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Spaces of polynomials with real roots of bounded multiplicity

By

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1. Introduction

The principal motivation for this paper derives from work of V. A. Vassiliev [16], [17], [18] and [19]. He describes a general method for calculating the cohomology of certain spaces of discriminants. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we denote by $\mathrm{P}^d_n(\mathbb{K})$ the space consisting of all monic polynomials

$$f(z) = z^d + a_1 z^{d-1} + \cdots + a_{d-1}z + a_d \in \mathbb{K}[z]$$

of degree d which have no n-fold real roots (but may have complex ones of arbitrary multiplicity!). As his typical example, he takes the space $\mathrm{P}^d_n(\mathbb{R})$ and in particular, he computes the cohomology of the space $\mathrm{P}^d_n(\mathbb{R})$.

Theorem 1.1 (Vassiliev [16] and [17]). *If* $n \geq 3$,

$$H^j(\mathrm{P}^d_n(\mathbb{R}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if} \quad j = k(n-2), \quad 0 \leq k \leq [d/n], \\ 0 & \text{otherwise}, \end{cases}$$

where $[x]$ *denotes the integer part of a real number* x.

There is a "jet map" $j^d_n = j^d_{n;\mathbb{R}} : \mathrm{P}^d_n(\mathbb{R}) \to \Omega\mathbb{R}\,\mathrm{P}^{n-1}$ given by

$$j^d_n(f)(t) = \begin{cases} [f(t) : f'(t) : f''(t) : \cdots : f^{(n-1)}(t)] & \text{if} \quad t \in \mathbb{R}, \\ [1 : 0 : 0 : \cdots : 0] & \text{if} \quad t = \infty, \end{cases}$$

for $f \in \mathrm{P}^d_n(\mathbb{R})$ and $t \in S^1 = \mathbb{R} \cup \infty$.

If $n \geq 3$ and $k \in \mathbb{Z}/2 = \pi_0(\Omega\mathbb{R}\,\mathrm{P}^{n-1})$, we denote by $\Omega_k\mathbb{R}\,\mathrm{P}^{n-1}$ the space consisting of all base point preserving maps $f : S^1 \to \mathbb{R}\,\mathrm{P}^{n-1}$ with $[f] = k$. Remark that $j^d_n(\mathrm{P}^d_n(\mathbb{R})) \subset \Omega_{[d]_2}\mathbb{R}\,\mathrm{P}^{n-1}$, where $[d]_2$ denotes the number d mod 2. So it is regarded as the map $j^d_n : \mathrm{P}^d_n(\mathbb{R}) \to \Omega_{[d]_2}\mathbb{R}\,\mathrm{P}^{n-1} \simeq \Omega S^{n-1}$. For $\mathbb{K} = \mathbb{C}$, we can also define the jet map $j^d_{n;\mathbb{C}} : \mathrm{P}^d_n(\mathbb{C}) \to \Omega_{[d]_2}(\mathbb{C}^n - \{0\})/\mathbb{R}^* \simeq \Omega S^{2n-1}$ in a similar way.

Vassiliev also obtains the following result.

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Theorem 1.2 (Vassiliev [16], [17], [18] and [19]).

(1) *Let* $s_d : \mathrm{P}^d_n(\mathbb{R}) \to \mathrm{P}^{d+1}_n(\mathbb{R})$ *denote the stabilization map given by adding point from the edge. Then for any* $n \geq 3$*, the induced map*

$$(s_d)_* : H_k(\mathrm{P}^d_n(\mathbb{R}), \mathbb{Z}) \xrightarrow{\cong} H_k(\mathrm{P}^{d+1}_n(\mathbb{R}), \mathbb{Z})$$

is an isomorphism for any $k \leq ([d/n]+1)(n-2)-1$.

(2) *If* $n \geq 4$*, the jet map* $j^d_n : \mathrm{P}^d_n(\mathbb{R}) \to \Omega S^{n-1}$ *is a homotopy equivalence up to dimension* $([d/n]+1)(n-2)-1$.

(3) *If* $n = 3$, $j^d_3 : \mathrm{P}^d_3(\mathbb{R}) \to \Omega S^2$ *is a homology equivalence up to dimension* $[d/3]$.

(4) *If* $n = 3$, $(j^d_n)_* : \pi_k(\mathrm{P}^d_3(\mathbb{R})) \to \pi_k(\Omega S^2)$ *is an isomorphism when* $k = 1$ *and* $d \geq 3$*, and it is a surjection when* $k = 2$ *and* $d \geq 6$.

Remark. We shall call a map $f : X \to Y$ is a *homotopy equivalence* (resp. *homology equivalence*) up to dimension N if the induced homomorphism $f_* : \pi_j(X) \to \pi_j(Y)$ (resp. $f_* : H_j(X, \mathbb{Z}) \to H_j(Y, \mathbb{Z})$) is bijective when $j < N$ and surjective when $j = N$.

A. Kozlowski and the author studied the homotopy type of spaces $\mathrm{P}^d_n(\mathbb{K})$ and classified their homotopy types explicitely in [11] except for the case $(\mathbb{K}, n) = (\mathbb{R}, 3)$. In this paper we hope to study the homotopy type of $\mathrm{P}^d_3(\mathbb{R})$.

Remark that the conjugation on $\mathbb{C}$ induces the $\mathbb{Z}/2$-action on $\mathrm{P}^d_n(\mathbb{C})$ and its fixed points set is $\mathrm{P}^d_n(\mathbb{C})^{\mathbb{Z}/2} = \mathrm{P}^d_n(\mathbb{R})$. Similarly, if we identify $S^{2n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{k=1}^n |z_k|^2 = 1\}$, we also regard S^{2n-1} as $\mathbb{Z}/2$-space whose action is given by $(z_1, \ldots, z_n) \to (\overline{z}_1, \ldots, \overline{z}_n)$. If we regard S^1 as trivial $\mathbb{Z}/2$-space, the jet map $j^d_{n;\mathbb{C}}$ may be considered as $\mathbb{Z}/2$-equivariant map. In this paper, we shall prove the following 2 results.

Theorem A. *The jet map induces a* $\mathbb{Z}/2$*-equivariant homotopy equivalence*

$$j^\infty_{3,\mathbb{C}} : \lim_{d\to\infty} \mathrm{P}^d_3(\mathbb{C}) \xrightarrow{\simeq} \Omega S^5,$$

where the limit is taken from stabilization maps $\mathrm{P}^d_3(\mathbb{C}) \to \mathrm{P}^{d+1}_3(\mathbb{C})$ *given by a adding point from the edge.*

Theorem B. *The jet map* $j^d_3 : \mathrm{P}^d_3(\mathbb{R}) \to \Omega S^2$ *is a homotopy equivalence up to dimension* $[d/3]$.

The motivation of this paper is as follows. For a complex projective variety $X \subseteqq \mathbb{C}\mathrm{P}^N$, let $\mathrm{Hol}^*_D(S^2, X)$ be the space consisting of all based holomorphic maps $f : S^2 \to X$ with $[f] = D \in \pi_2(X)$. G. Segal studies the topology of the space $\mathrm{Hol}^*_d(S^2, X)$ for $X = \mathbb{C}\mathrm{P}^N$ in [14] and shows that $\mathrm{Hol}_d(S^2, \mathbb{C}\mathrm{P}^N)$ is a finite dimensional model for the infinite dimensional space $\Omega^2\mathbb{C}\mathrm{P}^N$. This result is generalized for more wide several another complex varieties (e.g. [7], [9] and [10]). Recently R. Cohen, J. Jones and G. Segal ([3] and [4]) investigate the Floer homotopy type which is the inverse system of spectra derived from

Floer functions, and they obtain loop space models using the classifying spaces of flow categories. Both results suggest that Morse theoretic principle should hold for wide infinite dimensional manifolds and in particular, it should hold for infinite loop spaces $\Omega^k X$ with $k = 1, 2$. The author tries to obtain another finite dimensional model of ΩX for $X = \mathbb{K}\mathrm{P}^n$. This attempt is already well studied in [7], [11] and [20] except for the case $(\mathbb{K}, n) = (\mathbb{R}, 3)$ and we shall treat this case here.

The similar result of Theorem A was first proved in [7] except the case $(\mathbb{K}, n) = (\mathbb{R}, 3)$. For studying this case, we shall study the fundamental group $\pi_1(\mathrm{P}^d_3(\mathbb{R}))$ more carefully. The topology of $\mathrm{P}^d_n(\mathbb{R})$ was first investigated by Vassiliev well [16]. However, its proof has a gap for the case $n = 3$. He asserted there that (3) and (4) of Theorem 1.2 would imply Theorem B using Whitehead theorem. But it is not sufficient and in fact, he also admit this gap in [17]. So we shall give its corrected proof here.

This paper is organized as follows. In Section 2, we compute the fundamental group $\pi_1(\mathrm{P}^d_3(\mathbb{R}))$ and prove Theorem A. In Section 3, we consider the subspace $\mathrm{P}^d_3 \subset \mathrm{P}^d_3(\mathbb{R})$ and study the retraction map $R^d : \mathrm{P}^d_3 \to \mathbb{C}^*$ for an odd integer $d \geq 3$, which is the keypoint of the proof of Theorem B. Finally in Section 4, we study the universal covering of P^d_3 and prove Theorem B.

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2. Fundamental group and stability result

Lemma 2.1. *If $d \geq 3$, the stabilization map induces an isomorphism*

$$(s_d)_* : \pi_1(\mathrm{P}^d_3(\mathbb{R})) \xrightarrow{\cong} \pi_1(\mathrm{P}^{d+1}_3(\mathbb{R})).$$

In particlular, since $\mathrm{P}^3_3(\mathbb{R}) \simeq S^1$, $\pi_1(\mathrm{P}^d_3(\mathbb{R})) = \mathbb{Z}$ if $d \geq 3$.

Proof. Although the assertion easily follows from (1.2), we give another easy proof. Using a similar method given in appendix of [6], we can see that $\pi_1(\mathrm{P}^d_3(\mathbb{R}))$ is abelian. Consider the commutative diagram

$$\begin{array}{ccc}
\pi_1(\mathrm{P}^d_3(\mathbb{R})) & \xrightarrow{(s_d)_*} & \pi_1(\mathrm{P}^{d+1}_3(\mathbb{R})) \\
\cong\big\downarrow & & \cong\big\downarrow \\
H_1(\mathrm{P}^d_3(\mathbb{R}), \mathbb{Z}) & \xrightarrow[\cong]{(s_d)_\#} & H_1(\mathrm{P}^{d+1}_3(\mathbb{R}), \mathbb{Z}) \cong \mathbb{Z}.
\end{array}$$

It follows from (1.2) that $(s_d)_\#$ is bijective. Hence the assertions easily follow. $\square$

Consider the map

$$j_3^\infty = \lim_d j_3^d : \mathrm{P}_3^\infty(\mathbb{R}) = \lim_{d\to\infty} \mathrm{P}_3^d(\mathbb{R}) \to \lim_{d\to\infty} \Omega_{[d]_2}\mathbb{R}\,\mathrm{P}^2 \simeq \Omega S^2, \tag{2.2}$$

where the limit is taken by stabilization maps s_d. Then we obtain

Theorem 2.3. *The map $j_3^\infty : \mathrm{P}_3^\infty(\mathbb{R}) \xrightarrow{\simeq} \Omega S^2$ is a homotopy equivalence.*

Proof. Note that $\pi_1(\mathrm{P}_3^d(\mathbb{R})) = \mathbb{Z}$ and it is abelian. Hence the proof given in (3.3) of [7] equally works for the case $n = 3$ and the detail is omitted. $\square$

Now we can give the proof of Theorem A.

Proof of Theorem A. Consider the $\mathbb{Z}/2$-equivariant map

$$j_{3;\mathbb{C}}^\infty = \lim_d j_{3;\mathbb{C}}^d : \lim_{d\to\infty} \mathrm{P}_3^d(\mathbb{C}) \to \Omega S^5.$$

It follows from (3.5) of [7] that $j_{3;\mathbb{C}}^\infty$ is a homotopy equivalence. Remark that the restriction of $j_{3;\mathbb{C}}^\infty$ to the $\mathbb{Z}/2$-fixed point sets is just the map $(j_{3;\mathbb{C}}^\infty)^{\mathbb{Z}/2} = j_3^\infty$. So it is also a homotopy equivalence. Hence for any subgroup $H \subsetneqq \mathbb{Z}/2$, $(j_{3;\mathbb{C}}^\infty)^H$ is a homotopy equivalence. Then using equivariant Whitehead theorem, $j_{3;\mathbb{C}}^\infty$ is a $\mathbb{Z}/2$-equivariant homotopy equivalence. $\square$

Since we know that $j_{n;\mathbb{C}}^\infty$ is $\mathbb{Z}/2$-equivariant homotopy equivalence by (3.7) of [7] for $n \geq 4$, we also obtain the following.

Corollary 2.4. *For any $n \geq 3$, the induced map*

$$j_{n;\mathbb{C}}^\infty = \lim_d j_{n;\mathbb{C}}^d : \lim_{d\to\infty} \mathrm{P}_n^d(\mathbb{C}) \xrightarrow{\simeq} \Omega S^{2n-1}$$

is a $\mathbb{Z}/2$-equivarinat homotopy equivalence.

Let $\overline{\mathbb{H}} = \{w = x + \sqrt{-1}y \in \mathbb{C} : x \in \mathbb{R}, y \geq 0\}$ denote closed upper half space, let $\mathbb{H}_+, \mathbb{H}_-, \mathbb{H} \subset \overline{\mathbb{H}}$ be subspaces

$$\begin{cases} \overline{\mathbb{H}}_+ = \{w = x + \sqrt{-1}y \in \mathbb{C} : x > 0, y \geq 0\}, \\ \overline{\mathbb{H}}_- = \{w = x + \sqrt{-1}y \in \mathbb{C} : x < 0, y \geq 0\}, \\ \mathbb{H} = \{w = x + \sqrt{-1}y \in \mathbb{C} : x \in \mathbb{R}, y > 0\}, \end{cases}$$

and $\phi_+ : \overline{\mathbb{H}} \xrightarrow{\cong} \overline{\mathbb{H}}_+$, $\phi_- : \overline{\mathbb{H}} \xrightarrow{\cong} \overline{\mathbb{H}}_-$ be fixed homeomorphisms. Then define the map $\mu_{s,t} : \mathrm{P}_3^s(\mathbb{R}) \times \mathrm{P}_3^t(\mathbb{R}) \to \mathrm{P}_3^{s+t}(\mathbb{R})$ by

$$\mu_{s,t}(f, g) = \prod_{k=1}^{s}(z - \phi_-(\alpha_k)) \cdot \prod_{k=1}^{t}(z - \phi_+(\beta_k)) \tag{2.5}$$

for $(f, g) = (\prod_{k=1}^s (z - \alpha_k), \prod_{k=1}^t (z - \beta_k)) \in \mathrm{P}_3^s(\mathbb{R}) \times \mathrm{P}_3^t(\mathbb{R})$.

Let $\mathrm{P}_3^0(\mathbb{R}) = \{*\}$. Then $\{\mu_{s,t} : s, t \geq 0\}$ induces the monoid structure on $\coprod_{d\geq 0} \mathrm{P}_3^d(\mathbb{R})$ and it is easy to see the following result using the group completion theorem and Theorem 2.3.

Corollary 2.6. *There is a homotopy equivalence* $\Omega B(\coprod_{d\geq 0} \mathrm{P}^d_3(\mathbb{R})) \simeq \Omega S^2 \times \mathbb{Z}$.

3. The subspace space P^d_3

Let $\mathrm{P}^d_3 \subset \mathrm{P}^d_3(\mathbb{R})$ be the subspace consisting of all monic polynomials $f(z) \in \mathrm{P}^d_3(\mathbb{R})$ of the form

$$f(z) = z^d + a_{d-2}z^{d-2} + a_{d-3}z^{d-3} + \cdots + a_2z^2 + a_1z + a_0$$

(i.e. the coefficient of $z^{d-1} = 0$). It is known that

Lemma 3.1 ([1] and [14])**.** *The inclusion* $\mathrm{P}^d_3 \overset{\subsetneq}{\to} \mathrm{P}^d_3(\mathbb{R})$ *is a deformation retract. Hence it is a homotopy equivalence.*

Hence, from now on, we shall consider the space P^d_3. For example, it is easy to see that

$$\mathrm{P}^3_3 = \{z^3 + az + b : a, b \in \mathbb{R}, (a, b) \neq (0, 0)\} \tag{3.2}$$

and there is a homeomorphism $\mathrm{P}^3_3 \overset{\cong}{\to} \mathbb{C}^*$ given by $z^3 + az + b \mapsto a + \sqrt{-1}b$.

Remark that $f(z) = z^d + az + b \in \mathbb{R}[z]$ has no 3-fold real root if and only if $(a, b) \neq (0, 0)$. Hence, for any $d \geq 3$, we can define the map $i^d : \mathbb{C}^* = \mathbb{C} - \{0\} \to \mathrm{P}^d_3$ by

$$i^d(a + \sqrt{-1}\ b) = z^d + az + b \quad (\text{for} \quad a + \sqrt{-1}\ b \in \mathbb{C}^*).$$

Assume that d is an odd integer ≥ 3 and let $f(z) \in \mathrm{P}^d_3$. Since $\deg(f'')$ is an odd integer, the polynomial $f''(z)$ can be written as the form

$$f''(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_{t-1})(z - \alpha_t) \cdot g(z),$$

where $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_t$ and the polynomial $g(z) \in \mathbb{R}[z]$ has no real roots.

In this situation, we take

$$R^d(f) = \prod_{j=1}^{t} (f'(\alpha_j) + \sqrt{-1}\ f(\alpha_j))^{\epsilon(j)} \quad (\text{where} \quad \epsilon(j) = (-1)^{j-1}).$$

Since polynomials $f(z), f'(z), f''(z)$ have no common real roots, $R^d(f) \in \mathbb{C}^*$. Moreover, if $\alpha_{j-1} = \alpha_j$,

$$(f'(\alpha_{j-1}) + \sqrt{-1}\ f(\alpha_{j-1}))^{\epsilon(j-1)} \cdot (f'(\alpha_j) + \sqrt{-1}\ f(\alpha_j))^{\epsilon(j)} = 1.$$

Hence $R^d : \mathrm{P}^d_3 \to \mathbb{C}^*$ is a continuous map.

Remark. If d is an even integer, $f''(z) \in \mathrm{P}^d_3$ does not necessarily have a real root. Hence the map R^d is well-defined only when d is odd.

Easy computations show the following 2 results.

Lemma 3.3. *$R^3 : \mathrm{P}_3^3 \xrightarrow{\cong} \mathbb{C}^*$ is a homeomorphism. Hence $\mathrm{P}_3^3(\mathbb{R}) \simeq \mathrm{P}_3^3 \simeq S^1$.*

Proof. Let $f(z) = z^3+az+b \in \mathbb{R}[z]$. Then, by (3.2), $f(z) \in \mathrm{P}_3^3$ if and only if $(a,b) \neq (0,0)$. Since $f''(z) = 6z$, it has only a real root 0. Hence if $f(z) \in \mathrm{P}_3^3$, $R^3(f) = f'(0) + \sqrt{-1}\ f(0) = a + \sqrt{-1}\ b$. Hence R^3 is a homeomorphism. □

Lemma 3.4. *If $d \geq 5$ is odd, the map $R^d : \mathrm{P}_3^d \to \mathbb{C}^*$ is a retraction. More precisely, $R^d \circ i^d = id$.*

Proof. Let $\alpha = a + \sqrt{-1}b \in \mathbb{C}^*$ and we take $i^d(\alpha) = z^d + az + b = f_\alpha$. Then, since $f''_\alpha(z) = d(d-1)z^{d-2}$ has only a root $z = 0$,

$$R^d \circ i^d(\alpha) = R^d(f_\alpha) = f'_\alpha(0) + \sqrt{-1}\ f_\alpha(0) = a + \sqrt{-1}\ b = \alpha.$$ □

Remark 3.5. For an integer $d \geq 3$, let $\phi_d : S^1 \to \mathrm{P}_3^d$ be the map given by

$$\phi_d(e^{i\theta}) = z^d + z\cos\theta + \sin\theta \quad \text{for} \quad e^{\theta\sqrt{-1}} \in S^1.$$

Then by (2.3), (2.4) and (3.2), we see that $\phi_d : S^1 \to \mathrm{P}_3^d(\mathbb{R})$ represents the generator of $\pi_1(\mathrm{P}_3^d) \cong \pi_1(\mathrm{P}_3^d(\mathbb{R})) \cong \mathbb{Z}$.

4. Universal covering space

Let $ex : \mathbb{C} \to \mathbb{C}^*$ denote the universal covering projection given by $ex(\alpha) = e^{2\pi\alpha\sqrt{-1}}$ for $\alpha \in \mathbb{C}$. For an odd interger $d \geq 3$, let $\tilde{\mathrm{P}}_n^d$ be the space defined by

$$\tilde{\mathrm{P}}_n^d = \{(f(z), \alpha) \in \mathrm{P}_n^d \times \mathbb{C} : R^d(f) = e^{2\pi\alpha\sqrt{-1}}\}.$$

There is a pullback diagram

$$\begin{array}{ccc} \tilde{\mathrm{P}}_3^d & \xrightarrow{q_2} & \mathbb{C} \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle ex} \\ \mathrm{P}_3^d & \xrightarrow{R^d} & \mathbb{C}^* \end{array} \quad , \tag{4.1}$$

where q_1 denotes the projection to the first factor.

Lemma 4.2. *Let $d \geq 3$ be an odd integer. Then $q_1 : \tilde{\mathrm{P}}_3^d \to \mathrm{P}_3^d$ is a universal covering projection.*

Proof. It follows from (2.3) and (3.2) that $R_*^d : \pi_1(\mathrm{P}_3^d) \xrightarrow{\cong} \pi_1(\mathbb{C}^*) = \mathbb{Z}$ is an isomorphism. Hence the assertion easily follows from the diagram chasing of (4.1). □

Remark 4.3. Let $d \geq 3$ be an odd integer. Then it is easy to see that $\pi_1(\mathrm{P}^d_3) = \mathbb{Z}$ acts on the $\tilde{\mathrm{P}}^d_3$ by $(f,\alpha)\cdot n = (f, \alpha+n)$ for $((f,\alpha),n) \in \tilde{\mathrm{P}}^d_3 \times \mathbb{Z}$, and that there is a homeomorphism $\tilde{\mathrm{P}}^d_3/\pi_1(\mathrm{P}^d_3) \cong \mathrm{P}^d_3$. Let $\tau_n \in \mathrm{Aut}(\tilde{\mathrm{P}}^d_3)$ denote the automorphism of $\tilde{\mathrm{P}}^d_3$ over P^d_3 given by $\tau_n(f,\alpha) = (f,\alpha+n)$ for $(f,\alpha) \in \tilde{\mathrm{P}}^d_3$, $n \in \pi_1(\mathrm{P}^d_3) = \mathbb{Z}$. This gives the action of $\pi_1(\mathrm{P}^d_3)$ on $H_*(\tilde{\mathrm{P}}^d_3;\mathbb{Z})$ by $n \cdot x = (\tau_n)_*(x)$ for $x \in H_*(\tilde{\mathrm{P}}^d_3;\mathbb{Z})$.

Although there is a homotopy equivalence $\mathrm{P}^d_3 \simeq \mathrm{P}^d_3(\mathbb{R})$, it is difficult to define a canonical stabilization map $\mathrm{P}^d_3 \to \mathrm{P}^{d+1}_3$. So we shall consider the composite of stabilization maps $s_{d+1}\circ s_d : \mathrm{P}^d_3(\mathbb{R}) \to \mathrm{P}^{d+2}_3(\mathbb{R})$. In this case, there is a canonical stabilization map $\hat{s}_d : \mathrm{P}^d_3 \to \mathrm{P}^{d+2}_3$ by adding points from the edge such that the following diagram is commutative

$$\begin{array}{ccc}
\mathrm{P}^d_3 & \xrightarrow{\hat{s}_d} & \mathrm{P}^{d+2}_3 \\
\cap\big\downarrow\simeq & & \cap\big\downarrow\simeq \\
\mathrm{P}^d_3(\mathbb{R}) & \xrightarrow{s_{d+1}\circ s_d} & \mathrm{P}^{d+2}_3(\mathbb{R})
\end{array} .$$

Since $\tilde{\mathrm{P}}^d_3$ is simply connected and $q_1 : \tilde{\mathrm{P}}^{d+2}_3 \to \mathrm{P}^{d+2}_3$ is a universal covering, there is a map $\tilde{s}_d : \tilde{\mathrm{P}}^d_3 \to \tilde{\mathrm{P}}^{d+2}_3$ such that the following diagram is commutative.

(4.4)
$$\begin{array}{ccc}
\tilde{\mathrm{P}}^d_3 & \xrightarrow{\tilde{s}_d} & \tilde{\mathrm{P}}^{d+2}_3 \\
{\scriptstyle q_1}\big\downarrow & & {\scriptstyle q_1}\big\downarrow \\
\mathrm{P}^d_3 & \xrightarrow{\hat{s}_d} & \mathrm{P}^{d+2}_3
\end{array}$$

Theorem 4.5. *$\tilde{s}_d : \tilde{\mathrm{P}}^d_3 \to \tilde{\mathrm{P}}^{d+2}_3$ is a homotopy equivalence up to dimension $[d/3]$.*

Proof. Since $\tilde{\mathrm{P}}^d_3$ and $\tilde{\mathrm{P}}^{d+2}_3$ are simply connected, it suffices to show that $\tilde{s}_d$ is a homology equivalence up to dimension $[d/3]$.

Let $\mathrm{P}^d \cong \mathbb{R}^d$ denote the space consisting of all monic real coefficients polynomials of degree d. Then $\tilde{\mathrm{P}}^d_3 \subset \mathrm{P}^d \times \mathbb{C}$ and we denote by $\tilde{\Sigma}^d$ the complement $\tilde{\Sigma}^d = \mathrm{P}^d \times \mathbb{C} - \tilde{\mathrm{P}}^d_3$. Since $\mathrm{P}^d \times \mathbb{C} \cong \mathbb{R}^{d+2}$, it follows from Alexander duality that there is a natural isomorphism

(4.6)
$$H^j(\tilde{\mathrm{P}}^d_3;\mathbb{Z}) \cong H^c_{d+1-j}(\tilde{\Sigma}^d) \quad \text{for any} \quad 1 \leq j \leq d,$$

where $\overline{X}$ denotes the one-point compactification of a locally compact space X and $H^c_*(X) = H_*(\overline{X};\mathbb{Z})$ the Borel-Moore homology group.

Let $\Sigma^d = \mathrm{P}^d - \mathrm{P}^d_3$ and let $A^d, B^d \subset \mathrm{P}^d \times \mathbb{C}$ denote the subspaces given by

$$\begin{cases} A^d = \Sigma^d \times \mathbb{C}, \\ B^d = \tilde{\Sigma}^d - A^d = \{(f,\alpha) \in \mathrm{P}^d_3 \times \mathbb{C} : R^d(f) \neq e^{2\pi\sqrt{-1}\alpha}\}. \end{cases}$$

The map $\tilde{s}_d$ naturally extends to the open embedding $\mathrm{P}^d \times \mathbb{H} \to \mathrm{P}^{d+2}$, and we also obtain open maps by restrictions and their extensions to one-point compactifications

$$\begin{cases} \tilde{s}_d : \tilde{\Sigma}^d \times \mathbb{H} \to \tilde{\Sigma}^{d+2}, \\ s_d^A : A^d \times \mathbb{H} \to A^{d+2}, \\ s_d^B : B^d \times \mathbb{H} \to B^{d+2}, \end{cases} \qquad \text{and} \qquad \begin{cases} \overline{s}_d : \overline{\tilde{\Sigma}^{d+2}} \to \overline{\tilde{\Sigma}^d} \wedge S^2, \\ \overline{s}_d^A : \overline{A^{d+2}} \to \overline{A^d} \wedge S^2, \\ \overline{s}_d^B : \overline{B^{d+2}} \to \overline{B^d} \wedge S^2. \end{cases}$$

Consider the commutative diagram

(4.7)
$$\begin{array}{ccccccccc} \longrightarrow & H^c_{k+2}(A^{d+2}) & \longrightarrow & H^c_{k+2}(\tilde{\Sigma}^{d+2}) & \longrightarrow & H^c_{k+2}(B^{d+2}) & \longrightarrow & \\ & \Big\downarrow{\scriptstyle (\overline{s}_d^A)_*} & & \Big\downarrow{\scriptstyle (\overline{s}_d)_*} & & \Big\downarrow{\scriptstyle (\overline{s}_d^B)_*} & & \\ \longrightarrow & H^c_k(A^d) & \longrightarrow & H^c_k(\tilde{\Sigma}^d) & \longrightarrow & H^c_k(B^d) & \longrightarrow & , \end{array}$$

where horizontal sequences are exact.

We shall call a homomorphism $H^c_{k+2}(X) \to H^c_k(Y)$ is *stable* for any $k \geq N$ if it is bijective when $k > N$ and surjective when $k = N$.

Remark the following 2 results.

Lemma 4.8. *The homomorphism $(\overline{s}_d^A)_* : H^c_{k+2}(A^{d+2}) \to H^c_k(A^d)$ is stable for any $k \geq N_1(d) = d + 1 - [d/3]$.*

Lemma 4.9. *The homomorphism $(\overline{s}_d^B)_* : H^c_{k+2}(B^{d+2}) \to H^c_k(B^d)$ is stable for any $k \geq N_2(d) = d - 1 - [d/3]$.*

We postpone the proof of the above 2 results and complete the proof of Theorem 4.5.

Consider the diagram (4.7). Then it follows from (4.8), (4.9) and 5-lemma that $(\overline{s}_d)_* : H^c_{k+1}(\tilde{\Sigma}^{d+1}) \to H^c_k(\tilde{\Sigma}^d)$ is stable for $k \geq \max(N_1(d), N_2(d)) = d + 1 - [d/3]$.

On the other hand, it follows from (4.6) that there is a commutative diagram

$$\begin{array}{ccc} H^j(\tilde{\mathrm{P}}_3^{d+2}, \mathbb{Z}) & \xrightarrow{\tilde{s}_d^*} & H^j(\tilde{\mathrm{P}}_3^{d}, \mathbb{Z}) \\ {\scriptstyle \text{A.D.}}\Big\downarrow\cong & & {\scriptstyle \text{A.D.}}\Big\downarrow\cong \\ H^c_{d+3-j}(\tilde{\Sigma}^{d+2}) & \xrightarrow{(\overline{s}_d)_*} & H^c_{d+1-j}(\tilde{\Sigma}^d) \end{array}$$

Because $d + 1 - j \geq d + 1 - [d/3]$ if and only if $j \leq [d/3]$, the induced homomorphism $\tilde{s}_d^* : H^j(\tilde{\mathrm{P}}_3^{d+1}, \mathbb{Z}) \to H^j(\tilde{\mathrm{P}}_3^{d}, \mathbb{Z})$ is bijective when $j < [d/3]$ and is a surjective when $j = [d/3]$. Hence it follows from the universal coefficient theorem that $\tilde{s}_d$ is a homology equivalence up to dimension $[d/3]$. This completes the proof of Theorem 4.5. $\square$

Now we can also give the proof of Theorem B using Theorem 4.5.

Proof of Theorem B. It follows from theorems (4.2), (4.5) and the diagram (4.4) that $s_{d+1} \circ s_d : \mathrm{P}_3^d(\mathbb{R}) \to \mathrm{P}_3^{d+2}(\mathbb{R})$ is a homotopy equivalence up to dimension $[d/3]$. Hence using Theorem 2.4, $j_3^d : \mathrm{P}_3^d(\mathbb{R}) \to \Omega S^2$ is a homotopy equivalence up to dimension $[d/3]$. □

It remains to prove (4.8) and (4.9).

Proof of Lemma 4.8. It follows from Alexander duality and its naturality that there is a commutative diagram

$$\begin{array}{ccc}
H^c_{k+2}(A^{d+2}) & \xrightarrow{(\overline{s}_d^A)_*} & H^c_k(A^d) \\
\sigma \downarrow \cong & & \sigma \downarrow \cong \\
H^c_k(\Sigma^{d+2}) & \rightarrow & H^c_{k-2}(\Sigma^d) \\
\text{A.D.} \downarrow \cong & & \text{A.D.} \downarrow \cong \\
H^{d-k+1}(\mathrm{P}_3^{d+2}) & \rightarrow & H^{d-k+1}(\mathrm{P}_3^d) \\
\uparrow \cong & & \uparrow \cong \\
H^{d-k+1}(\mathrm{P}_3^{d+1}(\mathbb{R})) & \xrightarrow{(s_{d+1}\circ s_d)^*} & H^{d-k+1}(\mathrm{P}_3^d(\mathbb{R}))
\end{array} .$$

Remark that $s_d : \mathrm{P}_3^d(\mathbb{R}) \to \mathrm{P}_3^{d+1}(\mathbb{R})$ is a homology equivalence up to dimension $[d/3]$ and that $H^*(\mathrm{P}_3^d(\mathbb{R}))$ is torsion free by Theorems 1.1 and 1.2. Because $d-k+1 \leq [d/3]$ if and only if $k \geq d+1-[d/3]$, the assertion easily follows from the above diagram. □

Proof of Lemma 4.9. Remark that there is a homeomorphism $B^d \cong \mathrm{P}_3^d \times (\mathbb{C}-\mathbb{Z})$. Note also that $\hat{s}_d : \mathrm{P}_3^d \to \mathrm{P}_3^{d+2}$ and $s_{d+1}\circ s_d : \mathrm{P}_3^d(\mathbb{R}) \to \mathrm{P}_3^{d+2}(\mathbb{R})$ naturally extend to the open embeddings and corresponding maps between one-point compactifications

$$\begin{cases} \hat{s}_d : \mathrm{P}_3^d \times \mathbb{H} \to \mathrm{P}_3^{d+2}, \\ s_{d+1}\circ s_d : \mathrm{P}_3^d(\mathbb{R}) \times \mathbb{H} \to \mathrm{P}_3^{d+2}(\mathbb{R}), \end{cases}$$

$$\text{and} \quad \begin{cases} \overline{\hat{s}_d} : \overline{\mathrm{P}_3^{d+2}} \to \overline{\mathrm{P}_3^d} \wedge S^2, \\ \overline{s_{d+1}\circ s_d} : \overline{\mathrm{P}_3^{d+2}(\mathbb{R})} \to \overline{\mathrm{P}_3^d(\mathbb{R})} \wedge S^2. \end{cases}$$

Moreover there is a commutative diagram

$$\begin{array}{ccc}
B^d \times \mathbb{H} & \xrightarrow{s_d^B} & B^{d+2} \\
\cong \downarrow & & \cong \downarrow \\
\mathrm{P}_3^d \times \mathbb{H} \times (\mathbb{C}-\mathbb{Z}) & \xrightarrow{\hat{s}_d \times \mathrm{id}} & \mathrm{P}_3^{d+2} \times (\mathbb{C}-\mathbb{Z})
\end{array} .$$

Hence it follows from Künneth formula that it suffices to show that the induced homomorphism $\overline{\hat{s}_d}_* : H^c_{k+2}(\mathrm{P}_3^{d+2}) \to H^c_k(\mathrm{P}_3^d)$ is stable for any $k \geq$

$N_2(d)$. Remark that the inclusion $\mathrm{P}_3^d \to \mathrm{P}_3^d(\mathbb{R})$ naturally extends to the open embedding $i'_d : \mathrm{P}_3^d \times \mathbb{R} \to \mathrm{P}_3^d(\mathbb{R})$ such that the induced homomorphism $\overline{i'_{d*}} : H^c_{k+1}(\mathrm{P}_3^d(\mathbb{R})) \xrightarrow{\cong} H^c_k(\mathrm{P}_3^d)$ is bijective for any k, because the inclusion $\mathrm{P}_3^d \to \mathrm{P}_3^d(\mathbb{R})$ is a deformation retract and the codimension of $\mathrm{P}_3^d \subset \mathrm{P}_3^d(\mathbb{R})$ is one (cf. [1]). Since $\mathrm{P}_3^d(\mathbb{R}) \subset \mathrm{P}^d \cong \mathbb{R}^d$ is an open subset, it is an open manifold of dimension d. So the Poincaré duality $H^c_k(\mathrm{P}_3^d(\mathbb{R})) \cong H^{d-k}(\mathrm{P}_3^d(\mathbb{R}))$ holds. Hence there is a commutative diagram

$$\begin{tikzcd}
H^c_{k+2}(\mathrm{P}_3^{d+2}) \arrow[r, "\overline{\hat{s}_{d*}}"] & H^c_k(\mathrm{P}_3^d) \\
H^c_{k+3}(\mathrm{P}_3^{d+2}(\mathbb{R})) \arrow[r] \arrow[u, "\overline{i'_{d+2*}}", "\cong"'] \arrow[d, "\mathrm{P.D.}"', "\cong"] & H^c_{k+1}(\mathrm{P}_3^d(\mathbb{R})) \arrow[u, "\overline{i'_{d*}}", "\cong"'] \arrow[d, "\mathrm{P.D.}"', "\cong"] \\
H^{d-k-1}(\mathrm{P}_3^{d+2}(\mathbb{R})) \arrow[r, "(s_{d+1}\circ s_d)^*"] & H^{d-k-1}(\mathrm{P}_3^d(\mathbb{R}))
\end{tikzcd}.$$

Then it follows from (1.2) that $(s_{d+1} \circ s_d)^*$ is bijective if $d-k-1 < [d/3]$ and is surjective if $d-k-1 = [d/3]$. Hence $\overline{\hat{s}_{d*}} : H^c_{k+2}(\mathrm{P}_3^{d+2}) \to H^c_k(\mathrm{P}_3^d)$ is stable for any $k \geq d-1-[d/3] = N_2(d)$. $\square$

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