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A bifurcation phenomenon for the periodic solutions of a semilinear dissipative wave equation

By

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1. Introduction and main result

In this paper, we consider the time periodic solutions of a following semilinear dissipative wave equation

$$u_{tt} - c^2 u_{xx} + \mu u_t + u^3 = f(t,x), \quad t, x \in R, \tag{1.1}$$

with periodic boundary condition

$$u(t,x) = u(t, x+L), \quad t, x \in R. \tag{1.2}$$

Here c and μ are positive constants and $f(t,x)$ is a given external force, which is T-periodic in t. It is known that for any periodic external force, there exists at least one T-periodic solution of (1.1) with (1.2). Moreover, if $f(t,x)$ is suitably small, then any time periodic solution of (1.1) with (1.2) is unique and asymptotically stable. It was basically proved by P. H. Rabinowitz [7], [8]. However, in the case of relatively large external force, the numerical computations suggest not only the non-uniqueness of T-periodic solution, but also the existence of $2T$-periodic solution. In order to investigate these phenomena, we give one-parameter family of external force $\{f_\lambda\}_{\lambda \geq 0}$, where $f_0 = 0$, and consider the structure of periodic solution in the product space $\lambda \times u$. Here, λ is a positive parameter which somewhat represents the magnitude of external force. Then, as λ increase from 0, various bifurcation phenomena are observed by numerical computations. In particular, we can observe the period-doubling bifurcations which are known as very important phenomena along the route toward a so called "Chaos". However, for the nonlinear dissipative wave equation (1.1), there has been no rigorous proof on the existence of these bifurcation phenomena.

In order to attack this basic problem, we take the following strategy. At first, we construct a specific one parameter family of functions $\{u_\lambda\}_{\lambda>0}$. Next,

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we give the one parameter family of external forces $\{f_\lambda(t)\}_{\lambda>0}$ so that $\{u_\lambda\}$ become the exact solution of (1.1). It means that we insert the "probe" $\{u_\lambda\}$ from the origin in the product space. In this paper, we deal with more specific case and consider the structure of solution around the "probe".

Let's define $\{u_\lambda(t)\}_{\lambda>0}$ and $\{f_\lambda(t)\}_{\lambda>0}$ as follows

$$(1.3)\qquad \begin{cases} u_\lambda(t) := \lambda U(t), \quad U(t) : \text{given } T\text{-periodic smooth function,} \\ f_\lambda(t) := u_\lambda''(t) + \mu u_\lambda'(t) + u_\lambda^3(t). \end{cases}$$

Here we note that if the solution of (1.1) is uniform with respect to x, the solution satisfies the ordinary differential equation so called Duffing equation. For Duffing equation, we investigated the bifurcation phenomena by the same strategy [4], [5], and succeeded in showing the existence of bifurcation points. In particular, for some $\{f_\lambda\}_{\lambda>0}$, we proved the existence of countably many period-doubling bifurcation points. Since the solution of Duffing equation is also the solution of (1.1), we see that the bifurcation points of Duffing equation also become the bifurcation points of (1.1). Therefore we do focus our attention on the existence of spatially non-homogeneous bifurcation.

Now, we show the numerical results computed for the case $c = \mu = 1$, $L = 2\pi$, and $U(t) = \sin 2\pi t + 0.5$, by the second order central difference scheme, putting a suitable initial data. (See Figs. 1 through 4, details in Section 6) These numerical results indicate that when $\lambda = 1$ (Figs. 1 and 2), the trivial vibration $u_\lambda(t)$ is asymptotically stable, but when $\lambda = 1.5$ (Figs. 3 and 4), the $2T$-periodic solution (which is $L/2$-periodic for x) appears. The aim of this paper is giving a rigorous proof on the existence of a period-doubling bifurcation phenomenon from the time periodic vibration $u_\lambda(t)$. In order to describe main result, we prepare notations. For positive constants $\widetilde{T}$ and $\widetilde{L}$, $H^1_{per}((0,\widetilde{T})\times(0,\widetilde{L}))$ denotes the Banach space:

(1.4)

$$H^1_{per}((0,\widetilde{T})\times(0,\widetilde{L})) = \{u; u(0,\cdot) = u(\widetilde{T},\cdot), u(\cdot,0) = u(\cdot,\widetilde{L}), \|u\|_{H^1_{per}} < \infty\},$$

with the norm

$$(1.5)\qquad \|u\|^2_{H^1_{per}} = \int_0^{\widetilde{T}} \int_0^{\widetilde{L}} |u|^2 + |u_t|^2 + |u_x|^2 dxdt.$$

Main result of this paper is the following.

Theorem 1.1. *Suppose that the specific function and coefficients of* (1.1) *are given by*

$$(1.6)\qquad u_\lambda(t) = \lambda(\sin 2\pi t + 0.5), \qquad \mu = 1, \qquad \frac{2\pi c}{L} = 1.$$

Then there exists a period-doubling bifurcation point $\lambda_0 = 1.45\cdots$ *such that*

(1) $u_\lambda(t)$ *is asymptotically stable for any* $\lambda(0 < \lambda < \lambda_0)$,

(2) *There is a neighborhood* V *of* $(\lambda_0, u_{\lambda_0})$ *in* $R^+ \times H^1_{per}((0,2T)\times(0,L))$, *and an interval* $(-\delta,\delta)$ *such that the solution of* (1.1) *with* (1.2) *in* V *is given by*

(1.7)
$$\{(\lambda, u_\lambda(t)) : (\lambda, u_\lambda) \in V\} \cup \{(\tilde{\lambda}(\epsilon), u_\lambda(t) + \tilde{\lambda}(\epsilon)(\epsilon v(t,x) + \epsilon\psi(\epsilon)) : |\epsilon| < \delta)\}$$

where v *is belonging to* $H^1_{per}((0,2T)\times(0,L/2))$ *and* T*-anti periodic,* $\tilde{\lambda}$ *and* ψ *are continuous functions* $\tilde{\lambda} : (-\delta,\delta) \to R$, $\psi : (-\delta,\delta) \to H^1_{per}((0,2T)\times(0,L/2))$ *which satisfy that* $\tilde{\lambda}(0) = \lambda_0, \psi(0) = 0$.

In the next section, we show the existence of periodic solution of (1.1) with (1.2), applying Leray-Schauder's fixed point Theorem. In Section 3, we reformulate the problem in order to apply Crandall-Rabinowitz's Theorem [2] on bifurcation theory. In this process, the eigenvalue problem of linearized equation plays an essential role. To study it, we expand the solution by Fourier series in the space valuable. Because we consider the problem around $u_\lambda(t)$ which is uniform with respect to the space valuable, we can reduce to a linear system of ordinary differential equations. Since all of these equations have the same form of linearized Duffing equation, we can make use of the idea employed in the study of Duffing equation [4], [5]. Then we see the fundamental solutions of the corresponding Hill's equation are crucial, and we give a criterion of the existence of bifurcation points in Section 4. In Section 5, to check this criterion, we make a computer assisted proof by using softwares of the interval arithmetics made by Prof. H. Yosihara. Finally, we show the results of the numerical computations in Section 6.

2. Existence of the periodic solution

In this section, we show the existence of a T-periodic solution of (1.1) with periodic boundary condition (1.2) for any external force f with period T, applying Leray-Schauder's fixed point Theorem. In order to do it, we reduce (1.1) to an integral equation. Let's define the operator $\mathfrak{L}^{-1} : L^2_{per}((0,T)\times(0,L)) \to H^1_{per}((0,T)\times(0,L))$ by

(2.1)
$$\mathfrak{L}^{-1} : g \to u,$$

where $u \in H^1_{per}((0,T)\times(0,L))$ is a solution of the equation

(2.2)
$$u_{tt} - c^2 u_{xx} + \mu u_t + u = g.$$

Then $\mathfrak{L}^{-1}$ is well defined and a compact operator $L^2_{per}((0,T)\times(0,L)) \to L^p_{per}((0,T)\times(0,L))$ $(p \geq 2)$. In fact, we expand g and u by Fourier series as

(2.3)
$$g(t,x) = \sum_m \sum_n g_{m,n} e^{i2\pi mx/L} e^{i2\pi nt/T},$$
$$u(t,x) = \sum_m \sum_n u_{m,n} e^{i2\pi mx/L} e^{i2\pi nt/T}.$$

Then it satisfies

$$\sum_m \sum_n \left\{ \left\{ -\left(\frac{2\pi n}{T}\right)^2 + c^2 \left(\frac{2\pi m}{L}\right)^2 + \frac{i2\pi\mu n}{T} + 1 \right\} u_{m,n} - g_{m,n} \right\} \times e^{i2\pi mx/L} e^{i2\pi nt/T} = 0. \tag{2.4}$$

Since it holds

$$|u_{m,n}|^2 = \frac{|g_{m,n}|^2}{(-(\frac{2\pi n}{T})^2 + c^2(\frac{2\pi m}{L})^2 + 1)^2 + (\frac{2\pi\mu n}{T})^2}, \tag{2.5}$$

we have that if g belongs to $L^2_{per}((0,T)\times(0,L))$, then u belongs to $H^1_{per}((0,T)\times(0,L))$. From the embedding relation $H^1_{per}((0,T)\times(0,L)) \hookrightarrow L^p_{per}((0,T)\times(0,L))$ $(p \geq 2)$, we can see that $\mathfrak{L}^{-1}$ is a compact operator $L^2_{per}((0,T)\times(0,L)) \to L^p_{per}((0,T)\times(0,L))$ $(p \geq 2)$. Here we note that (1.1) is rewritten by

$$u = \mathfrak{L}^{-1}(u - u^3 + f), \tag{2.6}$$

where $\mathfrak{L}^{-1}(u - u^3 + f)$ is a compact operator in $L^6_{per}((0,T)\times(0,L))$. Therefore, in order to apply Leray-Schauder's fixed point Theorem, we only have to show that for any $\epsilon \in [0,1]$, there exists M such that any solution of the equation

$$u = \epsilon\mathfrak{L}^{-1}(u - u^3 + f) \tag{2.7}$$

satisfies

$$\|u\|_{L^6_{per}} < M, \tag{2.8}$$

where M is a positive constant independent of ϵ. Note that (2.7) is rewritten by

$$u_{tt} - c^2 u_{xx} + \mu u_t + (1-\epsilon)u + \epsilon u^3 = \epsilon f. \tag{2.9}$$

Since we can easily have the following estimates from (2.9)

$$\int_0^T \int_0^L \mu u_t^2 dxdt \leq \int_0^T \int_0^L f^2 dxdt \tag{2.10}$$

and

$$\int_0^T \int_0^L c^2 u_x^2 + (1-\epsilon)u^2 + \frac{3}{4}\epsilon u^4 dxdt \leq \int_0^T \int_0^L \frac{3\epsilon}{4}|f|^{4/3} + u_t^2 dxdt, \tag{2.11}$$

we see $\|u\|_{H^1_{per}} < \widetilde{M}$, where

$$\begin{aligned} \widetilde{M} = \max\Bigg(& \int_0^T \int_0^L 4f^2 + \frac{9}{4}|f|^{4/3} dxdt, \\ & \int_0^T \int_0^L 2f^2 + \frac{3}{2}|f|^{4/3} dxdt + \sqrt{TL}\left(\int_0^T \int_0^L \frac{8}{3}f^2 + 2|f|^{4/3} dxdt\right)^{1/2}\Bigg). \end{aligned} \tag{2.12}$$

Using the fact $\|u\|_{L^6_{per}} \le C\|u\|_{H^1_{per}}$, we have the estimates corresponding to (2.8). Thus we can show the existence of periodic solution of (1.1) with (1.2).

3. Reformulation of the problem

We first note that any periodic solution of (1.1) should have the period $\widetilde{T} = mT$ for an $m \in N$. Hence, for any fixed $m \in N$, we look for the periodic solution of (1.1) in the form:

$$u(t,x) = u_\lambda(t,x) + \lambda v(t,x), \tag{3.1}$$

where $v(t,x)$ is a $\widetilde{T}$-periodic in t. Then $v(t,x)$ must satisfy the periodic problem

$$\begin{cases} v_{tt} - c^2 v_{xx} + \mu v_t + \Lambda\left(U^2 v + U v^2 + \dfrac{1}{3}v^3\right) = 0, \\ v(t+\widetilde{T},x) = v(t,x), \quad t,x \in R, \end{cases} \tag{3.2}$$

where we set $\Lambda = 3\lambda^2$. To study the bifurcation problem to (3.2) around the trivial solution $v = 0$, we make use of a following bifurcation Theorem in Crandall-Rabinowitz [2].

Theorem 3.1 (Crandall and Rabinowitz). *Let X, Y be Banach spaces, V a neighborhood of 0 in X and*

$$F : (0,\infty) \times V \to Y$$

have the properties for a $\Lambda_0 > 0$

(a) $F(\Lambda,0) = 0$ *for* $\Lambda \in (0,\infty)$,
(b) *The partial derivatives* F_Λ, F_x *and* $F_{\Lambda x}$ *exist and are continuous,*
(c) $N(F_x(\Lambda_0,0))$ *and* $Y/R(F_x(\Lambda_0,0))$ *are one dimensional,*
(d) $F_{\Lambda x}(\Lambda_0,0)x_0 \notin R(F_x(\Lambda_0,0))$, *for a non trivial* $x_0 \in N(F_x(\Lambda_0,0))$.

Let Z be any complement of $N(F_x(\Lambda_0,0))$ in X. Then there is a neighborhood U of $(\Lambda_0,0)$ in $R\times X$, an interval $(-\delta,\delta)$, and continuous functions $\varphi : (-\delta,\delta) \to R$, $\psi : (-\delta,\delta) \to Z$ such that $\varphi(0) = \Lambda_0$, $\psi(0) = 0$ and

$$F^{-1}(0) \cap U = \{(\varphi(\epsilon),\, \epsilon x_0 + \epsilon\psi(\epsilon)) \,:\, |\epsilon| < \delta\} \cup \{(\Lambda,0) \,:\, (\Lambda,0) \in U\}. \tag{3.3}$$

That is, $(\varphi(\epsilon),\, \epsilon x_0 + \epsilon\psi(\epsilon))$ is the solution of the equation $F(\Lambda,v) = 0$, and the solution $v = \epsilon x_0 + \epsilon\psi(\epsilon)$ bifurcates from trivial solution $v = 0$ at $\Lambda = \Lambda_0$.

Similar to the previous section, we define the operator $\widetilde{\mathfrak{L}}^{-1} : L^2_{per}((0,\widetilde{T}) \times (0,L)) \to H^1_{per}((0,\widetilde{T}) \times (0,L))$ by

$$\widetilde{\mathfrak{L}}^{-1} : g \to u, \tag{3.4}$$

where $u \in H^1_{per}((0,\widetilde{T}) \times (0,L))$ is a unique solution of the equation

$$u_{tt} - c^2 u_{xx} + \mu u_t + u = g. \tag{3.5}$$

Using this operator, (3.2) is rewritten by

$$v = \widetilde{\mathfrak{L}}^{-1}\left(v - \Lambda\left(U^2 v + U v^2 + \frac{1}{3}v^3\right)\right), \tag{3.6}$$

where $\Lambda = 3\lambda^2$. Let's define the operator $F : (0,\infty) \times H^1_{per}((0,\widetilde{T}) \times (0,L)) \to H^1_{per}((0,\widetilde{T}) \times (0,L))$ by

$$F(\Lambda, v) = v - \widetilde{\mathfrak{L}}^{-1}\left(v - \Lambda\left(U^2 v + U v^2 + \frac{1}{3}v^3\right)\right). \tag{3.7}$$

Then we have next lemma.

Lemma 3.2. *In the problem* (3.7), *the hypotheses* (a)–(d) *of Theorem* 3.1 *reduce the following three conditions.*

(i) $\Lambda = \Lambda_0$ *is a positive eigenvalue of the following linearized eigenvalue problem of* (3.2) *at* $v = 0$:

$$\begin{cases} v_{tt} - c^2 v_{xx} + \mu v_t + \Lambda U^2 v = 0, \\ v(t, x+L) = v(t,x), \\ v(t+\widetilde{T}, x) = v(t,x). \end{cases} \tag{3.8}$$

(ii) *The eigenspace of* (3.8) *is one dimensional.*

(iii)

$$\int_0^L \int_0^{\widetilde{T}} v_0(t,x) v_0^*(t,x) U^2(t)\, dt dx \neq 0, \tag{3.9}$$

where $v_0(t,x)$ *is an eigenfunction of* (3.8) *with* $\Lambda = \Lambda_0$ *and* $v_0^*(t,x)$ *is a non-trivial solution of the adjoint problem to* (3.8) *with* $\Lambda = \Lambda_0$

$$\begin{cases} v_{tt} - c^2 v_{xx} - \mu v_t + \Lambda U^2 v = 0, \\ v(t, x+L) = v(t,x), \\ v(t+\widetilde{T}, x) = v(t,x). \end{cases} \tag{3.10}$$

Proof. It's clear that $F(\Lambda, 0) = 0$ for any $\Lambda \in (0,\infty)$. Moreover, $F_v(\Lambda, 0)$, $F_{\Lambda v}(\Lambda, 0)$ are given by

$$F_v(\Lambda, 0)w = w - \widetilde{\mathfrak{L}}^{-1}(w - \Lambda U^2 w), \tag{3.11}$$

$$F_{\Lambda v}(\Lambda, 0)w = -\widetilde{\mathfrak{L}}^{-1}(-U^2 w). \tag{3.12}$$

Therefore, we have $N(F_v(\Lambda_0, 0))$ coincides with the eigenspace of (3.8) with $\Lambda = \Lambda_0$. Since it holds from the Fredholm's alternative Theorem that

$$\mathrm{codim}(R(F_v(\Lambda_0, 0))) = \dim(N(F_v^*(\Lambda_0, 0)) = \dim(N(F_v(\Lambda_0, 0))),$$

we see that the condition (c) in Theorem 3.1 reduces to the condition (ii). Finally, we consider the condition (d). If $F_{\Lambda v}(\Lambda_0, 0)v_0 \in R(F_v(\Lambda_0, 0))$, then there exists w such that

$$w_{tt} - c^2 w_{xx} + \mu w_t + \Lambda_0 U^2 w = U^2 v_0,$$

which implies $\int_0^L \int_0^{\widetilde{T}} v_0(t,x) v_0^*(t,x) U^2(t) dt dx = 0$. Therefore the condition (d) reduces to the condition (iii). Thus the proof is completed. □

The condition (iii) means that the eigenvalue $\Lambda = \Lambda_0$ is simple. Since our problem here is not self-adjoint, its condition is not trivial at all.

4. Eigenvalue problem of the linearized equation

In this section, we investigate the eigenvalue problem (3.8) in details. We expand the solution by Fourier series as

$$w(t,x) = \sum_{n=-\infty}^{\infty} w_n(t) e^{\frac{i2\pi nx}{L}},$$

then $w_n(t)$ satisfies

(4.1)

$$w_n''(t) + \mu w_n'(t) + \left(\frac{2\pi nc}{L}\right)^2 w_n(t) + \Lambda U^2(t) w_n(t) = 0, \qquad \text{for } n = 0, 1, 2, \dots .$$

We set $w_n(t) = e^{-\mu t/2} y_n(t)$, then (4.1) becomes

$$y_n''(t) + \left(-\frac{\mu^2}{4} + \left(\frac{2\pi nc}{L}\right)^2 + \Lambda U^2(t)\right) y_n(t) = 0, \tag{4.2}$$

which is a type of so called Hill's equation. The equation (4.2) also has the matrix form

$$\begin{pmatrix} y_n \\ y_n' \end{pmatrix}' = \begin{pmatrix} 0, & 1 \\ \frac{\mu^2}{4} - (\frac{2\pi nc}{L})^2 - \Lambda U^2(t), & 0 \end{pmatrix} \begin{pmatrix} y_n \\ y_n' \end{pmatrix}. \tag{4.3}$$

To consider the original problem (3.8), we may seek the solution of (4.2) in the form $e^{\mu t/2}\tilde{y}(t)$, where $\tilde{y}$ is periodic of period $\widetilde{T} = mT$. Let $\Phi_n(\Lambda, t)$ be a fundamental matrix for (4.3),

$$\Phi_n(\Lambda, t) = \begin{pmatrix} \phi_1(\Lambda, n, t) & \phi_2(\Lambda, n, t) \\ \phi_1'(\Lambda, n, t) & \phi_2'(\Lambda, n, t) \end{pmatrix} \tag{4.4}$$

where $\{\phi_i(\Lambda, n, t)\}_{i=1}^2$ are given by the solutions of initial value problem to (4.2) with initial data $\Phi_n(\Lambda, 0) = E$. By the Floquet's Theory, we can see that the equation (4.1) has an mT-periodic solution if and only if $\Phi_n(\Lambda, T)$ has a characteristic root $e^{\mu T/2}\omega_m$, where ω_m is a primitive m-th root of 1. Note that $\det \Phi_n(\Lambda,t) = 1$ for $t \geq 0$, because the trace of the coefficient matrix of (4.3) is zero. Then, the characteristic roots of $\Phi_n(\Lambda,T)$ are given by the roots of characteristic equation

$$\sigma^2 - \Delta(\Lambda,n)\sigma + 1 = 0, \tag{4.5}$$

where $\Delta(\Lambda,n)$ is a trace of $\Phi_n(\Lambda,T)$, that is, $\Delta(\Lambda,n) = \phi_1(\Lambda,n,T) + \phi_2'(\Lambda,n,T)$. Therefore, we can divide the characteristic roots $\{\sigma_i\}$ into the following three types:

$$\begin{cases} |\sigma_i|_{i=1,2} \leq 1 & \text{if} \quad |\Delta(\Lambda,n)| \leq 2, \\ \sigma_1 = e^{z(\Lambda,n)T}, \ \sigma_2 = e^{-z(\Lambda,n)T} & \text{if} \quad \Delta(\Lambda,n) > 2, \\ \sigma_1 = -e^{z(\Lambda,n)T}, \sigma_2 = -e^{-z(\Lambda,n)T} & \text{if} \quad \Delta(\Lambda,n) < -2. \end{cases} \tag{4.6}$$

Here $z(\Lambda,n)$ is explicitly given by the formula for $|\Delta(\Lambda,n)| > 2$

$$z(\Lambda,n) = \frac{1}{T}\cosh^{-1}\frac{\Delta(\Lambda,n)}{2} = \frac{1}{T}\log\frac{|\Delta| + \sqrt{|\Delta|^2 - 4}}{2}. \tag{4.7}$$

We also define $z(\Lambda,n) = 0$ for $|\Delta(\Lambda,n)| \leq 2$. Then, $z(\Lambda,n)$ coincides with so called "Lyapunov exponent" of the solution of (4.1). By these consideration above, we have next lemma.

Lemma 4.1. *For the linearized equation* (3.8), *it holds the followings.*
(i) $mT (m \geq 3)$*-periodic solution does not exist.*
(ii) T*-periodic solution exists at* $\Lambda = \Lambda_0$ *if and only if there exists* n_0 *such that* $\Delta(\Lambda_0,n_0) = e^{\mu T/2} + e^{-\mu T/2}$.
(iii) $2T$*-periodic solution exists at* $\Lambda = \Lambda_0$ *if and only if there exists* n_0 *such that* $\Delta(\Lambda_0,n_0) = -(e^{\mu T/2} + e^{-\mu T/2})$. *This* $2T$*-periodic solution is* T*-anti-periodic solution, i.e.* $u(t,x) = -u(t+T,x)$ *for* $t,x \in R$.

Now, we further investigate the properties of $\Delta(\Lambda,n)$ and $z(\Lambda,n)$. Define

$$\Sigma = \{\Lambda \in R; \quad |\Delta(\Lambda)| \leq 2\},$$

and let K be a operator in $L^2_{U^2}(R)$ defined by

$$K = \frac{1}{U^2}\left(-\frac{d^2}{dt^2} + \left(\frac{\mu^2}{4} - \left(\frac{2\pi nc}{L}\right)^2\right)\right),$$

where $L^2_{U^2}$ denotes the weighted L^2-space defined by

$$L^2_{U^2}(R) = \left\{h(t); \int_R |h(s)|^2 U^2(s)ds < \infty\right\}.$$

Then, we can see that K is a self-adjoint operator in $L^2_{U^2}$, the spectrum of K coincides with Σ, and the resolvent set coincides with $R \setminus \Sigma$. In particular, if

$\Lambda \notin \Sigma$, by the above argument on $\Phi_\Lambda(T)$ and $\Delta(\Lambda)$, there are two independent solution of (4.2) $w_\Lambda^\pm(t)$ $(t \in R)$ such that $w_\Lambda^+(t)$ (resp. $w_\Lambda^-(t)$) decays at the rate $e^{-z(\Lambda)t}$ (resp. $e^{z(\Lambda)t}$) as $t \to +\infty$ (resp. $t \to -\infty$), and ${}^t(w_\Lambda^\pm(0), w_\Lambda^{\pm'}(0))$ is an eigenvector of $\Phi_\Lambda(T)$. Then, the solution of

$$(K - \Lambda I)g = f \quad \text{in} \quad L^2_{U^2} \tag{4.8}$$

which is equivalent to

$$-\frac{d^2 g}{dt^2} + \left(\frac{\mu^2}{4} - \left(\frac{2\pi nc}{L}\right)^2 - \Lambda U^2\right) g = U^2 f \tag{4.9}$$

is concretely constructed by the Green function in the form

$$g(t) = \int_R G_\Lambda(t,s)U^2(s)f(s)ds, \tag{4.10}$$

where

$$G_\Lambda(t,s) = G_\Lambda(s,t) = \frac{w_\Lambda^+(t)w_\Lambda^-(s)}{[w_\Lambda^+, w_\Lambda^-]}; \quad t \geq s,$$

and $[w_\Lambda^+, w_\Lambda^-]$ is the Wronskian.

In [5], the Lyapunov exponent $z(\Lambda, n)$ were investigated precisely. In our case, the same property holds.

Lemma 4.2. *For $\Lambda \notin \Sigma$, $dz/d\Lambda$ can be represented in the form*

$$\frac{dz}{d\Lambda} = -\frac{1}{T}\int_0^T G_\Lambda(\tau,\tau)U^2(\tau)d\tau. \tag{4.11}$$

According to the expansion theory by generalized eigen-functions established by Weyl, Stone, Titschmarsh and Kodaira, $G_\Lambda(s,t)$ has the following representation;

$$G_\Lambda(s,t) = \int_\Sigma \frac{\Sigma_{1\leq i,j\leq 2}\phi_i(s,\xi)\phi_j(t,\xi)\sigma_{ij}(d\xi)}{\xi - \Lambda}, \tag{4.12}$$

where $\{\sigma_{ij}\}$ is a matrix valued Stieltjes measure which is nonnegative definite.

From 4.2 and (4.12), we have following two lemmas.

Lemma 4.3.

$$\frac{d^2 z}{d\Lambda^2} = -\int_\Sigma \frac{\sigma(d\xi)}{(\xi - \Lambda)^2} < 0 \tag{4.13}$$

for $\Lambda \notin \Sigma$, that is, $z(\Lambda, n)$ is a convex function on $R \setminus \Sigma$.

Lemma 4.4. *For any eigenvalues $\Lambda = \Lambda_0$ of (4.1) with $n = n_0$, it holds that*

$$\frac{dz}{d\Lambda}(\Lambda_0, n_0) \neq 0 \Longleftrightarrow \int_0^T w_{n_0}(t)w_{n_0}^*(t)U^2(t)dt \neq 0, \tag{4.14}$$

where w_{n_0} is a solution of (4.1) with $n = n_0$ and $w_{n_0}^*$ is a solution of the adjoint equation of (4.1) with $n = n_0$.

By these lemmas, we have the following Theorem.

Theorem 4.5. (I) *Suppose that there exists* (Λ_0, n_0) *satisfying the next conditions*:

(i) $\Delta(\Lambda_0, n_0) = e^{\mu T/2} + e^{-\mu T/2}$ (*resp.* $-(e^{\mu T/2} + e^{-\mu T/2})$),

(ii) $\Delta(\Lambda_0, n) \neq \pm(e^{\mu T/2} + e^{-\mu T/2})$ *for* $n \neq n_0$,

(iii) $(d\Delta/d\Lambda)(\Lambda_0, n_0) \neq 0$.

Then T*-periodic* (*resp.* $2T$*-periodic*) *solution bifurcates at* $\Lambda = \Lambda_0$. *Moreover, there is a neighborhood* V *of* $(\Lambda_0, 0)$ *in* $R^+ \times H^1_{per}((0, \widetilde{T}) \times (0, L))$, *and an interval* $(-\delta, \delta)$ *such that the solution of* (3.2) *in* V *is given by*

$$\{(\Lambda, 0) : (\Lambda, 0) \in V\} \cup \{(\tilde{\Lambda}(\epsilon), \epsilon v_0(t, x) + \epsilon\psi(\epsilon)) : |\epsilon| < \delta\}. \tag{4.15}$$

Here $\tilde{\Lambda}$ *and* ψ *are continuous functions* $\tilde{\Lambda} : (-\delta, \delta) \to R$, $\psi : (-\delta, \delta) \to H^1_{per}((0, \widetilde{T}) \times (0, L)$ *which satisfy that*

$$\tilde{\Lambda}(0) = \Lambda_0, \quad \psi(0) = 0, \tag{4.16}$$

Furthermore, we have

$$\begin{aligned} v_0(t + T, x) &= v_0(t, x) \quad (\textit{resp.} - v_0(t, x)), \\ v_0(t, x + L/n_0) &= v_0(t, x). \end{aligned} \tag{4.17}$$

(II) *If* $\Lambda = \Lambda_0$ *is the smallest eigenvalue of* (3.8), *which satisfys the above conditions* (i)–(iii), *then trivial solution alternates the asymptotic stability at* $\Lambda = \Lambda_0$.

Proof. From Lemmas 4.1 and 4.4, if it holds the condition (i)–(iii), then all hypotheses of Lemma 3.2 are satisfied. Therefore, if it holds the condition (i)–(iii), T-periodic (resp. $2T$-periodic) solution bifurcates at $\Lambda = \Lambda_0$. Considering the problem in $H^1_{per}((0, \widetilde{T}) \times (0, L/n_0))$ instead of $H^1_{per}((0, \widetilde{T}) \times (0, L))$, we get (4.17). Moreover, we see that the sign of $z(\Lambda, n_0) - \mu T/2$ changes from negative to positive at $\Lambda = \Lambda_0$, which implies that the stability of $u_\lambda(t)$ alternates at $\Lambda = \Lambda_0$. □

Remark 1. Concerning the function $\tilde{\Lambda} : (-\delta, \delta) \to R$, where $\tilde{\Lambda}(0) = \Lambda_0$, we have the following properties.

$$\operatorname{sign}\left\{\frac{d\tilde{\Lambda}}{d\epsilon}|_{\epsilon=0}\right\} = \operatorname{sign}\left\{\frac{-\int_0^L \int_0^{\tilde{T}} U(t) v_0^2(t, x) v_0^*(t, x) dt dx}{\int_0^L \int_0^{\tilde{T}} U^2(t) v_0(t, x) v_0^*(t, x) dt dx}\right\} \tag{4.18}$$

$$\operatorname{sign}\left\{\frac{d^2\tilde{\Lambda}}{d\epsilon^2}|_{\epsilon=0}\right\} = \operatorname{sign}\left\{\frac{-\int_0^L \int_0^{\tilde{T}} v_0^3(t, x) v_0^*(t, x) dt dx}{\int_0^L \int_0^{\tilde{T}} U^2(t) v_0(t, x) v_0^*(t, x) dt dx}\right\} \tag{4.19}$$

Here, we note that if $\Lambda = \Lambda_0$ is a period-doubling bifurcation point, then

$(d\tilde{\Lambda}/d\epsilon)|_{\epsilon=0} = 0$, because we have

$$\begin{aligned}
&\int_0^L \int_0^{2T} U(t)v_0^2(t,x)v_0^*(t,x)dtdx \\
&= \int_0^L \int_0^{T} U(t)v_0^2(t,x)v_0^*(t,x)dtdx \\
&+ \int_0^L \int_0^{T} U(t+T)v_0^2(t+T,x)v_0^*(t+T,x)dtdx \\
&= 0.
\end{aligned} \tag{4.20}$$

5. Proof of Main Theorem

In the previous section, we have a criterion of the existence of bifurcation points, which is given by a trace of fundamental solution. In this section, we show the existence of λ_0 which satisfies a criterion, analyzing a trace of fundamental solution for a system of the following equations:

$$\frac{d\Phi}{dt} = A(t)\Phi, \quad \Phi(0) = I, \tag{5.1}$$

where

$$A(t) = \begin{pmatrix} 0, & 1 \\ \frac{\mu^2}{4} - (\frac{2\pi nc}{L})^2 - \Lambda U^2(t), & 0 \end{pmatrix}, \qquad n = 0, 1, 2, \dots . \tag{5.2}$$

To prove the existence of bifurcation point, we take the following steps.

(1) We decide the interval $(0, \overline{\lambda})$ and look for the bifurcation point in this interval. This $\overline{\lambda}$ is properly picked up by observation of rather rough numerical computations.

(2) We determine $n_0 \in N$ depending on $\overline{\lambda}$ such that if $n > n_0$ then the equation (4.1) does not have nontrivial solution for $\lambda \in (0, \overline{\lambda})$. Then, we investigate the fundamental solutions of (5.1) only for $0 \leq n \leq n_0$.

(3) We decide the target interval $[\lambda_1, \lambda_2] \subset (0, \overline{\lambda})$, and make a detailed numerical computations of (5.1), in particular on the trace $\Delta(\lambda, n)$, at $\lambda = \lambda_1$ and $\lambda = \lambda_2$. Because we'll show in Lemma 5.2 that $|\Delta(\lambda, n) - \Delta(\lambda_0, n)| \leq C|\lambda - \lambda_0|$, for any $\lambda, \lambda_0 \in [\lambda_1, \lambda_2]$. For numerical computations, we use the following fourth order Taylor finite difference scheme

$$\begin{cases} \Phi_{k+1} = R^{(k)}\Phi_k & k = 0, 1, 2, \cdots, K \\ \qquad := \Phi_k + \Delta t A_k \Phi_k + \dfrac{(\Delta t)^2}{2} B_k \Phi_k + \dfrac{(\Delta t)^3}{6} C_k \Phi_k + \dfrac{(\Delta t)^4}{24} D_k \Phi_k, \\ \Phi_0 = I, \end{cases} \tag{5.3}$$

where $\Delta t = T/K,\ A_k = A(k\Delta t), B_k = B(k\Delta t), C_k = C(k\Delta t), D_k = D(k\Delta t)$, and

$$
(5.4)\qquad
\begin{aligned}
B(t) &= \frac{dA(t)}{dt} + A^2(t),\\
C(t) &= \frac{d^2A(t)}{dt^2} + 2\frac{dA(t)}{dt}A(t) + A(t)\frac{dA(t)}{dt} + A^3(t),\\
D(t) &= \frac{d^3A(t)}{dt^3} + 3\frac{d^2A(t)}{dt^2}A(t) + 3\left(\frac{dA(t)}{dt}\right)^2 + 3\frac{dA(t)}{dt}A^2(t)\\
&\quad + 2A(t)\frac{dA(t)}{dt}A(t) + A(t)^2\frac{dA(t)}{dt} + A(t)\frac{d^2A(t)}{dt^2} + A^4(t).
\end{aligned}
$$

(4) We estimate the difference $||\Phi(T) - \overline{\Phi_K}||$, where $\overline{\Phi_K}$ is a value of fundamental solution by the numerical computation using the scheme (5.3). The estimate will be given by Proposition 5.4, Lemmas 5.5 and 5.6. To do that, we have to prepare some softwares which perform the interval arithmetics for the computations. Here, we used softwares for computations of double precision by the Sun Workstation, supported by Prof. Hideaki Yosihara.

Taking the step (1)–(4), we prove the existence of bifurcation point $\lambda_0 \in (\lambda_1, \lambda_2)$. Moreover, in order to prove that λ_0 is a first bifurcation point, we need to add the following step.

(5) We show that $\Delta(\lambda, n) \neq \pm(e^{1/2} + e^{-1/2})$ for $0 < \lambda < \lambda_1$, $0 \leq n \leq n_0$.

Proof of Main Theorem.

Preparation. At first, we determine $\overline{\lambda}$, n_0, and the target interval as follows.

(1) $\overline{\lambda} = 1.5$,

(2) $n_0 = 8$ (see Fig. 5),

(3) Target interval is $[1.4525, 1.4550]$.

Next, we compute the fundamental solution using (5.3) with double precision. The number of mesh-points is $K = 1024$, that is, $\Delta t = 1/1024$.

(4) We estimate the difference

$$
||\Phi(T) - \overline{\Phi_K}|| < 0.0000041, \tag{5.5}
$$

from Proposition 5.4, Lemmas 5.5 and 5.6. The details are postponed until the end of this section.

Verification. Here, we verify the hypotheses of Theorem 4.5.

(I) (i) From the numerical computation and (5.5), we have

$$
\Delta(\lambda, 2) > -2.2540897 + 0.0000082, \qquad \lambda = 1.4525, \tag{5.6}
$$

$$
\Delta(\lambda, 2) < -2.2571220 - 0.0000082, \qquad \lambda = 1.4550. \tag{5.7}
$$

From the fact $2.25525193 - 10^{-10} < e^{1/2} + e^{-1/2} < 2.25525193 + 10^{-10}$, and the Intermediate theorem, we can see that there exists an eigenvalue of the linearized equation $\lambda_0 (\lambda_0 \in (1.4525, 1.4550))$ such that $\Delta(\lambda_0, 2) = -(e^{1/2} + e^{-1/2})$.

(I) (ii) Computing the fundamental solution for $n = 0, 1, 3, \cdots 8$, we have

$$\sup_{\lambda\in[1.4525,1.4550],n\neq 2} |\Delta(\lambda, n)| < 2.006967 + 0.000021, \tag{5.8}$$

which implies that $\Delta(\lambda_0, n) \neq \pm(e^{1/2} + e^{-1/2})$ for $n \neq 2$.

(I) (iii) Suppose that $(d\Delta/d\lambda)(\lambda_0, 2) = 0$. According to the convexity of $z(\Lambda, n)$ by Lemma 4.3, there exist $\tilde{\lambda} \in (\lambda_0, 1.4550)$ and $\tilde{\tilde{\lambda}} \in (\tilde{\lambda}, 1.4550)$ such that $\Delta(\tilde{\lambda}, 2) = 2$ and $\Delta(\tilde{\tilde{\lambda}}, 2) = -2$. Then we have

$$\max_{\lambda} |\Delta(\lambda, 2) - \Delta(\hat{\lambda}, 2)| > 2 + e^{1/2} + e^{-1/2}, \qquad \hat{\lambda} = (\lambda_1 + \lambda_2)/2. \tag{5.9}$$

On the other hand, by Lemma 5.2 and (5.54),

$$\max_{\lambda} |\Delta(\lambda, 2) - \Delta(\hat{\lambda}, 2)| < 1. \tag{5.10}$$

It contradicts (5.9). Therefore we have $(d\Delta/d\lambda)(\lambda_0, 2) \neq 0$.

Thus, we can obtain the existence of the period-doubling bifurcation point $\lambda_0 (\lambda_0 \in (1.4525, 1.4550))$.

(II) By Lemma 5.3, we see that $\Delta(\lambda, n) \neq \pm(e^{1/2} + e^{-1/2})$ for any $\lambda \in (0, 0.068]$, $n \in N$. And we have

$$\sup_{\lambda\in[0.068,1.4525],n=2} |\Delta(\lambda, n)| < 2.2540897 + 0.0000082, \tag{5.11}$$

$$\sup_{\lambda\in[0.068,1.4525],n\neq 2} |\Delta(\lambda, n)| < 2.2481613 + 0.0000366, \tag{5.12}$$

which imply that $\Delta(\lambda, n) \neq \pm(e^{1/2} + e^{-1/2})$ for $0 < \lambda < 1.4525$, $0 \leq n \leq 8$. Therefore, λ_0 is the first bifurcation point.

Thus, we can verify all hypotheses of Theorem 4.5. □

From here, we show some lemma needed in the proof of main theorem. First, we show the lemma used in step (2), which is important for reductions to the problem of finite number of ordinary differential equations.

Lemma 5.1. *For any fixed* $\overline{\Lambda}$, *if it holds*

$$n > \frac{\sqrt{\overline{\Lambda}}L}{4\pi c\mu} \sup_t |U'(t)|, \tag{5.13}$$

then $w_n(t) \equiv 0$ *for* $0 < \Lambda \leq \overline{\Lambda}$.

Proof. From (4.1), we have the equalities

$$\int_0^T \mu(w_n')^2 dt = \int_0^T \Lambda U U' w_n^2 dt, \tag{5.14}$$

and

$$\int_0^T \left(\frac{c2\pi n}{L}\right)^2 w_n^2 + \Lambda U^2 w_n^2 dt = \int_0^T (w_n')^2 dt. \tag{5.15}$$

Combining (5.14) and (5.15), we have

$$\int_0^T \left(\frac{c2\pi n}{L}\right)^2 w_n^2 dt \leq \frac{\Lambda}{4\mu}(\sup_t |U'(t)|)^2 \int_0^T w_n^2 dt. \tag{5.16}$$

Therefore, if $c2\pi n/L > \sqrt{\overline{\Lambda}} \sup_t |U'(t)|/(2\mu)$, then $w_n(t) \equiv 0$ for $0 < \Lambda \leq \overline{\Lambda}$. □

From this lemma, we know the relation between n and λ such that the equation (4.1) does not have a non-trivial solution. Especially, in the case $\mu = 1$, $2\pi c/L = 1$, $U(t) = \sin 2\pi t + 0.5$, we obtain that if $n > 3\sqrt{3}\pi/2$ $(8.162\cdots)$ then $w_n(t) \equiv 0$ for $0 < \Lambda \leq 1.5$. The following figure shows the relation n and λ.

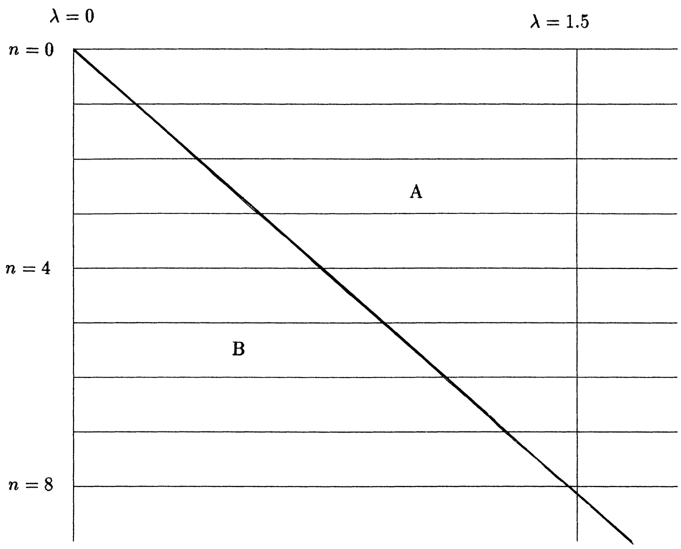


Figure 8

We see that (4.1) does not have a non-trivial solution in the area (B). Therefore, we only have to consider the eigenvalue problem in the area (A). In order to compute the fundamental solution of (5.1) in the area (A), we prepare the following lemma relating the mesh size of Λ.

Lemma 5.2. *Let $\Delta(\Lambda, n)$ and $\Delta(\Lambda_0, n)$ be the trace of the fundamental solutions for $A(t, \Lambda, n)$ and $A(t, \Lambda_0, n)$. Then we have*

$$|\Delta(\Lambda, n) - \Delta(\Lambda_0, n)| \leq C_3 |\Lambda - \Lambda_0|, \tag{5.17}$$

where C_3 is a positive constant, depend on n.

In fact, C_3 is explicitly given by Lemma 5.6.

Now, we consider the case $(\Lambda, n) = (0, 0)$. In this case, the linearized equation (4.1)

$$w_n'' + w_n' = 0 \tag{5.18}$$

has nontrivial solution $w_n \equiv C$, where C is a nonzero constant. Since $\Delta(0,0) = e^{\mu T} + e^{-\mu T}$, we can't use the computer aided proof for the neighborhood of $(\Lambda, n) = (0, 0)$. As concerns the behavior for the neiborhood of $\Lambda = 0$, we have following lemma.

Lemma 5.3. *If it holds*

$$0 < \Lambda \leq \frac{\mu^2}{32(\sup_t |U(t)|)^2}, \tag{5.19}$$

then (1.1) *has no periodic solution except for* $u_\lambda(t)$.

Proof. Instead of (3.1), we set $u(t,x) = u_\lambda(t) + p(t,x)$. Then $p(t,x)$ satisfies

$$p_{tt} - c^2 p_{xx} + \mu p_t + 3\lambda^2 U^2 p + 3\lambda U p^2 + p^3 = 0. \tag{5.20}$$

Multiplying p_t or p and integrated by parts, we have

$$\int_0^L \int_0^T \mu p_t^2 dt dx = -\int_0^L \int_0^T 3\lambda^2 U^2 p p_t + 3\lambda U p^2 p_t dt dx, \tag{5.21}$$

and

$$\int_0^L \int_0^T p_t^2 dt dx = \int_0^L \int_0^T c^2 p_x^2 + 3\lambda^2 U^2 p^2 + 3\lambda U p^3 + p^4 dt dx. \tag{5.22}$$

Combining (5.21) and (5.22), we have

$$\begin{aligned} \int_0^L \int_0^T \frac{3}{16}(\lambda^2 U^2 + p^2) p^2 dt dx &\leq \int_0^L \int_0^T p_t^2 dt dx \\ &\leq \int_0^L \int_0^T \frac{18}{\mu^2} \lambda^2 U^2 (\lambda^2 U^2 + p^2) p^2 dt dx. \end{aligned} \tag{5.23}$$

Therefore, if $18\lambda^2 \sup_t |U(t)|^2/\mu^2 \leq 3/16$, then $p(t,x) \equiv 0$. In the case $\mu = 1$, $2\pi c/L = 1$, $U(t) = \sin 2\pi t + 0.5$, we obtain that if $\lambda \leq \sqrt{6}/36$ $(0.068\cdots)$ then $p(t,x) \equiv 0$. ◻

At the step (4), we have to estimate $||\Phi(T) - \overline{\Phi_K}||$. To do that, we need the theory of pseudo trajectory. Let $\{\overline{\Phi_k}\}_{k=0}^{K+1}$ be the computed value using the scheme (5.3) which is so called a pseudo trajectory of (5.3) which contains the round-off error at each step:

$$||\overline{\Phi_{k+1}} - R^{(k)}\overline{\Phi_k}|| \leq \alpha, \quad k = 0, 1, 2, \ldots K, \tag{5.24}$$

where the round-off error α depends on the complexity computation (5.3) and also on the computer and its software of floating point arithmetics. To estimate the round-off error, we have to have prepared some softwares which perform the interval arithmetics for the computations.

Proposition 5.4 (Nishida, Teramoto and Yosihara [6]). *We have the estimate for the difference between* $\Phi(k\Delta t)$ *and* $\overline{\Phi_k}$:

$$||\Phi(k\Delta t) - \overline{\Phi_k}|| \leq C_1 k\Delta t C_{10}(\Delta t)^3, \tag{5.25}$$

where

$$C_1 = \max_{0\leq s\leq t\leq T} ||L(t,s)||, \tag{5.26}$$

(5.27)
$$C_{10} = \frac{\alpha}{(\Delta t)^4} + \overline{C_1}\frac{\beta}{(\Delta t)^3}\left(1 + \frac{\Delta t}{2} + \frac{(\Delta t)^2}{6} + \frac{(\Delta t)^3}{24}\right) + \frac{(\Delta t)}{120}\overline{C_1}C_5\exp(A\Delta t),$$

$$\overline{C_1} = \max_{0\leq j\leq k} ||\overline{\Phi_j}||, \tag{5.28}$$

$$C_5 = \max_{0\leq t\leq k\Delta t} ||E(t)||, \quad \frac{d^5\Phi}{dt^5} = E(t)\Phi(t), \tag{5.29}$$

(5.30)
$$\beta = \max_{0\leq j\leq k}\{||A(j\Delta t) - \overline{A_j}||, ||B(j\Delta t) - \overline{B_j}||, ||C(j\Delta t) - \overline{C_j}||, ||D(j\Delta t) - \overline{D_j}||\}.$$

Here, we see that C_5 is the cut off error of the scheme (5.3), β is the error of the coefficients of (5.1).

Proof. Let's set $Q_k = \overline{\Phi_k} - \Phi(k\Delta t)$. First, we show the equality:

(5.31)
$$Q_k = \sum_{j=0}^{k} L(k\Delta t, j\Delta t)V_j, \qquad \text{where} \quad V_j = \overline{\Phi_j} - L(j\Delta t, (j-1)\Delta t)\overline{\Phi}_{j-1}.$$

Suppose that the above equality is true. Then it follows that

(5.32)
$$\begin{aligned} Q_{k+1} &= \overline{\Phi}_{k+1} - \Phi((k+1)\Delta t) \\ &= \overline{\Phi}_{k+1} - L((k+1)\Delta t, k\Delta t)\overline{\Phi}_k + L((k+1)\Delta t, k\Delta t)\{\overline{\Phi}_k - \Phi(k\Delta t)\} \\ &= \overline{\Phi}_{k+1} - L((k+1)\Delta t, k\Delta t)\overline{\Phi}_k + \sum_{j=0}^{k} L((k+1)\Delta t, j\Delta t)V_j \\ &= \sum_{j=0}^{k+1} L((k+1)\Delta t, j\Delta t)V_j. \end{aligned}$$

From an induction, we show the equality (5.31). Therefore, we have only to estimate V_k.

$$
\begin{aligned}
||V_k|| &\leq ||\overline{\Phi_k} - R^{(k)}\overline{\Phi_{k-1}}|| + ||R^{(k)}\overline{\Phi_{k-1}} - L(k\Delta t,(k-1)\Delta t)\overline{\Phi}_{k-1}|| \\
&\leq \alpha + ||R^{(k)}\overline{\Phi_{k-1}} - L(k\Delta t,(k-1)\Delta t)\overline{\Phi}_{k-1}||.
\end{aligned} \tag{5.33}
$$

The second term of right hand side has the following estimate:

$$
\begin{aligned}
&||R^{(k)}\overline{\Phi_{k-1}} - L(k\Delta t,(k-1)\Delta t)\overline{\Phi}_{k-1}|| \\
&\quad \leq \Delta t||\overline{\Phi_{k-1}}|| \left\{ ||A(j\Delta t) - \overline{A_j}|| + \frac{\Delta t}{2}||B(j\Delta t) - \overline{B_j}|| \right. \\
&\qquad \left. + \frac{(\Delta t)^2}{6}||C(j\Delta t) - \overline{C_j}|| + \frac{(\Delta t)^3}{24}||D(j\Delta t) - \overline{D_j}|| \right\} \\
&\qquad + \frac{(\Delta t)^5}{120}\max_t ||E(t)|| \max_{(k-1)\Delta t \leq s \leq k\Delta t} ||L(s,(k-1)\Delta t)\overline{\Phi}_{k-1}|| \\
&\quad \leq \overline{C_1}\beta\Delta t\left(1 + \frac{\Delta t}{2} + \frac{(\Delta t)^2}{6} + \frac{(\Delta t)^3}{24}\right) + \frac{(\Delta t)^5}{120}\overline{C_1}C_5\exp(||A||\Delta t).
\end{aligned} \tag{5.34}
$$

Therefore, we have

$$
\begin{aligned}
Q_k \leq C_1 k\{&\alpha + \overline{C_1}\beta\Delta t\left(1 + \frac{\Delta t}{2} + \frac{(\Delta t)^2}{6} + \frac{(\Delta t)^3}{24}\right) \\
&+ \frac{(\Delta t)^5}{120}\overline{C_1}C_5\exp(||A||\Delta t)\}.
\end{aligned} \tag{5.35}
$$

Thus the proof of Proposition 5.4 is completed. □

Now, we have the relation between C_1 and $\overline{C_1}$ by the following lemma.

Lemma 5.5. *$C_1, \overline{C_1}$ satisfy the estimate*

$$
\overline{C_1} \leq \frac{(\exp(2||A||\Delta t) + \alpha T/\Delta t)C_1}{1 - C_1 T(\beta(1+\Delta t) + (\Delta t)^4 C_5 \exp(||A||\Delta t)/120)}. \tag{5.36}
$$

Proof. For any (s,t), we take $(l,k) \in N \times N$ such that $(l-1)\Delta t < s \leq l\Delta t \leq k\Delta t \leq t < (k+1)\Delta t$. Then it holds that

$$
L(t,s) = L(t,k\Delta t)L(k\Delta t,l\Delta t)L(l\Delta t,s). \tag{5.37}
$$

According to Proposition 5.4, we have

$$
||L(t,s)|| \leq \exp(-||A||\Delta t)(\overline{C_1} - C_1 C_{10} k(\Delta t)^4)\exp(-||A||\Delta t). \tag{5.38}
$$

Dividing C_{10} into two parts: $C_{10} = I_1 + I_2\overline{C_1}$, where

(5.39)

$$
I_1 = \frac{\alpha}{(\Delta t)^4}, \quad I_2 = \frac{\beta}{(\Delta t)^3}\left(1 + \frac{\Delta t}{2} + \frac{(\Delta t)^2}{6} + \frac{(\Delta t)^3}{24}\right) + \frac{\Delta t}{120}C_5\exp(||A||\Delta t),
$$

we have

$$\overline{C_1} \leq \frac{(\exp(2||A||\Delta t) + I_1 k(\Delta t)^4)C_1}{1 - I_2 C_1 k(\Delta t)^4}. \tag{5.40}$$

□

Moreover, C_1 is explicitly given by the following lemma.

Lemma 5.6. *Suppose that* $\mu = 1$, $2\pi c/L = 1$. *Let* y_n *be the solution of the equation* (4.2). *Then it follows for any* $\epsilon > 0$, $r^2 \geq 0$ *that*

$$\begin{aligned}&((y_n' + \epsilon y_n)^2 + (n^2 + r^2)y_n^2 + \Lambda U^2 y_n^2)(t)\\ &\leq e^{\nu(t-s)}((y_n' + \epsilon y_n)^2 + (n^2 + r^2)y_n^2 + \Lambda U^2 y_n^2)(s),\end{aligned} \tag{5.41}$$

where

$$\nu = \max\left\{2\epsilon, \epsilon + \frac{1 + 4r^2}{4\epsilon}, \frac{\epsilon/2 - 2\epsilon n^2 + \Lambda \max_t |U'|}{(\epsilon^2 + n^2 + r^2)}, \max_t |U'| - 2\epsilon\right\}. \tag{5.42}$$

Proof. From (4.2), we have

$$\left(\frac{(y_n')^2}{2} + n^2\frac{y_n^2}{2} + \frac{\Lambda U^2 y_n^2}{2}\right)' = \frac{1}{4}y_n y_n' + \Lambda U U' y_n^2, \tag{5.43}$$

$$(y_n y_n')' = (y_n')^2 + \frac{1}{4}y_n^2 - n^2 y_n^2 - \Lambda U^2 y_n^2. \tag{5.44}$$

Combining (5.43) and (5.44), we have

$$\begin{aligned}&\left(\frac{(y_n')^2}{2} + \epsilon y_n y_n' + \frac{n^2 y_n^2}{2} + \frac{\Lambda U^2 y_n^2}{2}\right)'\\ &= \frac{1}{4}y_n y_n' + \Lambda U U' y_n^2 + \epsilon (y_n')^2 + \frac{\epsilon}{4}y_n^2 - \epsilon n^2 y_n^2 - \epsilon \Lambda U^2 y_n^2.\end{aligned} \tag{5.45}$$

Therefore, it holds for any $r^2 \geq 0$ that

(5.46)

$$\begin{aligned}&((y_n' + \epsilon y_n)^2 + (n^2 + r^2)y_n^2 + \Lambda U^2 y_n^2)'\\ &= 2\epsilon(y_n')^2 + \left(\frac{1}{2} + 2\epsilon^2 + 2r^2\right) y_n y_n' + \left(\frac{\epsilon}{2} - 2\epsilon n^2\right) y_n^2 - 2\epsilon\Lambda U^2 y_n^2 + 2\Lambda U U' y_n^2\\ &\leq 2\epsilon(y_n')^2 + \left(\frac{1}{2} + 2\epsilon^2 + 2r^2\right) y_n y_n'\\ &\quad + (\epsilon/2 - 2\epsilon n^2 + \max_t \Lambda |U'|)y_n^2 + (\max_t |U'| - 2\epsilon)\Lambda U^2 y_n^2\\ &\leq \nu((y_n' + \epsilon y_n)^2 + (n^2 + r^2)y_n^2 + \Lambda U^2 y_n^2),\end{aligned}$$

which implies (5.41).

□

Putting $s = 0$ in this lemma, we have

(5.47)
$$\begin{cases} ((\phi_1' + \epsilon\phi_1)^2 + (n^2 + r^2)\phi_1^2 + \Lambda U^2\phi_1^2)(T) \le e^{\nu T}(\epsilon^2 + (n^2 + r^2 + \Lambda U^2(0)), \\ ((\phi_2' + \epsilon\phi_2)^2 + (n^2 + r^2)\phi_2^2 + \Lambda U^2\phi_2^2)(T) \le e^{\nu T}, \end{cases}$$

which gives the bound of C_1 and C_3,

(5.48)
$$C_1 \le e^{\nu T/2}\sqrt{\{1 + \epsilon^2 + n^2 + r^2 + \Lambda U^2(0)\}\left\{\frac{1}{n^2 + r^2} + \left(1 + \frac{\epsilon}{\sqrt{n^2 + r^2}}\right)^2\right\}},$$
$$C_3 \le e^{\nu T}\left(\frac{\sqrt{\epsilon^2 + n^2 + r^2 + \Lambda U^2(0)} + \epsilon}{n^2 + r^2} + \frac{1}{\sqrt{n^2 + r^2}}\right).$$

Now recall the basic numerical data $c = \mu = 1$, $L = 2\pi$, $\overline{\lambda} = 1.5$, $K = 1024$, $\Delta t = 1/1024$, $U(t) = \sin 2\pi t + 0.5$. In our numerical computations with double precision, the round off error is known to satisfy

$$\alpha < 1.0 \times 10^{-14}, \tag{5.49}$$

and we performed the computations so that the error of coefficients satisfy

$$\beta < 1.0 \times 10^{-10}. \tag{5.50}$$

Then Lemma 5.6 gives

(5.51)
$$C_1 \le e^{\epsilon}\sqrt{\left(1 + \epsilon^2 + n^2 + r^2 + \frac{9}{4}\frac{27}{4}\right)\left(\frac{1}{n^2 + r^2} + \left(1 + \frac{\epsilon}{\sqrt{n^2 + r^2}}\right)^2\right)},$$

$$C_3 \le \frac{9}{4}e^{2\epsilon}\left(\frac{\sqrt{\epsilon^2 + n^2 + r^2 + \frac{9}{4}\frac{27}{4}} + \epsilon}{n^2 + r^2} + \frac{1}{\sqrt{n^2 + r^2}}\right), \tag{5.52}$$

where

$$\begin{cases} \epsilon = 2.25,\ r^2 = \epsilon^2 - 1/4,\ n = 0, \\ \epsilon = 2.15,\ r^2 = \epsilon^2 - 1/4,\ n = 1, \\ \epsilon = 1.65,\ r^2 = \epsilon^2 - 1/4,\ n = 2, \\ \epsilon = \pi/2,\ r^2 = 0, \qquad n \ge 3. \end{cases} \tag{5.53}$$

Especially, in the case $n = 2$, we have

$$\begin{matrix} C_1 < 44.5, & \overline{C_1} < 46.3, \\ C_3 < 87, & C_5 < 2.5 \times 10^5. \end{matrix} \tag{5.54}$$

Thus, we can estimate the difference

$$||\Phi(T) - \overline{\Phi_K}|| < 0.0000041, \tag{5.55}$$

by Proposition 5.4, and can obtain all concrete value used in the proof of Main Theorem.

6. Numerical computations

In this section, we show some results of numerical computations. First, we made the numerical computations for the equation (1.1) with (1.2), by the second order central difference scheme, putting a suitable initial data. In the case $c = \mu = 1$, $L = 2\pi$, and $U(t) = \sin 2\pi t + 0.5 (T = 1)$, the numerical results are given in Figs. 1 through 4. When $\lambda = 1$, Fig. 1 shows the trajectory of $(x, u(t,x) - 0.5)$ from t_0 until $t_0 + T$, where t_0 is sufficiently large time, and the trajectory $(t, u(t,x) - 0.5)$ at $x = 3L/8$ is given in Fig. 2. These trajectories indicate that the trivial vibration $u_\lambda(t)$ is asymptotically stable. On the other hand, when $\lambda = 1.5$, the trajectory $(x, 1.5(u(t,x) - 0.5))$ from t_0 until $t_0 + T$ is given in Fig. 3, and the trajectory $(t, 1.5(u(t,x) - 0.5))$ at $x = 3L/8$ is given in Fig. 4. Figure 3 shows that the solution is not uniformly and $L/2$-periodic for x, moreover Fig. 4 shows that the solution is $2T$-periodic. In fact, we know the existence of a period-doubling bifurcation point $\lambda_0 (\lambda_0 \in (1.4525, 1.4550))$, from Example 1.

However, we mathematically studied that the conditions for the existence of bifurcation points reduct the condition for $\Delta(\lambda, n)$, which is a trace of the fundamental solution of (5.1). Therefore we made the numerical computations of $\Delta(\lambda, n)$, using the forth order Taylor finite difference scheme, but not using the software of floating point arithmetics. The graph of $(\lambda, \Delta(\lambda, n))$ for $0 \le \lambda \le 5$ is given in Fig. 6. The curve of $n = 0$ shows that $\Delta(0, 0) = e^{T/2} + e^{-T/2}$, which is consistent with the fact (5.18). The curve of $n = 2$ intersects the line $\Delta = -(e^{T/2} + e^{-T/2})$ at $\lambda_0 (\lambda_0 \in (1.4, 1.5))$, and the other curve don't intersect the line $\Delta = -(e^{T/2} + e^{-T/2})$ in $\lambda \in (0, 1.5)$.

Finally, we investigated the relations between the bifurcation points and the amplitude of periodic vibrations $u_\lambda(t) = \lambda(\sin 2\pi t + \epsilon)$ $0 \le \epsilon \le 1$. We made a numerical computations of $\Delta(\lambda, n)$, using the forth order Taylor finite difference scheme, (not using the software of floating point arithmetics) and plot the points which satisfy the condition of Proposition 3.5. Figure 7 shows the relation (λ, ϵ) for $0 \le \lambda \le 3$, $0 \le \epsilon \le 1$, where the mesh sizes of λ, ϵ are respectively 0.01, 0.0125. This picture point out as follows.

(i) When $0 \le \epsilon \le 0.1375$, the solution which is T-periodic and $L/5$ periodic for x, bifurcates at first.

(ii) When $0.15 \le \epsilon \le 0.1625$, the solution which is $2T$-periodic and uniform in x, bifurcates at first.

(iii) When $0.175 \le \epsilon \le 0.2$, the solution which is $2T$-periodic and L-periodic for x, bifurcates at first.

(iv) When $0.2125 \le \epsilon \le 1.0$, the solution which is $2T$-periodic and $L/2$-periodic for x, bifurcates at first.

(v) The bifurcation occurs at smaller λ as ϵ becomes large.

In particular, in the case $\epsilon = 0$, we can prove that the period-doubling bifurcation point does not exist at all. Since the period of $U^2(t)$ is $T/2$, the argument of Lemma 4.1 implies the period of any bifurcation points is $T/2$ or T.

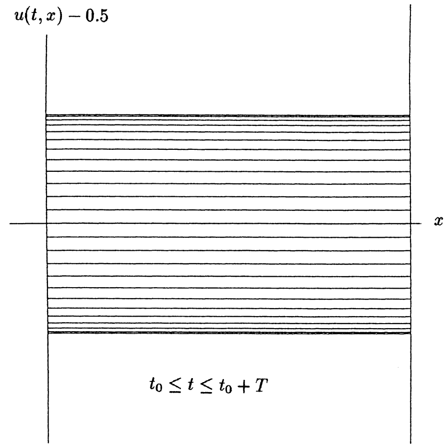
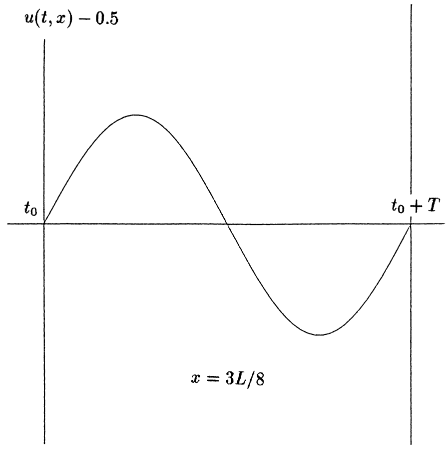


Figure 1



Figrure 2

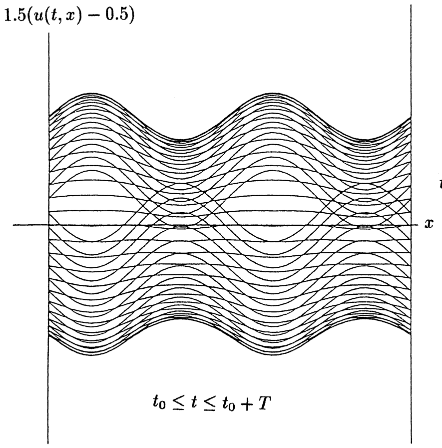
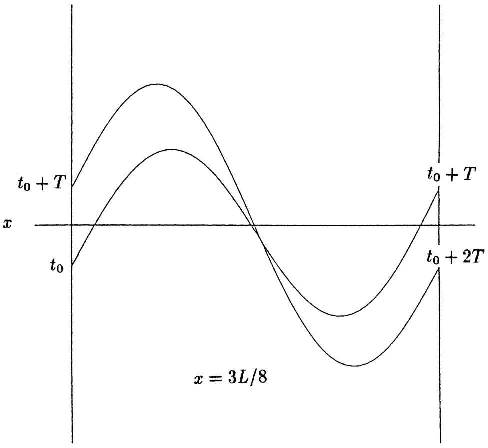


Figure 3



Figrure 4

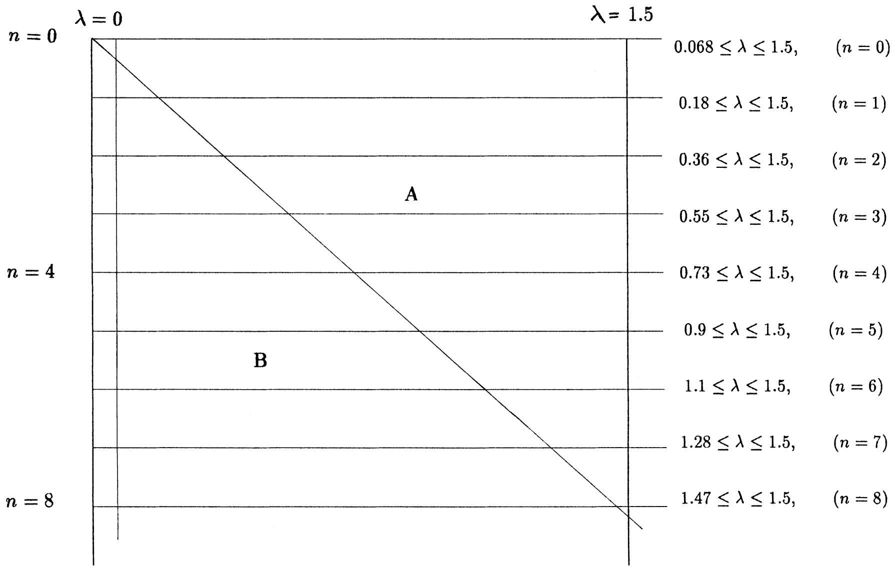


Figure 5

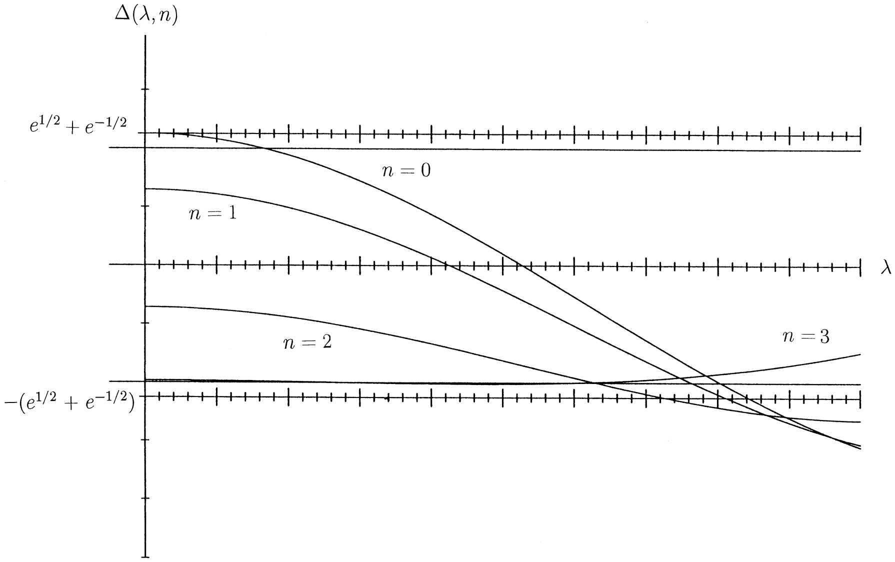


Figure 6

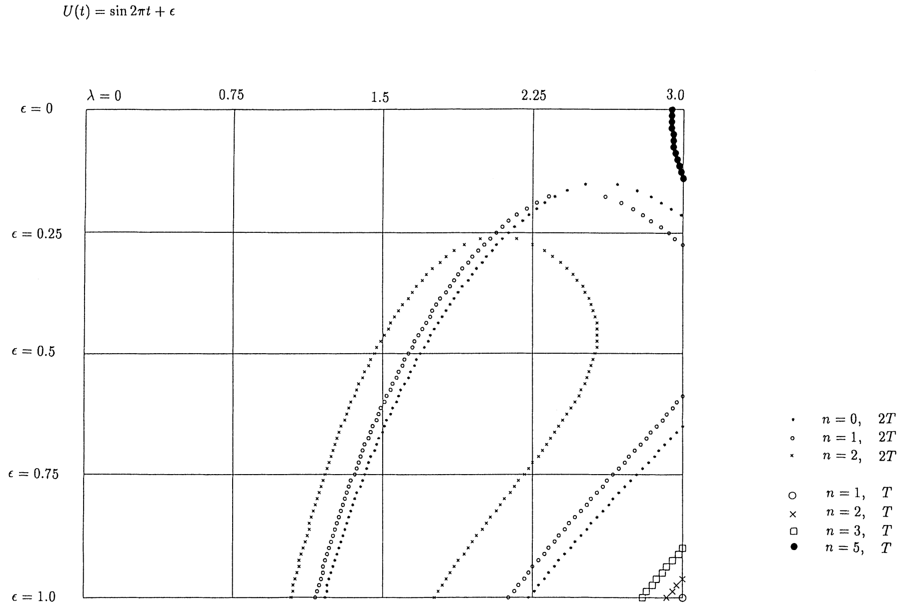


Figure 7

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