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Extension of Thomas' result and upper bound on the spectral gap of $d(\geq 3)$-dimensional Stochastic Ising models

By

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1. Introduction

Let us consider the Glauber dynamics at low temperature (large β) which evolves on a cube

$$\Lambda(l,d) = \left(-\frac{l}{2}, \frac{l}{2}\right]^d \cap \mathbf{Z}^d$$

whose side-length is $l \in \mathbf{N}$ with a boundary condition ω. By $\text{gap}(\Lambda(l,d),\omega)$, we will denote the spectral gap corresponding to a boundary condition ω. Especially, By $\text{gap}(\Lambda(l,d),\phi)$ and $\text{gap}(\Lambda(l,d),+)$, we will mean spectral gaps corresponding to free and + boundary conditions, respectively. When $\beta > \beta_c(d)$, it is known that $\text{gap}(\Lambda(l,d),\omega)$ shrinks to zero as $l \nearrow \infty$. For $d = 2$, it is known that the speed at which $\text{gap}(\Lambda(l,d),+)$ shrinks to zero as $l \nearrow \infty$ is different from the one at which $\text{gap}(\Lambda(l,d),\phi)$ does (see [Mar94]). It is known (Theorem 5 in Section 3 of [Sch94]) that the spectral gap has the following general lower bound for any $d \geq 2$ and any $\beta > 0$:

$$\underline{c}(\beta,d)l^{-d}\exp\Big(-4\beta\sum_{i=1}^{d-1} l^i\Big) \leq \inf_{\omega\in\Omega_{\text{b.c.}}} \text{gap}(\Lambda(l,d),\omega) \qquad \text{for any} \quad l \in \mathbf{N}. \tag{1.1}$$

On the other hand, L. E. Thomas proved in [Tho89] that

$$\text{gap}(\Lambda(l,d),\phi) \leq B\exp(-\beta C l^{d-1}) \qquad \text{for any} \quad l \in \mathbf{N} \tag{1.2}$$

for any $d \geq 2$ and sufficiently large β, where $B = B(\beta,d) > 0$ and $C = C(d) > 0$. In this note we will make an attempt to extend the class of boundary conditions in the case that $d \geq 3$ (see [HY97] and [AY99]) for which the estimate

$$\text{gap}(\Lambda(l,d),\omega) \leq B\exp(-\beta C l^{d-1}) \qquad \text{for any} \quad l \in \mathbf{N}$$

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holds for sufficiently large β and some $B = B(\omega, \beta, d) > 0,\ C = C(\omega, d) > 0$. In order to do this, we want to maximize $k\lambda$ for which the estimate

$$\frac{|\gamma \cap \mathrm{int}\Lambda(l,d)|}{|\gamma|} \geq \lambda \qquad \text{for any} \quad \gamma \in \mathrm{C}(l,d),\ |\gamma| \leq kl^{d-1}$$

holds (see Sections 3 and 5). But it is difficult if we consider all contours in $\mathrm{C}(l,d)$. For this reason, by introducing the notion of simple contours (see the final paragraph of Sections 1 and 3), we will refine the Thomas' argument on contour.

For example, we will consider the boundary conditions $\omega_\delta \in \Omega^+_{\text{b.c.}}$ which are defined for all $\delta \in [0,1]$ by

$$\omega_\delta(x) = \begin{cases} +1 & \text{if} \quad x_d = \left[\dfrac{l}{2}\right] \text{ and } \dfrac{-\delta l}{2} < x_i \leq \dfrac{\delta l}{2}\ (i \neq d), \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

From the consequences for $d = 2$ (see [Mar94]), we can expect not only that $\mathrm{gap}(\Lambda(l,d), \omega_\delta)$ for each $\delta < 1$ behaves like (1.2) as $l \nearrow \infty$, but also that the behavior of $\mathrm{gap}(\Lambda(l,d), \omega_1)$ as $l \nearrow \infty$ is different from that of $\mathrm{gap}(\Lambda(l,d), \omega_\delta)$ for any $\delta < 1$. Unfortunately, we can not prove it. But we can show that for example, for $d = 3$, $\mathrm{gap}(\Lambda(l,d), \omega_\delta)$ for each $\delta < 3/4$ shrinks to zero as $l \nearrow \infty$ like (1.2).

Basic Definitions

The lattice. For $x = (x_i)_{i=1}^d \in \mathbf{Z}^d$, we will use the l_1-norm $\|x\|_1 = \sum_{i=1}^d |x_i|$ and l_∞-norm $\|x\|_\infty = \max_{1 \leq i \leq d} |x_i|$. We will also use the partial order $x \geq y$ if and only if $x_i \geq y_i$ for all $i \leq d$. Let $p = 1$ or $p = \infty$. A set $\Lambda \subset \mathbf{Z}^d$ is said to be l_p-connected if for each distinct $x, y \in \Lambda$, we can find some $\{z_0, \dots, z_m\} \subset \Lambda$ with $z_0 = x,\ z_m = y$ and $\|z_i - z_{i-1}\|_p = 1$ for any $i \leq m$. The interior and exterior boundaries of a set $\Lambda \subset \mathbf{Z}^d$ will be denoted respectively by

$$\partial_{in}\Lambda = \{x \in \Lambda; \|x - y\|_1 = 1 \text{ for some } y \notin \Lambda\},$$
$$\partial_{ex}\Lambda = \{y \notin \Lambda; \|x - y\|_1 = 1 \text{ for some } x \in \Lambda\}.$$

The number of points contained in a set $\Lambda \subset \mathbf{Z}^d$ will be denoted by $|\Lambda|$. We will use the notation $\Lambda \subset\subset \mathbf{Z}^d$ to indicate that $\Lambda \subset \mathbf{Z}^d$ and $|\Lambda| < \infty$ at the same time.

The configurations and the Gibbs states. In addition to the usual spin configuration spaces

$$\Omega_\Lambda = \{\sigma = (\sigma(x))_{x \in \Lambda}; \sigma(x) = +1 \text{ or } -1\}, \quad \Lambda \subset \mathbf{Z}^d,$$

we will introduce a configuration space $\Omega^+_{\text{b.c.}}$ for boundary conditions

$$\Omega^+_{\text{b.c.}} = \{\omega = (\omega(x))_{x \in \mathbf{Z}^d}; \omega(x) = +1 \text{ or } 0\}.$$

We define $\phi,\ + \in \Omega^+_{\text{b.c.}}$ by

$$\phi(x) = 0 \quad \text{for all} \quad x \in \mathbf{Z}^d \quad \text{and} \quad +(x) = +1 \quad \text{for all} \quad x \in \mathbf{Z}^d, \quad \text{respectively.}$$

The set of all real functions on Ω_Λ will be denoted by C_Λ. For $\Lambda \subset\subset \mathbf{Z}^d$ and $\omega \in \Omega^+_{\text{b.c.}}$, the *Hamiltonian* $H^\omega_\Lambda \in \mathrm{C}_\Lambda$ is defined by

$$H^\omega_\Lambda(\sigma) = -\frac{1}{2} \sum_{\substack{x,y\in\Lambda \\ \|x-y\|_1=1}} \sigma(x)\sigma(y) - \sum_{\substack{x\in\Lambda,\ y\notin\Lambda \\ \|x-y\|_1=1}} \sigma(x)\omega(y).$$

A Gibbs state on $\Lambda \subset\subset \mathbf{Z}^d$ with a boundary condition $\omega \in \Omega^+_{\text{b.c.}}$ and inverse temperature $\beta > 0$ is the probability distribution μ^ω_Λ such that the probability of each configuration $\sigma \in \Omega_\Lambda$ is given by

$$\mu^\omega_\Lambda(\{\sigma\}) = \frac{1}{\mathrm{Z}^\omega_\Lambda} \exp\{-\beta H^\omega_\Lambda(\sigma)\},$$

where $\mathrm{Z}^\omega_\Lambda$ is the normalization constant.

Stochastic Ising models. For $\Lambda \subset\subset \mathbf{Z}^d$ and $\beta > 0$, we consider a function $c_\Lambda : \Lambda \times \Omega_\Lambda \times \Omega^+_{\text{b.c.}} \longrightarrow (0, \infty)$ which satisfies the following conditions:

(i) Boundedness. There exist constants $\underline{c}(\beta, d) > 0,\ \overline{c}(\beta, d) > 0$ such that

$$\underline{c}(\beta, d) \le c_\Lambda(x, \sigma, \omega) \le \overline{c}(\beta, d)$$

for all $\Lambda \subset\subset \mathbf{Z}^d$ and all $(x, \sigma, \omega) \in \Lambda \times \Omega_\Lambda \times \Omega^+_{\text{b.c.}}$.

(ii) The detailed balance condition. It holds that

$$c_\Lambda(x, \sigma, \omega) \exp\{-\beta H^\omega_\Lambda(\sigma)\} = c_\Lambda(x, \sigma^x, \omega) \exp\{-\beta H^\omega_\Lambda(\sigma^x)\} \tag{1.4}$$

for all $\Lambda \subset\subset \mathbf{Z}^d$ and all $(x, \sigma, \omega) \in \Lambda \times \Omega_\Lambda \times \Omega^+_{\text{b.c.}}$, where σ^x is the configuration obtained from σ by replacing $\sigma(x)$ with $-\sigma(x)$.

An example of functions c_Λ is given by

$$\begin{aligned} c_\Lambda(x, \sigma, \omega) &= \exp\left\{-\frac{\beta}{2}(H^\omega_\Lambda(\sigma^x) - H^\omega_\Lambda(\sigma))\right\} \\ &= \exp\Big\{-\beta\sigma(x)\Big(\sum_{y\in\Lambda:\|x-y\|_1=1} \sigma(y) + \sum_{y\notin\Lambda:\|x-y\|_1=1} \omega(y)\Big)\Big\}. \end{aligned}$$

The generator of a stochastic Ising model is a linear operator $A^\omega_\Lambda : \mathrm{C}_\Lambda \longrightarrow \mathrm{C}_\Lambda$ for $\Lambda \subset\subset \mathbf{Z}^d$ and $\omega \in \Omega^+_{\text{b.c.}}$ given by

$$A^\omega_\Lambda f(\sigma) = \sum_{x\in\Lambda} c_\Lambda(x, \sigma, \omega)[f(\sigma^x) - f(\sigma)], \quad f \in \mathrm{C}_\Lambda.$$

It can be seen by (1.4) that for any $f, g \in \mathrm{C}_\Lambda$

$$\begin{aligned} -\mu^\omega_\Lambda(f A^\omega_\Lambda g) &= -\mu^\omega_\Lambda(g A^\omega_\Lambda f) \\ &= \frac{1}{2} \sum_{x\in\Lambda} \sum_{\sigma\in\Omega_\Lambda} \mu^\omega_\Lambda(\sigma) c_\Lambda(x, \sigma, \omega)[f(\sigma^x) - f(\sigma)][g(\sigma^x) - g(\sigma)]. \end{aligned}$$

Finally, we define

$$\text{gap}(\Lambda,\omega)=\inf\Big\{\frac{-\mu_\Lambda^\omega(fA_\Lambda^\omega f)}{\mu_\Lambda^\omega(|f-\mu_\Lambda^\omega(f)|^2)};f\in \mathrm{C}_\Lambda\Big\}, \tag{1.5}$$

which is the smallest positive eigenvalue of $-A_\Lambda^\omega$ and hence it is called the *spectral gap*.

Main Result

Theorem 1.1. *Let $d\geq 3$. Consider a stochastic Ising model on the square $\Lambda(l,d)$. Suppose that a boundary condition $\omega\in\Omega_{\text{b.c.}}^+$ is such that*

$$\limsup_{l\to\infty}\frac{|F_l^+(\omega)|}{l^{d-1}}<\delta<\frac{27}{16d}, \tag{1.6}$$

where

$$F_l^+(\omega)=\{y\in\partial_{ex}\Lambda(l,d);\omega(y)=+1\}.$$

Then, there exists $\beta_0=\beta_0(\delta,d)>0$ such that for any $\beta\geq\beta_0$ and any $l\in\mathbf{N}$

$$\text{gap}(\Lambda(l,d),\omega)\leq B\exp(-\beta C l^{d-1}), \tag{1.7}$$

where $B=B(\omega,\beta,\delta,d)>0$ and $C=C(\delta,d)>0$. Especially, if there exists some $\delta\in(0,27/16d)$ such that $|F_l^+(\omega)|\leq\delta l^{d-1}$ for any $l\in\mathbf{N}$, then we can take B in (1.7) *as a constant independent of ω.*

A better bound can be obtained for $d=3$ case by a slight modification of the argument in the proof of Theorem 1.1.

Theorem 1.2. *Suppose that a boundary condition $\omega\in\Omega_{\text{b.c.}}^+$ is such that*

$$\limsup_{l\to\infty}\frac{|F_l^+(\omega)|}{l^2}<\delta<\frac{3}{4}. \tag{1.8}$$

Then, there exists $\beta_0'=\beta_0'(\delta)>0$ such that (1.7) *holds for any $\beta\geq\beta_0'$ and any $l\in\mathbf{N}$. Especially, if there exists some $\delta\in(0,3/4)$ such that $|F_l^+(\omega)|\leq\delta l^2$ for any $l\in\mathbf{N}$, then we can take B in* (1.7) *as a constant independent of ω.*

For $d=4$ or 5, we have a little better result than Theorem 1.1. We will present it in Appendix with its proof (see Theorem A.1).

Contours, l_∞-Contours and Simple Contours

Here we will introduce the notion of simple contours which will play an important role in this note. The set $\mathbf{B}$ of bonds in $\mathbf{Z}^d$ is defined by

$$\mathbf{B}=\{\{x,y\}\subset\mathbf{Z}^d;\|x-y\|_1=1\}.$$

For $\Lambda\subset\mathbf{Z}^d$, we also define

$$\partial\Lambda=\{\{x,y\}\in\mathbf{B};(x,y)\in\Lambda\times\Lambda^c\}.$$

For a bond $b = \{x, y\}$, we will consider a $(d-1)$-dimensional unit cell $b^* = Q(x) \cap Q(y)$, where $Q(x) = \prod_{i=1}^{d} [x_i - (1/2), x_i + (1/2)] \subset \mathbf{R}^d$. For a finite set $V \subset \mathbf{R}^d$, we put

$$Q(V) = \cup_{x \in V} Q(x) \subset \mathbf{R}^d$$

and $\partial Q(V)$ will indicate the set of $(d-1)$-dimensional unit cells constituting the boundary of $Q(V)$. Two bonds b_1 and b_2 are said to be adjacent if $b_1^* \cap b_2^* \neq \phi$. A set $E \subset \mathbf{B}$ is said to be connected if for each distinct $b, b' \in E$, we can find some $\{b_0, \dots, b_m\} \subset E$ such that b_i and b_{i-1} are adjacent for any $i \leq m$ with $b_0 = b$ and $b_m = b'$.

Clusters. For $\sigma \in \Omega_{\Lambda(l,d)}$, we define

$$\Lambda(l,d)(\sigma,+) = \{x \in \Lambda(l,d); \sigma(x) = +1\},$$
$$\Lambda(l,d)(\sigma,-) = \{x \in \Lambda(l,d); \sigma(x) = -1\},$$

and let $\{\Lambda_i^+(\sigma)\}$ and $\{\Lambda_i^-(\sigma)\}$ be the decomposition of $\Lambda(l,d)(\sigma,+)$ and $\Lambda(l, d)(\sigma,-)$ into l_1-connected components, respectively. We will call an element of $\{\Lambda_i^+(\sigma)\}$ and $\{\Lambda_i^-(\sigma)\}$ a $(+)$-cluster at σ and a $(-)$-cluster at σ, respectively. When $\sigma(x) = +1$, there exists the unique $(+)$-cluster including $x \in \Lambda(l,d)$, which will be denoted by $C_x^+(\sigma)$. We define $C_x^-(\sigma)$ similarly.

Contours and l_∞-Contours. A contour (an l_∞-contour) γ is a union of $(d-1)$-dimensional unit cells with the following properties: There exists $\Theta \subset\subset \mathbf{Z}^d$ such that

(i) Θ is $l_1(l_\infty)$-connected and Θ^c is l_∞-connected, and

(ii) $\gamma = \cup_{b \in \partial\Theta} b^*$.

The set $\Theta \subset\subset \mathbf{Z}^d$ is uniquely determined by a contour (an l_∞-contour) γ and hence will be denoted by $\Theta(\gamma)$. We can see that $\gamma = \partial Q(\Theta(\gamma))$. The $(d-1)$-dimensional Lebesgue measure of an l_∞-contour γ will be denoted by $|\gamma|$ and the d-dimensional Lebesgue measure of $Q(\Theta(\gamma))$ will be denoted by $|Q(\Theta(\gamma))|$. Since an l_∞-contour corresponds to a connected set of bonds, it follows that for each $b \in \mathbf{B}$ and each $n \in \mathbf{N}$,

$$\sharp\{\gamma; \gamma \text{ is an } l_\infty\text{-contour with } |\gamma| = n \text{ and } \gamma \ni b^*\} \leq \kappa(d)^{n-1}, \tag{1.9}$$

where $\kappa(d) > 0$ is a constant which depends only on d ((4.24) in [Gri89]). If $\Theta(\gamma)$ is a subset of $\Lambda \subset \mathbf{Z}^d$, γ is said to be a contour (an l_∞-contour) in Λ. The set of all contours in $\Lambda(l,d)$ will be denoted by $\mathrm{C}(l,d)$. The set of all l_∞-contours in $\Lambda(l,d)$ will be denoted by $\overline{\mathrm{C}}(l,d)$. For $\sigma \in \Omega_{\Lambda(l,d)}$, a contour γ is said to be a $(+)$-contour at σ if it satisfies the following properties:

(i) There exists a $(+)$-cluster $\Lambda_i^+(\sigma) \subset \Theta(\gamma)$, and

(ii) $\gamma \in \{\partial\Lambda_{i,j}^+(\sigma)\}$, where $\{\partial\Lambda_{i,j}^+(\sigma)\}$ is the decomposition of $\partial Q(\Lambda_i^+(\sigma))$ into connected components.

The $(+)$-cluster $\Lambda_i^+(\sigma)$ is uniquely determined by a $(+)$-contour γ at σ and hence will be denoted by $C^+(\sigma,\gamma)$. Similarly, we define $(-)$-contours at σ. By a contour at σ, we will mean either a $(+)$-contour at σ or a $(-)$-contour at σ.

Simple contours in $\Lambda(l,d)$. We define for $i = 1, \dots, d$

$$\mathrm{C}_i(l,d) = \left\{ \gamma \in \mathrm{C}(l,d);\ \begin{array}{l} \text{if } x \in \Theta(\gamma) \text{ and } y \in \mathrm{L}^i(l,d)(x) \\ \text{with } y_i \le x_i, \text{ then } y \in \Theta(\gamma) \end{array} \right\},$$

where $\mathrm{L}^i(l,d)(x) = \{y = (y_j)_{j=1}^d \in \Lambda(l,d); y_j = x_j \text{ for any } j \ne i\}$. A contour $\gamma \in \cap_{i=1}^d \mathrm{C}_i(l,d)$ is said to be a *simple contour* in $\Lambda(l,d)$. The set of all simple contours in $\Lambda(l,d)$ will be denoted by $\mathrm{S}(l,d)$. The main idea of this note is to reduce analysis of contours to that of simple contours.

2. Outline of the proof of Theorems 1.1 and 1.2

Our proof of Theorems 1.1 and 1.2 is based on the ways in [Tho89], [HY97] and [AY99]. First, we will explain an outline of the proof of Theorem 1.1.

For $\sigma \in \Omega_{\Lambda(l,d)}$, we define

$$C_l(\sigma) = \{\gamma; \gamma \text{ is a } (+)\text{-contour in } \Lambda(l,d) \text{ at } \sigma \text{ with } |\gamma| \ge 9l^{d-1}/2\}. \tag{2.1}$$

Let $\chi_l : \Omega_{\Lambda(l,d)} \longrightarrow \{0,1\}$ be the indicator function of the event Γ_l which is defined by

$$\Gamma_l = \{\sigma \in \Omega_{\Lambda(l,d)}; C_l(\sigma) \ne \phi\}. \tag{2.2}$$

Then, we have by (1.5) that

$$\begin{aligned} \mathrm{gap}(\Lambda(l,d),\omega) &\le \frac{-\mu^\omega_{\Lambda(l,d)}(\chi_l A^\omega_{\Lambda(l,d)}\chi_l)}{\mu^\omega_{\Lambda(l,d)}(|\chi_l - \mu^\omega_{\Lambda(l,d)}(\chi_l)|^2)} \\ &\le \frac{\overline{c}(\beta,d)}{\mu^\omega_{\Lambda(l,d)}(\Gamma_l)\mu^\omega_{\Lambda(l,d)}(\Gamma_l^c)} \sum_{x\in\Lambda(l,d)} \sum_{\sigma\in\Gamma_l,\ \sigma^x\notin\Gamma_l} \mu^\omega_{\Lambda(l,d)}(\sigma). \end{aligned} \tag{2.3}$$

To bound the RHS of (2.3) from above, we will use the following two lemmas.

Lemma 2.1. *Suppose that* $\omega \in \Omega^+_{\mathrm{b.c.}}$. *Then, there exist* $\beta_1 = \beta_1(d) > 0$ *and* $l_1 = l_1(d) > 0$ *such that for any* $\beta \ge \beta_1$

$$\inf_{l \ge l_1} \mu^\omega_{\Lambda(l,d)}(\Gamma_l) \ge \frac{1}{3}. \tag{2.4}$$

Lemma 2.2. *Suppose that a boundary condition* $\omega \in \Omega^+_{\mathrm{b.c.}}$ *satisfies* (1.6). *Then, there exist* $\beta_2 = \beta_2(\delta,d) > 0$ *and* $l_2 = l_2(\omega,\delta,d) > 0$ *such that for any* $\beta \ge \beta_2$ *and any* $l \ge l_2$

$$\sum_{x\in\Lambda(l,d)} \sum_{\sigma\in\Gamma_l,\ \sigma^x\notin\Gamma_l} \mu^\omega_{\Lambda(l,d)}(\sigma) \le \mu^\omega_{\Lambda(l,d)}(\Gamma_l^c) B \exp(-\beta C l^{d-1}), \tag{2.5}$$

where $B = B(\beta,\delta,d) > 0$ *and* $C = C(\delta,d) > 0$.

From (2.3), (2.4) and (2.5), we have that for any $\beta \geq \max\{\beta_1, \beta_2\}$ and any $l \geq \max\{l_1, l_2\}$

$$\mathrm{gap}(\Lambda(l,d),\omega) \leq 3\overline{c}(\beta,d)B\exp(-\beta C l^{d-1}),$$

which proves Theorem 1.1.

To prove Theorem 1.2, we have only to use the following lemma instead of Lemma 2.2.

Lemma 2.3. *Let $d = 3$. Suppose that a boundary condition $\omega \in \Omega^+_{\mathrm{b.c.}}$ satisfies* (1.8)*. Then, there exist $\beta_3 = \beta_3(\delta) > 0$ and $l_3 = l_3(\omega,\delta) > 0$ such that* (2.5) *holds for any $\beta \geq \beta_3$ and any $l \geq l_3$.*

3. Simple contours

To reduce analysis of contours to that of simple contours, we will introduce a lemma which asserts that

$$\inf_{\gamma \in \mathrm{C}(l,d)} \frac{|\gamma \cap \mathrm{int}\Lambda(l,d)|}{|\gamma|} \geq \min\Big\{\frac{1}{2}, \inf_{\gamma \in \mathrm{S}(l,d)} \frac{|\gamma \cap \mathrm{int}\Lambda(l,d)|}{|\gamma|}\Big\}, \tag{3.1}$$

where $\gamma \cap \mathrm{int}\Lambda(l,d) = \gamma \backslash \partial Q(\Lambda(l,d))$. From now on, we will use the notations

$$b_{\pm i}(x) = \{x, x \pm e_i\}, \qquad i = 1, \dots, d$$

to specify $2d$ bonds including $x \in \mathbf{Z}^d$, where $\{e_i\}_{i=1}^d$ are the canonical unit vectors in $\mathbf{Z}^d$. We also define the following notations to specify $2d$ sides of $Q(\Lambda(l,d))$:

$$F_{\pm i}(l,d) = \{b^* \in \partial Q(\Lambda(l,d)); b = b_{\pm i}(x) \text{ for some } x \in \partial_{in}\Lambda(l,d)\}, \qquad i = 1, \dots, d.$$

Lemma 3.1. *For each $i = 1, \dots, d$, consider the map $\varphi_i : \mathrm{C}(l,d) \ni \gamma \longmapsto \varphi_i(\gamma) \in \mathrm{C}_i(l,d)$ which satisfies that for any $\bar{x} \in \Lambda(l,d)$*

$$|\Theta(\gamma) \cap \mathrm{L}^i(l,d)(\bar{x})| = |\Theta(\varphi_i(\gamma)) \cap \mathrm{L}^i(l,d)(\bar{x})\}|. \tag{3.2}$$

Then, for each $i = 1, \dots, d$ and any $\gamma \in \mathrm{C}(l,d)$ with $|\gamma \cap \mathrm{int}\Lambda(l,d)|/|\gamma| \leq 1/2$ it holds that

$$|\gamma| \geq |\varphi_i(\gamma)|, \tag{3.3}$$

$$\frac{|\gamma \cap \mathrm{int}\Lambda(l,d)|}{|\gamma|} \geq \frac{|\varphi_i(\gamma) \cap \mathrm{int}\Lambda(l,d)|}{|\varphi_i(\gamma)|}. \tag{3.4}$$

Proof. It suffices to prove (3.3) and (3.4) for $i = d$. From the definition of the map φ_d, we can see that for any $\bar{x} \in \Lambda(l,d)$

$$\left|\left\{b^* \in \gamma; \begin{array}{c} b = b_{+d}(x) \text{ or } b = b_{-d}(x) \\ \text{for some } x \in \Theta(\gamma) \cap \mathrm{L}^d(l,d)(\bar{x}) \end{array}\right\}\right| \geq \left|\left\{b^* \in \varphi_d(\gamma); \begin{array}{c} b = b_{+d}(x) \text{ or } b = b_{-d}(x) \\ \text{for some } x \in \Theta(\varphi_d(\gamma)) \cap \mathrm{L}^d(l,d)(\bar{x}) \end{array}\right\}\right|. \tag{3.5}$$

For $j = 1, \dots, d-1$, we can also see that for any $\bar{x} \in \Lambda(l,d)$ and $\bar{y} = \bar{x} + e_j$

(3.6)

$$\begin{aligned}
&\left|\left\{ b^* \in \gamma \cap \mathrm{int}\Lambda(l,d);\ \begin{array}{l} b = b_{+j}(x) \text{ for some } x \in \Theta(\gamma) \cap \mathrm{L}^d(l,d)(\bar{x}) \text{ or} \\ b = b_{-j}(y) \text{ for some } y \in \Theta(\gamma) \cap \mathrm{L}^d(l,d)(\bar{y}) \end{array} \right\}\right| \\
&= |\{x \in \Theta(\gamma); x \in \mathrm{L}^d(l,d)(\bar{x})\}| + |\{y \in \Theta(\gamma); y \in \mathrm{L}^d(l,d)(\bar{y})\}| \\
&\quad - 2|\{x \in \Theta(\gamma); x \in \mathrm{L}^d(l,d)(\bar{x}) \text{ and } x + e_j \in \Theta(\gamma)\}| \\
&\geq \Big| |\{x \in \Theta(\gamma); x \in \mathrm{L}^d(l,d)(\bar{x})\}| - |\{y \in \Theta(\gamma); y \in \mathrm{L}^d(l,d)(\bar{y})\}| \Big| \\
&= \left|\left\{ b^* \in \varphi_d(\gamma) \cap \mathrm{int}\Lambda(l,d);\ \begin{array}{c} b = b_{+j}(x) \text{ for some} \\ x \in \Theta(\varphi_d(\gamma)) \cap \mathrm{L}^d(l,d)(\bar{x}) \text{ or} \\ b = b_{-j}(y) \text{ for some} \\ y \in \Theta(\varphi_d(\gamma)) \cap \mathrm{L}^d(l,d)(\bar{y}) \end{array} \right\}\right|,
\end{aligned}$$

and that for any $\bar{x} \in \partial_{in}\Lambda(l,d)$

(3.7)

$$\begin{aligned}
&\left|\left\{ b^* \in \gamma \cap \partial Q(\Lambda(l,d));\ \begin{array}{c} b = b_{+j}(x) \text{ or } b = b_{-j}(x) \\ \text{for some } x \in \Theta(\gamma) \cap \mathrm{L}^d(l,d)(\bar{x}) \end{array} \right\}\right| \\
&= \left|\left\{ b^* \in \varphi_d(\gamma) \cap \partial Q(\Lambda(l,d));\ \begin{array}{c} b = b_{+j}(x) \text{ or } b = b_{-j}(x) \\ \text{for some } x \in \Theta(\varphi_d(\gamma)) \cap \mathrm{L}^d(l,d)(\bar{x}) \end{array} \right\}\right|.
\end{aligned}$$

From (3.5), (3.6) and (3.7), we have that

$$|\gamma| \geq |\varphi_d(\gamma)|. \tag{3.8}$$

Moreover, note that

(3.9)

$$\begin{aligned}
|\varphi_d(\gamma) \cap \partial Q(\Lambda(l,d))| &= |\gamma \cap \partial Q(\Lambda(l,d))| \\
&\quad + \left|\left\{ x \in \Lambda(l,d);\ \begin{array}{l} b^*_{-d}(x) \notin \gamma \cap F_{-d}(l,d) \text{ and} \\ b^*_{-d}(x) \in \varphi_d(\gamma) \cap F_{-d}(l,d) \end{array} \right\}\right| \\
&\quad - \left|\left\{ x \in \Lambda(l,d);\ \begin{array}{l} b^*_{+d}(x) \in \gamma \cap F_{+d}(l,d) \text{ and} \\ b^*_{+d}(x) \notin \varphi_d(\gamma) \cap F_{+d}(l,d) \end{array} \right\}\right|,
\end{aligned}$$

and that

(3.10)

$$\begin{aligned}
&\frac{1}{2}\Big(|\gamma| - |\varphi_d(\gamma)|\Big) \\
&\geq \left|\left\{ x \in \Lambda(l,d);\ \begin{array}{l} x_d = [\frac{l}{2}] \text{ and there exist } x', x'' \in \Theta(\gamma) \cap \mathrm{L}^d(l,d)(x) \\ \text{such that } b^*_{-d}(x'),\ b^*_{+d}(x'') \in \gamma \cap \mathrm{int}\Lambda(l,d) \end{array} \right\}\right| \\
&\geq \left|\left\{ x \in \Lambda(l,d);\ \begin{array}{l} b^*_{+d}(x) \in \gamma \cap F_{+d}(l,d) \text{ and} \\ b^*_{+d}(x) \notin \varphi_d(\gamma) \cap F_{+d}(l,d) \end{array} \right\}\right| \\
&\quad - \left|\left\{ (x \in \Lambda(l,d);\ \begin{array}{l} b^*_{-d}(x) \notin \gamma \cap F_{-d}(l,d) \text{ and} \\ b^*_{-d}(x) \in \varphi_d(\gamma) \cap F_{-d}(l,d) \end{array} \right\}\right|.
\end{aligned}$$

Therefore, we have from (3.8), (3.9) and (3.10) that for any $\gamma \in \mathrm{C}(l,d)$ with $|\gamma \cap \mathrm{int}\Lambda(l,d)|/|\gamma| \leq (1/2)$

$$\frac{|\gamma \cap \mathrm{int}\Lambda(l,d)|}{|\gamma|} \geq \frac{|\varphi_d(\gamma) \cap \mathrm{int}\Lambda(l,d)|}{|\varphi_d(\gamma)|},$$

since the following inequality holds:

$$\frac{t_1}{t_2} \geq \frac{t_1 - p_1}{t_2 - p_2} \qquad \text{for any } t_1, t_2, p_1, p_2 > 0 \text{ with } t_1 > p_1,\ t_2 > p_2 \text{ and } \frac{p_1}{p_2} \geq \frac{t_1}{t_2}.$$

□

4. Proof of Lemma 2.1

For $\gamma \in \mathrm{C}(l,d)$ at $\sigma \in \Omega_{\Lambda(l,d)}$, we set

$$\begin{aligned} \text{(4.1)} \qquad \Delta_\gamma H^\omega_{\Lambda(l,d)}(\sigma) &= H^\omega_{\Lambda(l,d)}(\sigma) - H^\omega_{\Lambda(l,d)}(T_\gamma \sigma) \\ &= 2\Big(|\gamma \cap \mathrm{int}\Lambda(l,d)| - |\partial_{ex}\Theta(\gamma) \cap F_l^+(\omega)|\Big), \end{aligned}$$

where the map $T_\gamma : \Omega_{\Lambda(l,d)} \longrightarrow \Omega_{\Lambda(l,d)}$ is defined by

$$T_\gamma \sigma(x) = \begin{cases} -\sigma(x) & \text{if} \quad x \in \Theta(\gamma), \\ \sigma(x) & \text{if} \quad x \notin \Theta(\gamma). \end{cases}$$

For $\gamma \in \mathrm{S}(l,d)$ and $i = 1, \dots, d$, we put

$$S_{+i}(\gamma) = \gamma \cap F_{+i}(l,d) \qquad \text{and} \qquad S_{-i}(\gamma) = \gamma \cap F_{-i}(l,d).$$

Note that there exists i_m such that

$$\text{(4.2)} \qquad |S_{-i_m}(\gamma)| \leq \frac{1}{2d}|\gamma|.$$

Lemma 4.1. *Let $\rho \in (0, 1/2)$ and $\gamma \in \mathrm{C}(l,3)$ be a contour with $|\Theta(\gamma)| < (1-2\rho)l^3$. Then, there exists $l_0 > 0$ such that for any $l \geq l_0$*

$$\text{(4.3)} \qquad \frac{|\gamma \cap \mathrm{int}\Lambda(l,3)|}{|\gamma|} \geq \frac{\rho}{6}.$$

Lemma 4.2. *Let $\rho' \in (0,3)$ and $\gamma \in \mathrm{C}(l,d)$ be a contour at $\sigma \in \Omega_{\Lambda(l,d)}$ with $|\Theta(\gamma)| < \rho' l^d/d$. Then, there exist $\varepsilon = \varepsilon(\rho', d) > 0$ and $l_0 = l_0(d) > 0$ such that for any $l \geq l_0$*

$$\text{(4.4)} \qquad \Delta_\gamma H^\phi_{\Lambda(l,d)}(\sigma) \geq \varepsilon|\gamma|.$$

Proof of Lemma 4.2. Since we can see from (4.1) that (4.4) holds for any $\gamma \in \mathrm{C}(l,d)$ with $|\gamma \cap \mathrm{int}\Lambda(l,d)|/|\gamma| \geq 1/2$, we have only to show that

$$\text{(4.5)} \qquad \frac{|\gamma \cap \mathrm{int}\Lambda(l,d)|}{|\gamma|} \geq \frac{\min\{\rho', 3-\rho'\}}{2d^2(d-1)}$$

for any $\gamma \in \mathrm{S}(l,d)$ with $|\Theta(\gamma)| < \rho' l^d/d$ by (3.1) and (3.2). Assuming that Lemma 4.1 is true, we will prove (4.5) by induction. For $d = 3$, (4.5) holds from (4.3). Thus, we assume that (4.5) holds for $d = n \geq 3$ and consider $\gamma \in \mathrm{S}(l, n+1)$ with $|\Theta(\gamma)| < \rho' l^{n+1}/(n+1)$. From the definition of simple contours we can see that $|S_{+i}(\gamma)| \cdot l < |Q(\Theta(\gamma))| = |\Theta(\gamma)|$ for $i = 1, \ldots, n+1$, which implies that

$$|S_{+i}(\gamma)| < \rho' l^n/(n+1), \qquad i = 1, \ldots, n+1. \tag{4.6}$$

If there exists some i_0 such that $|S_{-i_0}(\gamma)| \geq \rho' l^n/n$, then, we have that

$$\begin{aligned} \frac{|\gamma \cap \mathrm{int}\Lambda(l, n+1)|}{|\gamma|} &\geq \frac{1}{2(n+1)l^n}\Big(\frac{\rho'}{n} - \frac{\rho'}{n+1}\Big) l^n \\ &= \frac{\rho'}{2(n+1)^2 n}, \end{aligned} \tag{4.7}$$

since $|\gamma \cap \mathrm{int}\Lambda(l, n+1)| \geq |S_{-i_0}(\gamma)| - |S_{+i_0}(\gamma)|$. Otherwise, we have that $|S_{-i}(\gamma)| < \rho' l^n/n$ for $i = 1, \ldots, n+1$. We consider the n-dimensional hyperplanes $\mathrm{H}(t)$ of integer height, which are defined by

$$\mathrm{H}(t) = \{z \in \mathbf{R}^{n+1}; z_{i_m} = t\}, \quad t \in \mathbf{Z}.$$

Here i_m satisfies (4.2). Then, for any integer $t \in (-l/2, l/2]$ we can regard the intersection of γ and $\mathrm{H}(t)$ as a simple contour in $\mathrm{S}(l,n)$. Let $\gamma'(t) = \gamma \cap \mathrm{H}(t)$ and $Q'(t) = Q(\Theta(\gamma)) \cap \mathrm{H}(t)$. By $|S_{-i_m}(\gamma)| < \rho' l^n/n$, we have that $|Q'(t)| < \rho' l^n/n$. Then, by the hypothesis of the induction, we have that for any integer $t \in (-l/2, l/2]$

$$\frac{|\gamma'(t) \cap \mathrm{int}\Lambda(l, n+1)|}{|\gamma'(t)|} \geq \frac{\min\{\rho', 3-\rho'\}}{2n^2(n-1)}. \tag{4.8}$$

Therefore, we have from (4.2) and (4.8) that

$$\begin{aligned} \frac{|\gamma \cap \mathrm{int}\Lambda(l, n+1)|}{|\gamma|} &\geq \frac{1}{|\gamma|}\Big((|\gamma| - 2|S_{-i_m}(\gamma)|) \cdot \frac{\min\{\rho', 3\rho'\}}{2n^2(n-1)}\Big) \\ &\geq \frac{n}{n+1} \cdot \frac{\min\{\rho', 3-\rho'\}}{2n^2(n-1)} \\ &= \frac{\min\{\rho', 3-\rho'\}}{2(n+1)n(n-1)}. \end{aligned} \tag{4.9}$$

From (4.7) and (4.9), we can show that (4.5) holds for $d = n+1$. □

Proof of Lemma 4.1. By the same reason as in the proof of the previous lemma, we have only to show that (4.3) holds for any $\gamma \in \mathrm{S}(l,3)$ with $|\Theta(\gamma)| < (1-2\rho)l^3$ by (3.1) and (3.2). Thus, we assume that $\gamma \in \mathrm{S}(l,3)$ with $|\Theta(\gamma)| < (1-2\rho)l^3$. Note that

$$|S_{+i}(\gamma)| < (1-2\rho)l^2, \qquad i = 1, 2, 3. \tag{4.10}$$

If $|S_{-i_m}(\gamma)| \geq (1-\rho)l^2$, then we have from (4.10) that

$$\frac{|\gamma \cap \mathrm{int}\Lambda(l,3)|}{|\gamma|} \geq \frac{\rho}{6}. \tag{4.11}$$

Otherwise, we consider the 2-dimensional hyper-planes $\mathrm{H}(t)$ of integer height and $\gamma'(t) = \gamma \cap \mathrm{H}(t)$ in the same way as in the proof of Lemma 4.2. Then, by $|S_{-i_m}(\gamma)| < (1-\rho)l^2$, we have that for any integer $t \in (-l/2, l/2]$

$$\frac{|\gamma'(t) \cap \mathrm{int}\Lambda(l,3)|}{|\gamma'(t)|} \geq \frac{\sqrt{\rho}}{2}. \tag{4.12}$$

Therefore, we have from (4.2) and (4.12) that

$$\begin{aligned} \frac{|\gamma \cap \mathrm{int}\Lambda(l,3)|}{|\gamma|} &\geq \frac{1}{|\gamma|}\Big((|\gamma| - 2|S_{-i_m}(\gamma)|) \cdot \frac{\sqrt{\rho}}{2}\Big) \\ &\geq \frac{\sqrt{\rho}}{3}. \end{aligned} \tag{4.13}$$

From (4.11) and (4.13), we can show that (4.3) holds. □

Proof of Lemma 2.1. We define a set $\Gamma'_l \subset \Omega_{\Lambda(l,d)}$ by

$$\Gamma'_l = \{\sigma \in \Omega_{\Lambda(l,d)}; \text{there exists some } (+)\text{-contour } \gamma \text{ at } \sigma \text{ with } |\Theta(\gamma)| \geq 9l^d/4d\}.$$

We can see that $\Gamma'_l \subset \Gamma_l$ as follows. From (3.2) and (3.3), we have only to show that $|\gamma| \geq 9l^{d-1}/2$ for any $\gamma \in \mathrm{S}(l,d)$ with $|\Theta(\gamma)| \geq 9l^d/4d$. Take such a simple contour $\gamma \in \mathrm{S}(l,d)$, and then we will prove that

$$|S_{-i}(\gamma)| \geq 9l^{d-1}/4d, \qquad i = 1, \dots, d, \tag{4.14}$$

which implies that $|\gamma| \geq 9l^{d-1}/2$. To see (4.14), we assume for example that $|S_{+1}(\gamma)| < 9l^{d-1}/4d$. Then, by the definition of simple contours we have that $|Q(\Theta(\gamma))| = |\Theta(\gamma)| < 9l^d/4d$, which contradicts $|\Theta(\gamma)| \geq 9l^d/4d$.

By FKG inequality, we have that

$$\begin{aligned} \mu^{\omega}_{\Lambda(l,d)}(\Gamma_l) &\geq \mu^{\omega}_{\Lambda(l,d)}(\Gamma'_l) \\ &\geq \mu^{\phi}_{\Lambda(l,d)}(\Gamma'_l) \\ &\geq \mu^{\phi}_{\Lambda(l,d)}(\{\sigma(0) = +1\} \cap \Gamma'_l) \\ &= \frac{1}{2} - \mu^{\phi}_{\Lambda(l,d)}(\{\sigma(0) = +1\} \cap \Gamma'^{c}_l). \end{aligned} \tag{4.15}$$

At a configuration $\sigma \in \{\sigma(0) = +1\} \cap \Gamma'^{c}_l$, the origin is enclosed by a (+)-contour γ with $|\Theta(\gamma)| < 9l^d/4d$. Therefore, we have by Lemma 4.2 with $\rho' = 9/4$ that

$$\Delta_\gamma H^{\phi}_{\Lambda(l,d)}(\sigma) \geq \varepsilon|\gamma| \qquad \text{for any} \quad l \geq l_0. \tag{4.16}$$

By this and the standard Peierls' argument, we have that for any $l \geq l_0$

(4.17)
$$\begin{aligned}\mu^{\phi}_{\Lambda(l,d)}(\{\sigma(0)=+1\}\cap \Gamma_l'^{c}) &\leq \sum_{\gamma} \mu^{\phi}_{\Lambda(l,d)}\{\gamma \text{ appears and } \Delta_\gamma H^{\phi}_{\Lambda(l,d)}(\sigma) \geq \varepsilon|\gamma|\} \\ &\leq \sum_{\gamma} \exp(-\beta\varepsilon|\gamma|),\end{aligned}$$

where $\sum_\gamma$ stands for the summation over all contours γ with $\Theta(\gamma) \ni 0$. By using the counting inequality (1.9), it is not difficult to see that for sufficiently large β

$$\lim_{\beta \nearrow \infty} \sum_{\gamma} \exp(-\beta\varepsilon|\gamma|) = 0,$$

which together with (4.15) and (4.17) implies Lemma 2.1. □

Remark 4.3. At a configuration $\sigma \in \{\sigma(0) = +1\} \cap \Gamma_l^c$, the origin is enclosed by a (+)-contour γ with $|\gamma| < 9l^{d-1}/2$, which does not necessarily imply that the estimate

$$\Delta_\gamma H^{\omega}_{\Lambda(l,d)}(\sigma) \geq \varepsilon|\gamma|$$

holds. Therefore, we replace the boundary conditions ω with ϕ by using FKG inequality for $\Gamma_l' \subset \Gamma_l$. For this reason, boundary conditions ω are restricted to ones belonging to $\Omega^+_{\text{b.c.}}$.

5. Proof of Lemmas 2.2 and 2.3

For $\alpha \in (0,1]$ and $k \in (0,6)$, we say that the condition $\mathrm{P}(k,l,\alpha;d)$ holds if it holds that

$$\frac{|\gamma \cap \mathrm{int}\Lambda(l,d)|}{|\gamma|} \geq \frac{\alpha}{d} \qquad \text{for any} \quad \gamma \in \mathrm{C}(l,d) \quad \text{with} \quad |\gamma| < kl^{d-1}.$$

Lemma 5.1. *If* $\mathrm{P}(k,l,\alpha;d)$ *holds, then* $\mathrm{P}(k,l,\alpha \wedge d/(2(d+1));d+1)$ *holds.*

Proof. Since we have only to show that

$$\frac{|\gamma \cap \mathrm{int}\Lambda(l,d+1)|}{|\gamma|} \geq \frac{1}{d+1}\Big(\alpha \wedge \frac{d}{2(d+1)}\Big) \tag{5.1}$$

for any $\gamma \in \mathrm{C}(l,d+1)$ with $|\gamma \cap \mathrm{int}\Lambda(l,d+1)|/|\gamma| < 1/2$, we can suppose by (3.1) and (3.3) that $\gamma \in \mathrm{S}(l,d+1)$ with $|\gamma| < kl^d$. Note that

$$|S_{-i_m}(\gamma)| < kl^d/2(d+1). \tag{5.2}$$

For any integer $t \in (-l/2, l/2]$, we consider $\gamma'(t) = \gamma \cap \mathrm{H}(t)$, $Q'(t) = Q(\Theta(\gamma)) \cap \mathrm{H}(t)$ in the same way as in the proof of Lemma 4.2. We can define $S_{+j}(\gamma'(t))$

for any $j \leq d+1$, $j \neq i_m$ in the same way as the definitions of $S_{+j}(\gamma')$ for $\gamma' \in \mathrm{S}(l,d)$. If $|\gamma'(t)| < kl^{d-1}$, we have by the hypothesis of the induction that

$$\frac{|\gamma'(t) \cap \mathrm{int}\Lambda(l,d+1)|}{|\gamma'(t)|} \geq \frac{\alpha}{d}. \tag{5.3}$$

Otherwise, from (5.2) we can suppose that $|\gamma'(t)| \geq kl^{d-1}$ and $|Q'(t)| < kl^d/2(d+1)$. Then, we can see that

$$\sum_{j \neq i_m} |S_{+j}(\gamma'(t))| < kdl^{d-1}/2(d+1).$$

Therefore, we have that

$$\begin{aligned} \frac{|\gamma'(t) \cap \mathrm{int}\Lambda(l,d+1)|}{|\gamma'(t)|} &\geq \frac{1}{|\gamma'(t)|}\Big(\frac{|\gamma'(t)|}{2} - \sum_{j \neq i_m} |S_{+j}(\gamma'(t))|\Big) \\ &\geq \frac{1}{2} - \frac{d}{2(d+1)} = \frac{1}{2(d+1)}. \end{aligned} \tag{5.4}$$

From (4.2), (5.2), (5.3) and (5.4), we can conclude that

$$\begin{aligned} \frac{|\gamma \cap \mathrm{int}\Lambda(l,d+1)|}{|\gamma|} &\geq \frac{1}{|\gamma|}\left((|\gamma| - 2|S_{-i_m}(\gamma)|) \cdot \left(\frac{\alpha}{d} \wedge \frac{1}{2(d+1)}\right)\right) \\ &= \frac{1}{d+1}\Big(\alpha \wedge \frac{d}{2(d+1)}\Big), \end{aligned}$$

which implies that $\mathrm{P}(k,l,\alpha \wedge d/(2(d+1)); d+1)$ holds. □

Lemma 5.2. *Let $\gamma \in \mathrm{C}(l,3)$ be a contour with $|\gamma| < 9l^2/2$. Then, it holds that*

$$\frac{|\gamma \cap \mathrm{int}\Lambda(l,3)|}{|\gamma|} \geq \frac{1}{6}. \tag{5.5}$$

Proof. Let $\rho \in (0,1)$. Since we have only to show that (5.5) holds for any $\gamma \in \mathrm{C}(l,3)$ with $|\gamma \cap \mathrm{int}\Lambda(l,3)|/|\gamma| < 1/2$, we can suppose by (3.1) and (3.3) that $\gamma \in \mathrm{S}(l,3)$ with $|\gamma| < 6(1-\rho)l^2$. Then, we have from (4.2) that $|S_{-i_m}(\gamma)| < (1-\rho)l^2$. For any integer $t \in (-l/2, l/2]$, we consider $\gamma'(t) = \gamma \cap \mathrm{H}(t)$ and $Q'(t) = Q(\Theta(\gamma)) \cap \mathrm{H}(t)$ in the same way as in the proof of Lemma 4.2. We can see that

$$\frac{|\gamma'(t) \cap \mathrm{int}\Lambda(l,3)|}{|\gamma'(t)|} \geq \min\left\{\frac{1}{4}, \frac{\sqrt{\rho}}{2}\right\}, \tag{5.6}$$

since we have that $|Q'(t)| < (1-\rho)l^2$. Therefore, we have from (4.2) that

$$\begin{aligned} \frac{|\gamma \cap \mathrm{int}\Lambda(l,3)|}{|\gamma|} &\geq \frac{1}{|\gamma|}\left((|\gamma| - 2|S_{-i_m}(\gamma)|) \cdot \min\left\{\frac{1}{4}, \frac{\sqrt{\rho}}{2}\right\}\right) \\ &\geq \frac{2}{3} \cdot \min\left\{\frac{1}{4}, \frac{\sqrt{\rho}}{2}\right\}, \end{aligned}$$

which with $\rho = 1/4$ implies (5.5). □

Corollary 5.3. *Let $d \geq 3$ and $\gamma \in \mathrm{C}(l,d)$ be a contour at $\sigma \in \Omega_{\Lambda(l,d)}$ with $|\gamma| < 9l^{d-1}/2$. Then, it holds that*

$$\Delta_\gamma H^\omega_{\Lambda(l,d)}(\sigma) \geq 2\Big(\frac{3}{8d}|\gamma| - |\partial_{ex}\Theta(\gamma) \cap F_l^+(\omega)|\Big). \tag{5.7}$$

Proof. Since $\mathrm{P}(9/2, l, 1/2; 3)$ holds from (5.5), $\mathrm{P}(9/2, l, 3/8; d)$ holds for any $d \geq 3$ by Lemma 5.1. From this and (4.1), we can obtain (5.7). □

Corollary 5.4. *Let $\gamma \in \mathrm{C}(l,3)$ be a contour at $\sigma \in \Omega_{\Lambda(l,3)}$ with $|\gamma| < 9l^2/2$. Then, it holds that*

$$\Delta_\gamma H^\omega_{\Lambda(l,d)}(\sigma) \geq 2\Big(\frac{1}{6}|\gamma| - |\partial_{ex}\Theta(\gamma) \cap F_l^+(\omega)|\Big). \tag{5.8}$$

Proof. (5.8) follows from (4.1) and (5.5). □

Proof of Lemma 2.2. We will prove Lemma 2.2 in the following four steps for sufficiently large l. Before we proceed to the first step, we will introduce definitions and notations. For $m \leq 3^d - 1$, let $\underline{\gamma} = \{\gamma_i\}_{i=1}^m$ be a set of l_∞-contours in $\Lambda(l,d)$ such that γ_i and γ_j have no common $(d-1)$-dimensional unit cells for $i \neq j$. We set

$$\Delta_{\underline{\gamma}} H^\omega_{\Lambda(l,d)}(\sigma) = H^\omega_{\Lambda(l,d)}(\sigma) - H^\omega_{\Lambda(l,d)}(T_{\gamma_1} \circ \cdots \circ T_{\gamma_m}\sigma),$$

where the map $T_{\gamma_i} : \Omega_{\Lambda(l,d)} \longrightarrow \Omega_{\Lambda(l,d)}$ for each $i \leq m$ is defined by

$$T_{\gamma_i}\sigma(x) = \begin{cases} -\sigma(x) & \text{if} \quad x \in \Theta(\gamma_i), \\ \sigma(x) & \text{if} \quad x \notin \Theta(\gamma_i). \end{cases}$$

For a $(+)$-contour γ at σ, we define the map $T_\gamma^{\mathrm{cluster}} : \Omega_{\Lambda(l,d)} \longrightarrow \Omega_{\Lambda(l,d)}$ by

$$T_\gamma^{\mathrm{cluster}}\sigma(x) = \begin{cases} -\sigma(x) & \text{if} \quad x \in C^+(\sigma,\gamma), \\ \sigma(x) & \text{if} \quad x \notin C^+(\sigma,\gamma). \end{cases}$$

First, we will prove that if $\sigma \in \Gamma_l$, $\sigma^x \notin \Gamma_l$ for some $x \in \Lambda(l,d)$ and $\sigma(x) = +1$, then there exist a $(+)$-contour $\gamma \in \mathrm{C}(l,d)$ at σ with $|\gamma| \geq 9l^{d-1}/2$ and at most $(3^d - 2)$ l_∞-contours $\{\alpha_i\}_{i=1}^m \subset \overline{\mathrm{C}}(l,d)$ (we understand that $\{\alpha\}_{i=1}^0 = \phi$) such that

$$C_l(\sigma) = \{\gamma\} \qquad \text{and} \qquad Q(\gamma) \ni x, \tag{5.9}$$

$$Q(\alpha_i) \ni x \qquad \text{for any} \quad i \leq m, \tag{5.10}$$

$$\Delta_{\underline{\gamma}} H^\omega_{\Lambda(l,d)}(\sigma) \geq \varepsilon|\underline{\gamma}| = \varepsilon\left(|\gamma| + \sum_{i=1}^m |\alpha_i|\right), \tag{5.11}$$

where $\varepsilon = \varepsilon(\delta,d) > 0$ and $\underline{\gamma} = \{\gamma\} \cup \{\alpha_i\}_{i=1}^m$. (5.9), (5.10) and (5.11) can be seen as follows. Flipping $\sigma(x)$ to -1 does not change the shapes of $(+)$-clusters at σ not including x. Thus, the case where the transition from $\sigma \in \Gamma_l$

to $\sigma^x \notin \Gamma_l$ occurs is the one where there exists the unique (+)-contour γ such that $\{\gamma\} = C_l(\sigma)$ and the flipping of $\sigma(x)$ shortens γ or makes γ break into contours which do not belong to $C_l(\sigma^x)$. This is possible only when (5.9) is satisfied. Let $\{C_i\}_{i=1}^n$ be the decomposition of $C_x^+(\sigma)\backslash\{x\}$ into l_1-connected components. Then, there exist (+)-contours $\{\gamma_i'\}_{i=1}^n \subset \mathrm{C}(l,d)$ at σ^x such that $C_i = C^+(\sigma^x, \gamma_i')$ for each $i \leq n$. Since $\sigma^x \notin \Gamma_l$, we can see that $|\gamma_i'| < 9l^{d-1}/2$ for any $i \leq n$. Note that

(5.12)
$$\{b \in \mathbf{B}; b \in \partial C_x^+(\sigma)\} \ominus \{b \in \mathbf{B}; b \in \partial C_i \text{ for some } i \leq n\} \subset \{b_{\pm i}(x)\}_{i=1}^d,$$

where $\ominus$ stands for the symmetric difference of two sets. Let $\{\eta_i\}$ and $\{\xi_i\}$ be the decomposition of $\{\partial Q(C_x^+(\sigma))\}$ and $\{\partial Q(C_i)\}_{i=1}^n$ into connected components, respectively. Then, we can show that

$$\{\eta_i; \eta_i \cap \partial Q(x) = \phi\} = \{\xi_i\}\backslash\{\gamma_i'\}_{i=1}^n \tag{5.13}$$

as follows. Let us suppose that $\eta \in \{\eta_i; \eta_i \cap \partial Q(x) = \phi\}$. For any $b^* \in \eta$, there exist $u, v \in \mathbf{Z}^d$ such that $b = b(u,v),\ u \in C_x(\sigma),\ \sigma(v) = -1$ and $\|v - x\|_\infty \geq 2$. Then, there exists $\xi \in \{\xi_i\}\backslash\{\gamma_i'\}_{i=1}^n$ such that $b^* \in \xi$. Therefore, we have from the definitions of $\{\eta_i\}$ and $\{\xi_i\}$ that

$$\{\eta_i; \eta_i \cap \partial Q(x) = \phi\} \subset \{\xi_i\}\backslash\{\gamma_i'\}_{i=1}^n. \tag{5.14}$$

Let us suppose that $\xi \in \{\xi_i\}\backslash\{\gamma_i'\}_{i=1}^n$. For any $b^* \in \xi$, there exist $u, v \in \mathbf{Z}^d$ such that $b = b(u,v),\ u \in C_i$ for some $i \leq n,\ \sigma^x(v) = -1$ and $\|v - x\|_\infty \geq 2$. Then, there exists $\eta \in \{\eta_i; \eta_i \cap Q(x) = \phi\}$ such that $b^* \in \eta$. Therefore, we also have that

$$\{\eta_i; \eta_i \cap \partial Q(x) = \phi\} \supset \{\xi_i\}\backslash\{\gamma_i'\}_{i=1}^n. \tag{5.15}$$

From (5.14) and (5.15), we can conclude that (5.13) holds. Therefore, putting $\underline{\gamma} = \{\eta_i; \eta_i \cap Q(x) \neq \phi\}$, we have that

$$\begin{aligned}
&H^\omega_{\Lambda(l,d)}(T^{\text{cluster}}_\gamma \sigma) - H^\omega_{\Lambda(l,d)}(T_{\underline{\gamma}}\sigma) \\
&= H^\omega_{\Lambda(l,d)}(T^{\text{cluster}}_{\gamma_1'} \circ \cdots \circ T^{\text{cluster}}_{\gamma_n'} \sigma^x) - H^\omega_{\Lambda(l,d)}(T_{\gamma_1'} \circ \cdots \circ T_{\gamma_n'} \sigma^x),
\end{aligned}$$

which implies that

$$\begin{aligned}
&H^\omega_{\Lambda(l,d)}(\sigma) - H^\omega_{\Lambda(l,d)}(T_{\underline{\gamma}}\sigma) \\
&= H^\omega_{\Lambda(l,d)}(\sigma) - H^\omega_{\Lambda(l,d)}(\sigma^x) + H^\omega_{\Lambda(l,d)}(T^{\text{cluster}}_\gamma \sigma) - H^\omega_{\Lambda(l,d)}(T_{\underline{\gamma}}\sigma) \\
&\quad + H^\omega_{\Lambda(l,d)}(T^{\text{cluster}}_{\gamma_1'} \circ \cdots \circ T^{\text{cluster}}_{\gamma_n'} \sigma^x) - H^\omega_{\Lambda(l,d)}(T^{\text{cluster}}_\gamma \sigma) \\
&\quad + H^\omega_{\Lambda(l,d)}(T_{\gamma_1'} \circ \cdots \circ T_{\gamma_n'} \sigma^x) - H^\omega_{\Lambda(l,d)}(T^{\text{cluster}}_{\gamma_1'} \circ \cdots \circ T^{\text{cluster}}_{\gamma_n'} \sigma^x) \\
&\quad + H^\omega_{\Lambda(l,d)}(\sigma^x) - H^\omega_{\Lambda(l,d)}(T_{\gamma_1'} \circ \cdots \circ T_{\gamma_n'} \sigma^x) \\
&= H^\omega_{\Lambda(l,d)}(\sigma) - H^\omega_{\Lambda(l,d)}(\sigma^x) \\
&\quad + H^\omega_{\Lambda(l,d)}(T^{\text{cluster}}_{\gamma_1'} \circ \cdots \circ T^{\text{cluster}}_{\gamma_n'} \sigma^x) - H^\omega_{\Lambda(l,d)}(T^{\text{cluster}}_\gamma \sigma) \\
&\quad + H^\omega_{\Lambda(l,d)}(\sigma^x) - H^\omega_{\Lambda(l,d)}(T_{\gamma_1'} \circ \cdots \circ T_{\gamma_n'} \sigma^x) \\
&\geq H^\omega_{\Lambda(l,d)}(\sigma^x) - H^\omega_{\Lambda(l,d)}(T_{\gamma_1'} \circ \cdots \circ T_{\gamma_n'} \sigma^x) - 4d.
\end{aligned} \tag{5.16}$$

From (5.12) and (5.13), we also have that

$$|\underline{\gamma}| - 2d \le \sum_{i=1}^{n} |\gamma_i'| \le |\underline{\gamma}| + 2d. \tag{5.17}$$

Since each contour in $\{\gamma_i'\}_{i=1}^n$ satisfies the condition for (5.7), we have from (5.16) and (5.17) that

$$\begin{aligned} H^{\omega}_{\Lambda(l,d)}(\sigma) - H^{\omega}_{\Lambda(l,d)}(T_{\underline{\gamma}}\sigma) &\ge 2\sum_{i=1}^{n}\Big(\frac{3}{8d}|\gamma_i'| - |\partial_{ex}\Theta(\gamma_j)\cap F_l^+(\omega)|\Big) - 4d \\ &\ge 2\Big(\frac{3}{8d}(|\underline{\gamma}| - 2d) - \delta l^{d-1}\Big) - 4d \\ &\ge 2\left(\frac{3}{8d} - \frac{2}{9}\delta\right)|\underline{\gamma}| - \frac{3}{2} - 4d, \end{aligned} \tag{5.18}$$

which implies (5.10) and (5.11).

Second, we will prove that if $\sigma \in \Gamma_l$, $\sigma^x \notin \Gamma_l$ for some $x \in \Lambda(l,d)$ and $\sigma(x) = -1$, then there exist a (+)-contour $\gamma \in \mathrm{C}(l,d)$ at σ with $|\gamma| \ge 9l^{d-1}/2$ and at most $(3^d - 2)$ contours $\{\alpha_i\}_{i=1}^m \subset \mathrm{C}(l,d)$ (we understand that $\{\alpha_i\}_{i=1}^0 = \phi$) such that

$$C_l(\sigma) \ni \gamma \qquad \text{and} \qquad Q(\gamma) \ni x, \tag{5.19}$$

$$Q(\alpha_i) \ni x \qquad \text{for any} \quad i \le m, \tag{5.20}$$

$$\Delta_{\underline{\gamma}} H^{\omega}_{\Lambda(l,d)}(\sigma) \ge \varepsilon|\underline{\gamma}|, \tag{5.21}$$

where $\varepsilon = \varepsilon(\delta, d) > 0$ and $\underline{\gamma} = \{\gamma\} \cup \{\alpha_i\}_{i=1}^m$. If $Q(\gamma) \not\ni x$, flipping $\sigma(x)$ to $+1$ does not change the shape of the (+)-contour γ. Thus, we can see that (5.19) holds. (5.20) and (5.21) can be seen by similar argument in the first step. There exist (+)-contours $\{\gamma_i\}_{i=1}^n$ at σ and a (+)-contour $\gamma' \in \mathrm{C}(l,d)$ at σ^x with $|\gamma'| < 9l^{d-1}/2$ such that

$$C^+(\sigma,\gamma) \cup (\cup_{i\le n} C^+(\sigma,\gamma_i)) \cup \{x\} = C^+(\sigma^x,\gamma').$$

Hence, we can see that there exist at most $(3^d - 2)$ l_∞-contours $\{\alpha_i'\}_{i=1}^r \overline{\mathrm{C}}(l,d)$ (we understand that $\{\alpha_i'\}_{i=1}^0 = \phi$) such that

(5.22)

$$H^{\omega}_{\Lambda(l,d)}(\sigma^x) - H^{\omega}_{\Lambda(l,d)}(T_{\underline{\gamma'}}\sigma^x) \le H^{\omega}_{\Lambda(l,d)}(\sigma) - H^{\omega}_{\Lambda(l,d)}(T_{\gamma_1}\circ\cdots\circ T_{\gamma_n}\circ T_{\gamma}\sigma) + 4d,$$

where $\underline{\gamma'} = \{\gamma'\} \cup \{\alpha_i'\}_{i=1}^r$. Since $\Theta(\alpha_i') \subset \Theta(\gamma')$ for each $i \le r$, α_i' satisfies the condition for (5.7). Thus, we can prove (5.21) by the same argument in (5.18).

Third, we will prove that if $\sigma \in \Gamma_l$ and $C_l(\sigma) \ni \gamma$ for some $\gamma \in \mathrm{C}(l,d)$, then it follows that

$$T_{\underline{\gamma}}\sigma \in \Gamma_l^c \text{ or } \left\{ \begin{array}{l} C_l(T_{\underline{\gamma}}\sigma) = \{\tilde{\gamma}_i\}_{i=1}^m \text{ with } \Theta(\tilde{\gamma}_i) \subset \Lambda(l,d)\backslash\partial_{in}\Lambda(l,d) \\ \text{for any } i \le m \text{ and } T_{\tilde{\gamma}_m}\circ\cdots\circ T_{\tilde{\gamma}_1}\circ T_{\underline{\gamma}}\sigma \in \Gamma_l^c \end{array} \right., \tag{5.23}$$

where $\underline{\gamma}$ is the one defined in (5.11) or (5.21). To see (5.23), let us suppose that $T_{\underline{\gamma}}\sigma \in \Gamma_l$. Then, by $C_l(\sigma) \ni \gamma$ there exists some $\tilde{\gamma} \in C_l(T_{\underline{\gamma}}\sigma)$ such that $\Theta(\tilde{\gamma}) \subset \Theta(\underline{\gamma})$ and $\tilde{\gamma}$ is a $(-)$-contour at σ. Moreover, we can see that $T_{\tilde{\gamma_m}} \circ \cdots \circ T_{\tilde{\gamma_1}} \circ T_{\underline{\gamma}}\sigma \in \Gamma_l^c$ as follows. If γ' is a $(+)$-contour at $T_{\tilde{\gamma_m}} \circ \cdots \circ T_{\tilde{\gamma_1}} \circ T_{\underline{\gamma}}\sigma$, then we can see that γ' is a $(+)$-contour at either σ or $T_{\underline{\gamma}}\sigma$. Therefore, we can see that $|\gamma'| < 9l^{d-1}/2$. Hence, (5.23) holds.

Fourth, we will prove (2.5) to finish the proof of Lemma 2.2. From (5.9), (5.10), (5.11), (5.19), (5.20) and (5.21), we have that

(5.24)

$$\begin{aligned}
&\sum_{x\in\Lambda(l,d)} \sum_{\sigma\in\Gamma_l,\ \sigma^x\notin\Gamma_l} \mu^{\omega}_{\Lambda(l,d)}(\sigma) \\
&\le \sum_{x\in\Lambda(l,d)} \sum_{\gamma}{}^{l} 1_{\{Q(\gamma)\ni x\}} \mu^{\omega}_{\Lambda(l,d)}\{C_l(\sigma)=\{\gamma\} \text{ and } \Delta_\gamma H^{\omega}_{\Lambda(l,d)}(\sigma) \ge \varepsilon|\gamma|\} \\
&\quad + \sum_{x\in\Lambda(l,d)} \sum_{\gamma}{}^{l} \sum_{m=1}^{3^d-2} \sum_{\{\gamma_i\}_{i=1}^m} 1_{\{Q(\gamma)\ni x\}} \prod_{i=1}^m 1_{\{Q(\gamma_i)\ni x\}} \\
&\qquad \times \mu^{\omega}_{\Lambda(l,d)} \left\{ \begin{array}{l} C_l(\sigma)\ni\gamma,\ \{\gamma_i\}_{i=1}^m \subset \overline{\mathrm{C}}(l,d),\ \Delta_{\underline{\gamma}} H^{\omega}_{\Lambda(l,d)}(\sigma) \ge \varepsilon|\underline{\gamma}| \\ \text{for } \underline{\gamma} = \{\gamma\}\cup\{\gamma_i\}_{i=1}^m \text{ and } T_{\underline{\gamma}}\sigma \in \Gamma_l^c \end{array} \right\},
\end{aligned}$$

where $\sum_{\gamma}^{l}$ and $\sum_{\{\gamma_i\}_{i=1}^m}$ stand for the summation over all contours $\gamma \in \mathrm{C}(l,d)$ with $|\gamma| \ge 9l^{d-1}/2$ and the summation over all sets $\{\gamma_i\}_{i=1}^m \subset \overline{\mathrm{C}}(l,d)$, respectively. By the standard Peierls' argument and (5.23), we have for fixed $\gamma \in \mathrm{C}(l,d)$ that

(5.25)

$$\begin{aligned}
&\mu^{\omega}_{\Lambda(l,d)}\{C_l(\sigma)=\{\gamma\} \text{ and } \Delta_\gamma H^{\omega}_{\Lambda(l,d)}(\sigma) \ge \varepsilon|\gamma|\} \\
&\le e^{-\beta\varepsilon|\gamma|}\Bigg(\mu^{\omega}_{\Lambda(l,d)}(\Gamma_l^c) \\
&\qquad + \sum_{n=1}^{\infty} \sum_{\{\gamma_i\}_{i=1}^n}{}^{l} \mu^{\omega}_{\Lambda(l,d)} \left\{ \begin{array}{l} C_l(\sigma)=\{\gamma\}_{i=1}^n,\ T_{\gamma_n}\circ\cdots\circ T_{\gamma_1}\sigma \in \Gamma_l^c \text{ and} \\ \Delta_{\gamma_i} H^{\omega}_{\Lambda(l,d)}(\sigma) \ge 2|\gamma_i| \text{ for any } i \le n \end{array} \right\} \Bigg) \\
&\le \mu^{\omega}_{\Lambda(l,d)}(\Gamma_l^c) e^{-\beta\varepsilon|\gamma|} \Big(1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \sum_{\gamma_i}{}^{l} e^{-2\beta|\gamma_i|} \Big),
\end{aligned}$$

where $\sum_{\{\gamma_i\}_{i=1}^n}^{l}$ and $\sum_{\gamma_i}^{l}$ stand for the summation over all sets $\{\gamma_i\}_{i=1}^n \subset \mathrm{C}(l,d)$ such that $|\gamma_i| \ge 9l^{d-1}/2$ for any $i \le n$ and the summation over all contours $\gamma_i \in \mathrm{C}(l,d)$ with $|\gamma_i| \ge 9l^{d-1}/2$ for any $i \le n$, respectively. By using the counting inequality (1.9), it is not difficult to see that for sufficiently large β

$$\sum_{n=1}^{\infty}\prod_{i=1}^{n}\sum_{\gamma_i}{}^{l} e^{-2\beta|\gamma_i|} \le \sum_{n=1}^{\infty}\Big(B_1 l^d \kappa(d)^{2l^{d-1}} e^{-4\beta l^{d-1}}\Big)^n \le B_2 l^d \kappa(d)^{2l^{d-1}} e^{-4\beta l^{d-1}} < \infty, \tag{5.26}$$

where $B_1 = B_1(\beta, d) > 0$ and $B_2 = B_2(\beta, d) > 0$. Therefore, we have from (5.25) and (5.26) that for sufficiently large β

$$\mu^{\omega}_{\Lambda(l,d)}\{C_l(\sigma) = \{\gamma\} \text{ and } \Delta_\gamma H^{\omega}_{\Lambda(l,d)}(\sigma) \ge \varepsilon|\gamma|\} \le B_3 \mu^{\omega}_{\Lambda(l,d)}(\Gamma^c_l) e^{-\beta\varepsilon|\gamma|}, \tag{5.27}$$

where $B_3 = B_3(\beta, d) > 0$. Similarly, we have that for sufficiently large β

$$\mu^{\omega}_{\Lambda(l,d)}\left\{\begin{array}{l} C_l(\sigma) \ni \gamma,\ \{\gamma_i\}_{i=1}^m \subset \overline{\mathrm{C}}(l,d),\ \Delta_{\underline{\gamma}} H^{\omega}_{\Lambda(l,d)}(\sigma) \ge \varepsilon|\underline{\gamma}| \\ \text{for } \underline{\gamma} = \{\gamma\} \cup \{\gamma_i\}_{i=1}^m \text{ and } T_{\underline{\gamma}}\sigma \in \Gamma^c_l \end{array}\right\} \le B_3 \mu^{\omega}_{\Lambda(l,d)}(\Gamma^c_l) e^{-\beta\varepsilon|\underline{\gamma}|}. \tag{5.28}$$

From (5.24), (5.27) and (5.28), we can see that for sufficiently large β

$$\sum_{x\in\Lambda(l,d)} \sum_{\sigma\in\Gamma_l,\ \sigma^x\in\Gamma^c_l} \mu^{\omega}_{\Lambda(l,d)}(\sigma) \le 2^d B_3 \mu^{\omega}_{\Lambda(l,d)}(\Gamma^c_l)\left(\sum_{\gamma}{}^{l} |\gamma| e^{-\beta\varepsilon|\gamma|} + \sum_{\gamma}{}^{l} |\gamma| e^{-\beta\varepsilon|\gamma|} \sum_{m=1}^{3^d-2} \sum_{\{\gamma_i\}_{i=1}^m}{}' \prod_{i=1}^{m} e^{-\beta\varepsilon|\gamma_i|}\right), \tag{5.29}$$

where $\sum'_{\{\gamma_i\}_{i=1}^m}$ stands for the summation over all sets $\{\gamma_i\}_{i=1}^m \subset \overline{\mathrm{C}}(l,d)$ with $Q(\gamma_i) \ni 0$ for any $i \le m$. Then, by using the counting inequality (1.9) again we have that for sufficiently large β

$$\sum_{\gamma}{}^{l} |\gamma| e^{-\beta\varepsilon|\gamma|} \le B_4 \exp(-\beta C l^{d-1}), \tag{5.30}$$

$$\sum_{m=1}^{3^d-1} \sum_{\{\gamma_i\}_{i=1}^m}{}' \prod_{i=1}^{m} e^{-\beta\varepsilon|\gamma_i|} < \infty, \tag{5.31}$$

where $B_4 = B_4(\beta, \delta, d) > 0$ and $C = C(\delta, d) > 0$. Thus, we can conclude (2.5) from (5.29), (5.30) and (5.31). □

We can similarly prove Lemma 2.3 by using Corollary 5.4 instead of Corollary 5.3.

Appendix

In this section we will prove the following theorem.

Theorem A.1. *For $d \geq 3$, consider a stochastic Ising model on the square $\Lambda(l,d)$. Suppose that a boundary condition $\omega \in \Omega^+_{\text{b.c.}}$ is such that*

$$\limsup_{l\to\infty} \frac{|F_l^+(\omega)|}{l^{d-1}} < \delta < 3(d-1)2^{-d}. \tag{A.1}$$

Then, there exists $\beta_0'' = \beta_0''(\delta,d) > 0$ such that (1.7) *holds for any $\beta \geq \beta_0''$ and any $l \in \mathbf{N}$. Especially, if there exists some $\delta \in [0, 3(d-1)2^{-2})$ such that $|F_l^+(\omega)| \leq \delta l^{d-1}$ for any $l \in \mathbf{N}$, then we can take B in* (1.7) *as a constant independent of ω.*

Note that (A.1) is a better bound than (1.6) if $d = 4$ or 5. The proof of Theorem A.1 is similar to that of Theorem 1.1. We replace definitions of $C_l(\sigma)$, Γ_l and Γ_l' with

$$\begin{aligned} C_l(\sigma) &= \{\gamma; \gamma \text{ is a } (+)\text{-contour in } \Lambda(l,d) \text{ at } \sigma \text{ with } |\gamma| \geq 3d\cdot 2^{-(d-2)}l^{d-1}\},\\ \Gamma_l &= \{\sigma \in \Omega_{\Lambda(l,d)}; C_l(\sigma) \neq \phi\},\\ \Gamma_l' &= \{\sigma \in \Omega_{\Lambda(l,d)}; \text{there exists some } (+)\text{-contour } \gamma \text{ at } \sigma \text{ with } |\Theta(\gamma)|\\ &\qquad \geq 3\cdot 2^{-(d-1)}l^d\}. \end{aligned}$$

Since we can still use Lemma 4.2, we can obtain the same estimate in (2.4). We use the following lemma instead of Lemma 2.2.

Lemma A.2. *Suppose that a boundary condition $\omega \in \Omega^+_{\Lambda(l,d)}$ satisfies*

$$\limsup_{l\to\infty} \frac{|F_l^+(\omega)|}{l^{d-1}} < \delta < 3(d-1)2^{-d}. \tag{A.2}$$

Then, there exist $\beta_4 = \beta_4(\delta,d) > 0$ and $l_4 = l_4(\omega,\delta,d) > 0$ such that (2.5) *holds for any $\beta \geq \beta_4$ and any $l \geq l_4$.*

Proof. We assume that for $\gamma \in \mathrm{C}(l,d)$ with $|\gamma| < a_d l^{d-1}$,

$$\frac{|\gamma \cap \mathrm{int}\Lambda(l,d)|}{|\gamma|} \geq \lambda_d. \tag{A.3}$$

Let $\gamma \in \mathrm{S}(l,d+1)$ with $|\gamma| < a_{d+1}l^d$. For any integer $t \in (-l/2, l/2]$, we can consider $\gamma'(t)$, $Q'(t)$ and $S_{+i}(\gamma'(t))$ for any $i \leq d+1$, $i \neq i_m$. If $|\gamma'(t)| < a_d l^{d-1}$, then we have from (A.3) that

$$\frac{|\gamma'(t) \cap \mathrm{int}\Lambda(l,d+1)|}{|\gamma'(t)|} \geq \lambda_d. \tag{A.4}$$

Otherwise, we can suppose that $|\gamma'(t)| \geq a_d l^{d-1}$ and from (4.2) that $|Q'(t)| < a_{d+1}l^d/2(d+1)$. Then, we have by $|S_{+i}(\gamma'(t))| < a_{d+1}l^{d-1}/2(d+1)$ for any $i \leq d+1$, $i \neq i_m$ that

$$\begin{aligned} \frac{|\gamma'(t) \cap \mathrm{int}\Lambda(l,d+1)|}{|\gamma'(t)|} &= \frac{1}{|\gamma'(t)|}\left(\frac{|\gamma'(t)|}{2} - \sum_{i\neq i_m} |S_{+i}(\gamma'(t))|\right)\\ &\geq \frac{1}{2} - \frac{da_{d+1}}{2(d+1)a_d}. \end{aligned} \tag{A.5}$$

Therefore, from (4.2), (A.4) and (A.5), we have that

$$\frac{|\gamma \cap \mathrm{int}\Lambda(l, d+1)|}{|\gamma|} = \frac{d}{d+1} \min\Big\{\lambda_d, \frac{1}{2} - \frac{d a_{d+1}}{2(d+1)a_d}\Big\}, \tag{A.6}$$

from which we can obtain an estimate like (5.11). Thus, we can prove Lemma A.2 in the similar way to that of the proof of Lemma 2.2. □

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References

[AY99] K. S. Alexander and N. Yoshida, The spectral gap of the 2-D stochastic Ising model with mixed boundary conditions, J. Stat. Phys, **104** (2001), to appear.

[CGMS96] F. Cesi, G. Guadagni, F. Martinelli and R. H. Schonmann, On the Two-Dimensional Stochastic Ising Model in the Phase Coexistence Region Near the Critical Point, J. Stat. Phys., **85** (1996), 55–102.

[Gri89] G. Grimmett, Percolation, Springer-Verlag, 1989.

[HY97] Y. Higuchi and N. Yoshida, Slow relaxation of 2-D stochastic Ising models with random and non-random boundary conditions, New trends in stochastic analysis (K. D. Elworthy, S. Kusuoka and I. Shigekawa ed.), World Scientific Publishing, 1997, 153–167.

[Mar94] F. Martinelli, On the two dimensional dynamical Ising model in the phase coexistence region, J. Stat. Phys., **76** (1994), 1179–1246.

[Sch94] R. H. Schonmann, Slow droplet-driven relaxation of stochastic Ising Models in the vicinity of the phase coexistence region, Commun. Math. Phys., **161** (1994), 1–49.

[Tho89] L. E. Thomas, Bound on the mass gap for finite volume stochastic Ising models at low temperature, Commun. Math. Phys., **126** (1989), 1–11.