J. Math. Kyoto Univ. (JMKYAZ)
41-4 (2002), 769–793

Mod 3 homotopy uniqueness of BF_4

By

Antonio Viruel*

1. Introduction

Let F_4 be the exceptional compact Lie group of rank 4, and denote by BF_4 its classifying space. Previous work about homotopy uniqueness of classifying spaces of compact Lie groups by Dwyer-Miller-Wilkerson [6], and Notbohm [16], shows that this classifying space is determined, up to completion, by its mod p cohomology at primes greater than 3, that is, if X is a p-complete space ($p > 3$) such that $H^*(X; \mathbb{F}_p)$ is isomorphic to $H^*(BF_4; \mathbb{F}_p)$ as $\mathcal{A}_p$-algebras, then X is homotopy equivalent to BF_4 up to p-completion. At the prime 3, BF_4 has torsion and its mod 3 cohomology was calculated by Toda [20]. As an algebra:

$$H^*(BF_4; \mathbb{F}_3) = \mathbb{F}_3[t_4, t_8, t_{20}, t_{26}, t_{36}, t_{48}] \otimes \Lambda_{\mathbb{F}_3}(t_9, t_{21}, t_{25})/R,$$

where R is an ideal generated by t_4t_9, t_8t_9, t_4t_{21}, $t_9t_{20} + t_8t_{21}$, $t_9t_{20} + t_4t_{25}$, $t_{26}t_4 + t_{21}t_9$, t_8t_{25}, $t_{26}t_8 - t_{25}t_9$, $t_{20}t_{21}$, $t_{20}t_{25}$, $t_{26}t_{20} - t_{21}t_{25}$ and $t_{20}^3 - t_4^3t_{48} - t_8^3t_{36} + t_{20}^2t_8^2t_4$. In this note we prove that BF_4 is determined up to completion by its cohomology at the torsion prime 3, as well.

Theorem 1.1. *Let X be a 3-complete space such that $H^*(X; \mathbb{F}_3)$ is isomorphic to $H^*(BF_4; \mathbb{F}_3)$ as $\mathcal{A}_3$-algebras. Then X is homotopy equivalent to BF_4 up to 3-completion.*

Proof. See Section 2. □

A different question is whether or not a compact Lie group or p-compact group is N-determined (see [12] and [18]): Let X be a p-compact group and $j\colon N \longrightarrow X$ its maximal torus normalizer. Then X is said to be N-determined if any diagram

$$\begin{tikzcd} & N \arrow[dl, "j"'] \arrow[dr, "j'"] & \\ X \arrow[rr, dashed] & & X' \end{tikzcd}$$

Communicated by Prof. K. Ueno, March 24, 2000

*The author is partially supported by DGES grant PB97-1095.

where X' is another p-compact group with isomorphic maximal torus normalizer $j'\colon N \longrightarrow X'$, might be closed by an isomorphism of p-compact groups $X \longrightarrow X'$.

This essentially means that X is determined by its maximal torus normalizer in the sense that any other p-compact group with isomorphic maximal torus normalizer is actually isomorphic to X.

For a p-compact group X we denote $\operatorname{Out} X$ the group of homotopy classes of self homotopy equivalences of BX. Restriction to the maximal torus normalizer $j\colon N \longrightarrow X$ induces a homomorphism $\operatorname{Out}(X) \longrightarrow \operatorname{Out}(N)$ (see [12]). Then, a p-compact group X with maximal torus normalizer $j\colon N \longrightarrow X$ has N-determined automorphisms if $\operatorname{Out}(X) \longrightarrow \operatorname{Out}(N)$ is injective. A p-compact group is totally N-determined if it is N-determined and has N-determined automorphisms.

Møller and Notbohm have considered (although using a different language) the torsion free case: [14]. Here we have considered again the case of F_4 at the prime 3. Our result is

Theorem 1.2. *The exceptional Lie group F_4 is a totally N-determined 3-compact group.*

Proof. Compact Lie groups are known to have N-determined automomorphisms, hence all we have to prove is that F_4 is N-determined. See Section 7. □

Organization of the paper. The paper is organized as follows. In Section 2, we prove Theorem 1.1. The gaps left open in that section are filled in the following ones. In Section 3 we describe the 3-stubborn subgroups of F_4. In Section 4 we construct an inclusion of the maximal torus of F_4, into a 3-complete space X with nice properties, and extend that inclusion of the maximal torus to an inclusion of the torus normalizer in F_4, into X. In Section 5, we calculate the homotopy type of some mapping spaces involving the 3-stubborn subgroups of F_4. In Section 6, we contruct the mod 3 homotopy equivalence between BF_4 and X. In the last section we prove Theorem 1.2.

Notation. Here $\mathcal{A}_p$ is the mod p Steenrod algebra, all spaces are assumed to have the homotopy type of CW-complexes, and completion means Bousfield-Kan completion ([3]). Given a space Y, we write H^*Y for $H^*(Y;\mathbb{F}_3)$, and $Y_p^\wedge$ for the B-K $(\mathbb{Z}/p)_\infty$-completion of the space Y. The symbol $\mathcal{H}^*\mathcal{T}$ will denote the cohomological category of spaces (see Section 2). The symbol $\Lambda_\mathbb{F}$ is used to denote an exterior algebra over the coefficients field $\mathbb{F}$. Given a group W, we denote by $\mathcal{H}^*(W;M)$ the group cohomology with coefficients in the W-module M. Given a compact Lie group, the symbol $\mathcal{Q}_p(G)$ denotes the Quillen category of G at the prime p (see Section 7).

Acknowledgements. To finish the introduction, it is a pleasure to thank C. Broto and D. Notbohm for useful conversations and interesting suggestions, and SFB 170 at Göttingen for hospitality and economical support during part of this work.

2. Proof of Theorem 1.1

In what follows, we work in $\mathcal{H}^*\mathcal{T}$, the cohomological category of spaces. In this category, objects are topological spaces and morphisms are homomorphisms of their cohomological algebras over the Steenrod algebra. If a morphism is induced by a homotopy class of maps, it is denoted by a solid arrow. Dotted arrows denote morphisms which are not necessarily induced by a topological map. We say that a diagram commutes (in $\mathcal{H}^*\mathcal{T}$) if the subdiagram of solid arrows commutes homotopically and the total diagram of homomorphisms of cohomological algebras over the Steenrod algebra commutes.

Throughout this section, X is a 3-complete space with $H^*X \cong H^*BF_4$ as $\mathcal{A}_3$-algebras. The construction of the mod 3 equivalence $f : BF_4 \to X$ is based on the mod p approximation of the classifying space BG of a compact Lie group G via its p-stubborn subgroups that appears in [9].

Let p be a fixed prime. A p-toral group is a compact Lie group P whose component of the unit, P_0, is a torus and whose group of components P/P_0, is a finite p-group. Given a compact Lie group G, a p-toral subgroup P of G is said to be p-stubborn if the quotient $N(P)/P$, where $N(P)$ is the normalizer of P in G, is finite and does not contain any nontrivial normal p-subgroup.

Let G be a compact Lie group and $\mathcal{R}_p(G)$ denote the (topological) orbit category, whose objects are homogeneous spaces G/P with $P \subset G$ p-stubborn, and whose morphisms are given by G-equivariant maps. Then, if EG denotes a free G-CW-complex, the Borel construction defines a (continous) functor

$$\begin{array}{cccc} EG \times_G \_: & \mathcal{R}_p(G) & \longrightarrow & Top \\ & G/P & \mapsto & EG \times_G G/P \end{array}$$

and because the category $\mathcal{R}_p(G)$ is finite, it makes sense to consider

$$\operatorname{hocolim}_{\mathcal{R}_p(G)} EG \times_G \_.$$

Jackowski, McClure and Oliver proved [9]

Theorem 2.1. *For any compact Lie group G, the map*

$$\operatorname{hocolim}_{\mathcal{R}_p(G)} EG \times_G \_ \longrightarrow EG \times_G * \simeq BG$$

is a p-local equivalence, that is, induces an isomorphism in cohomology with $\mathbb{Z}_{(p)}$-coefficients.

To apply this result, we follow a simplified version of the programme developed by the author in [21].

Actually, we choose, for simplicity, a skeletal subcategory of $\mathcal{R}_3(F_4)$; that is, a full subcategory containing just one represetative for each isomorphism class of objects in $\mathcal{R}_3(F_4)$. This election is described in Proposition 3.6.

By abuse of language, we keep the same notation $\mathcal{R}_3(F_4)$ for such subcategory. In fact, since both categories are equivalent we can still write

$$\operatorname{hocolim}_{\mathcal{R}_3(F_4)} EF_4 \times_{F_4} F_4/P \xrightarrow{\simeq_3} BF_4$$

where $\simeq_3$ means homotopy equivalence up to 3-completion.

In order to get the desired map $f : BF_4 \to X$, we construct a collection of maps $EF_4 \times_{F_4} F_4/P \simeq BP \xrightarrow{f_P} X$ such that they fit together in a homotopy commutative diagram

$$\{BP\}_{\mathcal{R}_3(F_4)} \longrightarrow X,$$

and finally we check that the associated obstruction groups to extend the map to the homotopy colimit vanish.

The proof is essentially divided in the following series of propositions.

Write $\phi^* : H^*X \longrightarrow H^*BF_4$ for the given isomorphism of $\mathcal{A}_3$-algebras, which can be assumed to be the identity, and denote it by $\phi : (BF_4)^\wedge_3 \dashrightarrow X$ in $\mathcal{H}^*\mathcal{T}$. Let $T \xrightarrow{i} F_4$ be a maximal torus of F_4 and $N \xrightarrow{i} F_4$ be the normalizer of the torus in F_4. Then:

Proposition 2.2. *There exist maps $f_T : BT^\wedge_3 \longrightarrow X$ and $f_N : BN^\wedge_3 \longrightarrow X$ such that they fit in the diagram*

$$\begin{tikzcd}
& BT^\wedge_3 \arrow[d] \arrow[ddl, "Bi^\wedge_3"'] \arrow[ddr, "f_T"] & \\
& BNT^\wedge_3 \arrow[dl] \arrow[dr, "f_N"] & \\
(BF_4)^\wedge_3 \arrow[rr, dashed, "\phi"] & & X
\end{tikzcd}$$

which commutes in $\mathcal{H}^*\mathcal{T}$:

Proof. See Section 4. □

Now, note that for a given P, representative 3-stubborn subgroup of F_4, the standard map

$$BP \longrightarrow BF_4$$

factors through BNT

$$BP \xrightarrow{h} BNT \longrightarrow BF_4,$$

so we have a diagram

$$\{BP\}_{\mathcal{R}_3(F_4)} \xrightarrow{h} BNT$$

that is probably non commutative. We prove that the composition with the map f_N constructed in Proposition 2.2 commutes again up to homotopy. Note that the diagram

$$\{BP\}_{\mathcal{R}_3(F_4)} \overset{f_N \circ h}{\dashrightarrow} X$$

commutes in $\mathcal{H}^*\mathcal{T}$ by construction.

Proposition 2.3. *The diagram*

$$\{BP\}_{\mathcal{R}_3(F_4)} \xrightarrow{f_N \circ h} X \tag{2.1}$$

commutes up to homotopy.

Proof. See Section 6. □

The homotopy commutativity of the diagram (2.1) induces a map from the 1-skeleton of the homotopy colimit of $\{BP\}_{\mathcal{R}_3(F_4)}$ to X. The obstruction groups to extend this map to the total homotopy colimit are

$$\underset{\mathcal{R}_3(F_4)}{\varprojlim{}^i}\, \pi_j(\mathrm{map}(BP, X)_{f_P}),$$

where $\lim^i$ is the i-th derived functor of the inverse limit functor (see [3] and [23]).

To calculate those groups, we compare the functors

$$\Pi_j^X, \Pi_j^{F_4} : \mathcal{R}_3(F_4) \longrightarrow \mathcal{A}b$$

defined as

$$\begin{aligned} \Pi_j^X(F_4/P) &:= \pi_j(\mathrm{map}(BP, X)_{f_P}), \\ \Pi_j^{F_4}(F_4/P) &:= \pi_j(\mathrm{map}(BP, (BF_4)_3^\wedge)_{Bi_P}). \end{aligned}$$

The category $\mathcal{A}b$ is the category of abelian groups. These functors are well defined for $j \geq 2$ and note that according to [9], $\Pi_1^{F_4}(F_4/P) = \pi_1(BZ(P))$ which is an abelian group and therefore the functor is well defined in this case as well. The case of Π_1^X is similar (see Section 6).

The relation between those functors is given by the following proposition.

Proposition 2.4. *There exists a natural transformation*

$$\mathcal{T} : \Pi_j^{F_4} \longrightarrow \Pi_j^X$$

which is a natural equivalence.

Proof. See Section 6. □

So we have that:

$$\lim_{\overleftarrow{\mathcal{R}_3(F_4)}}{}^{i}\, \pi_j(\mathrm{map}(BP, X)_{f_P}) \cong \lim_{\overleftarrow{\mathcal{R}_3(F_4)}}{}^{i}\, \pi_j(\mathrm{map}(BP, (BF_4)^{\wedge}_3)_{Bi_P}),$$

and by [9]:

$$\lim_{\overleftarrow{\mathcal{R}_3(F_4)}}{}^{i}\, \pi_j(\mathrm{map}(BP, (BF_4)^{\wedge}_3)_{Bi_P}) = 0.$$

Hence we have a map $f : BF_4 \longrightarrow X$. To finish the proof we have to check that the map induced by f on cohomology is an isomorphism. This follows from the commutative diagram

$$\begin{tikzcd} & BNT \arrow[dl, "Bi"'] \arrow[dr, "f_N"] & \\ BF_4 \arrow[rr, "f"] & & X \end{tikzcd}$$

because both diagonal maps induce injective maps in cohomology, so f^* is injective too, and because H^*X and H^*BF_4 have the same Poincaré series, which implies that f^* is an isomorphism.

3. The 3-stubborn subgroups of F_4

In this section we deal with the theory relating to the 3-stubborn subgroups of F_4. First we calculate the conjugacy classes of elements of order 3 in F_4. The following lemma, that appears in [10], contains two general facts which are very useful when making computations.

Lemma 3.1. *Let G be a compact connected Lie group.*

(i) *If G is simply connected, then the centralizer of any element in G is connected.*

(ii) *Fix a maximal torus $T \subset G$ and an element $g \in T$. Let $W = N_G(T)/T$ and $W_g = N_{C(g)}(T)/T$ be the Weyl groups of G and of the centralizer $C(g)$, respectively. Then the number of elements in T conjugate (in G) to g is just the Weyl group index $[W : W_g]$.*

Next proposition describes the conjugacy classes of elements of order 3 in F_4, as well as their centralizers.

Proposition 3.2. *The group F_4 contains exactly 3 conjugacy classes of elements of order 3 listed below.*

Class	*Centralizer*	*Nr in T*
$3A$	A_2A_2	32
$3B$	B_3T_1	24
$3C$	C_3T_1	24

Proof. Because for any two elements in a simply connected compact Lie group there exits a maximal torus containing both of them, it is enough to calculate the conjugacy classes in a maximal torus T.

We can identify $\hat{T} = \hat{T}(F_4) \cong \mathbb{R}^4$ with the usual inner product such that the set of roots of F_4 is (see [2])

$$R' = \left\{ \pm x_i (1 \le i \le 4), \pm x_i \pm x_j (1 \le i < j \le 4), \frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4) \right\} \subset (\hat{T}')^*.$$

Given $\hat{g} \in \hat{T}$, and $g = \exp(\hat{g})$, by Lemma 3.1 (i), $C_{F_4}(g)$ is connected (and it clearly has maximal torus T). The roots of $C_{F_4}(g)$ are precisely those roots of F_4 which take integral values on $\hat{g}$, hence we can know the group type of $C_{F_4}(g)$.

Set $\hat{g}_1 = (1/3, 1/3, 1/3, 1) \in \hat{T}$ and let $g_1 = \exp(\hat{g}_1)$. Then g_1 has order 3 and $C_{F_4}(g_1)$ is a compact connected Lie group of rank 4 with roots

$$R_1 = \left\{ \pm(x_i - x_j)(1 \le i < j \le 3), \pm x_4, \pm\frac{1}{2}(x_1 + x_2 + x_3 \pm x_4) \right\},$$

hence $C_{F_4}(g_1)$ has type A_2A_2. In fact $C_{F_4}(g_1) = SU(3,3)$ (see [10]). Call $3A$ the conjugacy class of g_1.

Set $\hat{g}_2 = (0, 0, 0, 1/3) \in \hat{T}$ and let $g_2 = \exp(\hat{g}_2)$. Then g_2 has order 3 and $C_{F_4}(g_2)$ is a compact connected Lie group of rank 4 with roots

$$R_2 = \big\{ \pm x_i (1 \le i \le 3), \pm x_i \pm x_j (1 \le i < j \le 3) \big\},$$

hence $C_{F_4}(g_2)$ has type B_3T_1. Call $3B$ the conjugacy class of g_2.

Set $\hat{g}_3 = (1/3, 1/3, 0, 0) \in \hat{T}$ and let $g_3 = \exp(\hat{g}_3)$. Then g_3 has order 3 and $C_{F_4}(g_3)$ is a compact connected Lie group of rank 4 with roots

$$R_3 = \left\{ \pm x_3, \pm x_4, \pm x_3 \pm x_4, \pm(x_1 - x_2), \frac{1}{2}(\pm(x_1 - x_2) \pm x_3 \pm x_4) \right\},$$

hence $C_{F_4}(g_3)$ has type C_3T_1. Call $3C$ the conjugacy class of g_3.

Now, by Lemma 3.1 (ii), we know that the number of elements in T which are in the class

- $3A$ is $[W_{F_4} : W_{C(g_1)}] = 32$,
- $3B$ is $[W_{F_4} : W_{C(g_2)}] = 24$,
- $3C$ is $[W_{F_4} : W_{C(g_3)}] = 24$.

Because there are only 80 elements of order 3 in T, this means that those three classes are all the conjugacy classes of elements of order 3 in T. $\square$

If an elementary abelian 3-subgroup $V \subset F_4$ has all its elements (but the unit) in the conjugacy class $3X$, that is $V - \{1\} \subset 3X$, we will say that V is $3X$-pure.

Because every element of order three in F_4 is (up to conjugation) in $SU(3,3)$, we are also interested in the conjugacy classes of elements of order 3 in $SU(3,3)$.

Consider the following elements of $SU(3)$,

$$A = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad D = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi\omega & 0 \\ 0 & 0 & \xi\omega \end{pmatrix},$$

where $\xi, \omega \in \mathbb{C}$ such that $\omega^3 = 1$, $\omega \neq 1$ and $\xi^3 = \omega$.

Note that the class $(A, 1) = (1, A) \in SU(3, 3)$ is the generator of the center of $SU(3, 3)$ and the group generated by $(A, 1)$, (B, B) and (C, C) is the unique (up to conjugation) non toral elementary abelian 3-subgroup of $SU(3, 3)$.

The conjugacy classes of elements of order three in $SU(3, 3)$ are described in the following proposition.

Proposition 3.3. *$SU(3, 3)$ contains exactly 11 conjugacy classes of elements of order three listed below,*

Class	*Centralizer*	*Nr in T*
$3\alpha_i$, $i = 1, 2$	$SU(3, 3)$	1
$3\alpha_3$	$\langle T, (C, C)\rangle$	12
$3\beta_1$	$(S^1)^2 \times_{\mathbb{Z}/3} SU(3)$	6
$3\beta_2$	$SU(3) \times_{\mathbb{Z}/3} (S^1)^2$	6
$3\gamma_k$, $k = 1, \ldots, 6$	$S(S^1 \times U(2)) \times_{\mathbb{Z}/3} S(S^1 \times U(2))$	9

Proof. As $SU(3, 3)$ is a small group, the proof is done by a routine calculation. Note that B, B^2, AB, A^2B, AB^2 and A^2B^2 are conjugate each other in $SU(3)$ as well as D, ABD and AB^2D, therefore the representatives can be chosen:

- $(A^i, 1)$ for $3\alpha_i$, $i = 1, 2$,
- (B, B) for $3\alpha_3$,
- $(B, 1)$ for $3\beta_1$,
- $(1, B)$ for $3\beta_2$ and
- (A^sD^r, D^r) where $s = 0, 1, 2$ and $r = 1, 2$ for $3\gamma_{2s+r}$.

The centralizers are easily calculated and the number of elements in T which are in each class can be calculated by Lemma 3.1. □

Remark 3.4. According to Rector ([19], Proposition 7.5), the Weyl group of $V_2 = \langle(A, 1), (B, B)\rangle$ in F_4 is $GL_2(\mathbb{F}_3)$, hence all the elements of this subgroup are in the same conjugacy class of F_4, that is, $3\alpha_i \subset 3A$ for all $i = 1, 2, 3$. Moreover $3\beta_i \not\subset 3A$ for any $i = 1, 2$ because in that case $\sharp|3A - (\cup_j 3\alpha_j \cup 3\beta_i)| = 12$ and 12 can not be expressed as a sum of 9's and just one 6.

The class $3A$ in F_4 has a special role, as the following proposition shows.

Proposition 3.5. *F_4 contains exactly 3 conjugacy classes of elementary abelian $3A$-pure subgroups, with representatives V_1, V_2 and V_3 as listed below. They are all presented as subgroups of $SU(3,3) \subset F_4$ where the matrices A, B and C are those of above.*

V_i	$C_{F_4}(V)$	$N_{F_4}(C_{F_4}(V))/C_{F_4}(V)$
$V_1 = \langle (A,1) \rangle$	$SU(3,3)$	$\mathbb{Z}/2$
$V_2 = \langle (A,1), (B,B) \rangle$	$\langle T, (C,C) \rangle$	$GL_2(\mathbb{F}_3)$
$V_3 = \langle (A,1), (B,B), (C,C) \rangle$	V_3	$SL_3(\mathbb{F}_3)$

Proof. Let V_i an elementary abelian $3A$-pure subgroup of F_4 with $\operatorname{rk} V_i = i$. The case V_1 appears in [10] Lemma 3.3 (v).

Consider the case V_2. We can assume that V_2 is generated by $(A,1)$ and a second generator named X. If X is (up to conjugation) (B,B), that is, X is in the class $3\alpha_3 \subset SU(3,3)$, then by the remark above we are finished. Suppose now that $X \notin 3\alpha_3$, in that case we use again the remark above and $X \in 3\gamma_j$ for some j, but because V_2 is $3A$-pure, $X^r(A,1)^s \in 3A$ for all $r, s \in \mathbb{Z}$, that is, $3\gamma_j \subset 3A$ for all j. This is impossible because $54 = 6\sharp|3\gamma_j| \not\leq \sharp|3A| = 32$, hence V_2 is (up to conjugation) $\langle (A,1),(B,B) \rangle$ and we are finished in this case.

Consider the case of V_i for $i \geq 3$. If V_i is toral, similar arguments to those of above show that, $(V_i - 1) \subset \cup_j 3\alpha_j$, which is impossible because $3^i - 1 = \sharp|V_i - \{1\}| \not\leq \sharp| \cup_j 3\alpha_j| = 14$ for $i \geq 3$. Hence V_i can not be toral. In this case $i = 3$, and V_3 has to be (up to conjugation) $\langle (A,1),(B,B),(C,C) \rangle$, and by [19] 7.4, we are finished. $\square$

Now we have enough information to calculate the 3-stubborn subgroups of F_4.

Proposition 3.6. *The group F_4 contains exactly 7 conjugacy classes of 3-stubborn subgroups, with representatives $P_1, \dots, P_7$. They are all presented as subgroups of $SU(3,3) \subset F_4$. Also $Q \subset N \subset SU(3)$ are the subgroups $Q = \langle A, B, C \rangle$ (the non abelian 3-group of order 27 and exponent 3) and $N = \langle S^1 \times S^1, C \rangle$.*

P	$N(P)/P$
$P_1 = N \times_{\mathbb{Z}/3} N = N_3(T)$	$(\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2$
$P_2 = N \times_{\mathbb{Z}/3} Q$	$(\mathbb{Z}/2 \times Sp_2(\mathbb{F}_3)) \rtimes \mathbb{Z}/2$
$P_3 = Q \times_{\mathbb{Z}/3} N$	$(Sp_2(\mathbb{F}_3) \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2$
$P_4 = Q \times_{\mathbb{Z}/3} Q$	$(Sp_2(\mathbb{F}_3) \times Sp_2(\mathbb{F}_3)) \rtimes \mathbb{Z}/2$
$P_5 = \langle (S^1)^2 \times_{\mathbb{Z}/3} (S^1)^2, (C,C) \rangle \cong T \rtimes \mathbb{Z}/3$	$GL_2(\mathbb{F}_3)$
$P_6 = \langle (A,1),(B,B),(C,C) \rangle \cong (\mathbb{Z}/3)^3$	$SL_3(\mathbb{F}_3)$
$P_7 = T$	W_{F_4}

Proof. Fix a 3-stubborn subgroup $P \subset F_4$, and let $Z(P)_3$ be the 3-torsion subgroup of its center. Because every 3-stubborn subgroup of F_4 has a representative in $SU(3,3)$ and $C_{F_4}(P) \subset Z(P)$, we have that there exists at least one element in $Z(P)_3$ which is in the conjugacy class $3A$. We consider the following cases.

If $\operatorname{rk} Z(P)_3 = 1$, say $Z(P)_3 = \langle g \rangle \subset 3A$. Then $N_{F_4}(P) = N_{C(g)}(P) \rtimes \mathbb{Z}/2$, and so P is also 3-stubborn in $C(g) \cong SU(3,3)$. By [9] Proposition 1.6, the 3-stubborn subgroups of $SU(3,3)$ are precisely the groups of the form $P' \times_{\mathbb{Z}/3} P''$ where P' and P'' are 3-stubborn in $SU(3)$. Also, the only 3-stubborn subgroups of $SU(3)$ are Q and N. Conversely, if $P = P' \times_{\mathbb{Z}/3} P''$ where P' and P'' are 3-stubborn in $SU(3)$, then

$$N_{F_4}(P)/P = \big((N_{SU(3)}(P')/P') \times (N_{SU(3)}(P'')/P'')\big) \rtimes \mathbb{Z}/2,$$

and so P is a 3-stubborn subgroup of F_4. Note that the subgroups $N \times_{\mathbb{Z}/3} Q$ and $Q \times_{\mathbb{Z}/3} N$ are not conjugate in F_4, since the action of $\mathbb{Z}/2$ on each factor of $C(g) \cong SU(3,3)$ is via complex conjugation.

If $\operatorname{rk} Z(P)_3 = 2$, say $Z(P)_3 = \langle g, h \rangle$ where $\langle g \rangle \subset 3A$. In this case, we have two possibilities: if $h \in 3\gamma_i$ for $i = 1, \ldots, 6$, or if $h \in 3\beta_j$ for $j = 1, 2$.

If $h \in 3\gamma_i$ for any $i = 1, \ldots, 6$, then

$$P \subset C(g,h) \cong S(S^1 \times U(2)) \times_{\mathbb{Z}/3} S(S^1 \times U(2)),$$

and since P is 3-toral, $P \subset T$. The only possible 3-stubborn subgroup $P \subset T$ of F_4 is T and $\operatorname{rk} Z(T)_3 = 4 \neq 2$.

If $h \in 3\beta_1$ (the case $h \in 3\beta_2$ is similar) then

$$P \subset C(g,h) \cong (S^1)^2 \times_{\mathbb{Z}/3} SU(3)$$

and the only possible 3-stubborn subgroups are $P' = (S^1)^2 \times_{\mathbb{Z}/3} Q$ and $P'' = (S^1)^2 \times_{\mathbb{Z}/3} N$. But $N_{F_4}(P')/P' = (\Sigma_3 \times SP_2(\mathbb{F}_3)) \rtimes \mathbb{Z}/2$ and $N_{F_4}(P'')/P'' = \Sigma_3 \rtimes \mathbb{Z}/2$ which are not 3-reduced and therefore P' and P'' are not 3-stubborn subgroups of F_4. Therefore, the only chance is that $h \in 3\alpha_3$, that is, $Z(P)_3 = \langle g, h \rangle$ is an elementary abelian $3A$-pure subgroup of F_4, in that case we have a new 3-stubborn subgroup of F_4, $P = C(g,h) = \langle T, (C,C) \rangle$.

If $\operatorname{rk} Z(P)_3 = 3$, and $Z(P)_3$ is not $3A$-pure, then is toral and $P \subset C(Z(P)_3) \cong T$ which again produces no 3-stubborn subgroup $P \subset F_4$ such that $\operatorname{rk} Z(P)_3 = 3$. Hence if $\operatorname{rk} Z(P)_3 = 3$, $Z(P)_3$ has to be $3A$-pure and therefore $P = C(Z(P)_3) = V_3$.

Finally, if $\operatorname{rk} Z(P)_3 = 4$, then $P = T$, because W_{F_4} is 3-reduced. □

Note that,

Remark 3.7. Given P a 3-stubborn subgroup of F_4, define $P_T := P \cap T$. Then can be easily checked that the short exact sequence

$$P_T \to P \xrightarrow{\pi} P/P_T$$

has a section. Therefore we have that $B\pi^* : H^*B(P/P_T) \hookrightarrow H^*BP$.

Finally, two technical lemmas.

Lemma 3.8. *The cohomology group $\mathcal{H}^2(\mathbb{Z}/3; (LT_{PU(3)})^\wedge_3)$ is trivial.*

Proof. In [16], proof of Proposition 6.7, we see that $\mathcal{H}^2(\mathbb{Z}/3; (LT_{U(3)})^\wedge_3) = 0$. To finish the proof, we consider the exact sequence of $\mathbb{Z}/3$-modules:

$$0 \to \mathbb{Z}^\wedge_3 \to (LT_{U(3)})^\wedge_3 \to (LT_{PU(3)})^\wedge_3 \to 0. \qquad \square$$

Lemma 3.9. *Let $P \subset SU(3)$ be N or Q, and let T be the standard torus of $SU(3)$. Then, given the canonical inclusion $Bi : BP_T \longrightarrow BSU(3)$, there exists only one extension to a map $Bi : BP \longrightarrow BSU(3)$ up to homotopy, that is, there exists only one possible extension $i : P \longrightarrow SU(3)$ up to conjugation.*

Proof. The number of those extensions are calculated by the obstruction groups

$$\mathcal{H}^*(P/P_T; \pi_* \operatorname{map}(BP_T; BSU(3))_{Bi}) \cong \mathcal{H}^2(\mathbb{Z}/3; (LT_{PU(3)})^\wedge_3)$$

(notice that $C_{SU(3)}(P_T) = T$) and by Lemma 3.8 this group is trivial, hence we are finished. $\square$

Now, we can prove the next proposition which describes some properties of the 3-stubborn subgroups of F_4 that we need.

Proposition 3.10. *The representatives $i : P \hookrightarrow N_{F_4}(T) \hookrightarrow F_4$ defined in Proposition* 3.6 *satisfy the following conditions*:

(1) *P_T is a 3-toral group.*

(2) *$Z(SU(3,3)) \subset P_T$.*

(3) *$C_{F_4}(P_{iT}) = T$ for all $i \neq 6$. $C_{F_4}(P_{6T}) = P_5$.*

(4) *The canonical map*

$$\pi_0(\operatorname{map}(BP, (BF_4)^\wedge_3)_{B\alpha|BP_T=Bi}) \to \hom\big(H^*(BF_4; \mathbb{F}_3), H^*(BP; \mathbb{F}_3)\big)$$

is an injection.

Remark 3.11. Note that by (2), every element in $C_{F_4}(P_T)$ centralizes $Z(SU(3,3))$, so the statement (3) can be proved by calculating the centralizers in $SU(3,3)$.

Remark 3.12. By $\operatorname{map}(BP, (BF_4)^\wedge_3)_{B\alpha|_{BP_T}=Bi}$ we denote the components of the mapping space $\operatorname{map}(BP, (BF_4)^\wedge_3)$ given by maps $B\alpha : BP \to (BF_4)^\wedge_3$, such that $B\alpha|_{BP_T} \simeq Bi$. Hence (4) can be reformulated as "the extensions of the canonical inclusion $Bi|_{BP_T} : BP_T \to (BF_4)^\wedge_3$ to maps $B\alpha : BP \to (BF_4)^\wedge_3$ are controlled by cohomology".

Proof. The proof of the statements (1) to (3) is a straight forward calculation.

Let $\alpha, \beta : P \to F_4$ be two homomorphisms, such that $B\alpha^* \equiv B\beta^*$ and $B\alpha|_{BP_T} \simeq B\beta|_{BP_T}$, we see that $B\alpha \simeq B\beta$. That is trivial in the case of P_6 by Lannes' theory, but the other cases are more complicated and will be done in several steps.

Step 1. Both maps factor through $BSU(3,3)^\wedge_3$.

Consider the induced map between mapping spaces:

$$B\alpha, B\beta : \mathrm{map}(BZ(SU(3,3)), BP)_{Bi} \to \mathrm{map}(BZ(SU(3,3)), (BF_4)^\wedge_3)_{Bi}.$$

According to [8] and [15], and by (2), we get that $\mathrm{map}(BZ(SU(3,3)), BP)_{Bi} \simeq_3 BP^\wedge_3$ and $\mathrm{map}(BZ(SU(3,3)), (BF_4)^\wedge_2)_{Bi} \simeq BSU(3,3)^\wedge_3$ and the evaluation map allows us to reconstruct the original map. Hence we have two maps:

$$B\alpha, B\beta : BP \to BSU(3,3)^\wedge_3,$$

which when composed with the standard inclusion of $BSU(3,3)$ in BF_4, give the original ones. Moreover, by Lannes' T functor, we get that $B\alpha^* \equiv B\beta^*$.

Step 2. $\beta(x)(\alpha(x))^{-1} \in Z(SU(3,3))$ for all $x \in P$.

Let $\alpha, \beta : P \to SU(3,3)$ be two homomorphisms, such that $B\alpha|_{BP_T} \simeq B\beta|_{BP_T}$, we see that there exists $\gamma : P/P_T \to Z(SU(3,3))$ such that β is conjugate in $SU(3,3)$ to the homomorphism:

$$P \xrightarrow{\Delta} P \times P \xrightarrow{Id\times\pi} P \times P/P_T \xrightarrow{\alpha\times\gamma} SU(3,3) \times Z(SU(3,3)) \xrightarrow{\mu} SU(3,3),$$

where μ is the multiplication in $SU(3,3)$.

Proving the existence of γ is equivalent to proving that the induced map between the quotients by the center of $SU(3,3)$ are conjugate in $PU(3)^2$. We consider two different cases: when P/P_T has rank 1 or 2.

The first one is the case of P_5. In that case, we are interested in extensions of the standard inclusion of the torus $Bi : BT \to (BPU(3)^2)^\wedge_3$ to maps $B\alpha : B\overline{P_5} \to (BPU(3)^2)^\wedge_2$, where $\overline{P_5} := P_5/\mathbb{Z}/3$. The uniqueness of these extensions are classified by:

$$\mathcal{H}^*(\overline{P_5}/\overline{P_{5T}}; \pi_* \mathrm{map}(B\overline{P}_{5T}; (BPU(3)^2)^\wedge_3)_{Bi}) = \mathcal{H}^2(\mathbb{Z}/3; (LT_{PU(3)^2})^\wedge_3),$$

where the action is diagonal. To prove that this group is trivial, we apply Lemma 3.8 and the exact sequence of $\mathbb{Z}/3$ modules:

$$(LT_{PU(3)})^\wedge_3 \to (LT_{PU(3)^2})^\wedge_3 \to (LT_{PU(3)})^\wedge_3.$$

This finishes the proof of Step 2 for P_5.

The second case is the case of $P = P_1 \times_{\mathbb{Z}/3} P_2$ where P_i is 3-stubborn in $SU(3)$. Denote by $\overline{P_i}$ the quotient of P_i by the center of $SU(3)$. Then, the quotient of P by $Z(SU(3,3)$ is $\overline{P} = \overline{P_1} \times \overline{P_2}$, and therefore, any induced map $\overline{\alpha} : \overline{P} \longrightarrow PU(3)^2$ can be expressed as a matrix $\overline{\alpha} = (\overline{\alpha}_{i,j})$ where the morphism $\overline{\alpha}_{i,j}$ appears as the quotient by the center of the composition

$$P_i \hookrightarrow P \xrightarrow{\alpha} SU(3,3) \xrightarrow{\pi_j} PU(3)_j.$$

By construction, $\overline{\alpha}_{i,i}$ and $\overline{\beta}_{i,i}$ lift to $\tilde{\alpha}_{i,i}, \tilde{\beta}_{i,i} : P_i \longrightarrow SU(3)_i$ which are extensions of the canonical inclusion $(P_i)_T \longrightarrow SU(3)$, so by Lemma 3.9 $B\tilde{\alpha}_{i,i} \simeq B\tilde{\beta}_{i,i} \simeq Bi$, and therefore $B\overline{\alpha}_{i,i} \simeq B\overline{\beta}_{i,i}$.

Now, because the inclusion of $P_1 \hookrightarrow P$ commutes with the inclusion $P_2 \hookrightarrow P$, it forces to $\overline{\alpha}_{i,j} = \overline{\beta}_{i,j}$ and both are trivial. That implies $B\overline{\alpha} \simeq B\overline{\beta}$, which finishes the proof of Step 2.

Step 3. Different extensions are detected by cohomology.

We see that if $B\alpha^* \equiv B\beta^*$ then the homomorphism $\gamma : P/P_T \to Z(SU(3, 3))$ defined in Step 2 is trivial. By Lannes' theory it is enough to prove that $B\gamma^* \equiv 0$. Assume that it is non trivial, then $B\gamma^* : H^*BZ(SU(3,3)) \hookrightarrow H^*B(P/P_T)$. Note that also $B\pi^* : H^*B(P/P_T) \hookrightarrow H^*BP$ by Remark 3.7.

For $x_4 \in H^*BSU(3,3)$ (see [22]) we then have that:

$$\begin{aligned} B\alpha^*(x_4) &= B\beta^*(x_4) \\ &= B\alpha^*(x_4) + B\alpha^*(x_2)(B\pi^* \circ B\gamma^*)(v) + B\alpha^*(x_3)(B\pi^* \circ B\gamma^*)(u), \end{aligned}$$

where $H^*BZ(SU(3,3)) \cong \mathbb{F}_3[v] \otimes \Lambda_{\mathbb{F}_3}(u)$.

An easy analysis of the low dimensional cohomology of those 3-toral subgroups of $SU(3,3)$ shows that this equation cannot hold because, as we noted, $(B\pi^* \circ\ B\gamma^*)(u) \neq 0$ and $(B\pi^* \circ\ B\gamma^*)(v) \neq 0$.

So γ has to be trivial and therefore $B\alpha \simeq B\beta$. $\square$

4. The normalizer of the maximal torus

In this section, we prove Proposition 2.2, that is, we construct maps

$$BT_3^\wedge \xrightarrow{f_T} X$$

and

$$BNT_3^\wedge \xrightarrow{f_N} X$$

such that it fits in the following commutative diagram in $\mathcal{H}^*\mathcal{T}$:

$$\begin{tikzcd} & BT_3^\wedge \arrow[ddl, "Bi"'] \arrow[d] \arrow[ddr, "f_T"] & \\ & BNT_3^\wedge \arrow[dl] \arrow[dr, "f_N"] & \\ (BF_4)_3^\wedge \arrow[rr, dashed, "\phi"] & & X\ . \end{tikzcd} \tag{4.1}$$

First we fix some notation:

• V_T is the maximal toral elementary abelian subgroup of F_4, i.e. $V_T \cong (\mathbb{Z}/3)^4$,

• $SU(3,3)$ is the central product $SU(3) \times_{\mathbb{Z}/3} SU(3)$, which appears as a maximal connected subgroup of maximal rank in F_4 ([9]),

• V_S is the center of $SU(3,3)$, $V_S \cong \mathbb{Z}/3$, and it is identified as a subgroup of V_T,

• $\widetilde{BV_T}$ is a model of the classifying space of V_T such that the natural action of W_{F_4} on V_T induces an action on $\widetilde{BV_T}$ with a fixed-point.

Now, Lannes' theory provides a map $\widetilde{BV_T} \xrightarrow{f_{V_T}} X$ such that the following diagram is commutative in $\mathcal{H}^*\mathcal{T}$:

$$\begin{tikzcd} & \widetilde{BV_T} \arrow[dl] \arrow[dr, "f_{V_T}"] & \\ (BF_4)^\wedge_3 \arrow[rr, dashed, "\phi"] & & X \end{tikzcd} .$$

Moreover, Lannes' T functor shows that $\mathrm{map}(\widetilde{BV_T}, X)_{f_{V_T}} \simeq BT^\wedge_3$. Evaluation at the fixed-point of the W_{F_4}-action on $\widetilde{BV_T}$, provides a W_{F_4}-equivariant map

$$BT^\wedge_3 \simeq \mathrm{map}(\widetilde{BV_T}, X)_{f_{V_T}} \xrightarrow{f_T} X$$

with respect to the trivial W_{F_4}-action on X (moreover, $Bi^* = f_T^*$ by construction as we assumed $\phi^* = 1_{H^*BF_4}$). Therefore it produces a well defined map on the associated Borel construction $Y := (\mathrm{map}(\widetilde{BV_T}, X)_{f_{V_T}})_{hW_{F_4}}$ and an extension:

$$\begin{tikzcd} BT^\wedge_3 \arrow[d, "f"'] \arrow[dr, "f_T"] & \\ Y \arrow[r, "g"] \arrow[d] & X \\ BW_{F_4} & \end{tikzcd} . \tag{4.2}$$

We prove that Y is homotopy equivalent to $(BNT)^\circ_3$, the fibrewise completion (see [3]) of BNT via the fibration:

$$BT \to BNT \to BW_{F_4}. \tag{4.3}$$

Fibrations of the form $BT^\wedge_3 \to Z \to BW_{F_4}$ are determined (up to equivalence) by

• the W_{F_4}-action

• and a cohomological class in $\mathcal{H}^3(W_{F_4}; (LT_{F_4})^\wedge_3)$.

According to [17], there is just one possible lift of the W_{F_4}-action on V_T to the W_{F_4}-action on $T^\wedge_3$, hence fibrations (4.2) and (4.3) induce the same W_{F_4}-action on $BT^\wedge_3$. Therefore both fibrations are equivalent if and only if they are represented by the same cohomological class in $\mathcal{H}^3(W_{F_4}; (LT_{F_4})^\wedge_3)$. According to [1], this latest cohomology group is trivial, so fibrations (4.2) and (4.3) are equivalents. Hence $Y \simeq (BNT)^\circ_3$.

Finally, notice that

$$\mathcal{H}^3(W_{F_4}; (LT_{F_4})^\wedge_3) = \mathcal{H}^3(W_{F_4}; \pi_2 \operatorname{map}(BT^\wedge_3, (BF_4)^\wedge_3)_{Bi})$$

does also classify the possible extensions of the natural map $BT^\wedge_3 \longrightarrow (BF_4)^\wedge_3$ to maps $(BNT)^\circ_3 \longrightarrow (BF_4)^\wedge_3$, thus there is just one possible extension to the natural inclusion and it makes commutative:

$$\begin{tikzcd}
 & BT^\wedge_3 \arrow[ddl, "Bi"'] \arrow[d] \arrow[ddr, "f_T"] & \\
 & BNT^\wedge_3 \arrow[dl] \arrow[dr, "f_N"] & \\
(BF_4)^\wedge_3 & & X .
\end{tikzcd} \tag{4.4}$$

To obtain diagram (4.1), we need to show that it is possible to close diagram (4.4) (in $\mathcal{H}^*\mathcal{T}$) with ϕ. Consider $i : V_S \subset V_T$, then applying $\operatorname{map}(BV_S, \_)_{Bi\circ\_}$ to diagram 4.4, and noting that:

• $\operatorname{map}(BV_S, BT^\wedge_3)_{Bi} \simeq BT^\wedge_3$, $\operatorname{map}(BV_S, BNT^\wedge_3)_{Bi} \simeq (BN_ST)^\wedge_3$ (where $N_ST = N_{SU(3,3)}T$), and $\operatorname{map}(BV_S, (BF_4)^\wedge_3)_{Bi} \simeq BSU(3,3)^\wedge_3$ by [8], and

• $H^* \operatorname{map}(BV_S, X)_{Bi\circ f_T} = H^*BSU(3,3)$ by Lannes' T functor, thus according to [22], $\operatorname{map}(BV_S, X)_{Bi\circ f_T} \simeq BSU(3,3)^\wedge_3$,

we obtain a new diagram:

$$\begin{tikzcd}
 & BT^\wedge_3 \arrow[ddl, "Bi"'] \arrow[d] \arrow[ddr, "f_{T\#}"] & \\
 & (BN_ST)^\wedge_3 \arrow[dl] \arrow[dr, "f_{N\#}"] & \\
BSU(3,3)^\wedge_3 & & BSU(3,3)^\wedge_3 .
\end{tikzcd} \tag{4.5}$$

According to [4] and [22], $BSU(3,3)$ is a totally N-determined 3-compact group, hence there exists a map $BSU(3,3)^\wedge_3 \xrightarrow{g} BSU(3,3)^\wedge_3$ closing the diagram (4.5) and (together with the evaluation maps) giving rise to the commutative diagram in $\mathcal{H}^*\mathcal{T}$:

$$\begin{tikzcd}
BSU(3,3)^\wedge_3 \arrow[rr, "g"] \arrow[ddd] & & BSU(3,3)^\wedge_3 \arrow[ddd] \\
& BN_{SU(3,3)}T^\wedge_3 \arrow[ul] \arrow[ur] \arrow[d] & \\
& BNT^\wedge_3 \arrow[dl] \arrow[dr, "f_T"] & \\
(BF_4)^\wedge_3 \arrow[rr, dashed, "\widetilde{\phi}"] & & X\ .
\end{tikzcd}$$

where all the cohomological maps induced by the arrows are injective, and $\widetilde{\phi}^*$ is just the restriction of g^*. It is trivial that $\widetilde{\phi}$ close diagram (4.4), hence all we have to prove is that $\widetilde{\phi}^* = \phi^*$. It follows from the fact that $Bi^* = f_T^*$ and:

Lemma 4.1. *There exists just one unstable map* $H^*BF_4 \xrightarrow{\phi^*} H^*BF_4$ *such that the following diagram is commutative*:

$$\begin{tikzcd}
& H^*BT \arrow[dl, "Bi^*"'] \arrow[dr, "Bi^*"] & \\
H^*BF_4 \arrow[rr, "\phi^*"] & & H^*BF_4\ .
\end{tikzcd}$$

Proof. Recall that from [20], as an algebra:

$$H^*BF_4 = \mathbb{F}_3[t_4, t_8, t_{20}, t_{26}, t_{36}, t_{48}] \otimes \Lambda_{\mathbb{F}_3}(t_9, t_{21}, t_{25})/R,$$

where R is an ideal generated by t_4t_9, t_8t_9, t_4t_{21}, $t_9t_{20} + t_8t_{21}$, $t_9t_{20} + t_4t_{25}$, $t_{26}t_4 + t_{21}t_9$, t_8t_{25}, $t_{26}t_8 - t_{25}t_9$, $t_{20}t_{21}$, $t_{20}t_{25}$, $t_{26}t_{20} - t_{21}t_{25}$ and $t_{20}^3 - t_4^3t_{48} - t_8^3t_{36} + t_{20}^2t_8^2t_4$.

Also in [20] it is shown that

$$\ker Bi^* = \{t_9, t_{21}, t_{25}, t_{26}, t_{20}t_9, t_{21}t_9, t_{25}t_9, t_{26}t_{20}\}\mathbb{F}_3[t_{26}, t_{36}, t_{48}]$$

thus $\phi^*(t_i) = t_i$ for $i = 4, 8, 20, 36$, and 48. Now

- $\beta(t_8) = t_9$ thus $\phi^*(t_9) = \beta(\phi^*(t_8)) = t_9$,

- $\mathcal{P}^3(t_8) = t_{20} - t_8^2 t_4$, hence $\phi^*(t_{20}) = \mathcal{P}^3(\phi^*(t_8)) - \phi^*(t_8^2 t_4) = t_{20}$,
- $\beta(y_{20}) = y_{21}$, thus $\phi^*(t_{21}) = \beta(\phi^*(t_{20})) = t_{21}$,
- $\mathcal{P}^1(t_{21}) = t_{25}$, hence $\phi^*(t_{25}) = \mathcal{P}^1(\phi^*(t_{20})) = t_{25}$, and
- $\beta(t_{25}) = t_{26}$, thus $\phi^*(t_{26}) = \beta(\phi^*(t_{25})) = t_{26}$.

Therefore $\phi^* = 1_{H^*BF_4}$. $\square$

5. Some mapping spaces

In this section we determine the homotopy type of some mapping spaces relating the toral part of the 3-stubborn subgroups of F_4 and a 3-complete space whose mod 3 cohomology equals to that of BF_4.

Let X be a 3-complete space such that $H^*X \cong H^*F_4$ as algebras over the mod 3 Steenrod algebra. Let $BT \xrightarrow{f_T} X$ the map constructed in Section 4. Given P a 3-stubborn subgroup of F_4, we can consider $P_T := P \cap T$, the toral part of P and we have a well defined map $BP_T \xrightarrow{Bi} BT \xrightarrow{f_T} X$. We prove,

Proposition 5.1. *If $P \neq P_6$, then the natural map*

$$\mathrm{map}(BP_T, BT_3^\wedge)_{Bi} \to \mathrm{map}(BP_T, X)_{f_T Bi},$$

is a mod 3 equivalence. Therefore $\mathrm{map}(BP_T, X)_{f_T Bi} \simeq BT_3^\wedge$.

The rest of this section is devoted to the proof of this proposition. In what follows, P is a 3-stubborn subgroup of F_4 different from P_6,

We start introducing some notation. Denote by $A_k \subset P_T$ the finite subgroup of P_T of elements of order 3^k. Define $A_\infty := \cup A_k$, then the natural map $BA_\infty \to BP_T$ is a mod 3 equivalence, which implies that $\mathrm{map}(BA, X) \simeq \mathrm{holim}\,\mathrm{map}(BA_k, X)$ and $\mathrm{map}(BP_T, BT_3^\wedge) \simeq \mathrm{holim}\,\mathrm{map}(BA_k, BT_3)$. Therefore, Proposition 5.1 follows from,

Proposition 5.2. *Let $A \subset P_T$ be any 3-subgroup of P_T such that $A_1 \subset A$, and let $\mathrm{map}(BA, Y)_{\alpha|_{BA_1} = Bi}$ denote the components of the mapping space $\mathrm{map}(BA, Y)$ given by maps $\alpha : BA \to Y$, such that $\alpha|_{BA_1} \simeq Bi$. Then the natural map*

$$\mathrm{map}(BA, BT_3^\wedge)_{\alpha|_{BA_1} = Bi} \to \mathrm{map}(BA, BX)_{\alpha|_{BA_1} = f_T Bi},$$

is a homotopy equivalence and therefore $\mathrm{map}(BA, X)_{f_T Bi} \simeq BT_3^\wedge$.

Proof. The proof is done by induction on the order of A. For $A = A_1$ the proof is an easy application of the Lannes' T functor (note that A_1 is always a rank 3 elementary abelian 3-group) and [8].

Let $A \subset P_T$ be any 3-subgroup of P_T such that $A_1 \subset A$. We can choose a subgroup $A' \subset A$ of index 3 and get the exact sequence

$$0 \to A' \to A \to \mathbb{Z}/3 \to 0.$$

Define $\widetilde{BA'} := EA/A' \simeq BA'$ and consider,

$$\begin{aligned} MX &:= \operatorname{map}(BA, BX)_{\alpha|_{BA_1}=f_T Bi}, \\ MX_0 &:= \operatorname{map}(\widetilde{BA'}, BX)_{\alpha|_{BA_1}=f_T Bi}, \\ MT &:= \operatorname{map}(BA, BT_3^\wedge)_{\alpha|_{BA_1}=Bi}, \\ MT_0 &:= \operatorname{map}(\widetilde{BA'}, BT_3^\wedge)_{\alpha|_{BA_1}=Bi}. \end{aligned}$$

Therefore $MX \simeq (MX_0)^{h\mathbb{Z}/3}$ and $MT \simeq (MT_0)^{h\mathbb{Z}/3}$.

Now assume that the natural map $MT_0 \to MX_0$ is a homotopy equivalence and $MT_0 \simeq MX_0 \simeq BT_3^\wedge$. Because that map is induced by f_T, which is $\mathbb{Z}/3$-equivariant (the action on $BT_3^\wedge$ and X is trivial), we have a $\mathbb{Z}/3$-equivariant mod 3 equivalence between 1-connected spaces. Therefore it induces also a mod 3 equivalence between the homotopy fixed-point set, that is,

$$MT \simeq (MT_0)^{h\mathbb{Z}/3} \xrightarrow{\simeq} (MX_0)^{h\mathbb{Z}/3} \simeq MX,$$

which finishes the proof. □

6. The map $BF_4 \to X$

In this section we prove Propositions 2.3 and 2.4. It is here where we need the precise description of the 3-stubborn subgroups of F_4 computed in 3.6. Recall the situation:

Given a 3-complete space X with $H^*X \cong_{\mathcal{A}_3} H^*BF_4$, in Section 4, we constructed a diagram:

$$\begin{tikzcd} & BNT_3^\wedge \arrow[dl, "Bi_{NT}"'] \arrow[dr, "f_N"] & \\ (BF_4)_3^\wedge \arrow[rr, dashed, "\phi"] & & X \end{tikzcd}$$

that commutes in $\mathcal{H}^*\mathcal{T}$. For any conjugacy class of 3-stubborn subgroups of F_4, we have a representative P included in the normalizer of the torus $P \xrightarrow{i_P} NT \to F_4$ (see Proposition 3.6). Composition with the map f_N, produces a collection of maps:

$$f_P := f_N \circ Bi_P : BP \to BNT \to X,$$

that give us a diagram:

$$\{BP\}_{\mathcal{R}_3(F_4)} \xrightarrow{f} X$$

such that the induced diagram in cohomology commutes by construction. We prove that it commutes up to homotopy.

Proof of Proposition 2.3. As every morphism in $\mathcal{R}_3(F_4)$ is composition of one induced by an inclusion and one induced by conjugation in F_4, it is enough to consider those induced by conjugation. Let $c_g : F_4/P \to F_4/P$ be a map of $\mathcal{R}_3(F_4)$ given by conjugation. We have to prove that the diagram:

$$\begin{tikzcd} BP \arrow[r, "Bi_P"] \arrow[d, "Bc_g"'] & BNT \arrow[r, "f_N"] & X \arrow[d, equal] \\ BP \arrow[r, "Bi_P"] & BNT \arrow[r, "f_N"] & X \end{tikzcd}$$

commutes up to homotopy. Without loss of generality, we can assume that the subgroup P is the representative of the conjugacy class that appears in Proposition 3.6. Now define $\alpha := i_P \circ c_g$, we can reformulate the problem in the following way: given a homomorphism α such that $(f_N \circ B\alpha)^* \equiv (f_N \circ Bi_P)^*$ and α is conjugate to i_P in F_4, then show that $f_N \circ B\alpha \simeq f_N \circ Bi_P$.

The group $P_T = P \cap T$ is a 3-toral subgroup as we quoted in Proposition 3.10 (1). The restrictions $\alpha|_{P_T}$ and $i_P|_{P_T}$ are conjugated in F_4, and hence by Proposition 4.1 in [14], they are also conjugate in NT, that is, $f_N \circ B\alpha|_{BP_T} \simeq f_N \circ Bi_P|_{BP_T}$. The following proposition shows that the extensions of $f_N \circ Bi_P|_{BP_T}$ to maps $BP \longrightarrow X$ are classified by cohomology, which finishes the proof. $\square$

Proposition 6.1. *For any representative* 3*-stubborn subgroup* P *of* F_4*, the canonical map*

$$\pi_0(\mathrm{map}(BP, X)_{B\alpha|BP_T \simeq f_N Bi_P}) \longrightarrow \hom_{\mathcal{K}}(H^*X, H^*BP)$$

is an injection.

Proof. The case of P_6 is trivial by Lannes' theory as soon as it is an elementary abelian group. Here we consider the case $P \neq P_6$.

The quotient $Q := P/P_T$ acts on $\widetilde{BP_T} := EP/P_T \simeq BP_T$ and therefore on $\mathrm{map}(\widetilde{BP_T}, \_) \simeq \mathrm{map}(BP_T, \_)$, such that $\mathrm{map}(BP, \_) \simeq \mathrm{map}(\widetilde{BP_T}, \_)^{hQ}$. Define $i_{P_T} := i_P|_{P_T}$, and consider the induced maps

$$\begin{tikzcd} & \mathrm{map}(\widetilde{BP_T}, BNT_3^\wedge)_{Bi_{P_T}} \arrow[dl] \arrow[dr, "f_N"] & \\ \mathrm{map}(\widetilde{BP_T}, (BF_4)_3^\wedge)_{Bi_N Bi_{P_T}} & & \mathrm{map}(\widetilde{BP_T}, X)_{f_N Bi_{P_T}}. \end{tikzcd}$$

We show that the both maps are Q-equivariant homotopy equivalences.

First consider the case of the left one. By Proposition 3.10 (1), and [15], both mapping spaces are homotopy equivalent (up to 3-completion) to the classifying space of the centralizer in NT and F_4, of P_T via the indicated maps. From 3.10 (3), we get that both mapping spaces are homotopy equivalent to $BT_3^\wedge$.

The case of the right one follows from Proposition 5.1.

Both maps are Q-equivariant as soon as the action of Q on P_T is via conjugation by elements in NT.

The next step of the proof is to take homotopy fixed-points. We get that

(6.1)
$$
\begin{tikzcd}
 & \operatorname{map}(\widetilde{BP_T}, BNT_3^\wedge)^{hQ}_{Bi_{P_T}} \arrow[dl] \arrow[dr, "f_N"] & \\
\operatorname{map}(\widetilde{BP_T}, (BF_4)_3^\wedge)^{hQ}_{Bi_N Bi_{P_T}} & & \operatorname{map}(\widetilde{BP_T}, X)^{hQ}_{f_N Bi_{P_T}}.
\end{tikzcd}
$$

are again mod 3 equivalences because every equivariant mod 3 equivalence between 1-connected spaces induces a mod 3 equivalence between the homotopy fixed-point sets. Recall that:

1) The components of $\operatorname{map}(\widetilde{BP_T}, (BF_4)_3^\wedge)^{hQ}_{Bi_N Bi_{P_T}}$ are distinguished by mod 3 cohomology by Proposition 3.10 (4).

2) The diagram

$$
\begin{tikzcd}
 & BNT_3^\wedge \arrow[dl, "Bi_{NT}"'] \arrow[dr, "f_{NT}"] & \\
(BF_4)_3^\wedge \arrow[rr, dashed, "\phi"] & & X
\end{tikzcd}
$$

commutes in mod 3 cohomology.

3) Any map in $\operatorname{map}(\widetilde{BP_T}, X)^{hQ}_{f_N Bi_{P_T}}$ has a lift to $BNT_3^\wedge$.

4) The obstruction group which classifies the extensions up to homotopy is isomorphic to that associated to $(BF_4)_3^\wedge$, for:

$$
\begin{aligned}
\mathcal{H}^2(Q; \pi_2 \operatorname{map}(\widetilde{BP_T}, X)_{f_N Bi_{P_T}}) &\cong \mathcal{H}^2(Q; \pi_2 BT_3^\wedge) \\
&\cong \mathcal{H}^2(Q; \pi_2 \operatorname{map}(\widetilde{BP_T}, (BF_4)_3^\wedge)_{Bi_N Bi_{P_T}}).
\end{aligned}
$$

All this together implies that the components of

$$\operatorname{map}(\widetilde{BP_T}, X)^{hQ}_{f_N Bi_{P_T}} \simeq \operatorname{map}(BP, X)_{B\alpha|BP_T \simeq f_N Bi_P}$$

are also distinguished by mod 3 cohomology. $\square$

We finish the section by proving the Proposition 2.4.

Proof of Proposition 2.4. For any 3-stubborn subgroup P of F_4, we define an isomorphism

$$\Pi_i^{F_4}(F_4/P) \xrightarrow{\cong} \Pi_i^X(F_4/P)$$

which is compatible with the maps in $\mathcal{R}_3(F_4)$. In order to do that we use that the maps in (6.1) induce the homotopy equivalences

$$\begin{tikzcd}[column sep=tiny]
 & \operatorname{map}(BP, BNT_3^\wedge)_{Bi_P} \arrow[dl] \arrow[dr, "f_N"] & \\
\operatorname{map}(BP, (BF_4)_3^\wedge)_{Bi_P} & & \operatorname{map}(BP, X)_{f_P},
\end{tikzcd}$$

which depend on the chosen lift $BP \xrightarrow{Bi_P} BNT_3^\wedge$ of $BP \xrightarrow{Bi_P} (BF_4)_3^\wedge$. Two lifts differ by a conjugation c_g, hence we have to prove that the diagram

$$\begin{tikzcd}
 & \operatorname{map}(BP, BNT_3^\wedge)_{Bi_P Bc_g} \arrow[dl] \arrow[dr, "(f_N)_*"] & \\
\operatorname{map}(BP, (BF_4)_3^\wedge)_{Bi_P} & \operatorname{map}(BP, BNT_3^\wedge)_{Bi_P} \arrow[l] \arrow[r] & \operatorname{map}(BP, X)_{f_P}
\end{tikzcd}$$

commutes. It follows from the fact that the commutative diagram

$$\begin{tikzcd}
BC_{F_4}(P) \arrow[rr, equal] \arrow[dr, "\simeq_3"] \arrow[ddd, "Bc_g"', "\simeq"] & & BC_{NT}(P) \arrow[d, "\simeq_3"'] \\
 & \operatorname{map}(BP, (BF_4)_3^\wedge)_{Bi_P} \arrow[d, equal] & \operatorname{map}(BP, BNT_3^\wedge)_{Bi_P} \arrow[l] \\
 & \operatorname{map}(BP, (BF_4)_3^\wedge)_{Bi_P Bc_g} & \operatorname{map}(BP, BNT_3^\wedge)_{Bi_P Bc_g} \arrow[l] \\
BC_{F_4}(gPg^{-1}) \arrow[rr, equal] \arrow[ur, "\simeq_3"] & & BC_{NT}(gPg^{-1}) \arrow[u, "\simeq_3"]
\end{tikzcd}$$

can be glued to the similar one that can be obtained from Proposition 2.3

$$\begin{tikzcd}
BC_{NT}(P) \arrow[rr, equal] \arrow[d, "\simeq_3"'] & & BC_{F_4}(P) \arrow[dl, "\simeq_3"'] \arrow[ddd, "Bc_g"', "\simeq"] \\
\mathrm{map}(BP, BNT_3^{\wedge})_{Bi_P} \arrow[r] & \mathrm{map}(BP, X)_{f_N Bi_P} \arrow[d, equal] & \\
\mathrm{map}(BP, BNT_3^{\wedge})_{Bi_P Bc_g} \arrow[r] & \mathrm{map}(BP, X)_{f_N Bi_P Bc_g} & \\
BC_{NT}(gPg^{-1}) \arrow[rr, equal] \arrow[u, "\simeq_3"] & & BC_{F_4}(gPg^{-1}). \arrow[ul, "\simeq_3"']
\end{tikzcd}$$

□

7. The group F_4 is a N-determined 3-compact group

Here we prove Theorem 1.2. To make it so, we obtain some properties of the Quillen category of F_4 at the prime 3. Recall that the Quillen category of a group G at a prime p, in what follow denoted by $\mathcal{Q}_p(G)$, is defined as the category whose objects are pairs $(V, \alpha) \in \mathcal{A}b \times \mathrm{Mono}(V, G)$ such that V is a nontrivial elementary abelian p-group (sometimes, we will not distinguish between a group morphisms $\alpha : V \to G$ and its class $\alpha \in \mathrm{Mono}(V, G)$), and with morphisms $\mathrm{Mor}_{\mathcal{Q}_p(G)}\big((V_1, \alpha_1), (V_2, \alpha_2)\big)$, the set of group homomorphism $f : V_1 \to V_2$ such that $(V_1, \alpha_1) = (V_1, \alpha_2 f)$. The automorphism group of (V, α) in $\mathcal{Q}_p(G)$, $\mathrm{Mor}_{\mathcal{Q}_p(G)}\big((V, \alpha), (V, \alpha)\big)$, is denoted by $\mathcal{Q}_p(G)((V, \alpha))$.

If $N \xrightarrow{j} G$ is the maximal torus normalizer of G, and object $(V, \nu) \in \mathcal{Q}_p(N)$ is called a preferred lift of $(V, \alpha) \in \mathcal{Q}_p(G)$ if $j\nu = \alpha$ and $C_N(\nu) \xrightarrow{j_\sharp} C_G(\alpha)$ is the maximal torus normalizer of $C_G(\alpha)$. The set of preferred lifts of $\alpha \in \mathcal{Q}_p(G)$ is denoted by $\mathrm{SPL}(\alpha)$.

An element $\alpha \in \mathcal{Q}_p(G)$ is called oversized if for any $\nu \in \mathrm{SPL}(\alpha)$ the induced morphism $V \xrightarrow{\nu} N \longrightarrow W_G$ has kernel with nontrivial codimension. This is equivalent to say that α is oversized if it is non toral.

Indeed, we are interested in the full subcategory of $\mathcal{Q}_3(F_4)$, namely $\mathcal{Q}_3^{\leq 2}(F_4)$, whose objects are (V, α) with $\mathrm{rk}\, V \leq 2$.

Lemma 7.1. *For any object $(V, \alpha) \in \mathcal{Q}_3^{\leq 2}(F_4)$ the following hold:*
1. *The centralizer $C_{F_4}(\alpha)$ is totally N-determined.*
2. *α is not oversized.*

Proof. Let (V, α) be an element of $\mathcal{Q}_3(F_4)$ such that $\mathrm{rk}\, V = 1$. As $\mathrm{rk}\, V = 1$, and F_4 is connected then α verifies 7.1.2. Again, as $\mathrm{rk}\, V = 1$, then α represents a conjugacy class of elements of order 3 in F_4 thus $C_{F_4}(\alpha)$ is (up to 3-completion) one of the following 3-compact groups that appear in Proposition 3.2:

- $C_{F_4}(\alpha) \cong_3 SU(3,3)$ which is totally N-determined by [4] and [22].
- $C_{F_4}(\alpha) \cong_3 SO(7) \times S^1$ which is totally N-determined by [14].
- $C_{F_4}(\alpha) \cong_3 Sp(3) \times S^1$ which is totally N-determined by [14].

Therefore α always verifies 7.1.1.

Now, let (V, α) be an object in of $\mathcal{Q}_3(F_4)$ such that $\operatorname{rk} V = 2$. By Proposition 3.5 we know that (V, α) represents a conjugacy class of toral elementary abelian p-subgroups of F_4, thus (V, α) is not oversized. To finish with the proof, we have to prove the centralizer of (V, α) is totally N-determined. We consider two cases:

- If $\alpha(V)$ has an element $g \in \alpha(V)$ such that g is not in the conjugacy class $3A$, then $C_{F_4}(\alpha) = C_{C_{F_4}(\langle g \rangle)}(\alpha)$ and as $C_{F_4}(\langle g \rangle)$ has no torsion (Proposition 3.2), and as $\alpha(V)$ is toral $C_{F_4}(\alpha)$ has no torsion as well (see [5]). According to [14], torion free compact Lie groups are N-determined, and therefore totally N-determined.
- If every element in $\alpha(V)$ is in the class $3A$, then by Proposition 3.5 $C_{F_4}(\alpha) = T \rtimes \mathbb{Z}/3$ which is clearly totally N-determined. □

As a direct consequence of Lemma 7.1 we obtain

Proposition 7.2. *Let X be a 3-compact group with the same normalizer as F_3. Then $H^*BX = H^*BF_4$ as algebras over the Steenrod algebra.*

Proof. Acording to Lemma 7.1, X and F_4 fit the conditions of Lemma 3.3 in [12]. Hence $H^*BX = H^*BF_4$ as algebras over the Steenrod algebra. □

Therefore Theorem 1.2 is a direct consequence of Theorem 1.1 and the proposition above.

Departamento de Álgebra
Geometría y Topología
Universidad de Málaga
AP. 59
29080 Málaga, Spain

References

[1] K. K. S. Andersen, Cohomology of Weyl Groups with Applications to Topology, Master's thesis, Københavns Universitet, August 1997.

[2] N. Bourbaki, Groupes et algèbres de Lie, Chapitres 4–6, Hermann, 1968.

[3] A. Bousfield and D. Kan, Homotopy Limits, completion and localisation, SLNM 304, Springer Verlag, 1972.

[4] C. Broto and A. Viruel, Projective unitary groups are totally N-determined p-compact groups, preprint 1999.

[5] F. X. Dehon and J. Lannes, Sur les espaces fonctionnels dont la source est le classifiant d'un grupe de Lie compact commutatif, preprint.

[6] W. Dwyer, H. Miller and C. Wilkerson, The homotopical uniqueness of classifying spaces, Topology, **31** (1992), 29–45.

[7] W. Dwyer and C. Wilkerson, Homotopy fixed points methods for Lie groups and finite loop spaces, Ann. Math., **139** (1994), 395–442.

[8] W. Dwyer and A. Zabrodsky, Maps between classifying spaces, in Algebraic Topology, Barcelona, Lecture Notes in Math. 1298, Springer-Verlag, 1986, 106–119.

[9] S. Jackowski, J. McClure and R. Oliver, Homotopy classification of self maps of BG via G actions, Ann. Math., **135** (1992), 183–270.

[10] S. Jackowski, J. McClure and R. Oliver, Maps between classifying spaces revisited, Contemp. Mathemathics, **181** (1995), 263–298.

[11] J. Lannes, Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire, Inst. Hautes Études Sci. Publ. Math., **75** (1992), 135–244.

[12] J. M. Møller, N-Determined p-compact groups, preprint, June 18, 1999.

[13] J. M. Møller and D. Notbohm, Centers and finite coverings of finite loop spaces, J. Reine Angew. Math., **456** (1994), 99–133.

[14] J. M. Møller and D. Notbohm, Connected finite loop spaces with maximal tori, Trans. Amer. Math. Soc., **350** (1998), 3483–3504.

[15] D. Notbohm, Maps between classifying spaces, Math. Z., **207** (1991), 153–168.

[16] D. Notbohm, Homotopy uniqueness of classifying spaces of compact connected Lie groups at primes dividing the order of the Weyl group, Topology, **33** (1994), 271–330.

[17] D. Notbohm, p-adic Lattices of Pseudo Reflection Groups, in Algebraic Topology: New trends in Localization and periodicity, Prog. Math. 136, 337–352.

[18] D. Notbohm, *Classifying spaces of compact Lie groups*, Handbook of Algebraic Topology (I. M. James, ed.), North-Holland, 1995, pp. 1049–1094.

[19] D. L. Rector, Noetherian cohomology rings and finite loop spaces with torsion, J. Pure Appl. Algebra, **32** (1984), 191–217.

[20] H. Toda, Cohomology mod 3 of the classifying space BF_4 of the exceptional group F_4, J. Math. Kyoto Univ., **13** (1973), 97–115.

[21] A. Viruel, Homotopy uniqueness of BG_2, Manuscripta Math., **95** (1998), 471–497.

[22] A. Viruel, On the mod 3 homotopy type of the classifying space of a central product of $SU(3)$'s, J. Math. Kyoto Univ., **39** (1999), 249–275.

[23] Z. Wojtkowiak, On maps from holim F to Z, in Algebraic Topology, Barcelona, Lecture Notes in Math. 1298, Springer-Verlag, 1986, 227–236.