ and $\mathcal{H}$ are equivalent as in Definition* 1.7*. Let P be a proper right $\mathcal{G}$-manifold. Then $P \rtimes \mathcal{G}$ and $\Phi(P) \rtimes \mathcal{H}$ are equivalent.*

Proof. We define a left $P \rtimes \mathcal{G}$-action and a right $\Phi(P) \rtimes \mathcal{H}$-action on $P \times_{\mathcal{G}^{(0)}} Z$. Let

$$\rho' : P \times_{\mathcal{G}^{(0)}} Z \to (P \rtimes \mathcal{G})^{(0)} = P$$

be defined as $\rho'(p, z) = p$, and we let

$$\sigma' : P \times_{\mathcal{G}^{(0)}} Z \to (\Phi(P) \rtimes \mathcal{H})^{(0)} = \Phi(P)$$

be the projection. The actions are given by

$$(p, \gamma) \cdot (p, z) = (p\gamma^{-1}, \gamma z)$$
$$(p, z) \cdot ([p, z], \delta) = (p, z \cdot \delta)$$

where $(p, z) \in P \times_{\mathcal{G}^{(0)}} Z$, $(p, \gamma) \in P \rtimes \mathcal{G}$ and $([p, z], \delta) \in \Phi(P) \rtimes \mathcal{H}$. Then we can check that $P \rtimes \mathcal{G}$ and $\Phi(P) \rtimes \mathcal{H}$ are equivalent with $P \times_{\mathcal{G}^{(0)}} Z$ as a $(P \rtimes \mathcal{G}, \Phi(P) \rtimes \mathcal{H})$-equivalence. □

Corollary 3.4. *$T_{\mathcal{G}}P \rtimes \mathcal{G}$ and $T_{\mathcal{H}}\Phi(P) \rtimes \mathcal{H}$ are equivalent.*

Proof. This follows from Theorem 3.3 and from the fact that $\Phi(T_{\mathcal{G}}P) = T_{\mathcal{H}}\Phi(P)$. So $T_{\mathcal{G}}P \times_{\mathcal{G}^{(0)}} Z$ implements a $(T_{\mathcal{G}}P \rtimes \mathcal{G}, T_{\mathcal{H}}\Phi(P) \rtimes \mathcal{H})$-equivalence. □

Hence $C^*(P \rtimes \mathcal{G})$ and $C^*(\Phi(P) \rtimes \mathcal{H})$ are Morita equivalent. The $C^*(\Phi(P) \rtimes \mathcal{H})$-valued inner product on $C_c(P \times_{\mathcal{G}^{(0)}} Z)$ is given by

$$\langle \varphi, \psi \rangle ([p, z], \delta) = \int_{\mathcal{G}^{\tau(p)}} \overline{\varphi(p\gamma, \gamma^{-1})} \psi(p\gamma, \gamma^{-1} z\delta),$$

and the action of $C_c(P \rtimes \mathcal{G})$ on $C_c(P \times_{\mathcal{G}^{(0)}} Z)$ is given by

$$(f \cdot \varphi)(p, z) = \int_{\mathcal{G}^{\tau(p)}} f(p, \gamma)\varphi(p\gamma, \gamma^{-1} z).$$

These two induce a $C^*(P \rtimes \mathcal{G})$-$C^*(\Phi(P) \rtimes \mathcal{H})$ bimodule X, which is the closure of $C_c(P \rtimes \mathcal{G})$ under the norm induced by the $C^*(\Phi(P) \rtimes \mathcal{H})$-valued inner product.

Now we prove that the $\mathrm{Ind}_P^{\mathcal{G}}$ is invariant under the groupoid equivalence. Before proving that, we need to make a couple of observations.

Observation 3.5. (1) Suppose that $\mathcal{G}$ and $\mathcal{H}$ are equivalent as in Definition 1.7, and that $\mathcal{G}$ acts on P. Then the semi-direct products $T_{\mathcal{G}}P \overrightarrow{\rtimes} \mathcal{G}$ and $T_{\mathcal{H}}(\Phi(P)) \overrightarrow{\rtimes} \mathcal{H}$ are equivalent. We define actions on $T_{\mathcal{G}}P \times_{\mathcal{G}^{(0)}} Z$: $\rho'(w, z) = (0, \rho(z))$, $\sigma'(w, z) = ([0, z], \sigma(z))$ (we may identify $(0, \gamma)$ and $([0, z], \delta)$ with γ and δ respectively).

$$\begin{aligned} (v, \gamma) \cdot (w, z) &= (v + w \cdot \gamma^{-1}, \gamma \cdot z), \\ (w, z) \cdot ([v, z], \delta) &= (w + v, z \cdot \delta). \end{aligned}$$

With the above actions $T_{\mathcal{G}}P \times_{\mathcal{G}^0} Z$ is a $(T_{\mathcal{G}}P \overrightarrow{\rtimes} \mathcal{G}, T_{\mathcal{H}}(\Phi(P)) \overrightarrow{\rtimes} \mathcal{H})$-equivalence. Hence it gives us a $C^*(T_{\mathcal{G}}P \overrightarrow{\rtimes} \mathcal{G}) - C^*(T_{\mathcal{H}}(\Phi(P)) \overrightarrow{\rtimes} \mathcal{H})$-equivalence bimodule $\mathcal{E}$. Remember that

$$\begin{aligned} C^*(T_{\mathcal{G}}P \overrightarrow{\rtimes} \mathcal{G}) &\cong C^*((T^*_{\mathcal{G}}P) \rtimes \mathcal{G}), \text{ and} \\ C^*(T_{\mathcal{H}}(\Phi(P)) \overrightarrow{\rtimes} \mathcal{H}) &\cong C^*([T^*_{\mathcal{H}}(\Phi(P))] \rtimes \mathcal{H}). \end{aligned}$$

If we identify $T^*_{\mathcal{G}}P$ and $T^*_{\mathcal{H}}(\Phi(P))$ with $T_{\mathcal{G}}P$ and $T_{\mathcal{H}}(\Phi(P))$ respectively (by imposing invariant metrics), then $\mathcal{E}$ is the same as the $C^*(T_{\mathcal{G}}P \rtimes \mathcal{G}) - C^*(T_{\mathcal{H}}\Phi(P) \rtimes \mathcal{H})$-equivalence bimodule in Corollary 3.4.

(2) The $\mathcal{G}$ action on $P \times_\tau P$ given by $(p_1, p_2) \cdot \gamma = (p_1 \cdot \gamma, p_2 \cdot \gamma)$ satisfies the conditions of Definition 2.1. Hence we take $(P \times_\tau P) \overrightarrow{\rtimes} \mathcal{G}$. We define a $(P \times_\tau P) \overrightarrow{\rtimes} \mathcal{G}$ action on $(P \times_\tau P) \times_{\mathcal{G}^{(0)}} Z$, where

$$(P \times_\tau P) \times_{\mathcal{G}^{(0)}} Z = \{((p_1, p_2), z) : \ \tau(p_1) = \tau(p_2) = \rho(z)\}.$$

Let $\rho'((p_1, p_2), z) = ((p_1, p_1), \rho(z))$. So $((p_3, p_4), \gamma) \cdot ((p_1, p_2), z)$ is defined if and only if $p_4 = p_1 \cdot \gamma^{-1}$ and $s(\gamma) = \rho(z)$, and the action is given by

$$((p_3, p_1 \cdot \gamma^{-1}), \gamma) \cdot ((p_1, p_2), z) = ((p_3, p_2 \cdot \gamma^{-1}), \gamma \cdot z).$$

Now we define a $[\Phi(P) \times_{\tau'} \Phi(P)] \overrightarrow{\rtimes} \mathcal{H}$ action on $(P \times_\tau P) \times_{\mathcal{G}^{(0)}} Z$. Let $\sigma'((p_1, p_2), z) = (([p_2, z], [p_2, z]), \sigma(z))$. So $((p_1, p_2), z) \cdot (([p_3, z'], [p_4, z']), \delta)$ is defined if and only if $[p_2, z] = [p_3, z']$ and $\sigma(z) = r(\delta)$, and the action is given by

$$((p_1, p_2), z) \cdot (([p_2, z], [p_4, z]), \delta) = ((p_1, p_4), z \cdot \delta).$$

With these actions, $(P \times_\tau P) \times_{\mathcal{G}^{(0)}} Z$ is a $((P \times_\tau P) \overrightarrow{\rtimes} \mathcal{G}, (\Phi(P) \times_{\tau'} \Phi(P)) \overrightarrow{\rtimes} \mathcal{H})$-equivalence. We saw that $(P \times_\tau P) \overrightarrow{\rtimes} \mathcal{G}$ is equivalent to $\mathcal{G}$ with $P \times_{\mathcal{G}^{(0)}} \mathcal{G}$ as their equivalence. Also we have a $([\Phi(P) \times_\tau \Phi(P)] \overrightarrow{\rtimes} \mathcal{H}, \mathcal{H})$-equivalence $\Phi(P) \times_{\mathcal{H}^{(0)}} \mathcal{H}$. Hence we have the following four equivalences:

$$\begin{array}{ccccc} ((P \times_\tau P) \overrightarrow{\rtimes} \mathcal{G}) & \longrightarrow & P \times_{\mathcal{G}^{(0)}} \mathcal{G} & \longleftarrow & \mathcal{G} \\ \big\downarrow & & & & \big\downarrow \\ (P \times_\tau P) \times_{\mathcal{G}^{(0)}} Z & & & & Z \\ \big\uparrow & & & & \big\uparrow \\ [\Phi(P) \times_{\tau'} \Phi(P)] \overrightarrow{\rtimes} \mathcal{H} & \longrightarrow & \Phi(P) \times_{\mathcal{H}^{(0)}} \mathcal{H} & \longleftarrow & \mathcal{H}\ . \end{array}$$

Arrows do not mean mappings here, but instead, for instance, $\mathcal{G} \to Y$ means that $\mathcal{G}$ acts on Y from the left. The diagram gives us two equivalences between $(P \times_\tau P)\overrightarrow{\rtimes}\mathcal{G}$ and $\mathcal{H}$: one is obtained by passing through $\mathcal{G}$ and the other one by passing through $\Phi(P) \times_{\tau'} \Phi(P)$. But it is easy to see that these two are bi-equivariantly diffeomorphic. Hence the above diagram induces a commutative diagram:

$$\begin{array}{ccc} K_*(C^*((P \times_\tau P)\overrightarrow{\rtimes}\mathcal{G})) & \xrightarrow{\cong} & K_*(C^*(\mathcal{G})) \\ \cong\downarrow & & \downarrow\cong \\ K_*C^*((\Phi(P) \times_{\tau'} \Phi(P))\overrightarrow{\rtimes}\mathcal{H}) & \xrightarrow{\cong} & K_*(C^*(\mathcal{H})), \end{array}$$

where the isomorphisms are induced by groupoid equivalences.

Before we proceed, remember the definition of $\mathcal{G}'_P\overrightarrow{\rtimes}\mathcal{G}$ in Definition 2.2.

Theorem 3.6. *Suppose that $\mathcal{G}$ and $\mathcal{H}$ are equivalent as in Definition* 1.7 *and that P is a right $\mathcal{G}$-manifold. Then the following diagram commutes.*

$$\begin{array}{ccc} K_*(C^*(T_{\mathcal{G}}P \rtimes \mathcal{G})) & \xrightarrow{\mathrm{Ind}_P^{\mathcal{G}}} & K_*(C^*(\mathcal{G})) \\ \cong\downarrow & & \downarrow\cong \\ K_*(C^*(T_{\mathcal{H}}(\Phi(P)) \rtimes \mathcal{H})) & \xrightarrow{\mathrm{Ind}_{\Phi(P)}^{\mathcal{H}}} & K_*(C^*(\mathcal{H})), \end{array}$$

where the left vertical map is the isomorphism induced by the $(T_{\mathcal{G}}P\rtimes\mathcal{G}, T_{\mathcal{H}}(\Phi(P))\rtimes\mathcal{H})$-equivalence $T_{\mathcal{G}}P \times_{\mathcal{G}^{(0)}} Z$ in Corollary 3.4*, and the right vertical map is the isomorphism induced by the $(\mathcal{G}, \mathcal{H})$-equivalence Z.*

Proof. We have a left $\mathcal{G}'_P\overrightarrow{\rtimes}\mathcal{G}$-action and a right $\mathcal{H}'_{\Phi(P)}\overrightarrow{\rtimes}\mathcal{H}$-action on $\mathcal{G}'_P \times_{\mathcal{G}^{(0)}} Z$, where $\mathcal{G}'_P \times_{\mathcal{G}^{(0)}} Z$ is fibered over $[0, 1)$. The fiber over $t = 0$ is $T_{\mathcal{G}}P \times_{\mathcal{G}^{(0)}} Z$, and the space over $(0, 1)$ is $(0, 1) \times [(P \times_\tau P) \times_{\mathcal{G}^{(0)}} \mathcal{G}]$. The actions of $\mathcal{G}'_P\overrightarrow{\rtimes}\mathcal{G}$ and $\mathcal{H}'_{\Phi(P)}\overrightarrow{\rtimes}\mathcal{H}$ on $\mathcal{G}'_P \times_{\mathcal{G}^{(0)}} Z$ are "fiberwise". That is, for $t \in [0, 1)$, the actions restrict to the actions of the fibers of $\mathcal{G}'_P\overrightarrow{\rtimes}\mathcal{G}$ and $\mathcal{H}'_{\Phi(P)}\overrightarrow{\rtimes}\mathcal{H}$ on the fiber of $\mathcal{G}'_P \times_{\mathcal{G}^{(0)}} Z$ over t. For instance, when $t = 0$, we have the $(T_{\mathcal{G}}P\overrightarrow{\rtimes}\mathcal{G}, T_{\mathcal{H}}(\Phi(P))\overrightarrow{\rtimes}\mathcal{H})$-equivalence $T_{\mathcal{G}}P \times_{\mathcal{G}^0} Z$ described in (1) of Observation 3.5. For $t > 0$, we have the $((P \times_\tau P)\overrightarrow{\rtimes}\mathcal{G}, (\Phi(P) \times_{\tau'} \Phi(P))\overrightarrow{\rtimes}\mathcal{H})$-equivalence $(P \times_\tau P) \times_{\mathcal{G}^{(0)}} Z$ in (2) of Observation 3.5.

We have $\mathcal{G}'_P\overrightarrow{\rtimes}\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$, $\mathcal{H}'_{\Phi(P)}\overrightarrow{\rtimes}\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ where $\mathcal{G}_1 = T_{\mathcal{G}}P\overrightarrow{\rtimes}\mathcal{G}$, $\mathcal{G}_2 = (0, 1) \times ((P \times_\tau P)\overrightarrow{\rtimes}\mathcal{G})$, $\mathcal{H}_1 = T_{\mathcal{H}}(\Phi(P))\overrightarrow{\rtimes}\mathcal{H}$, and $\mathcal{H}_2 = (0, 1) \times (\Phi(P) \times_{\tau'} \Phi(P))$. It is easily checked that the following diagram is commutative:

$$\begin{array}{ccccc} K(C^*(\mathcal{G}_2)) & \longrightarrow & K(C^*(\mathcal{G}_P\overrightarrow{\rtimes}\mathcal{G})) & \longrightarrow & K(C^*(\mathcal{G}_1)) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ K(C^*(\mathcal{H}_2)) & \longrightarrow & K(C^*(\mathcal{H}_{\Phi(P)}\overrightarrow{\rtimes}\mathcal{H})) & \longrightarrow & K(C^*(\mathcal{H}_1)), \end{array}$$

where the vertical maps are the isomorphisms induced by the groupoid equivalences. By the naturality of the boundary maps, we have the following commutative diagram:

$$\begin{array}{ccccccc}
K_i(C^*(\mathcal{G}_2)) & \longrightarrow & K_i(C^*(\mathcal{G}_P \overrightarrow{\rtimes} \mathcal{G})) & \longrightarrow & K_i(C^*(\mathcal{G}_1)) & \xrightarrow{\partial_P} & K_{i-1}(C^*(\mathcal{G}_2)) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
K_i(C^*(\mathcal{H}_2)) & \longrightarrow & K_i(C^*(\mathcal{H}_{\widehat{P}} \overrightarrow{\rtimes} \mathcal{H})) & \longrightarrow & K_i(C^*(\mathcal{H}_1)) & \xrightarrow{\partial_{\widehat{P}}} & K_{i-1}(C^*(\mathcal{H}_2)),
\end{array}$$

where $\widehat{P} = \Phi(P)$. The boundary map ∂_P coincides with the map induced by the asymptotic homomorphism $(1 \otimes \varphi_t)$ associated to the deformation from $C_0(0,1) \otimes C^*(T_P\mathcal{G} \overrightarrow{\rtimes} \mathcal{G})$ to $C_0(0,1) \otimes C^*((P \times_\tau P) \overrightarrow{\rtimes} \mathcal{G})$ which is given by $C^*(\mathcal{G}'_P \overrightarrow{\rtimes} \mathcal{G})$. This is because the asymptotic homomorphism associated to the exact sequence

$$0 \to C^*(\mathcal{G}_2) \to C^*(\mathcal{G}'_P \overrightarrow{\rtimes} \mathcal{G}) \to C^*(\mathcal{G}_1) \to 0$$

induces the same map as the boundary map ∂_P. Note that $C^*(\mathcal{G}_2)$ is isomorphic to $C_0(0,1) \otimes C^*((P \times_\tau P) \overrightarrow{\rtimes} \mathcal{G})$. By the naturality of the Bott periodicity and Observations 3.5, we can conclude that ∂_P and $\partial_{\Phi(P)}$ coincide with $\mathrm{Ind}_P^{\mathcal{G}}$ and $\mathrm{Ind}_{\Phi(P)}^{\mathcal{H}}$ respectively, and that the diagram whose commutativity we want to prove is the right block of the above diagram. So the theorem is proved. □

Lemma 3.7. *Assume that smooth groupoids $\mathcal{G}$ and $\mathcal{H}$ are equivalent as in Definition* 1.7*. Suppose that $\mathcal{G}$ acts on another smooth groupoid $\mathcal{R}$ and that the action satisfies the conditions in Definition* 2.1*. Then the $\mathcal{H}$-manifold $\Phi(\mathcal{R})$ inherits a natural groupoid structure from $\mathcal{R}$. With this groupoid structure on $\Phi(\mathcal{R})$, the $\mathcal{H}$-action on $\Phi(\mathcal{R})$ satisfies the condition in Definition* 2.1*, and $\mathcal{R} \overrightarrow{\rtimes} \mathcal{G}$ and $\Phi(\mathcal{R}) \overrightarrow{\rtimes} \mathcal{H}$ are equivalent.*

Proof. The groupoid structure of $\Phi(\mathcal{R})$ is given as follows. $[\alpha, z]$ and $[\alpha', z']$ are composable if and only if $\sigma(z) = \sigma(z')$, i.e., $z' = \gamma z$ for some γ, and α, $\alpha'' = \alpha' \cdot \gamma^{-1}$ are composable. The composition is given by

$$[\alpha, z][\alpha'', z] = [\alpha\alpha'', z].$$

This is well-defined provided that the $\mathcal{G}$-action on $\mathcal{R}$ satisfies the conditions in Definition 2.1. It is also easy to check that the $\mathcal{H}$-action on $\Phi(\mathcal{R})$ also satisfies the conditions in Definition 2.1. So we can take the semi-direct product $\Phi(\mathcal{R}) \overrightarrow{\rtimes} \mathcal{H}$. We define a left $\mathcal{R} \overrightarrow{\rtimes} \mathcal{G}$-action and a right $\Phi(\mathcal{R}) \overrightarrow{\rtimes} \mathcal{H}$-action on $\mathcal{R} \times_{\mathcal{G}^{(0)}} Z$. $(\alpha, \gamma) \cdot (\alpha', z)$ is defined if and only if $s(\gamma) = \rho(z)$, and α, $\alpha' \cdot \gamma^{-1}$ are composable. Then

$$(\alpha, \gamma) \cdot (\alpha', z) = (\alpha(\alpha'\gamma^{-1}), \gamma z).$$

$(\alpha, z) \cdot ([\alpha', z], \delta)$ is defined if and only if α and α' are composable. It is defined as

$$(\alpha, z) \cdot ([\alpha', z], \delta) = (\alpha\alpha', z\delta).$$

With these actions, $\mathcal{R} \times_{\mathcal{G}^{(0)}} Z$ becomes a $(\mathcal{R} \overrightarrow{\rtimes} \mathcal{G},\ \Phi(\mathcal{R}) \overrightarrow{\rtimes} \mathcal{H})$-equivalence. □

Theorem 3.8. *Assume that $\mathcal{G}$ and $\mathcal{H}$ are equivalent as in Definition* 1.7. *Let $f : P_1 \to P_2$ be in $\mathcal{M}(\mathcal{G})$. Then we have the following commutative diagram:*

$$\begin{array}{ccc} K_*\left(C^*(T_\mathcal{G}P_1 \rtimes \mathcal{G})\right) & \xrightarrow{[df!]_\mathcal{G}} & K_*\left(C^*(T_\mathcal{G}P_2 \rtimes \mathcal{G})\right) \\ \cong\downarrow & & \downarrow\cong \\ K_*\left(C^*(T_\mathcal{H}\widehat{P_1} \rtimes \mathcal{H})\right) & \xrightarrow{[d\hat{f}!]_\mathcal{H}} & K_*\left(C^*(T_\mathcal{H}\widehat{P_2} \rtimes \mathcal{H})\right), \end{array}$$

where $\widehat{P_1} = \Phi(P_1)$, $\widehat{P_2} = \Phi(P_2)$, and $\hat{f} = \Phi(f)$. The vertical isomorphisms are induced by groupoid equivalences.

Proof. For the sake of simplifying the notations, we first consider a $\mathcal{G}$-equivariant map $h : V \to W$ as we did in the course of constructing $[df!]_\mathcal{G}$. Then its associated object $\Phi(h) : \Phi(V) \to \Phi(W)$ in $\mathcal{M}(\mathcal{H})$ gives us groupoids $\mathcal{R}^\mathcal{H}_{\Phi(h)}$ and $\mathcal{Q}^\mathcal{H}_{\Phi(h)}$ (see the paragraph which lies between Definitions 2.3 and 2.4). It is easy to check that $\mathcal{R}^\mathcal{H}_{\Phi(h)} = \Phi(\mathcal{R}^\mathcal{G}_h)$ and the groupoid structure of $\mathcal{R}^\mathcal{H}_{\Phi(h)}$ coincides with the one inherited from $\mathcal{R}^\mathcal{G}_h$. Hence $\mathcal{R}^\mathcal{G}_h \times_{\mathcal{G}^{(0)}} Z$ gives us a $(\mathcal{R}^\mathcal{G}_h \overrightarrow{\rtimes} \mathcal{G}, \mathcal{R}^\mathcal{H}_{\Phi(h)} \overrightarrow{\rtimes} \mathcal{H})$-equivalence. This equivalence is fiberwise. Each groupoid has simple form over $(0,1)$. Similar statements hold for $\mathcal{Q}^\mathcal{G}_h$ and $\mathcal{Q}^\mathcal{H}_{\Phi(h)}$.

As in Observation 3.5 we have the following four groupoid equivalences.

$$\begin{array}{ccccc} (W \times_{\mathcal{G}^{(0)}} (V \times_{\tau_V} V)) \overrightarrow{\rtimes} \mathcal{G} & \longrightarrow & W \times_{\mathcal{G}^{(0)}} V \times_{\mathcal{G}^{(0)}} \mathcal{G} & \longleftarrow & W \rtimes \mathcal{G} \\ \downarrow & & & & \downarrow \\ W \times_{\mathcal{G}^{(0)}} (V \times_{\tau_V} V) \times_{\mathcal{G}^{(0)}} Z & & & & W \times_{\mathcal{G}^{(0)}} Z \\ \uparrow & & & & \uparrow \\ \left[\widehat{W} \times_{\mathcal{H}^{(0)}} (\widehat{V} \times_{\tau'_V} \widehat{V})\right] \overrightarrow{\rtimes} \mathcal{H} & \longrightarrow & \widehat{W} \times_{\mathcal{H}^{(0)}} \widehat{V} \times_{\mathcal{H}^{(0)}} \mathcal{H} & \longleftarrow & \widehat{W} \rtimes \mathcal{H}, \end{array}$$

where $\widehat{V} = \Phi(V)$ and $\widehat{W} = \Phi(W)$. These equivalences induce the following commutative diagram:

$$\begin{array}{ccc} K_*\left(C^*\left([W \times_{\mathcal{G}^{(0)}} (V \times_{\tau_V} V)] \overrightarrow{\rtimes} \mathcal{G}\right)\right) & \xrightarrow{\cong} & K_*\left(C^*(W \rtimes \mathcal{G})\right) \\ \cong\downarrow & & \downarrow\cong \\ K_*\left(C^*\left(\left[\widehat{W} \times_{\mathcal{H}^{(0)}} \left(\widehat{V} \times_{\tau'_V} \widehat{V}\right)\right] \overrightarrow{\rtimes} \mathcal{H}\right)\right) & \xrightarrow{\cong} & K_*\left(C^*(\widehat{V} \rtimes \mathcal{H})\right), \end{array}$$

where the isomorphisms are induced by groupoid equivalences. Hence we have the following commutative diagram:

$$\begin{array}{ccc} K_*\left(C^*\left([T^*_\mathcal{G}V \oplus (dh)^*(T_\mathcal{G}W)] \rtimes \mathcal{G}\right)\right) & \xrightarrow{[h_{pr}!]_\mathcal{G}} & K_*(C^*(W \rtimes \mathcal{G})) \\ \cong\downarrow & & \cong\downarrow \\ K_*\left(C^*\left([T^*_\mathcal{H}\widehat{V} \oplus (d\hat{h})^*(T_\mathcal{H}\widehat{W})] \rtimes \mathcal{H}\right)\right) & \xrightarrow{[\hat{h}_{pr}!]_\mathcal{H}} & K_*(C^*(\widehat{W} \rtimes \mathcal{H})), \end{array}$$

where $\hat{h} = \Phi(h)$. In the case of $h = df$ (so $V = T_{\mathcal{G}}P_1$ and $W = T_{\mathcal{G}}P_2$) then the $\mathcal{G}$-equivariant Thom isomorphism between $K_*\left(C^*\left([T^*_{\mathcal{G}}V \oplus (dh)^*(T_{\mathcal{G}}W)] \rtimes \mathcal{G}\right)\right)$ and $K_*(C^*(V \rtimes \mathcal{G}))$ is given by $\mathrm{Ind}_F^{V\rtimes\mathcal{G}}$ as we saw in Section 2. Similarly $\mathrm{Ind}_{\Phi'(F)}^{\Phi(V)\rtimes\mathcal{H}}$ induces the $\mathcal{H}$-equivariant Thom isomorphism. (Φ' is the functor induced by the groupoid equivalence between $V \rtimes \mathcal{G}$ and $\Phi(V) \rtimes \mathcal{H}$.) Hence by Theorem 3.6, the diagram

$$\begin{array}{ccc} K_*\left(C^*(T_{\mathcal{G}}P_1 \rtimes \mathcal{G})\right) & \xrightarrow{[df!]_{\mathcal{G}}} & K_*\left(C^*(T_{\mathcal{G}}P_2 \rtimes \mathcal{G})\right) \\ \cong\downarrow & & \downarrow\cong \\ K_*\left(C^*(T_{\mathcal{H}}\widehat{P_1} \rtimes \mathcal{H})\right) & \xrightarrow{[d\hat{f}!]_{\mathcal{H}}} & K_*\left(C^*(T_{\mathcal{H}}\widehat{P_2} \rtimes \mathcal{H})\right) \end{array}$$

commutes. □

Theorem 3.8 gives us a homomorphism

$$\alpha : K^*_{top}(\mathcal{G}) \to K^*_{top}(\mathcal{H}).$$

Theorem 3.9. *For equivalent groupoids $\mathcal{G}$ and $\mathcal{H}$, $\alpha : K^*_{top}(\mathcal{G}) \to K^*_{top}(\mathcal{H})$ is an isomorphism.*

Proof. Surjectivity: For any $\mathcal{H}$-manifold P', there is a $\mathcal{H}$-equivariant smooth map $\Delta_{P'} : \Phi(\Psi(P')) \to P'$ which is an isomorphism in $\mathcal{M}(\mathcal{H})$. So any element in $K_*(C^*(T_{\mathcal{H}}P' \rtimes \mathcal{H}))$ can be identified with an element in $K_*(C^*(T_{\mathcal{H}}(\Phi(\Psi(P'))) \rtimes \mathcal{H}))$. So the surjectivity of α follows.

Injectivity: Let P_1 and P_2 be proper $\mathcal{G}$-manifolds. Suppose that we have a $\mathcal{H}$-equivariant map

$$h : \Phi(P_1) \to \Phi(P_2).$$

Since $\Delta_{P_1} : \widehat{\widehat{P_1}} \to P_1$ and $\Delta_{P_2} : \widehat{\widehat{P_2}} \to P_2$ are isomorphisms in $\mathcal{M}(\mathcal{G})$, h determines a $\mathcal{G}$-equivariant map f which makes the following diagram commute:

$$\begin{array}{ccc} P_1 & \xrightarrow{f} & P_2 \\ \Delta_{P_1}\downarrow & & \downarrow\Delta_{P_2} \\ \widehat{\widehat{P_1}} & \xrightarrow{\Psi(h)} & \widehat{\widehat{P_2}} \end{array}$$

Then it is easy to check that $h = \Phi(f)$. So the injectivity of α follows. □

Now Theorems 3.6 and 3.9 imply the main theorem.

Theorem 3.10. *The analytic assembly map is invariant under the groupoid equivalence. More explicitly, if $\mathcal{G}$ and $\mathcal{H}$ are equivalent as in Definition* 1.7, *then there is an isomorphism $\alpha : K^*_{top}(\mathcal{G}) \to K^*_{top}(\mathcal{H})$ which makes the following diagram commute*:

$$\begin{array}{ccc} K^*_{top}(\mathcal{G}) & \xrightarrow{\mu_{\mathcal{G}}} & K_*(C^*(\mathcal{G})) \\ \alpha\downarrow & & \downarrow\cong \\ K^*_{top}(\mathcal{H}) & \xrightarrow{\mu_{\mathcal{H}}} & K_*(C^*(\mathcal{H})), \end{array}$$

where the right vertical map is the isomorphism induced by the $(\mathcal{G}, \mathcal{H})$*-equivalence* Z.

We mention three easy consequences, which are well-known already.

Remark. (1) Let $\mathcal{G}_{\mathcal{F}}$ be the holonomy groupoid of a foliated space $(M, \mathcal{F})$ and T be a complete transversal. Since the etale groupoid $\mathcal{G}_T^T = \{\gamma : r(\gamma), s(\gamma) \in T\}$ is equivalent to $\mathcal{G}_{\mathcal{F}}$, the Baum-Connes conjecture for a foliation reduces to the conjecture for an etale groupoid.

(2) Let $M = \widetilde{B} \times_\Gamma F$ be the flat bundle associated to a discrete group Γ acting on F by diffeomorphims. Suppose that the fixed point set of any non-identity $g \in \Gamma$ has no interior. Then the etale groupoid associated to the transversal F is isomorphic to the groupoid $F \rtimes \Gamma$ ([13]). Hence the analytic assembly map for the foliated space $M = \widetilde{B} \times_\Gamma F$ is the same as the analytic assembly map for the action of Γ on F.

(3) Let $\mathcal{G}$ be a transitive groupoid. Then for any unit $u \in \mathcal{G}$, the Lie group $H = \mathcal{G}_u^u$ is equivalent to $\mathcal{G}$. So $\mathcal{G}$ and H have the same analytic assembly map.

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